Liouville Fields, Mahler Fields and Schanuel's Conjecture

By

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STATEMENT BY AUTHOR

Date: **K. Senthil Kumar**

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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List of Publications arising from the thesis

Journal

1. "Liouville numbers and Schanuel's conjecture", K. Senthil Kumar, R. Thangadurai and M. Waldschmidt, Arch. Math. (Basel), **2014**, 102 (1)., 59–70.

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Others

1. "Fields of Mahler's U-numbers", K. Senthil Kumar, communicated.

Date: **K. Senthil Kumar**

To my
Amma and Appa

We are what our thoughts have us;

so take care about what you think.

Words are secondary. Thoughts live;

they travel far...

Swami Vivekanada

Every good mathematician is at least half a

philosopher, and every good philosopher

is at least half a mathematician.

Friedrich Ludwig Gottlob Frege

To improve is to change;

To be perfect is to change often.

Winston Churchill

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K. Senthil Kumar

Contents

SYNOPSIS

The following summary contains some of my works on Liouville numbers and Mahler U-numbers during my stay at Harish-Chandra Research Institute as a research scholar.

In Section 2 (resp. in Section 3), we discuss some of the work done in the paper [24] (resp. in [26]). In Sections 1 and 4, we discuss some of the work done in the paper [25].

1. Baire's theorem and Liouville numbers

1.1 Definitions and some basic results. In this section, we discuss some consequences for Liouville numbers from Baire's Theorem. We start with the following:

Definition 0.0.1 (Liouville number). *A Liouville number is a real number ξ such that, for each integer $n \geq 1$, there exists a rational number p_n/q_n with $q_n \geq 2$ such that*

$$0 < \left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^n}\cdot$$

Following [17], any Liouville number is transcendental. The set of all Liouville numbers is denoted by $\mathbf{L}$. This set $\mathbf{L}$ is an uncountable, dense subset of $\mathbb{R}$ having Lebesgue measure 0.

Definition 0.0.2 (G_δ–subset). *A G_δ–subset of a topological space X is defined to be the countable intersection of dense open subsets of X.*

One can easily see that, $\mathbf{L}$ is a G_δ–subset of $\mathbb{R}$.

Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with at least two points.

Definition 0.0.3 (Nowhere locally constant). *A real function $f : \mathcal{I} \to \mathbb{R}$ is nowhere locally constant if, for every nonempty open interval $\mathcal{J}$ contained in $\mathcal{I}$, the restriction to $\mathcal{J}$ of f is not constant.*

Definition 0.0.4 (Algebraically dependent). *A set of complex numbers $x_1, \ldots, x_n$ $(n \geq 1)$ is said to be algebraically dependent if there exists a nonzero polynomial $P \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$.*

A subset S of $\mathbb{C}$ is said to be *algebraically independent* if no finite subset of S is algebraically dependent. Note that by our definition, the empty set $\emptyset$ is algebraically independent.

Definition 0.0.5 (Algebraic set). *A subset $X \subseteq \mathbf{R}^n$ is called an algebraic set if X is the zero locus of a finite set $\{P_1, P_2, \ldots, P_m\}$ of polynomials in n variables with real coefficients. We denote X by $Z(P_1, P_2, \ldots, P_m)$, if X is an algebraic set defined by the polynomials $P_1, P_2, \ldots, P_m$.*

The Baire's Theorem states the following:

Theorem (Baire's Theorem). *In a complete or locally compact space X, any G_δ-subset is dense.*

The main result of [4], which extended the earlier results of [21] and [23], deals with G_δ–subsets, and reads as follows:

Proposition 0.0.6 (Alniaçik–Saias). *Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior, G a G_δ–subset of $\mathbb{R}$ and $(f_n)_{n\geq 0}$ a sequence of real maps on $\mathcal{I}$, which are continuous and nowhere locally constant. Then*

$$\bigcap_{n\geq 0} f_n^{-1}(G)$$

is a G_δ–subset of $\mathcal{I}$.

As pointed out by the authors of [4], the proofs of several papers on this topic just reproduce the proof of Baire's Theorem. Here we use Baire's Theorem and deduce a number of consequences related with Liouville numbers in the next section.

Most results in this chapter are not specific to Liouville numbers: They hold with any G_δ–subset of $\mathbb{R}$ instead of $\mathbf{L}$. We pay attention to use the fact that $\mathbf{L}$ is a G_δ–subset of $\mathbb{R}$ rather than other Diophantine properties of $\mathbf{L}$.

The following Proposition 0.0.7 is a generalization of Proposition 0.0.6, where we can replace the interval $\mathcal{I}$ (resp. $\mathbb{R}$) in Proposition 0.0.6 by a topological space $\mathbf{X}$ (resp. an interval $\mathcal{J}$ of $\mathbb{R}$).

Proposition 0.0.7. *Let $\mathbf{X}$ be a complete, locally connected topological space, $\mathcal{J}$ an interval in $\mathbb{R}$ and $\mathcal{N}$ a set which is either finite or else countable. For each $n \in \mathcal{N}$, let G_n be a G_δ–subset of $\mathcal{J}$ and let $f_n : \mathbf{X} \to \mathcal{J}$ be a continuous function which is nowhere locally constant. Then $\bigcap_{n\in\mathcal{N}} f_n^{-1}(G_n)$ is a G_δ–subset of $\mathbf{X}$.*

Let $\mathbf{X}$ be a (nonempty) complete metric space without isolated point. Then the following Lemma says that, any G_δ–subset of X will remain a G_δ–subset after removing a countable subset from it.

Lemma 0.0.8. *Let $\mathbf{X}$ be a (nonempty) complete metric space without isolated point and let E be a G_δ–subset of $\mathbf{X}$. Let F be a countable subset of E. Then $E \setminus F$ is a G_δ–subset of $\mathbf{X}$.*

As a consequence we prove the following:

Corollary 0.0.9. *Let $\mathbf{X}$ be a (nonempty) complete metric space without isolated point and let E be a G_δ–subset of $\mathbf{X}$. Then E is uncountable.*

Note 0.0.10. *The Lemma 0.0.8 and Corollary 0.0.9 were quoted in [4].*

Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be non-empty open subsets of $\mathbb{R}$. Suppose that for each $j = 1, \dots, n$, let G_j be a G_δ–subset of $\mathcal{I}_j$. By Corollary 0.0.9, the sets G_j's are uncountable. Naturally one can ask the following question.

Question 1. *Can we find uncountably many n-tuples $(\xi_1, \dots, \xi_n) \in G_1 \times \cdots \times G_n$ such that $\xi_1, \dots, \xi_n$ are algebraically independent (over $\mathbb{Q}$)?*

We show that the answer to Question 1 is true. More precisely, we prove the following:

Lemma 0.0.11. *Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be non-empty open subsets of $\mathbb{R}$. For each $j = 1, \dots, n$, let G_j be a G_δ–subset of $\mathcal{I}_j$. Then there exists uncountably many $(\xi_1, \dots, \xi_n) \in G_1 \times \cdots \times G_n$ such that $\xi_1, \dots, \xi_n$ are algebraically independent (over $\mathbb{Q}$).*

1.2 Applications of Proposition 0.0.6 to Liouville numbers.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a nonconstant continuous function for each $n \in \mathbb{N}$. In this section, we are interested to study the following questions:

Question 2. Under what conditions on the functions $f_n : \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}$) which ensure the existence of uncountably many Liouville numbers ξ such that $f_n(\xi)$ is also a Liouville number for each $n \in \mathbb{N}$?

Question 3. Let $n \geq 2$ be an integer and let $P \in \mathbb{R}[X_1, X_2, \dots, X_n]$. Under what conditions on P which ensure that, the set $Z(P)$ contains uncountably many points $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ such that ξ_i's are Liouville numbers for $i = 1, 2, \dots, n$?

Since the set of Liouville numbers is a G_δ–subset in $\mathbb{R}$, a direct consequence of Proposition 0.0.6 and Corollary 0.0.9 answers Question 2 affirmatively:

Corollary 0.0.12. *Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $(f_n)_{n\geq 1}$ a sequence of real maps on $\mathcal{I}$, which are continuous and nowhere locally constant. Then there* *exists an uncountable subset E of $\mathcal{I} \cap \mathbf{L}$ such that $f_n(\xi)$ is a Liouville number for all $n \geq 1$ and all $\xi \in E$.*

We deduce some consequences of Corollary 0.0.12. By taking all the f_n's are same functions, say f, we get the following Corollary which says that, *nowhere locally constant functions takes uncountably many Liouville numbers to Liouville numbers.* More precisely,

Corollary 0.0.13. *Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $f : \mathcal{I} \to \mathbb{R}$ a continuous map which is nowhere locally constant. Then there exists an uncountable set of Liouville numbers $\xi \in \mathcal{I}$ such that $f(\xi)$ is a Liouville number.*

Simple examples of consequence of Corollary 0.0.13 are obtained with $\mathcal{I} = (0, +\infty)$ and either $f(x) = t - x$, for $t \in \mathbb{R}$, or else with $f(x) = t/x$, for $t \in \mathbb{R}^\times$, which yields Erdős [11] result on the decomposition of real number:

Theorem (Erdős)**.** *To each real number x $(x \neq 0)$ there correspond Liouville numbers ξ, η, ρ, ζ such that*

$$x = \xi + \eta = \rho\zeta.$$

Here after the “Erdős result ”we mean the above theorem “Theorem (Erdős)”.

We deduce also from Corollary 0.0.13 that *any positive real number t is the sum of two squares of Liouville numbers.* This follows by applying Corollary 0.0.13 with

$$\mathcal{I} = (0, \sqrt{t}) \quad \text{and} \quad f(x) = \sqrt{t - x^2}.$$

Similar examples can be derived from Corollary 0.0.13 involving transcendental functions: for instance, *any real number can be written $e^\xi + \eta$ with ξ and η Liouville numbers; any positive real number can be written $e^\xi + e^\eta$ with ξ and η Liouville numbers.*

Using the implicit function theorem, we deduce from Corollary 0.0.13 the following generalization of Erdős's result which answers the Question 3 for the case $n = 2$.

Corollary 0.0.14. *Let $P \in \mathbb{R}[X, Y]$ be an irreducible polynomial such that $(\partial/\partial X)P \neq 0$ and $(\partial/\partial Y)P \neq 0$. Assume that there exist two nonempty open intervals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbb{R}$ such that, for any $x \in \mathcal{I}$, there exists $y \in \mathcal{J}$ with $P(x, y) = 0$, and, for any $y \in \mathcal{J}$, there exists $x \in \mathcal{I}$ with $P(x, y) = 0$. Then there exist uncountably many pairs (ξ, η) of Liouville numbers in $\mathcal{I} \times \mathcal{J}$ such that $P(\xi, \eta) = 0$.*

The following Corollary 0.0.15 is a generalization of Corollary 0.0.14 to more than 2 variables, which answers the Question 3 for $n \geq 2$.

Corollary 0.0.15. *Let $\ell \geq 2$ and let $P \in \mathbb{R}[X_1, \dots, X_\ell]$ be an irreducible polynomial such that $(\partial/\partial X_1)P \neq 0$ and $(\partial/\partial X_2)P \neq 0$. Assume that there exist nonempty open subsets $\mathcal{I}_i$ of $\mathbb{R}$ ($i = 1, \dots, \ell$) such that, for any $i \in \{1, 2\}$ and any $(\ell - 1)$–tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell) \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_{i-1} \times \mathcal{I}_{i+1} \times \cdots \times \mathcal{I}_\ell$, there exists $x_i \in \mathcal{I}_i$ such that $P(x_1, \dots, x_\ell) = 0$. Then there exist uncountably many tuples $(\xi_1, \xi_2, \dots, \xi_\ell) \in \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell$ of Liouville numbers such that $P(\xi_1, \xi_2, \dots, \xi_\ell) = 0$.*

In [9], using a counting argument together with an application of Bézout's Theorem, E.B. Burger proved that *an irrational number t is transcendental if and only if there exist two $\mathbb{Q}$–algebraically independent Liouville numbers ξ and η such that $t = \xi + \eta$.* Extending the method of [9], we prove:

Proposition 0.0.16. *Let $F(X, Y) \in \mathbb{Q}[X, Y]$ be a nonconstant polynomial with rational coefficients and t a real number. Assume that there is an uncountable set of pairs of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$. Then the two following conditions are equivalent.*
(i) t is transcendental.
(ii) there exist two $\mathbb{Q}$–algebraically independent Liouville numbers (ξ, η) with $F(\xi, \eta) = t$.

One can easily see that: *For any nonzero rational number q and any Liouville number ξ, both the numbers $q+\xi$ and $q\xi$ are Liouville numbers.* Therefore, we have an uncountable subset of Liouville numbers ξ such that $q+\xi$ and $q\xi$ are Liouville numbers for each nonzero $q \in \mathbb{Q}$. One can ask the following question.

Question 4. *For any countable subset $\mathcal{E}$ of $\mathbb{R}$, can we find an uncountable subset S of $\mathbf{L}$ such that $t + \xi$ and $t\xi$ are Liouville numbers for all $t \in \mathcal{E}$ and $\xi \in \mathsf{S}$?*

As a consequence of Corollary 0.0.12, we answer Question 4 affirmatively. In fact, we prove a more general result.

Corollary 0.0.17. *Let $\mathcal{E}$ be a countable subset of $\mathbb{R}$. Then there exists an uncountable set of positive Liouville numbers ξ having simultaneously the following properties.*
(i) For any $t \in \mathcal{E}$, the number $\xi + t$ is a Liouville number.
(ii) For any nonzero $t \in \mathcal{E}$, the number ξt is a Liouville number.
(iii) Let $t \in \mathcal{E}$, $t \neq 0$. Define inductively $\xi_0 = \xi$ and $\xi_n = e^{t\xi_{n-1}}$ for $n \geq 1$. Then all

numbers of the sequence $(\xi_n)_{n\geq 0}$ are Liouville numbers.
(iv) For any rational number $r \neq 0$, the number ξ^r is a Liouville number.

In [19], É. Maillet gave a necessary and sufficient condition for a positive Liouville number ξ to have a p–th root (for a given positive integer $p > 1$) which is also a Liouville number: Among the convergents in the continued fraction expansion of ξ, infinitely many should be p–th powers. He provided explicit examples of Liouville numbers having a p–th root which is not a Liouville number.

Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ a continuous bijective map (hence φ is nowhere locally constant). Let $\psi : \mathcal{I} \to \mathcal{I}$ denotes the inverse bijective map of φ. For $n \in \mathbb{Z}$, we denote by φ^n the bijective map $\mathcal{I} \to \mathcal{I}$ defined inductively as usual: φ^0 is the identity, $\varphi^n = \varphi^{n-1} \circ \varphi$ for $n \geq 1$, and $\varphi^{-n} = \psi^n$ for $n \geq 1$. Using Corollary 0.0.12, we prove the following result which provides an uncountable set of elements ξ in $\mathcal{I}$ such that the orbit $\{\varphi^n(\xi) \mid n \in \mathbb{Z}\}$ consists only of Liouville numbers in $\mathcal{I}$.

Corollary 0.0.18. *Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ a continuous bijective map. Then the set of elements ξ in $\mathcal{I}$ such that the orbit $\{\varphi^n(\xi) \mid n \in \mathbb{Z}\}$ consists only of Liouville numbers in $\mathcal{I}$ is a G_δ–subset of $\mathcal{I}$, hence is uncountable.*

2. Liouville fields and some of its properties

2.1 Definitions and main results. By the result of Erdős [11], the set of sums $\xi + \eta$ (resp. $\xi\eta$) where $\xi, \eta \in \mathbf{L}$ coincide with $\mathbb{R}$ (resp. $\mathbb{R} \setminus \{0\}$). Our interest is to study the following:

Question 1. *Can we find a subset S of $\mathbf{L}$ which is closed under addition and multiplication and it is invariant under addition and multiplications by nonzero rational numbers?*

Since $\mathbb{Q} \bigcap \mathbf{L} = \emptyset$, therefore for any such S and for any $\xi \in S$, $q - \xi$ and $q\xi^{-1}$ does not belong to S for any $q \in \mathbb{Q}$, and this proves that no subset S of $\mathbf{L}$ will satisfy Question 1. It is therefore interesting to ask the following question.

Question 2. *Can we find a subset S of $\mathbf{L}$ such that, the numbers $\xi + \eta$ and $\xi\eta$ are either rational numbers or elements of S for all $\xi, \eta \in S$?*

We introduce the notion of a *Liouville set* which answer Question 2 affirmatively. In fact, these sets have a very nice algebraic structure namely, their union with the field of rational numbers forms a subfield of $\mathbf{R}$, which we call a Liouville field (which extend what was done by É. Maillet in Chap. III of [19]). Before going into the detail, we first give the definition of Liouville Set.

Definition 0.0.19 (Liouville set). *A Liouville set is a subset* S *of* $\mathbf{L}$ *for which there exists an increasing sequence* $(q_n)_{n\geq 1}$ *of positive integers having the following property: for any* $\xi \in \mathsf{S}$*, there exists a sequence* $\big(b_n\big)_{n\geq 1}$ *of positive rational integers and there exist two positive constants* κ_1 *and* κ_2 *such that, for any sufficiently large* n*,*

$$1 \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}}. \tag{1}$$

Remark 0.0.20. *In the definition of a Liouville set, it would not make a difference if we were requesting the inequalities* (1) *to hold for any* $n \geq 1$*; it suffices to change the constants* κ_1 *and* κ_2*. Also, if the assumption* (1) *is satisfied for some* κ_1*, then it is also satisfied with* κ_1 *replaced by any* $\kappa_1' > \kappa_1$*. Hence there is no loss of generality to assume* $\kappa_1 > 1$*. One could also add to* (1)*, the condition* $q_n \leq b_n$*.*

Definition 0.0.21 (Liouville field). *A Liouville field is a field of the form* $\mathbb{Q}(\mathsf{S})$*, where* S *is a Liouville set.*

Remark 0.0.22. *From the definitions, it follows that, for a real number* ξ*, the following conditions are equivalent:*

- ξ *is a Liouville number.*
- ξ *belongs to some Liouville set.*
- *The set* $\{\xi\}$ *is a Liouville set.*
- *The field* $\mathbb{Q}(\xi)$ *is a Liouville field.*

If we agree that the empty set is a Liouville set and that $\mathbb{Q}$ is a Liouville field, then any subset of a Liouville set is a Liouville set, and also (see Theorem 0.0.25) any subfield of a Liouville field is a Liouville field.

For any integer q and any real number $x \in \mathbb{R}$, we denote by $\|qx\| = \min_{m\in\mathbb{Z}} |qx - m|$, the distance of qx to the nearest integer.

Definition 0.0.23. *Let* $\underline{q} = (q_n)_{n\geq 1}$ *be an increasing sequence of positive integers and let* $\underline{u} = (u_n)_{n\geq 1}$ *be a sequence of positive real numbers such that* $u_n \to \infty$ *as* $n \to \infty$*. We* *defined by* $\mathsf{S}_{\underline{q},\underline{u}}$*, the set of* $\xi \in \mathbf{L}$ *for which there exist two positive constants* κ_1 *and* κ_2*, and, there exists a sequence* $\big(b_n\big)_{n\geq 1}$ *of positive rational integers such that*

$$1 \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 u_n}}.$$

Denote by $\underline{n}$ the sequence $\underline{u} = (u_n)_{n\geq 1} := (1, 2, 3, \dots)$ with $u_n = n$ $(n \geq 1)$. For any increasing sequence $\underline{q} = (q_n)_{n\geq 1}$ of positive integers, we denote by $\mathsf{S}_{\underline{q}}$ the set $\mathsf{S}_{\underline{q},\underline{n}}$. Hence, by definition, a Liouville set is a subset of $\mathsf{S}_{\underline{q}}$ for some $\underline{q}$.

The following lemma provides a large supply of Liouville sets.

Lemma 0.0.24. *For any increasing sequence* $\underline{q}$ *of positive integers and any sequence* $\underline{u}$ *of positive real numbers which tends to infinity, the set* $\mathsf{S}_{\underline{q},\underline{u}}$ *is a Liouville set.*

The sets $\mathsf{S}_{\underline{q},\underline{u}}$ have the following nice algebraic structure (compare with Theorem I_3 in [19]), namely:

Theorem 0.0.25. *For any increasing sequence* $\underline{q}$ *of positive integers and any sequence* $\underline{u}$ *of positive real numbers which tends to infinity, the set* $\mathbb{Q} \cup \mathsf{S}_{\underline{q},\underline{u}}$ *is a field.*

We denote this field by $\mathbb{Q}_{\underline{q},\underline{u}}$, and by $\mathbb{Q}_{\underline{q}}$ for the sequence $\underline{u} = \underline{n}$. From Theorem 0.0.25, it follows that a field is a Liouville field if and only if it is a subfield of $\mathbb{Q}_{\underline{q}}$ for some $\underline{q}$. Another consequence is that, if S is a Liouville set, then $\mathbb{Q}(\mathsf{S}) \setminus \mathbb{Q}$ is a Liouville set.

Note 0.0.26. *It is easily checked that if*

$$\liminf_{n\to\infty} \frac{u_n}{u'_n} > 0,$$

then $\mathbb{Q}_{\underline{q},\underline{u}}$ *is a subfield of* $\mathbb{Q}_{\underline{\mathrm{q}},\underline{\mathrm{u}}'}$*. In particular, if*

$$\liminf_{n\to\infty} \frac{u_n}{n} > 0,$$

then $\mathbb{Q}_{\underline{q},\underline{u}}$ *is a subfield of* $\mathbb{Q}_{\underline{q}}$*, while if*

$$\limsup_{n\to\infty} \frac{u_n}{n} < +\infty$$

then $\mathbb{Q}_{\underline{q}}$ *is a subfield of* $\mathbb{Q}_{\underline{q},\underline{u}}$*.*

Remark 0.0.27. *Since the field $\mathbb{Q}_{\underline{q},\underline{u}}$ does not contain irrational algebraic numbers, 2 is not a square in $\mathbb{Q}_{\underline{q},\underline{u}}$. For $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$, it follows that $\eta = 2\xi^2$ is an element in $\mathsf{S}_{\underline{q},\underline{u}}$ which is not the square of an element in $\mathsf{S}_{\underline{q},\underline{u}}$. According to [11], we can write $\sqrt{2} = \xi_1\xi_2$ with two Liouville numbers ξ_1, ξ_2, then the set $\{\xi_1, \xi_2\}$ is not a Liouville set.*

Remark 0.0.28. *Let N be a positive integer such that N cannot be written as a sum of two squares of an integer. Then, we show that, for $\varrho \in \mathsf{S}_{\underline{q},\underline{u}}$, the Liouville number $N\varrho^2 \in \mathsf{S}_{\underline{q},\underline{u}}$ is not the sum of two squares of elements in $\mathsf{S}_{\underline{q},\underline{u}}$. Therefore, if we write $N = \zeta^2 + \eta^2$ with two Liouville numbers ζ, η (which is possible by the result of Erdős [11]), then the set $\{\zeta, \eta\}$ is not a Liouville set.*

Remark 0.0.29. *If $R \in \mathbb{Q}(X_1, \dots, X_\ell)$ is a rational fraction and if $\xi_1, \dots, \xi_\ell$ are elements of a Liouville set S such that $\eta = R(\xi_1, \dots, \xi_\ell)$ is defined, then Theorem 0.0.25 implies that η is either a rational number or a Liouville number, and in the second case $\mathsf{S} \cup \{\eta\}$ is a Liouville set. For instance, if, in addition, R is not constant and $\xi_1, \dots, \xi_\ell$ are algebraically independent over $\mathbb{Q}$, then η is a Liouville number and $\mathsf{S} \cup \{\eta\}$ is a Liouville set. For $\ell = 1$, this yields:*

Corollary 0.0.30. *Let $R \in \mathbb{Q}(X)$ be a nonconstant rational fraction and let ξ be a Liouville number. Then $R(\xi)$ is a Liouville number and $\{\xi, R(\xi)\}$ is a Liouville set.*

Remark 0.0.31. *In [19], É Maillet introduced the definition of Liouville numbers corresponding to a given Liouville number. However this definition depends on the choice of a given sequence $\underline{q}$ giving the rational approximations. That is why we start with a sequence $\underline{q}$ instead of starting with a given Liouville number.*

By Theorem 0.0.25, we have an extension field $\mathbb{Q}_{\underline{q},\underline{u}}$ of $\mathbb{Q}$. A natural question is *whether $\mathbb{Q}_{\underline{q},\underline{u}}$ is a proper extension of $\mathbb{Q}$ (for any two sequences $\underline{q}$ and $\underline{u}$)*?; or, equivalently, *whether $\mathsf{S}_{\underline{q},\underline{u}}$ is a nonempty set (for any two sequences $\underline{q}$ and $\underline{u}$)*? We prove that the sets $\mathsf{S}_{\underline{q},\underline{u}}$ are either empty or else uncountable and we characterize such sets.

Theorem 0.0.32. *Let $\underline{q}$ be an increasing sequence of positive integers and $\underline{u} = (u_n)_{n\geq 1}$ be an increasing sequence of positive real numbers such that $u_{n+1} \geq u_n + 1$. Then the Liouville set $\mathsf{S}_{\underline{q},\underline{u}}$ is non empty if and only if*

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0.$$

Moreover, if the set $\mathsf{S}_{\underline{q},\underline{u}}$ is non empty, then it has the power of continuum.

The sets $\mathsf{S}_{\underline{q},\underline{u}}$ enjoy the following topological property:

Proposition 0.0.33. *The sets $\mathsf{S}_{\underline{q},\underline{u}}$ are not G_δ subsets of $\mathbb{R}$. If they are non empty, then they are dense in $\mathbb{R}$.*

2.2 Interrelations among the Liouville sets. In this section, we study some of the relations among the Liouville sets. We first define,

Definition 0.0.34. *For any two strictly increasing sequence of real numbers $\underline{x}$ and $\underline{y}$, their union $\underline{z}$ of $\underline{x}$ and $\underline{y}$ written as $\underline{x} \vee \underline{y}$, is defined to be an increasing sequence, whose terms belong either to $\underline{x}$ or to $\underline{y}$.*

Notation 0.0.35. *For any two sequences $\underline{x}$ and $\underline{y}$, the symbol $\underline{x} \prec \underline{y}$, we mean $\underline{x}$ is a subsequence of $\underline{y}$.*

The intersection of two nonempty Liouville sets may be empty. More generally, we show that there are uncountably many Liouville sets $\mathsf{S}_{\underline{q}}$ with pairwise empty intersections.

Proposition 0.0.36. *For $0 < \tau < 1$, define $\underline{q}^{(\tau)}$ as the sequence $(q_n^{(\tau)})_{n\geq 1}$ with*

$$q_n^{(\tau)} = 2^{n!\lfloor n^\tau \rfloor} \qquad (n \geq 1).$$

Then the sets $\mathsf{S}_{\underline{q}^{(\tau)}}$, $0 < \tau < 1$, are nonempty (hence uncountable) and pairwise disjoint.

If $\underline{q}' \prec \underline{q}$, one can easily see that $\mathsf{S}_{\underline{q}} \subseteq \mathsf{S}_{\underline{q}'}$. But, one may expect that $\mathsf{S}_{\underline{q}'}$ may often contain strictly $\mathsf{S}_{\underline{q}}$. Here is an example.

Proposition 0.0.37. *Define the sequences $\underline{q}$, $\underline{q}'$ and $\underline{q}''$ by*

$$q_n = 2^{n!}, \quad q_n' = q_{2n} = 2^{(2n)!} \quad \textit{and} \quad q_n'' = q_{2n+1} = 2^{(2n+1)!} \qquad (n \geq 1),$$

so that $\underline{q} = \underline{q}' \vee \underline{q}''$. Let λ_n be a sequence of positive integers such that

$$\lim_{n\to\infty} \lambda_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\lambda_n}{n} = 0.$$

Then the number

$$\xi := \sum_{n\geq 1} \frac{1}{2^{(2n-1)!\lambda_n}}$$

belongs to $\mathsf{S}_{\underline{q}'}$ *but not to* $\mathsf{S}_{\underline{q}}$*. Moreover*

$$\mathsf{S}_{\underline{q}} = \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}.$$

If $\underline{q} = \underline{q}' \vee \underline{q}''$, then by the observation preceding to the Proposition 0.0.37, we see that $\mathsf{S}_{\underline{q}} \subseteq \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. Proposition 0.0.36 gives an example where $\mathsf{S}_{\underline{q}'} \neq \emptyset$ and $\mathsf{S}_{\underline{q}''} \neq \emptyset$, while $\mathsf{S}_{\underline{q}}$ is the empty set. In the example from Proposition 0.0.37, the set $\mathsf{S}_{\underline{q}}$ coincides with $\mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. This is not always the case.

Proposition 0.0.38. *There exist two increasing sequences* $\underline{q}'$ *and* $\underline{q}''$ *of positive integers such that* $\mathsf{S}_{\underline{q}' \vee \underline{q}''}$ *is a strict nonempty subset of* $\mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$*.*

Note that, if we have a chain of integer sequences: $\underline{q}_1 \prec \underline{q}_2 \prec \cdots \prec \underline{q}_n \prec \cdots$, then correspondingly we have a decreasing chain of Liouville sets: $\mathsf{S}_{\underline{q}_1} \supseteq \mathsf{S}_{\underline{q}_2} \supseteq \cdots \supseteq \mathsf{S}_{\underline{q}_n} \supseteq \cdots$ We are interested to know whether the chain $\mathsf{S}_{\underline{q}_1} \supseteq \mathsf{S}_{\underline{q}_2} \supseteq \cdots$ is stationary after sometime. In general this is not true. For example, we prove that given any increasing sequence $\underline{q}$ of integers, there exists a subsequence $\underline{q}'$ of $\underline{q}$ such that $\mathsf{S}_{\underline{q}}$ is a strict subset of $\mathsf{S}_{\underline{q}'}$. More generally, we prove the following:

Proposition 0.0.39. *Let* $\underline{u} = (u_n)_{n \geq 1}$ *be a sequence of positive real numbers such that for every* $n \geq 1$*, we have* $\sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1}$*. Then any increasing sequence* $\underline{q}$ *of positive integers has a subsequence* $\underline{q}'$ *for which* $\mathsf{S}_{\underline{q}',\underline{u}}$ *strictly contains* $\mathsf{S}_{\underline{q},\underline{u}}$*. In particular, for any increasing sequence* $\underline{q}$ *of positive integers has a subsequence* $\underline{q}'$ *for which* $\mathsf{S}_{\underline{q}'}$ *is strictly contains* $\mathsf{S}_{\underline{q}}$*.*

3. Mahler fields and some of its properties

3.1 Introduction. We denote the set of Mahler U-numbers by $\mathbf{U}$, the set of Mahler U_m-numbers by $\mathbf{U}_m$, and the set of algebraic numbers by $\mathbb{A}$.

It is well known that algebraically dependent numbers belongs to the same Mahler class. It follows that, $\alpha\xi \in \mathbf{U}$ for all $\xi \in \mathbf{U}$ and for all nonzero $\alpha \in \mathbb{A}$. Naturally, one can ask the following:

Question 1. *Let* ξ *be a* U*-number such that* $\xi \in \mathbf{U}_m$ *for some* $m \geq 1$*. Is it true that* $\alpha\xi \in \mathbf{U}_m$ *for all nonzero* $\alpha \in \mathbb{A}$*?*

In [2], K. Alniacik proved that, if α is a nonzero algebraic number of degree m and $\xi \in \mathbf{U}_1$ is a strong Liouville number, then $\alpha\xi \in \mathbf{U}_m$. Therefore, in general, it is not true that for a fixed $m \geq 1$, $\xi \in \mathbf{U}_m$ implies $\alpha\xi \in \mathbf{U}_m$. By the same reasoning even the weaker question namely, *"whether $\alpha\xi \in \mathbf{U}_m$, for all nonzero α in a number field (i.e., a finite extension of $\mathbb{Q}$) K and for all $\xi \in \mathbf{U}_m$?"* does not hold. Therefore, one can modify the Question 1 as follows:

Question 2. *Given a number field K, can we find a proper subset Y of $\mathbf{U}$ such that $\alpha\xi \in Y$ for all $\xi \in Y$ and for all nonzero $\alpha \in K$?*

Let K be a number field. We define subsets $\mathsf{S}_{K,\underline{q},\underline{u}}$ of $\mathbf{U}$ (for any increasing sequence $\underline{q} = (q_n)_{n\geq 1}$ of positive integers, and for any unbounded sequence $\underline{u} = (u_n)_{n\geq 1}$ of positive real numbers), which answer Question 2 affirmatively. In fact, these subsets have a very nice algebraic structure: namely the union $K \bigcup \mathsf{S}_{K,\underline{q},\underline{u}}$ forms a subfield of $\mathbb{C}$ (see Theorem 0.0.45), which extends the results proved in [24]. We denote this field by $K_{\underline{q},\underline{u}}$, and we call $K_{\underline{q},\underline{u}}$, a *Mahler field* of degree m.

Once we have a field structure on $K_{\underline{q},\underline{u}}$, the next very beginning question is the following:

Question 3. *What are all the algebraic extensions of $K_{\underline{q},\underline{u}}$?*

We study about algebraic extensions of the field $K_{\underline{q},\underline{u}}$ and we prove that, numbers which are algebraic over $K_{\underline{q},\underline{u}}$ are either an algebraic number or a U-number (see Theorem 0.0.53)

For any given algebraic extension K of $\mathbb{Q}$, by Lindemann - Weierstrass theorem, we see that, $K \cap \exp(K^\times) = \emptyset$. It is interesting to ask the following question.

Question 4. *Is there any transcendental extension E of $\mathbb{Q}$ such that $E \cap \exp(E^\times) = \emptyset$?*

We prove that there are such extensions. In fact, we prove that, $e^\xi \notin K_{\underline{q},\underline{u}}$ for all $\xi \in K_{\underline{q},\underline{u}} \setminus \{0\}$ once if the sequence $(\frac{u_n}{\log q_n})_{n\geq 1}$ is unbounded, and therefore, the extensions $K_{\underline{q},\underline{u}}$ of $\mathbb{Q}$ answer Question 4 affirmatively (see Theorem 0.0.55).

Since $K_{\underline{q},\underline{u}}$ is a subfield of $\mathbb{C}$ which contains K, $P(\xi) \in K_{\underline{q},\underline{u}}$ for all $\xi \in K_{\underline{q},\underline{u}}$ and for all polynomial $P \in K[X]$. Our interest is to study whether the similar result hold for power series with algebraic coefficients. More generally, we study the following question.

Question 5. *Let $G(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series defined over K. What can we say about the transcendence nature of $G(\xi)$ for $\xi \in K_{\underline{q},\underline{u}}$?*

We prove that, under some conditions of the sequence $(c_n)_{n\geq 1}$, $G(\xi)$ is either an element of K or an element of $\mathbf{U}$ for all $\xi \in K_{\underline{q},\underline{u}}$ (see Theorem 0.0.56).

3.2 Definitions and main results.

Definition 0.0.40 (Height of an algebraic number)**.** *Let α be an algebraic number with the minimal polynomial $f(X) = a_0X^m + \cdots + a_m$ over $\mathbb{Z}$, and let $\{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_m\}$ be the set of all Galois conjugates of α. Then, the absolute logarithmic Weil height $h(\alpha)$, of α is given by*

$$h(\alpha) = \frac{1}{m}\left(\log a_0 + \sum_{i=1}^{m} \log \max\{1, |\alpha_i|\}\right).$$

while

$$H(\alpha) = H(f) = \max_{0 \le i \le m} |a_i|$$

is called the usual height of α (and of the polynomial f).

For more details on heights, we refer to Chapter 3 of [30].

Definition 0.0.41 (U_m-number)**.** *Let ξ be a complex number and let m be a positive integer. The number ξ is called a U-number of degree m or a U_m-number if there exists a sequence $(\alpha_n)_{n\ge1}$ of distinct algebraic numbers of degree m such that*

$$0 < |\xi - \alpha_n| < H(\alpha_n)^{-\omega_n} \tag{2}$$

with $\omega_n \to \infty$ as $n \to \infty$, and, for some $r \ge 1$, and for all sufficiently large n, we have

$$H(\alpha_n) < H(\alpha_{n+1}) < H(\alpha_n)^{r\omega_n}. \tag{3}$$

Remark 0.0.42. *In general, all the α_n's in the Definition 0.0.41 of U_m-number need not lie in a fixed number field. We are interested to study about those U-numbers for which all the α_n's are in some fixed number field K.*

Throughout this section, K denote a number field of degree m, $\underline{q} := (q_n)_{n\ge1}$ denote a strictly increasing sequence of positive integers, and $\underline{u} := (u_n)_{n\ge1}$ denote a sequence of positive real numbers such that $\lim_{n\to\infty} u_n = \infty$.

Definition 0.0.43 ($\mathbb{A}_t$-sequence)**.** *An $\mathbb{A}_t$-sequence for an U-number ξ over K with respect to the sequences $\underline{q}$ and $\underline{u}$ is a sequence $(\alpha_n)_{n\ge1}$ of distinct elements of K with same degree t over $\mathbb{Q}$, and two positive constants κ_1 and κ_2 such that for sufficiently large n, we have*

$$H(\alpha_n) \le q_n^{\kappa_1} \text{ and } 0 < |\xi - \alpha_n| \le \frac{1}{q_n^{\kappa_2 u_n}}. \tag{4}$$

Notation 0.0.44. *We denote by* $\mathsf{S}_{K,\underline{q},\underline{u}}$*, the set of* U*-numbers having an* $\mathbb{A}_t$*-sequence over* K *with respect to the sequences* $\underline{q}$ *and* $\underline{u}$*, and, for some positive integer* t *(note that* $t \le m$*). Let* $K_{\underline{q},\underline{u}} := K \cup \mathsf{S}_{K,\underline{q},\underline{u}}$*. The situation when* $\underline{u} = \underline{n}$*, we denote* $\mathsf{S}_{K,\underline{q},\underline{u}}$ *by* $\mathsf{S}_{K,\underline{q}}$ *and* $K_{\underline{q},\underline{u}}$ *by* $K_{\underline{q}}$*. Also, when* $K = \mathbb{Q}$*, we denote* $\mathsf{S}_{K,\underline{q},\underline{u}}$ *by* $\mathsf{S}_{\underline{q},\underline{u}}$*; these sets were first studied in [24] and the following theorem generalizes a result obtained in [24].*

Theorem 0.0.45. *For any increasing sequence* $\underline{q}$ *of positive integers and any sequence* $\underline{u}$ *of positive real numbers which tends to infinity, the set* $K_{\underline{q},\underline{u}}$ *is a subfield of* $\mathbb{C}$*.*

The fields $K_{\underline{q},\underline{u}}$ are called Mahler fields of degree m.

Remark 0.0.46. *Note that if a complex number* ξ *belongs to a Mahler field* $K_{\underline{q},\underline{u}}$ *of degree* m*, then it satisfies one of the following statements:*

1. *It is either an algebraic number of degree at most* m*, or*

2. *a* U*-number of degree at most* m*.*

Example 0.0.47. *Let* $K = \mathbb{Q}(\mathfrak{i})$*, where* $\mathfrak{i} = \sqrt{-1}$*. Then* $K_{\underline{q},\underline{u}} = \{\xi + \mathfrak{i}\eta \mid \quad \xi, \eta \in \mathbb{Q}_{\underline{q},\underline{u}}\}$*, and hence* $[K_{\underline{q},\underline{u}} : \mathbb{Q}_{\underline{q},\underline{u}}] = [K : \mathbb{Q}] = 2$*.*

For any non-constant polynomial $P(X) \in \mathbb{Q}[X]$, and any $\xi \in \mathbf{U}_m$ (with $m \ge 1$), is it true that $P(\xi) \in \mathbf{U}_m$? This result is true for $m = 1$; for $m > 1$, in general, it is not true. For example, for each prime number p, set $\eta_p = \frac{1}{p} + \sum_{n\ge1} \frac{1}{10^{n!}}$. Then one can easily see that $\eta_p^{\frac{1}{m}}$ is a $\mathbf{U}_m$-number. Therefore, if we set $\xi_p = \eta_p^{\frac{1}{m}}$ and $P(X) = X^m$, then $\xi_p \in \mathbf{U}_m$ but $P(\xi_p) = \eta_p \in \mathbf{U}_1$, see [6, p. 91] for more details. However, we prove the following result.

Corollary 0.0.48. *Let* ξ *be a Liouville number and let* K *be a number field of degree* m*. Then for any non-constant polynomial* $P(X) \in K[X]$*, we have* $P(\xi) \in \bigcup_{k=1}^{m} \mathbf{U}_k$*.*

By the result of Erdős [11] that nonzero real number can be written as a product of two Liouville numbers, we have, for any nonzero real algebraic number α, there exists a Liouville number ξ such that $\alpha\xi \in \mathbf{L}$. But, if α and ξ satisfies some additional conditions, then we prove that $\alpha\xi \in \mathbf{U}_{\deg(\alpha)}$.

Corollary 0.0.49. *Let* m *be a positive integer. Let* $\xi \in \mathbf{L}$ *and let* α *be an algebraic number of degree* m *over* $\mathbb{Q}$*. If there exists a sequence* $\left(\dfrac{a_n}{b_n}\right)_{n\ge1}$ *of rational numbers which satisfies*

(2) *and* (3) *such that for some* $\kappa > 0$, $H\left(\dfrac{\alpha a_n}{b_n}\right) \geq a_n^{\kappa}$ *for all sufficiently large* n, *then* $\alpha\xi \in \mathbf{U}_m$.

In [16], LeVeque proved that the sets $\mathbf{U}_m$ are nonempty for each positive integer m. The following example gives another proof of this fact.

Example 0.0.50. *Let* α *be an mth root of* 2 *and let* $\xi = 1 + \sum_{n=1}^{\infty} \frac{1}{3^{n!}}$. *Then,* $\alpha\xi \in \mathbf{U}_m$.

Remark 0.0.51. *By the technique employed in the proof of Theorem* 0.0.45, *we observe that, for any* U*-number* ξ *of degree* m *and for any nonzero algebraic number* α *of degree* n *over* $\mathbb{Q}$, *the* U*-number* $\alpha\xi$ *has degree at most* mn, *that is,* $\alpha\xi \in \bigcup_{k=1}^{mn} \mathbf{U}_k$.

Remark 0.0.52. *By Theorem* 2 *in [24], the set* $\mathsf{S}_{\underline{q},\underline{u}}$ *is not empty if and only if* $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0$. *Moreover, if the set* $\mathsf{S}_{\underline{q},\underline{u}} \neq \emptyset$, *then it has the power of continuum. Since* $\mathsf{S}_{\underline{q},\underline{u}} \subseteq \mathsf{S}_{K,\underline{q},\underline{u}}$, *we see that* $\mathsf{S}_{K,\underline{q},\underline{u}}$ *has the power of continuum, if* $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0$. *It follows that, if an algebraic number* α *of degree* m *over* $\mathbb{Q}$ *together with an element* $\xi \in S_{\underline{q},\underline{u}}$ *satisfies Corollary* 0.0.49 *then* $S_{K,\underline{q},\underline{u}} \bigcap \mathbf{U}_m \neq \emptyset$, *where* $K = \mathbb{Q}(\alpha)$.

By Theorem 0.0.45, $K_{\underline{q},\underline{u}}$ is a field. Our next interest is going to study about algebraic extensions of $K_{\underline{q},\underline{u}}$. Since algebraic extensions of $K_{\underline{q},\underline{u}}$ are separable, all the finite extensions of $K_{\underline{q},\underline{u}}$ are simple extensions. Thus, any finite extensions of $K_{\underline{q},\underline{u}}$ is of the form $K_{\underline{q},\underline{u}}(\eta)$ for some complex number η which is algebraic over $K_{\underline{q},\underline{u}}$. We prove that such an η is either an algebraic number or a U-number. More precisely, we prove the following:

Theorem 0.0.53. *A complex number* η *is algebraic over a Mahler field* $K_{\underline{q},\underline{u}}$ *of degree* m *if and only if it is either*

(i) an algebraic number, or

(ii) a U*-number for which there exists a sequence* $(\beta_i)_{i\geq 1}$ *of complex numbers which are algebraic of same degree over* K *with minimal polynomial* $P_i(X) \in K[X]$ *which converges coefficient-wise to a polynomial* $P(X) \in K_{\underline{q},\underline{u}}[X]$ *such that for all sufficiently large* i,

$$|\eta - \beta_i| \leq \frac{1}{H(\beta_i)^{\omega_i}},$$

with $\lim_{i\to\infty} \omega_i = \infty$.

Remark 0.0.54. *By Theorem 0.0.53, the algebraic closure of a Mahler field $K_{\underline{q},\underline{u}}$ is a subset of $\mathbb{A} \cup \mathbf{U}$.*

Regarding Question 4, we prove the following:

Theorem 0.0.55. *Let $K_{\underline{q},\underline{u}}$ be a Mahler field. Suppose that the sequences $\underline{q} = (q_n)_{n\geq 1}$ and $\underline{u} = (u_n)_{n\geq 1}$ satisfying the property that, the sequence $\underline{v} = (v_n)_{n\geq 1}$ defined by $v_n = \frac{u_n}{\log q_n}$, is unbounded. Then for all $\xi \in K_{\underline{q},\underline{u}}$, we have $e^{\xi} \notin K_{\underline{q},\underline{u}}$. As a consequence, $K_{\underline{q},\underline{u}} \cap \exp(K^{\times}_{\underline{q},\underline{u}}) = \emptyset$.*

Let K be a number field of degree m and let $\underline{c} = (c_n)_{n\geq 1}$ be a sequence of non-zero elements of K such that $H(c_n) \leq {q_n}^{\kappa}$ for some real number κ with $0 < \kappa < 1$. Let $G(z) = \sum_{n=1}^{\infty} \left(\frac{c_n}{q_n}\right) z^{n-1}$ be a power series over K. In [13], G. Karadeniz Gözeri studied the transcendence of the values of $G(z)$ for Liouville numbers. We study about the transcendence of the values of the power series $G(z)$ for $\xi \in S_{K,\underline{q},\underline{u}}$. The following theorem generalize the Theorem 5 in [13].

Theorem 0.0.56. *Let $K_{\underline{q},\underline{u}}$ be a Mahler field of degree m. Suppose that the sequences $\underline{q}, \underline{u}$ and $\underline{c}$ satisfies the following conditions:*

(i) $\liminf_{n\to\infty} \dfrac{\log q_{n+1}}{\log q_n} = \lambda > 1$,

(ii) $\limsup_{n\to\infty} \dfrac{\log q_{n+1}}{\log q_n} = \infty$,

(iii) $\lim_{n\to\infty} \dfrac{\log q_n}{n} = \infty$.

(iv) For all sufficiently large n, $H(c_n) \leq q_n^{\kappa}$ for some real number κ with $0 < \kappa < 1$.

Then, for all $\xi \in S_{K,\underline{q},\underline{u}}$, either $G(\xi) \in K$ or $G(\xi) \in \bigcup_{i=1}^{m} \mathbf{U}_i$.

4. Schanuel's conjecture and U-numbers

4.1 Preliminaries

Definition 0.0.57 (Transcendence degree)**.** *For any subfield L of $\mathbb{C}$, the transcendence degree of L (over $\mathbb{Q}$) is defined to be the cardinality of a maximal algebraically independent subset of L. We denote the transcendence degree of the field L by $\mathrm{trdeg}_{\mathbb{Q}}\mathsf{L}$.*

The famous Schanuel's conjeture states the following:

Conjecture (Schanuel's Conjecture)**.** *Given $\mathbb{Q}$–linearly independent complex numbers $x_1, \dots, x_n$, the transcendence degree over $\mathbb{Q}$ of the field*

$$\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) \tag{5}$$

is at least n.

One may ask whether the transcendence degree is at least $n+1$ when the following additional assumption is made: For each $i = 1, \dots, n$, one at least of the two numbers x_i, e^{x_i} is a U-number. We first proceed the case where: for each $i = 1, \dots, n$, one at least of the two numbers x_i, e^{x_i} is a Liouville number.

We show that for each pair of integers (n, m) with $n \geq m \geq 1$, there exist uncountably many tuples $\xi_1, \dots, \xi_n$ consisting of $\mathbb{Q}$–linearly independent real numbers, such that the numbers $\xi_1, \dots, \xi_n, e^{\xi_1}, \dots, e^{\xi_n}$ are all Liouville numbers, and the transcendence degree of the field (5) is $n+m$. For a survey on algebraic independence results related with Liouville numbers we refer to [29].

4.2 Main results

Theorem 0.0.58. *Let $n \geq 1$ and $1 \leq m \leq n$ be given integers. Then there exist uncountably many n-tuples $(\alpha_1, \dots, \alpha_n) \in \mathbf{L}^n$ such that $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, $e^{\alpha_i} \in \mathbf{L}$ for all $i = 1, 2, \dots, n$ and*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n}) = n + m.$$

Remark 0.0.59. *Theorem 0.0.58, is tight when $n = 1$: The result does not hold for $m = 0$. Indeed, since the set of α in $\mathbf{L}$ such that α and e^{α} are algebraically dependent over $\mathbb{Q}$ is countable, one cannot get uncountably many $\alpha \in \mathbf{L}$ such that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha, e^{\alpha}) = 1$.*

By an application of Theorem 0.0.58, we have the following analogous result for U-numbers:

Theorem 0.0.60. *Let $n \geq 1$ and $1 \leq m \leq n$ be given integers. Then there exist uncountably many n-tuples $(\alpha_1, \dots, \alpha_n)$ of U-numbers of degree > 1 such that $e^{\alpha_i} \in \mathbf{U} \smallsetminus \mathbf{U}_1$ for all $i = 1, 2, \dots, n$ and*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n}) = n + m.$$

Chapter 1

Some Consequences of Baire's Theorem to Liouville Numbers

1.1 Introduction

A complex number α is said to be an **algebraic number** if there exists a nonzero polynomial $P(X) \in \mathbb{Z}[X]$ such that $P(\alpha) = 0$. We denote the set of algebraic numbers by $\mathbb{A}$. The rational numbers are clearly algebraic numbers (for example, every rational number a/b is a root of the polynomial $bX - a$ over $\mathbb{Z}$). There are algebraic irrational numbers (for example, $\sqrt{2}$). This set $\mathbb{A}$ contains some complex numbers which are not real numbers (for example, the roots of the polynomial equation $X^2 + 1 = 0$). A complex number that is not algebraic is called a **transcendental number.**

By definition, every algebraic number is a complex number. A natural question is *whether every complex number is an algebraic number?*. This question was left open until 1844, and the first proof of the existence of transcendental number was given by J. Liouville in 1844. He observed that real algebraic numbers cannot be too "well approximated by rational numbers" in the following sense.

Theorem 1.1.1 (Liouville, [17])**.** *For any real algebraic number α of degree $n > 1$, there exist a positive constant $C = C(\alpha)$ such that the inequality $|\alpha - \frac{a}{b}| > \frac{C}{b^n}$ holds for all rational numbers $\frac{a}{b}$.*

Using this result, Liouville was able to exhibit an explicit example of transcendental numbers. Indeed, by the above theorem, if ξ is a real number such that for each positive integer n, there exists a rational number a_n/b_n $(b_n \geq 2)$ with

$$0 < \left| \xi - \frac{a_n}{b_n} \right| < \frac{1}{b_n^n}, \tag{1.1}$$

then ξ is a transcendental number. Any such number ξ is commonly called, a *Liouville number.* More precisely:

Definition 1.1.2. *A* **Liouville number** *is a real number ξ such that, for each positive integer n, there exists a rational number p_n/q_n with $q_n \geq 2$ and*

$$0 < \left| \xi - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^n}. \tag{1.2}$$

For example, Liouville himself showed that for any positive integer $a \geq 2$, the number $\sum_{n=1}^{\infty} a^{-n!}$ is a Liouville number, and hence a transcendental number. The set of all Liouville numbers are denoted by $\mathbf{L}$. This set $\mathbf{L}$ is an uncountable, dense subset of $\mathbb{R}$ having Lebesgue measure 0, and

$$\mathbf{L} = \bigcap_{n \geq 1} U_n \quad \text{with} \quad U_n = \bigcup_{q \geq 2} \bigcup_{p \in \mathbb{Z}} \left(\frac{p}{q} - \frac{1}{q^n}\, , \, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\}.$$

Each U_n is dense, since each $p/q \in \mathbb{Q}$ belongs to the closure of U_n. Hence $\mathbf{L}$ is a countable intersection of dense open subsets of $\mathbb{R}$.

Definition 1.1.3. *A G_δ–***subset** *of a topological space X is defined to be the countable intersection of dense open subsets of X.*

Thus, $\mathbf{L}$ is a G_δ–subset of $\mathbb{R}$. The classic theorem of R. L. Baire about G_δ–subsets in a complete or locally compact space X states the following:

Theorem (Baire's Theorem)**.** *In a complete or locally compact space X, any G_δ-subset is dense.*

A G_δ–subset is also defined as a set whose complement is **meager**. In our case, this complement $\mathbf{L}^{\mathsf{c}}$ is the set of non–Liouville numbers

$$\mathbf{L}^{\mathsf{c}} = \Big\{ x \in \mathbb{R} \;\; | \;\; \text{there exists } \kappa > 0 \text{ such that} \\ \left| x - \frac{p}{q} \right| \geq \frac{1}{q^\kappa} \text{ for all } \frac{p}{q} \in \mathbb{Q} \text{ with } q \geq 2 \Big\},$$

which has full Lebesgue measure.

In [11], P. Erdős proved that *every real number* t *can be written as* $t = \xi + \eta$ *for some Liouville numbers* ξ *and* η. He gave two proofs of this result. The first one is elementary and constructive: he splits the binary expansion of t into two parts, giving rise to binary expansions of two real numbers ξ and η, whose sum is t. The splitting is done in such a way that both binary expansions of ξ and η have long sequences of 0's. The second proof is not constructive, as it relies on Baire's Theorem. In the same paper, P. Erdős gives also in the same way two proofs, a constructive one and another depending on Baire's Theorem, that *every non-zero real number* t *can be written as* $t = \xi\eta$, *where* ξ *and* η *are in* $\mathbf{L}$. From each of these proofs, it follows that there exist uncountably many such representations $t = \xi + \eta$ (resp. $t = \xi\eta$) for a given t. Many authors extended this result in various ways: G.J. Rieger in [21], W. Schwarz in [23], K. Alniaçik in [3], K. Alniaçik and É. Saias in [4], E.B. Burger in [8, 9]. In [8], Burger extended Erdős's result to a very large class of functions, including $f(x, y) = x + y$ and $g(x, y) = xy$. Here we use Baire's theorem and deduce many consequences related with Liouville numbers.

In Section 1.2, we generalize the main result in [4]. In the same section, we prove that *G_δ–subsets of complete metric spaces without isolated points are uncountable.* In Section 1.3, we prove that *nowhere locally constant continuous functions takes uncountably many Liouville numbers to Liouville numbers.* In Section 1.4, we generalize some of the result obtained in [8]. In Section 1.4, we also study about zeros (whose co-ordinates are all Liouville numbers) of polynomials in more variables. In Section 1.5, we prove that *for any countable subset* $\mathcal{E}$ *of* $\mathbb{R}$, *there exists an uncountable subset* S *of* $\mathbf{L}$ *such that* $t + \xi$ *and* $t\xi$ *are Liouville numbers for all* $t \in \mathcal{E}$ *and* $\xi \in \mathsf{S}$.

Most results in this chapter are not specific to Liouville numbers. They hold with any G_δ–subset of $\mathbb{R}$ instead of $\mathbf{L}$. We pay attention to use the fact that $\mathbf{L}$ is a G_δ–subset of $\mathbb{R}$ rather than other Diophantine properties of $\mathbf{L}$.

1.2 Nowhere locally constant functions and G_δ–subsets

Throughout this chapter, $\mathcal{I}$ will denote an interval of $\mathbb{R}$ with nonempty interior.

Definition 1.2.1. *A function* $f : \mathcal{I} \to \mathbb{R}$ *is* **nowhere locally constant** *if, for every nonempty open interval* $\mathcal{J}$ *contained in* $\mathcal{I}$, *the restriction to* $\mathcal{J}$ *of* f *is not constant.*

The main result of [4], which extends the earlier results of [21] and [23], deals with G_δ–subsets, and reads as follows:

Proposition 1.2.2 (Alniaçik–Saias)**.** *Let G be a G_δ–subset of $\mathbb{R}$, and $(f_n)_{n\geq 0}$ a sequence of functions from $\mathcal{I}$ to $\mathbb{R}$, which are continuous and nowhere locally constant. Then*

$$\bigcap_{n\geq 0} f_n^{-1}(G)$$

is a G_δ–subset of $\mathcal{I}$.

As pointed out by the authors of [4], the proofs of several papers on this topic just reproduce the proof of Baire's Theorem. Here we use Baire's Theorem and deduce a number of consequences related with Liouville numbers in subsequent sections.

The following theorem generalizes Proposition 1.2.2.

Theorem 1.2.3. *Let $\mathbf{X}$ be a complete, locally connected topological space, $\mathcal{J}$ an interval in $\mathbb{R}$ and $\mathcal{N}$ a countable set. For each $n \in \mathcal{N}$, let G_n be a G_δ–subset of $\mathcal{J}$ and let $f_n : \mathbf{X} \to \mathcal{J}$ be a continuous function which is nowhere locally constant. Then*

$$\bigcap_{n\in\mathcal{N}} f_n^{-1}(G_n)$$

is a G_δ–subset of $\mathbf{X}$.

Taking $X = \mathbb{R}$, $\mathcal{J} = \mathbb{R}$, and f_n is the identity function for each n, we recover Baire's Theorem for $\mathbb{R}$.

Proof. Since $\mathcal{N}$ is at most countable, it is enough to prove for any $n \in \mathcal{N}$ that $f_n^{-1}(G_n)$ is a G_δ–subset of $\mathbf{X}$.

Since f_n is continuous, $f_n^{-1}(G_n)$ is a countable intersection of open sets in $\mathbf{X}$. To prove it is a G_δ–subset of $\mathbf{X}$, we need to prove that $f_n^{-1}(G_n)$ is dense in $\mathbf{X}$. Let V be a connected open subset of $\mathbf{X}$. Since f_n is continuous, $f_n(V)$ is a connected subset of $\mathcal{J}$. Since f_n is nowhere locally constant, $f_n(V)$ consists of at least two elements. Therefore, there exists an interval $(a, b) \subset \mathcal{J}$ with non-empty interior such that $(a, b) \subset f_n(V)$. Since G_n is a dense subset of $\mathcal{J}$, $(a, b) \cap G_n \neq \emptyset$ and hence $V \cap f_n^{-1}(G_n) \neq \emptyset$, which completes the proof of Theorem 1.2.3. $\square$

1.2.1 G_δ–subsets are uncountable

In this section, we prove that G_δ–subsets of complete metric spaces without isolated points are uncountable.

Lemma 1.2.4. *Let* $\mathbf{X}$ *be a (nonempty) complete metric space without isolated point and let* E *be a* G_δ*–subset of* $\mathbf{X}$*. Let* F *be a countable subset of* E*. Then* $E \setminus F$ *is a* G_δ*–subset of* $\mathbf{X}$*.*

Proof. We have

$$E \setminus F = \bigcap_{y \in F} E \setminus \{y\}$$

where each $E \setminus \{y\}$ is a G_δ–subset of $\mathbf{X}$ (since $\mathbf{X}$ has no isolated point). □

Theorem 1.2.5. *Let* $\mathbf{X}$ *be a (nonempty) complete metric space without isolated point and let* E *be a* G_δ*–subset of* $\mathbf{X}$*. Then* E *is uncountable.*

Proof. If E were countable then by Lemma 1.2.4, $E \setminus E$ is a G_δ–subset of $\mathbf{X}$, which is absurd. □

Note 1.2.6. *The Lemma 1.2.4 and the Theorem 1.2.5 were quoted in [4].*

1.3 Liouville numbers and nowhere locally constant functions

Since the set of Liouville numbers is a G_δ–subset in $\mathbb{R}$, a direct consequence of Proposition 1.2.2 and Theorem 1.2.5 is the following:

Theorem 1.3.1. *Let* $(f_n)_{n\geq 1}$ *be a sequence of functions from* $\mathcal{I}$ *to* $\mathbb{R}$*, which are continuous and nowhere locally constant. Then there exists an uncountable subset* E *of* $\mathcal{I} \cap \mathbf{L}$ *such that* $f_n(\xi)$ *is a Liouville number for all* $n \geq 1$ *and all* $\xi \in E$*.*

We deduce some consequences of Theorem 1.3.1. We first consider the special case, where all the f_n's are the same.

Corollary 1.3.2. *Let* $\varphi : \mathcal{I} \to \mathbb{R}$ *be a continuous map which is nowhere locally constant. Then there exists an uncountable set of Liouville numbers* $\xi \in \mathcal{I}$ *such that* $\varphi(\xi)$ *is a Liouville number.*

1(1) [1]Simple examples of consequences of Corollary 1.3.2 are obtain with $\mathcal{I} = (0, +\infty)$ and either $\varphi(x) = t - x$, for $t \in \mathbb{R}$, or else with $\varphi(x) = t/x$, for $t \in \mathbb{R}^\times$, which yield Erdős above mentioned result on the decomposition of any real number (resp. any nonzero real number) t as a sum (resp. a product) of two Liouville numbers.

1(2) We deduce also from Corollary 1.3.2 that any positive real number t is the sum of two squares of Liouville numbers. This follows by applying Corollary 1.3.2 with

$$\mathcal{I} = (0, \sqrt{t}) \quad \text{and} \quad \varphi(x) = \sqrt{t - x^2}.$$

1(3) Similar examples can be obtain from Corollary 1.3.2 involving transcendental functions: for instance, any real number can be written $e^\xi + \eta$ for some Liouville numbers ξ and η; any positive real number can be written $e^\xi + e^\eta$ for some Liouville numbers ξ and η.

Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ be a continuous bijective map (hence φ is nowhere locally constant). Let $\psi : \mathcal{I} \to \mathcal{I}$ denote the inverse bijective map of φ. For $n \in \mathbb{Z}$, we denote by φ^n the bijective map $\mathcal{I} \to \mathcal{I}$ defined inductively as usual: φ^0 is the identity, $\varphi^n = \varphi^{n-1} \circ \varphi$ for $n \geq 1$, and $\varphi^{-n} = \psi^n$ for $n \geq 1$. Then by Theorem 1.3.1 we have the following:

Corollary 1.3.3. *Let $\mathcal{I}$ be an interval of $\mathbb{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ be a continuous bijective map. Then the set of elements ξ in $\mathcal{I}$, such that the orbit $\{\varphi^n(\xi) \mid n \in \mathbb{Z}\}$ consists only of Liouville numbers in $\mathcal{I}$, is a G_δ–subset of $\mathcal{I}$ (and hence uncountable).*

Proof. This follows by taking $X = \mathcal{I}$, $\mathcal{N} = \mathbb{Z}$, $G_n = \mathbf{L} \cap \mathcal{I}$ and $f_n = \varphi^n$ for each $n \in \mathbb{Z}$ in Theorem 1.2.3. $\square$

1.4 Liouville numbers and polynomials

1(4) Let $n \geq 2$ be an integer and let $P \in \mathbb{R}[X_1, X_2, \ldots, X_n]$. *Under what conditions on P which ensure the existence of uncountably many n-tuples $(\xi_1, \xi_2, \ldots, \xi_n)$ with $\xi_1, \ldots, \xi_n$ are Liouville numbers such that $P(\xi_1, \xi_2, \ldots, \xi_n) = 0$?* In this section we

[1]Throughout this thesis the notation $m(n)$, where m and n are positive integers, refers Remark n in Chapter m.

give some affirmative answer to this question. First we consider the case where $n = 2$.

Theorem 1.4.1. *Let $P(X,Y) = \sum_{i\in\Lambda} f_i(Y)X^i$ be a polynomial in $\in \mathbb{R}[X,Y]$, where Λ is a finite index set whose cardinality is at least two. Assume that the polynomials $f_i(Y)$ $(i \in \Lambda)$ are relatively prime in $\mathbb{R}[Y]$. Suppose that there exists a point $(x_0, y_0) \in \mathbb{R}^2$ such that $P(x_0, y_0) = 0$ and $(\partial/\partial Y)P(x_0, y_0)$ is different from 0. Then there exist uncountably many pairs (ξ, η) of Liouville numbers such that $P(\xi, \eta) = 0$.*

Proof. We use the implicit function Theorem. By implicit function theorem, there exist neighborhoods U of x_0, V of y_0 and a unique function $\varphi : U \to V$ such that $\varphi(x_0) = y_0$ and

$$P(x, \varphi(x)) = 0 \text{ for all } x \in U. \tag{1.3}$$

First we shall show that φ is nowhere locally constant on U. Suppose not, then there exists an open interval $\mathcal{I}$ contained in U such that the restriction of φ to $\mathcal{I}$ is constant, say, c. But by (1.3), $P(x, c) = 0$ for all $x \in \mathcal{I}$. And therefore, $P(x, c)$ is the zero polynomial. Hence, c is a root of the polynomials $f_i(Y)$ $(i \in \Lambda)$. This contradicts the fact that the polynomials $f_i(Y)$ $(i \in \Lambda)$ are relatively prime. Hence φ is nowhere locally constant on U. And therefore, Theorem 1.4.1 now follows from Corollary 1.3.2. □

1(5) Erdős's result on $t = \xi + \eta$ for $t \in \mathbb{R}$ follows from Theorem 1.4.1 with $P(X,Y) = X + Y - t$, while his result on $t = \xi\eta$ for $t \in \mathbb{R}^\times$ follows with $P(X,Y) = XY - t$.

1(6) Also, the above mentioned fact (6) that any positive real number t is the sum of two squares of Liouville numbers follows by applying Theorem 1.4.1 to the polynomial $X^2 + Y^2 - t$.

1(7) One could also deduce, under the hypotheses of Theorem 1.4.1, the existence of one pair of Liouville numbers (ξ, η) with $P(\xi, \eta) = 0$ by applying Theorem 1 of [8] with $f(x, y) = P(x, y)$ and $\alpha = 0$. The proof we gave produces an uncountable set of solutions.

We extend Theorem 1.4.1 to more than 2 variables:

Corollary 1.4.2. *Let $\ell \geq 2$ and let $P(X_1, \ldots, X_\ell) = \sum_{i\in\Lambda} f_i(X_2, \ldots, X_\ell)X_1^i$ be a polynomial in $\mathbb{R}[X_1, \ldots, X_\ell]$, where Λ is a finite index set of cardinality at least two. Assume* *that there exist Liouville numbers $\xi'_3, \dots, \xi'_\ell$ such that the polynomials $f_i(X_2, \xi'_3, \dots, \xi'_\ell)$ $i \in \Lambda$ are relatively prime in $\mathbb{R}[X_2]$. Suppose that there exists a point $(x_0, y_0) \in \mathbb{R}^2$ with $P(x_0, y_0, \xi'_3, \dots, \xi'_\ell) = 0$ and $(\partial/\partial X_2)P(x_0, y_0, \xi'_3, \dots, \xi'_\ell)$ is different from 0. Then there exist uncountably many ℓ-tuples $(\xi_1, \dots, \xi_\ell)$ of Liouville numbers such that $P(\xi_1, \dots, \xi_\ell) = 0$.*

Proof. When $\ell = 2$, this is Theorem 1.4.1. For $\ell \geq 3$, apply Theorem 1.4.1 to the polynomial $P(X_1, X_2, \xi'_3, \dots, \xi'_\ell) \in \mathbb{R}[X_1, X_2]$. $\square$

1(8) In [9], using a counting argument together with an application of Bézout's Theorem, E.B. Burger proved that *an irrational number t is transcendental if and only if there exist two $\mathbb{Q}$–algebraically independent Liouville numbers ξ and η such that $t = \xi + \eta$.*

Extending the method of [9], we prove the following:

Theorem 1.4.3. *Let $F(X, Y) \in \mathbb{Q}[X, Y]$ be a non-constant polynomial with rational coefficients and t be a real number. Assume that there is an uncountable set of pairs of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$. Then the two following conditions are equivalent.*
(a) t is transcendental.
(b) there exist two $\mathbb{Q}$–algebraically independent Liouville numbers (ξ, η) so that $F(\xi, \eta) = t$.

Proof. Assume t is algebraic. Therefore there exists $P(X) \in \mathbb{Q}[X] \setminus \{0\}$ such that $P(t) = 0$. For any pair of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$, we have $P(F(\xi, \eta)) = 0$. Since $P \circ F \in \mathbb{Q}[X, Y] \setminus \{0\}$, we deduce that the numbers ξ and η are algebraically dependent.

Conversely, assume that for any pair of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$, the numbers ξ and η are algebraically dependent. Since $\mathbb{Q}[X, Y]$ is countable and since there is an uncountable set of such pairs of Liouville numbers (ξ, η), there exists a nonzero polynomial $A \in \mathbb{Q}[X, Y]$ such that $A(X, Y)$ and $F(X, Y) - t$ have infinitely many common zeros (ξ, η). We use Bézout's Theorem. We decompose $A(X, Y)$ into irreducible factors in $\overline{\mathbb{Q}}[X, Y]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. One of these factors, say $B(X, Y)$, divides $F(X, Y) - t$ in $\overline{\mathbb{Q}(t)}[X, Y]$, where $\overline{\mathbb{Q}(t)}$ denotes the algebraic closure of $\mathbb{Q}(t)$. Assume now that t is transcendental. Write $F(X, Y) - t = B(X, Y)C(X, Y)$ where $C \in \overline{\mathbb{Q}(t)}[X, Y]$. The coefficient of a monomial X^iY^j in C is

$$\left(\frac{\partial^{i+j}}{\partial X^i \partial Y^j}\right)\left(\frac{F(X, Y) - t}{B(X, Y)}\right)(0, 0),$$

hence $C \in \overline{\mathbb{Q}}[t, X, Y]$ and C has degree 1 in t, say $C(X, Y) = D(X, Y) + tE(X, Y)$, with D and E in $\overline{\mathbb{Q}}[t, X, Y]$. Therefore $B(X, Y)E(X, Y) = -1$, contradicting the fact that $B(X, Y)$ is irreducible. □

1.5 Liouville numbers and countable subsets of $\mathbb{R}$

1(9) One can easily see that, for any nonzero rational number q and any Liouville number ξ, both the numbers $q + \xi$ and $q\xi$ are Liouville numbers. Therefore, we have an uncountable set of Liouville numbers ξ such that $q + \xi$ and $q\xi$ are Liouville numbers for each nonzero $q \in \mathbb{Q}$. One can ask that: *for any countable subset $\mathcal{E}$ of $\mathbb{R}$, can we find an uncountable subset S of $\mathbf{L}$ such that $t + \xi$ and $t\xi$ are Liouville numbers for all $t \in \mathcal{E}$ and $\xi \in \mathsf{S}$?* As a consequence of Theorem 1.3.1, we answer this Question affirmatively. In fact, we prove more general result. More precisely:

Theorem 1.5.1. *Let $\mathcal{E}$ be a countable subset of $\mathbb{R}$. Then there exists an uncountable set of positive Liouville numbers ξ having simultaneously the following properties.*
(i) For any $t \in \mathcal{E}$, the number $\xi + t$ is a Liouville number.
(ii) For any nonzero $t \in \mathcal{E}$, the number ξt is a Liouville number.
(iii) Let $t \in \mathcal{E}$, $t \neq 0$. Define inductively $\xi_0 = \xi$ and $\xi_n = e^{t\xi_{n-1}}$ for $n \geq 1$. Then all terms of the sequence $(\xi_n)_{n\geq 0}$ are Liouville numbers.
(iv) For any rational number $r \neq 0$, the number ξ^r is a Liouville number.

Proof. Each of the four following sets of continuous real maps defined on $\mathcal{I} = (0, +\infty)$ is countable, hence their union is countable.
The first set consists of the maps $x \mapsto x + t$ for $t \in \mathcal{E}$.
The second set consists of the maps $x \mapsto xt$ for $t \in \mathcal{E}$, $t \neq 0$.
The third set consists of the maps φ_n defined inductively by $\varphi_0(x) = x$, $\varphi_n(x) = e^{t\varphi_{n-1}(x)}$ $(n \geq 1)$.
The fourth set consists of the maps $\psi_r(x) = x^r$ for any rational number $r \neq 0$.
We enumerate the elements of the union and we apply Theorem 1.3.1. □

Chapter 2

Liouville Sets and Liouville Fields

2.1 Introduction

We introduce the notions of a *Liouville set* and of a *Liouville field*. They extend what was done by É. Maillet in Chap. III of [19]. In Section 2.3 (respectively in 2.4), we give an uncountable family of Liouville sets (respectively, Liouville fields). In Section 2.5, we prove that Liouville sets are either empty or uncountable, and we characterize all such sets. We also study Liouville sets corresponding to subsequence and to the union of two sequences.

Throughout this thesis, an **admissible pair of sequences** (or simply an **admissible pair**), we mean a pair $(\underline{q}, \underline{u})$ of sequences $\underline{q}$ and $\underline{u}$, where $\underline{q} := (q_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers and $\underline{u} := (u_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim_{n\to\infty} u_n = \infty$. For any integer q and any real number x, we denote by $\|qx\| = \min_{m\in\mathbb{Z}} |qx - m|$, the distance of qx to the nearest integer.

2.2 Liouville sets and Liouville fields

The Liouville sets generalize the notion of a Liouville number.

Definition 2.2.1. *A* **Liouville set** *is a subset* S *of* $\mathbf{L}$ *for which there exists an increasing sequence* $(q_n)_{n\geq 1}$ *of positive integers having the following property: for any* $\xi \in \mathsf{S}$, *there exists a sequence* $\left(b_n\right)_{n\geq 1}$ *of positive rational integers and there exist two positive constants*

$\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ such that, for any sufficiently large n,

$$b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}} \cdot \tag{2.1}$$

2(1) From Definition 2.2.1, it follows that *an irrational number ξ is a Liouville number if and only if the set $\{\xi\}$ is a Liouville set.*

2(2) In the definition of Liouville sets, it would not make a difference if we were requesting the inequalities (2.1) to hold for any $n \geq 1$; it suffices to change the constants κ_1 and κ_2. Also, if the assumption (2.1) is satisfied for some κ_1, then it is also satisfied with κ_1 replaced by any $\kappa_1' > \kappa_1$. Hence there is no loss of generality to assume $\kappa_1 > 1$.

2(3) One could also add to (2.1), the condition $q_n \leq b_n$. Indeed, if, for some n, $b_n < q_n$, then we set

$$b_n' = \left\lceil \frac{q_n}{b_n} \right\rceil b_n,$$

so that $q_n \leq b_n' \leq q_n + b_n \leq 2q_n$. (Here $\lceil x \rceil$ denotes the smallest integer not less than x.)

Denote by a_n the nearest integer to $b_n\xi$ and set $a_n' = \left\lceil \frac{q_n}{b_n} \right\rceil a_n$. Then, for $\kappa_2' < \kappa_2$ and, for sufficiently large n,

$$|b_n'\xi - a_n'| = \left\lceil \frac{q_n}{b_n} \right\rceil |b_n\xi - a_n| \leq \frac{q_n}{q_n^{\kappa_2 n}} \leq \frac{1}{(q_n)^{\kappa_2' n}} \cdot$$

Hence condition (2.1) can be replaced by

$$q_n \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}} \cdot \tag{2.2}$$

2(4) In [19], É Maillet introduces the definition of Liouville numbers corresponding to a given Liouville number. However this definition depends on the choice of a given sequence $\underline{q}$ giving the rational approximations. That is why we start with a sequence $\underline{q}$ instead of starting with a given Liouville number.

Example 2.2.2. *The set $\mathsf{S} = \left\{\xi \mid \quad \xi = \sum_{m=1}^{\infty} a^{-m!} \text{ for some integer } a \geq 2\right\}$ is a Liouville set.*

To prove S is a Liouville set, by definition, we have to find a strictly increasing sequence $(q_n)_{n\geq 1}$ of positive integers such that: for each element $\xi \in \mathsf{S}$ there exists a sequence $\big(b_n\big)_{n\geq 1}$ of positive integers and there exist two positive constants κ_1 and κ_2 (depends only on ξ) such that, for any sufficiently large n,

$$b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}}. \tag{2.3}$$

How to find all these recipes? For, first let $\xi \in \mathsf{S}$. Then $\xi = \sum_{m=1}^{\infty} a^{-m!}$, for some integer $a \geq 2$. For each integer $n \geq 1$, let $b_n = a^{(n+1)!}$. Then, for $n \geq 1$, we have $b_n\xi = a_n + r_n$ with $a_n = \sum_{m=1}^{n+1} a^{n!-m!}$ and $r_n = \sum_{m\geq n+2} a^{n!-m!}$.

Clearly a_n is an integer, and

$$r_n = \sum_{m\geq n+2} a^{n!-m!} \leq a^{-(n+1)(n+1)!}\left(1 + a^{-1} + a^{-2} + \cdots\right) < 2^{-n(n!)}.$$

Thus $\|b_n\xi\| = r_n$. This discussion shows that:

(i) we can take $(q_n)_{n\geq 1}$ to be the sequence, where $q_n = 2^{n!}$ for each $n \geq 1$,

(ii) and for each $\xi = \sum_{m=1}^{\infty} a^{-m!} \in \mathsf{S}$, one can take $\big(b_n\big)_{n\geq 1}$ to be the sequence where $b_n = a^{n!}$ for each $n \geq 1$, and

(iii) $\kappa_1 = \kappa_1(a)$ to be a real number such that $a \leq 2^{\kappa_1}$ and $\kappa_2 = 1$.

Thus we have proved that S is a Liouville set.

Definition 2.2.3. *A* **Liouville field** *is a field of the form* $\mathbb{Q}(\mathsf{S})$, *where* S *is a Liouville set.*

2(5) Note that *an irrational number ξ is a Liouville number if and only if the field $\mathbb{Q}(\xi)$ is a Liouville field.* If we agree that the empty set is a Liouville set and that $\mathbb{Q}$ is a Liouville field, then any subset of a Liouville set is a Liouville set, and also (see Theorem 2.4.1) any subfield of a Liouville field is a Liouville field.

2(6) Let S be the Liouvile set in Example 2.2.2. By Theorem 2 of Adams [1], the set $\left\{\xi \mid \quad \xi = \sum_{m=1}^{\infty} p^{-m!} \text{ for some prime number } p\right\}$ is algebraically independent (see Chapter 4) over $\mathbb{Q}$. Therefore, the transcendence degree (see Chapter 4) of the Liouville field $\mathbb{Q}(\mathsf{S})$ over $\mathbb{Q}$ is infinite.

2.3 The family of Liouville sets $(\mathsf{S}_{\underline{q},\underline{u}})_{\underline{q},\underline{u}}$

Definition 2.3.1. *Let $(\underline{q},\underline{u})$ be an admissible pair. Then $\mathsf{S}_{\underline{q},\underline{u}}$ denote the set of Liouville numbers ξ for which there exist two positive constants $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ and there exists a sequence $(b_n)_{n\geq 1}$ of positive rational integers with*

$$b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 u_n}}.$$

2(7) We denote by $\underline{n}$ the sequence $\underline{u} = (u_n)_{n\geq 1}$ with $u_n = n$ $(n \geq 1)$. For any strictly increasing sequence $\underline{q} = (q_n)_{n\geq 1}$ of positive integers, we denote the set $\mathsf{S}_{\underline{q},\underline{n}}$ by $\mathsf{S}_{\underline{q}}$. Thus, *a subset S of $\mathbf{L}$ is a Liouville set if and only if it is a subset of $\mathsf{S}_{\underline{q}}$ for some $\underline{q}$.*

Lemma 2.3.2. *For all admissible pair $(\underline{q},\underline{u})$ of sequences $\underline{q}$ and $\underline{u}$, the set $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set.*

Proof. Let $(\underline{q},\underline{u})$ be an admissible. To prove $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set, we shall show that $\mathsf{S}_{\underline{q},\underline{u}}$ is a subset of $\mathsf{S}_{\underline{q}'}$ for some $\underline{q}'$ (and hence $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set by Remark 2(7)). For that, we define inductively a sequence of positive integers $(m_n)_{n\geq 1}$ as follows. Let m_1 be the least integer $m \geq 1$ such that $u_m > 1$. Once $m_1, \ldots, m_{n-1}$ are known, define m_n to be the least integer $m > m_{n-1}$ for which $u_m > n$. Consider the subsequence $\underline{q}'$ of $\underline{q}$ defined by $q'_n = q_{m_n}$. Then $\mathsf{S}_{\underline{q},\underline{u}} \subset \mathsf{S}_{\underline{q}'}$, hence $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set. □

Example 2.3.3. *Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers with $u_1 \geq 1$ and $u_{n+1} \geq u_n + 1$ for each $n \geq 1$. Define a function $f : \mathbb{N} \to \mathbb{R}_{>0}$ by $f(1) = 1$ and*

$$f(n) = u_1 u_2 \cdots u_{n-1} \qquad (n \geq 2),$$

so that $f(n+1)/f(n) = u_n$ for $n \geq 1$. We define the sequence $\underline{q} = (q_n)_{n\geq 1}$ by $q_n = \lfloor 2^{f(n)} \rfloor$. Then, for any real number $t > 1$, the number

$$\xi_t = \sum_{n\geq 1} \frac{1}{\lfloor t^{f(n)} \rfloor}$$

belongs to $\mathsf{S}_{\underline{q},\underline{u}}$. The set $\{\xi_t \mid t > 1\}$ has the power of continuum, since $\xi_{t_1} < \xi_{t_2}$ for $t_1 > t_2 > 1$.

Proof. For each positive integer $m \geq 1$, write $\frac{a_m}{b_m} = \sum_{n=1}^{m} \frac{1}{\lfloor t^{f(n)} \rfloor}$. Since $u_m \geq m$ for each m, and since $\lfloor x \rfloor^k \leq \lfloor x^k \rfloor$ for positive real numbers x and k, we have

$$\begin{aligned} b_m &\leq \lfloor t^{f(1)} \rfloor \dots \lfloor t^{f(m-1)} \rfloor \lfloor t^{f(m)} \rfloor \leq \lfloor t^{f(m-1)} \rfloor^{m-1} \lfloor t^{f(m)} \rfloor \\ &\leq \lfloor t^{(m-1)f(m-1)} \rfloor \lfloor t^{f(m)} \rfloor \leq \lfloor t^{u_{m-1}f(m-1)} \rfloor \lfloor t^{f(m)} \rfloor = \lfloor t^{f(m)} \rfloor^2. \end{aligned}$$

Choose a positive real number $\kappa = \kappa(t) > 0$ such that $t < 2^\kappa$. Let M be a positive integer such that $\kappa f(M) \geq 2$. Then by the use of the inequality $\lfloor x \rfloor \leq \lfloor x \rfloor^2$, which holds for $x \geq 2$, we have

$$b_m \leq \lfloor t^{f(m)} \rfloor^2 \leq t^{2f(m)} \leq 2^{2\kappa f(m)} \leq \lfloor 2^{f(m)} \rfloor^{4\kappa} \tag{2.4}$$

holds for all sufficiently large $m \geq M$.

Once again the use of the inequality $\lfloor x \rfloor^k \leq \lfloor x^k \rfloor$, which holds for positive real numbers x and k, we have

$$\lfloor t^{f(m+1)} \rfloor \geq \lfloor t^{f(m)} \rfloor^{u_m}$$

for each $m \geq 1$. We therefore have,

$$\begin{aligned} \left| \xi_t - \frac{a_m}{b_m} \right| &= \frac{1}{\lfloor t^{f(m+1)} \rfloor} + \frac{1}{\lfloor t^{f(m+2)} \rfloor} + \cdots \\ &\leq \frac{1}{\lfloor t^{f(m+1)} \rfloor} + \frac{1}{\lfloor t^{f(m+1)} \rfloor^{u_{m+1}}} + \cdots \\ &\leq \frac{1}{\lfloor t^{f(m+1)} \rfloor} \left(1 + \frac{1}{t} + \frac{1}{t^2} + \cdots \right) \\ &\leq \frac{1}{\lfloor t^{f(m)} \rfloor^{u_m}} (1 - 1/t)^{-1} \end{aligned}$$

Now choose a positive integer $\kappa' = \kappa'(t)$ such that

$$\frac{1}{\lfloor t^{f(m)} \rfloor^{u_m}} (1 - 1/t)^{-1} \leq \frac{1}{\lfloor t^{f(m)} \rfloor^{\kappa' u_m}}.$$

Finally, let $c = c(t) > 0$ be a real number such that $t \geq 2^c$. We then have

$$\left| \xi_t - \frac{a_m}{b_m} \right| \leq \frac{1}{\lfloor 2^{cf(m)} \rfloor^{\kappa' u_m}} \leq \frac{1}{\lfloor 2^{f(m)} \rfloor^{c\kappa' u_m}}, \tag{2.5}$$

since $\lfloor 2^{cf(m)} \rfloor \geq \lfloor 2^{f(m)} \rfloor^c$. By (2.4) and (2.5), we have $\xi_t = \sum_{n \geq 1} \frac{1}{\lfloor t^{f(n)} \rfloor}$ belongs to $\mathsf{S}_{\underline{q},\underline{u}}$. $\square$

2.4 The family of Liouville fields $(\mathbb{Q}_{\underline{q},\underline{u}})_{\underline{q},\underline{u}}$

2.4.1 The fields $\mathbb{Q}_{\underline{q},\underline{u}}$

Let $(\underline{q}, \underline{u})$ be an admissible pair. We denote the union $\mathbb{Q} \bigcup \mathsf{S}_{\underline{q},\underline{u}}$ by $\mathbb{Q}_{\underline{q},\underline{u}}$ (respectively by $\mathbb{Q}_{\underline{q}}$ when $\underline{u} = \underline{n}$). The set $\mathbb{Q}_{\underline{q},\underline{u}}$ has the following property (compare with Theorem I_3 in [19]), namely:

Theorem 2.4.1. *For all admissible pair $(\underline{q}, \underline{u})$, the set $\mathbb{Q}_{\underline{q},\underline{u}}$ is a field.*

2(8) From Theorem 2.4.1, it follows that a field is a Liouville field if and only if it is a subfield of $\mathbb{Q}_{\underline{q}}$ for some $\underline{q}$. Another consequence is that, if S is a Liouville set, then $\mathbb{Q}(\mathsf{S}) \setminus \mathbb{Q}$ is a Liouville set.

2(9) Let $R \in \mathbb{Q}(X_1, \ldots, X_\ell)$ be a rational fraction and let $\xi_1, \ldots, \xi_\ell$ be elements of a Liouville set S such that $\eta = R(\xi_1, \ldots, \xi_\ell)$ is defined. Then by Theorem 2.4.1, η is either a rational number or a Liouville number, and in the second case $\mathsf{S} \cup \{\eta\}$ is a Liouville set. For instance, if, in addition, R is not constant and $\xi_1, \ldots, \xi_\ell$ are algebraically independent over $\mathbb{Q}$, then η is a Liouville number and $\mathsf{S} \cup \{\eta\}$ is a Liouville set. For $\ell = 1$, this yields the following:

Corollary 2.4.2. *Let $R \in \mathbb{Q}(X)$ be a nonconstant rational fraction and let ξ be a Liouville number. Then $R(\xi)$ is a Liouville number and $\{\xi, R(\xi)\}$ is a Liouville set.*

2(10) As a consequence of Corollary 2.4.2, if S is a Liouville set then for any $\xi \in \mathsf{S}$, the set $\mathsf{S} \cup \{1/\xi\}$ is a Liouville set.

2.4.2 Proof of Theorem 2.4.1

We first prove the following:

Lemma 2.4.3. *Let $(\underline{q}, \underline{u})$ be an admissible pair. Then for any element $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$, we have $1/\xi \in \mathsf{S}_{\underline{q},\underline{u}}$.*

Proof. By definition, there exists a sequence of positive integers $(b_n)_{n\geq 1}$ and two positive real numbers $\kappa = \kappa(\xi)$ and $\kappa' = \kappa'(\xi)$ such that

$$b_n \leq q_n^{\kappa} \quad \text{and} \quad 0 < \|b_n\xi\| \leq \frac{1}{q_n^{\kappa' u_n}}.$$

For each $n \geq 1$, write $\|b_n\xi\| = |b_n\xi - a_n|$ with $a_n \in \mathbb{Z}$. Since $\xi \notin \mathbb{Q}$, we have $b_n\xi \notin \mathbb{Q}$ for all large values of n; in particular, for sufficiently large n, $a_n \neq 0$. Clearly we have $\left|\frac{a_n}{b_n}\right| < |\xi| + \frac{1}{2} < b_n$, for sufficiently large n. And hence

$$|a_n| \leq b_n^2 \leq q_n^{2\kappa}. \tag{2.6}$$

Write $\frac{1}{\xi} - \frac{b_n}{a_n} = \frac{1}{\xi a_n}(-b_n\xi + a_n)$. Since the sequence $(q_n)_{n\geq 1}$ is strictly increasing, $|\xi^{-1}| < q_n$ for sufficiently large n. Finally,

$$\left|a_n\xi^{-1} - b_n\right| < \left|\xi^{-1}\right| |b_n\xi - a_n| \leq \frac{1}{q_n^{\kappa' u_n - 1}}. \tag{2.7}$$

By (2.6) and (2.7), one can easily see that $1/\xi \in \mathsf{S}_{\underline{q},\underline{u}}$. $\square$

Proof of Theorem 2.4.1. To prove $\mathbb{Q}_{\underline{q},\underline{u}}$ is a field, by Lemma 2.4.3, it suffices to show that for ξ and ξ' in $\mathbb{Q}_{\underline{q},\underline{u}}$, both $\xi - \xi'$ and $\xi\xi'$ belongs to $\mathbb{Q}_{\underline{q},\underline{u}}$. Let $\xi^+ = \xi - \xi'$ and $\xi^* = \xi\xi'$. We require to prove that both ξ^+ and ξ^* belongs to $\mathbb{Q}_{\underline{q},\underline{u}}$.

If both ξ and ξ' are rational numbers, then the result follows. Assume that at least one of ξ and ξ', say, ξ is an element of $\mathsf{S}_{\underline{q},\underline{u}}$. Then there are two positive real numbers κ_1 and κ_2 depends only on ξ, and there are sequences of rational integers $\big(a_n\big)_{n\geq 1}$ and $\big(b_n\big)_{n\geq 1}$ such that

$$1 \leq b_n \leq q_n^{\kappa_1} \quad \text{and} \quad 0 < \left|b_n\xi - a_n\right| \leq \frac{1}{q_n^{\kappa_2 u_n}}. \tag{2.8}$$

Now ξ' has two possibilities: either ξ' is a rational number, or an element of $\mathsf{S}_{\underline{q},\underline{u}}$.

The case where ξ' is a rational number. Let $\xi' = \frac{r}{s}$ with r and $s \in \mathbb{Z}$, $s > 0$. We shall show that ξ^+ and ξ^* belongs to $\mathsf{S}_{\underline{q},\underline{u}}$. The idea is to prove that ξ^+ is approximated by $(sa_n - b_n)/b_n$, and ξ^* is approximated by ra_n/sb_n.

For each $n \geq 1$, write

$$\begin{aligned} b_n^+ &= b_n^* = sb_n, \\ a_n^+ &= sa_n - rb_n, \\ a_n^* &= ra_n. \end{aligned}$$

Let c be a positive real number such that $\max\{|r|, |s|\} \leq q_n^c$ for sufficiently large n. Then, clearly we have, $1 \leq b_n^+ \leq q_n^{\kappa_1 + c}$ and $1 \leq b_n^* \leq q_n^{\kappa_1 + c}$ holds for sufficiently large n. Moreover,

$$\begin{aligned} 0 < \left| b_n^+ \xi^+ - a_n^+ \right| = s \left| b_n \xi - a_n \right| \leq \frac{1}{q_n^{\tilde{\kappa_2} u_n}}, \\ 0 < \left| b_n^* \xi^* - a_n^* \right| = |r| \left| b_n \xi - a_n \right| \leq \frac{1}{q_n^{\tilde{\kappa_2} u_n}}. \end{aligned}$$

This shows that ξ^+ and ξ^* belongs to $\mathsf{S}_{\underline{q},\underline{u}}$.

The case where ξ' is an element of $\mathsf{S}_{\underline{q},\underline{u}}$. There are constants κ_3, κ_4 depends on ξ', and there are sequences of rational integers $\big(c_n\big)_{n \geq 1}$ and $\big(d_n\big)_{n \geq 1}$ such that

$$1 \leq d_n \leq q_n^{\kappa_3} \quad \text{and} \quad 0 < \left| d_n \xi' - c_n \right| \leq \frac{1}{q_n^{\kappa_4 u_n}}. \tag{2.9}$$

If ξ^+ and ξ^* are rational numbers, then there is nothing to prove. Assume that they are irrational numbers. With this assumption we shall show that, ξ^+ and $\xi^* \in \mathsf{S}_{\underline{q},\underline{u}}$. The idea is to show that: ξ^+ is approximated by $(a_n d_n - c_n b_n)/b_n d_n$ and ξ^* is approximated by $a_n c_n / b_n d_n$.

Let $\kappa_5 = \max\{\kappa_1, \kappa_3\}$ and $\kappa_6 = \min\{\kappa_2, \kappa_4\}$. For each $n \geq 1$, set

$$\begin{aligned} b_n^+ &= b_n^* = b_n d_n, \\ a_n^+ &= a_n d_n - b_n c_n, \\ a_n^* &= a_n c_n. \end{aligned}$$

We then have,

$$1 \leq b_n^+ \leq q_n^{2\kappa_5} \text{ and } 1 \leq b_n^* \leq q_n^{2\kappa_5}, \tag{2.10}$$

and

$$\begin{aligned} b_n\xi^+ - a_n^+ &= d_n(b_n\xi - a_n) - b_n(d_n\xi' - c_n), \\ b_n^*\xi^* - a_n^* &= b_n\xi(d_n\xi' - c_n) + c_n(b_n\xi - a_n). \end{aligned}$$

By (2.8) and (2.9), we have

$$\left|b_n^+\xi^+ - a_n^+\right| \leq |d_n|\,|b_n\xi - a_n| + |b_n|\,|d_n\xi' - c_n| \leq \frac{2q_n^{\kappa_5}}{q_n^{\kappa_6 u_n}}. \tag{2.11}$$

This inequality together with (2.10) shows that $\xi^+ \in \mathsf{S}_{\underline{q},\underline{u}}$.

Finally we prove that $\xi^* \in \mathsf{S}_{\underline{q},\underline{u}}$. Note that, since ξ' is the limit of the sequence $(\frac{c_n}{d_n})_{n\geq 1}$, for sufficiently large n, $\left|\frac{c_n}{d_n}\right| \leq |\xi| + 1 \leq |d_n|$. We then have $|c_n| \leq |d_n^2| \leq q_n^{2\kappa_3}$. From this inequality, we have

$$\begin{aligned} |b_n^*\xi^* - a_n^*| &= |b_n\xi(d_n\xi' - c_n) + c_n(b_n\xi - a_n)| \\ &\leq |b_n|\,|\xi|\,\frac{1}{q_n^{\kappa_4 u_n}} + |c_n|\,\frac{1}{q_n^{\kappa_2 u_n}} \\ &\leq \frac{(|\xi|+1)q_n^{2\kappa_5}}{q_n^{\kappa_6 u_n}}. \end{aligned} \tag{2.12}$$

By (2.10) and (2.12), one can easily see that $\xi^* \in \mathsf{S}_{\underline{q},\underline{u}}$. This completes the proof of Theorem 2.4.1.

2.4.3 Integers which are not sum of squares of elements of $\mathbb{Q}_{\underline{q},\underline{u}}$

Let N be a positive integer which is not a square of an integer. Let $(\underline{q},\underline{u})$ be an admissible pair of sequences $\underline{q}$ and $\underline{u}$. Since the field $\mathbb{Q}_{\underline{q},\underline{u}}$ does not contain irrational algebraic numbers, N is not a square in $\mathbb{Q}_{\underline{q},\underline{u}}$. For a given $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$, it follows that $\eta = N\xi^2$ is an element in $\mathsf{S}_{\underline{q},\underline{u}}$ which is not the square of an element in $\mathsf{S}_{\underline{q},\underline{u}}$. According to [11], we can write $\sqrt{N} = \xi_1\xi_2$ with two Liouville numbers ξ_1, ξ_2, then the set $\{\xi_1, \xi_2\}$ is not a Liouville set.

Theorem 2.4.4. *Let N be a positive integer such that N cannot be written as a sum of two squares of integers. Let $(\underline{q},\underline{u})$ be an admissible pair of sequences $\underline{q}$ and $\underline{u}$. Then for all $\varrho \in \mathsf{S}_{\underline{q},\underline{u}}$, the Liouville number $N\varrho^2 \in \mathsf{S}_{\underline{q},\underline{u}}$ is not the sum of two squares of elements in $\mathsf{S}_{\underline{q},\underline{u}}$. In particular, N cannot be written as a sum of two squares of elements of $\mathsf{S}_{\underline{q},\underline{u}}$.*

Proof. Let N be a positive integer such that N cannot be written as a sum of two squares of integers. We shall show that for each element $\varrho \in \mathsf{S}_{\underline{q},\underline{u}}$, the Liouville number $N\varrho^2 \in \mathsf{S}_{\underline{q},\underline{u}}$ is not the sum of two squares of elements in $\mathsf{S}_{\underline{q},\underline{u}}$. Dividing by ϱ^2, we reduce to show that the equation $N = \xi^2 + (\xi')^2$ has no solution (ξ, ξ') in $\mathsf{S}_{\underline{q},\underline{u}} \times \mathsf{S}_{\underline{q},\underline{u}}$. For otherwise, we would have, for suitable positive constants κ_1 and κ_2 depends on ξ and ξ' such that

$$\left|\xi - \frac{a_n}{b_n}\right| \leq \frac{1}{q_n^{\kappa_2 u_n + 1}}, \qquad 1 \leq b_n \leq q_n^{\kappa_1},$$
$$\left|\xi' - \frac{a'_n}{b'_n}\right| \leq \frac{1}{q_n^{\kappa_2 u_n + 1}}, \qquad 1 \leq b'_n \leq q_n^{\kappa_1}.$$

Hence
$$\left|\xi^2 - \frac{a_n^2}{b_n^2}\right| \leq \frac{2|\xi| + 1}{q_n^{\kappa_2 u_n + 1}} \qquad \text{and} \qquad \left|(\xi')^2 - \frac{(a'_n)^2}{(b'_n)^2}\right| \leq \frac{2|\xi'| + 1}{q_n^{\kappa_2 u_n + 1}}.$$

Thus,
$$\left|\xi^2 + (\xi')^2 - \frac{\left(a_n b'_n\right)^2 + \left(a'_n b_n\right)^2}{\left(b_n b'_n\right)^2}\right| \leq \frac{2(|\xi| + |\xi'| + 1)}{q_n^{\kappa_2 u_n + 1}}.$$

Using $\xi^2 + (\xi')^2 = N$, we deduce that

$$\left|N\left(b_n b'_n\right)^2 - \left(a_n b'_n\right)^2 - \left(a'_n b_n\right)^2\right| < 1.$$

The left hand side is an integer, hence it is 0. This implies that

$$N\left(b_n b'_n\right)^2 = \left(a_n b'_n\right)^2 + \left(a'_n b_n\right)^2.$$

This is impossible, since the equation $x^2 + y^2 = Nz^2$ has no solution in positive rational integers. □

2.5 Cardinality of the Liouville sets $\mathsf{S}_{\underline{q},\underline{u}}$

2(11) Let $(\underline{q}, \underline{u})$ be an admissible pair. By Theorem 2.4.1, $\mathbb{Q}_{\underline{q},\underline{u}}$ is a field extension of $\mathbb{Q}$. A natural question is: *under what conditions on the pair $(\underline{q}, \underline{u})$ which ensures that $\mathbb{Q}_{\underline{q},\underline{u}}$ a proper extension of $\mathbb{Q}$? or equivalently, when the set $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty?* We prove that the set $\mathsf{S}_{\underline{q},\underline{u}}$ is either empty or uncountable, if the sequence $\underline{u} = (u_n)$ satisfies $u_{n+1} \geq u_n + 1$ for all $n \geq 1$. We also characterize such sets.

Theorem 2.5.1. *Let $(\underline{q},\underline{u})$ be an admissible pair such that $u_{n+1} \geq u_n + 1$ for each $n \geq 1$. Then the Liouville set $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty if and only if*

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0. \tag{2.13}$$

Moreover, if the set $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty, then it has the power of continuum.

To prove Theorem 2.5.1, we need a lemma from the next section.

2.5.1 A lemma on approximation by rational numbers

Lemma 2.5.2. *Let ξ and u be two real numbers with $u > 0$, and let q and s be positive integers. Assume that there exist rational integers p and r such that $p/q \neq r/s$ and*

$$|q\xi - p| \leq \frac{1}{q^{\kappa u}}, \quad |s\xi - r| \leq \frac{1}{s^{\kappa(u+1)}},$$

for some positive real number κ with $\kappa u > 2$. Then we have either $2s > q^{\kappa u}$ or $q > s^{\frac{\kappa}{2}u}$.

Proof. From the assumptions, and since $\frac{p}{q} - \frac{r}{s} = \frac{ps-qr}{qs}$, we have

$$\frac{1}{qs} \leq \frac{|ps - rq|}{qs} \leq \left|\xi - \frac{p}{q}\right| + \left|\xi - \frac{r}{s}\right| \leq \frac{1}{q^{\kappa u+1}} + \frac{1}{s^{\kappa(u+1)+1}}.$$

And hence, $q^{\kappa u}s^{\kappa(u+1)} \leq s^{\kappa(u+1)+1} + q^{\kappa u+1}$. If $q < s$, then

$$q^{\kappa u} \leq s + \frac{q}{s^\kappa} < 2s.$$

Assume now $q \geq s$. We then have,

$$q^{\kappa u}s^{\kappa(u+1)} \leq s^{\kappa u}s^{1+\kappa} + q^{\kappa u+1} \leq s^{1+\kappa}q^{\kappa u} + q^{\kappa u+1}.$$

From this we deduce that $s^\kappa\left(s^{\kappa u} - s\right) \leq q$. Since $s \geq 1$, we have

$$q \geq s^{\kappa u} - s > s^{\kappa u} - s^{\frac{\kappa u}{2}} > s^{\frac{\kappa u}{2}}.$$

This completes the proof of Lemma 2.5.2. □

2.5.2 Proof of Theorem 2.5.1.

Suppose that $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0$. Then, we get, $\lim_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0$. Suppose that $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty, and let $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$. Then by Remark 2(3), there exists a sequence $\big(b_n\big)_{n\geq 1}$ of positive integers and there exist two positive constants $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ such that, for any sufficiently large n,

$$q_n \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq q_n^{-\kappa_2 u_n}. \tag{2.14}$$

Let N be an integer such that $\kappa_2 u_n \geq 2$ for each $n \geq N$, and the inequalities (2.14) are valid for $n \geq N$. Since $\lim_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0$, choose an integer $M \geq N$ such that

$$q_{n+1}^{\kappa_1} < q_n^{\frac{\kappa_2}{2} u_n} \tag{2.15}$$

for each $n \geq M$. By the choice of M, we deduce that

$$2b_{n+1} \leq b_{n+1}^2 \leq q_{n+1}^{2\kappa_1} < q_n^{\kappa_2 u_n} \leq b_n^{\kappa_2 u_n}$$

for all $n \geq M$. Moreover, since both the sequence $\underline{q}$ and $\underline{u}$ are increasing, by (2.15),

$$b_n \leq q_n^{\kappa_1} < q_{n+1}^{\kappa_1} \leq q_{n+1}^{\frac{\kappa_2}{2} u_n} \leq b_{n+1}^{\frac{\kappa_2}{2} u_n}$$

for sufficiently large values of n.

Denote by a_n (respectively, a_{n+1}) the nearest integer to ξb_n (respectively, ξb_{n+1}). Lemma 2.5.2 with q replaced by b_n and s by b_{n+1} implies that for large values of n,

$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}}.$$

This contradicts the assumption that ξ is irrational. This proves that $\mathsf{S}_{\underline{q},\underline{u}} = \emptyset$.

Conversely, assume that

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0.$$

Then there exists $\vartheta > 0$ and there exists a sequence $(N_\ell)_{\ell\geq 1}$ of positive integers such that

$$q_{N_\ell} > q_{N_\ell - 1}^{\vartheta(u_{N_\ell - 1})}$$

for all $\ell \geq 1$. Define a sequence $(c_\ell)_{\ell\geq 1}$ of positive integers by

$$2^{c_\ell} \leq q_{N_\ell} < 2^{c_\ell+1}.$$

Let $\underline{e} = (e_\ell)_{\ell\geq 1}$ be a sequence of elements in $\{-1, 1\}$. Define

$$\xi_{\underline{e}} = \sum_{\ell\geq 1} \frac{e_\ell}{2^{c_\ell}}\cdot$$

It remains to check that $\xi_{\underline{e}} \in \mathsf{S}_{\underline{q},\underline{u}}$ and that distinct $\underline{e}$ produce distinct $\xi_{\underline{e}}$.

For each sufficiently large n, let ℓ be the integer such that $N_{\ell-1} \leq n < N_\ell$, and set

$$b_n = 2^{c_{\ell-1}}, \quad a_n = \sum_{h=1}^{\ell-1} e_h 2^{c_{\ell-1}-c_h}, \quad r_n = \frac{a_n}{b_n}\cdot$$

We then have

$$|\xi_{\underline{e}} - r_n| = \left| \xi_{\underline{e}} - \sum_{h\geq\ell} \frac{e_h}{2^{c_h}} \right| \leq \frac{2}{2^{c_\ell}}\cdot$$

Let κ_2 be a positive real number such that $\kappa_2 < \vartheta$. Since n is sufficiently large and $n \leq N_\ell - 1$,

$$\begin{aligned} 4q_n^{\kappa_2 u_n} &\leq 4q_{N_\ell-1}^{\kappa_2 u_{N_\ell-1}} = \frac{4q_{N_\ell-1}^{\vartheta u_{N_\ell-1}}}{q_{N_\ell-1}^{(\vartheta-\kappa_2)u_{N_\ell-1}}} \\ &\leq \frac{4q_{N_\ell}}{q_{N_\ell-1}^{(\vartheta-\kappa_2)u_{N_\ell-1}}} \leq q_{N_\ell}. \end{aligned}$$

And hence

$$\frac{2}{2^{c_\ell}} < \frac{4}{q_{N_\ell}} < \frac{1}{q_n^{\kappa_2 u_n}}$$

for sufficiently large n. Also, choosing $\kappa_1 = 1$, we have $b_n = 2^{c_{\ell-1}} < q_{N_{\ell}-1} \leq q_n$. This proves that $\xi_{\underline{e}} \in \mathsf{S}_{\underline{q},\underline{u}}$, and hence $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty.

Finally, if $\underline{e}$ and $\underline{e}'$ are two elements of $\{-1,+1\}^{\mathbb{N}}$ for which $e_h = e'_h$ for $1 \leq h < \ell$ and, say, $e_\ell = -1$, $e'_\ell = 1$, then

$$\xi_{\underline{e}} < \sum_{h=1}^{\ell-1} \frac{e_h}{2^{c_h}} < \xi_{\underline{e}'},$$

hence $\xi_{\underline{e}} \neq \xi_{\underline{e}'}$. This completes the proof of Theorem 2.5.1.

2.6 Liouville sets and subsequences

Proposition 2.6.1. *Let $(\underline{q}, \underline{u})$ be an admissible pair, and let $\underline{q}'$ be a subsequence of $\underline{q}$. Then $\mathsf{S}_{\underline{q},\underline{u}}$ is a subset of $\mathsf{S}_{\underline{q}',\underline{u}}$ (and hence $\mathbb{Q}_{\underline{q},\underline{u}}$ is a subfield of $\mathbb{Q}_{\underline{q}',\underline{u}}$).*

Proof. By the definition of a subsequence, there is a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $\underline{q}' = (q_{f(n)})_{n\geq 1}$. Thus $f(n) \geq n$ for each $n \geq 1$. Let $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$. Then there are two positive real numbers $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ and there are sequences of rational integers $\big(a_n\big)_{n\geq 1}$ and $\big(b_n\big)_{n\geq 1}$ such that

$$1 \leq b_n \leq q_n^{\kappa_1} \quad \text{and} \quad 0 < \big|b_n\xi - a_n\big| \leq \frac{1}{q_n^{\kappa_2 u_n}}.$$

Consider the subsequence $(b_{f(n)})_{n\geq 1}$ of $\big(b_n\big)_{n\geq 1}$. Then for each $n \geq 1$,

$$1 \leq \; b_{f(n)} \leq \; q_{f(n)}^{\kappa_1} \text{ and } \parallel b_{f(n)}\xi \parallel \leq \frac{1}{q_{f(n)}^{\kappa_2 u_{f(n)}}} \leq \frac{1}{q_{f(n)}^{\kappa_2 u_n}}.$$

Thus we have, $\xi \in \mathsf{S}_{\underline{q}',\underline{u}}$. □

2(12) It is easily checked that if $\liminf_{n\to\infty} \frac{u_n}{u'_n} > 0$, then $\mathsf{S}_{\underline{q},\underline{u}}$ is a subset of $\mathsf{S}_{\underline{q},\underline{u}'}$ (and hence $\mathbb{Q}_{\underline{q},\underline{u}}$ is a subfield of $\mathbb{Q}_{\underline{\mathrm{q}},\underline{\mathrm{u}}'}$). In particular, if $\liminf_{n\to\infty} \frac{u_n}{n} > 0$, then $\mathsf{S}_{\underline{q},\underline{u}}$ is a subset of $\mathsf{S}_{\underline{q}}$, while if $\limsup_{n\to\infty} \frac{u_n}{n} < +\infty$ then $\mathsf{S}_{\underline{q}}$ is a subset of $\mathsf{S}_{\underline{q},\underline{u}}$.

2(13) Let $(\underline{q}, \underline{u})$ be an admissible pair, and let $\underline{q}'$ be a subsequence of $\underline{q}$. By Proposition 2.6.1, $\mathsf{S}_{\underline{q},\underline{u}} \subseteq \mathsf{S}_{\underline{q}',\underline{u}}$. However, if $\underline{q}'$ is obtained by removing finitely many terms from $\underline{q}$, then $\mathsf{S}_{\underline{q}',\underline{u}} = \mathsf{S}_{\underline{q},\underline{u}}$. The question is: *given any admissible pair $(\underline{q}, \underline{u})$, can we find a subsequence $\underline{q}'$ of $\underline{q}$ such that $\mathsf{S}_{\underline{q}',\underline{u}}$ strictly contains $\mathsf{S}_{\underline{q},\underline{u}}$*? We answers this question affirmatively under some assumptions on the sequence $\underline{u}$.

First we look at the following:

Example 2.6.2. *Let $\underline{q} = (q_n)_{n\geq 1}$ be the sequence, where $q_n = 2^{n!}$, for each $n \geq 1$; and let $\underline{q}' = (q'_n)_{n\geq 1}$ be the subsequence of $\underline{q}$ consists of those terms indexed by even positive integers, that is, $q'_n = q_{2n} = 2^{(2n)!}$ for each $n \geq 1$. Then $\mathsf{S}_{\underline{q}}$ is a strict subset of $\mathsf{S}_{\underline{q}'}$.*

Proof. The method is to construct explicitly an element $\xi \in \mathsf{S}_{\underline{q}'}$ such that $\xi \notin \mathsf{S}_{\underline{q}}$. Let λ_n be a sequence of positive integers such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \frac{\lambda_n}{n} = 0$ (for example, one can take $\lambda_n = \lfloor \sqrt{n} \rfloor$). Let

$$\xi := \sum_{m\geq 1} \frac{1}{2^{(2m-1)!\lambda_m}}.$$

We shall show that ξ belongs to $\mathsf{S}_{\underline{q}'}$ but not to $\mathsf{S}_{\underline{q}}$. For each sufficiently large n, we define

$$a_n = \sum_{m=1}^{n} 2^{(2n)!-(2m-1)!\lambda_m}.$$

Then,

$$\begin{aligned}
\frac{1}{q_{2n}^{(2n+1)\lambda_{n+1}}} &< \xi - \frac{a_n}{q_{2n}} = \sum_{m\geq n+1} \frac{1}{2^{(2m-1)!\lambda_m}} \\
&\leq \frac{1}{q_{2n}^{(2n+1)\lambda_{n+1}}}(1 + 1/2 + 1/2^2 + \cdots) \\
&= \frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}} \leq \frac{1}{q_{2n}^{n\lambda_{n+1}}}.
\end{aligned}$$

The right hand side inequality together with the lower bound $\lambda_{n+1} \geq 1$ proves that $\xi \in \mathsf{S}_{\underline{q}'}$. Now we shall show that $\xi \notin \mathsf{S}_{\underline{q}}$. Let κ_1 and κ_2 be positive real numbers, n a sufficiently large integer, s is an integer in the interval $q_{2n+1} \leq s \leq q_{2n+1}^{\kappa_1}$ and r an integer. Since $\lambda_{n+1} < \kappa_2 n$ for sufficiently large n, we have

$$q_{2n}^{(2n+1)\lambda_{n+1}} < q_{2n}^{\kappa_2 n(2n+1)} = q_{2n+1}^{\kappa_2 n} \leq s^{\kappa_2 n}.$$

Therefore, if $r/s = a_n/q_{2n}$, then

$$\left|\xi - \frac{r}{s}\right| = \left|\xi - \frac{a_n}{q_{2n}}\right| > \frac{1}{q_{2n}^{(2n+1)\lambda_{n+1}}} > \frac{1}{s^{\kappa_2(2n+1)}}.$$

On the other hand, for $r/s \neq a_n/q_{2n}$, we have

$$\left|\xi - \frac{r}{s}\right| \geq \left|\frac{a_n}{q_{2n}} - \frac{r}{s}\right| - \left|\xi - \frac{a_n}{q_{2n}}\right| \geq \frac{1}{q_{2n}s} - \frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}}.$$

Since $\lambda_n \to \infty$, for sufficiently large n,

$$4q_{2n}s \leq 4q_{2n}q_{2n+1}^{\kappa_1} = 4q_{2n}^{1+\kappa_1(2n+1)} \leq q_{2n}^{(2n+1)\lambda_{n+1}}$$

and hence

$$\frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}} \leq \frac{1}{2q_{2n}s}.$$

Further

$$2q_{2n}s < sq_{2n+1} \leq s^2 \leq s^{\kappa_2 n}.$$

Therefore

$$\left|\xi - \frac{r}{s}\right| \geq \frac{1}{2q_{2n}s} > \frac{1}{s^{\kappa_2 n}},$$

and hence $\xi \notin \mathsf{S}_{\underline{q}''}$. □

Theorem 2.6.3. *Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers such that for every $n \geq 1$, we have $\sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1}$. Then any increasing sequence $\underline{q}$ of positive integers has a subsequence $\underline{q}'$ for which $\mathsf{S}_{\underline{q}',\underline{u}}$ strictly contains $\mathsf{S}_{\underline{q},\underline{u}}$. In particular, for any increasing sequence $\underline{q}$ of positive integers has a subsequence $\underline{q}'$ for which $\mathsf{S}_{\underline{q}'}$ is strictly contains $\mathsf{S}_{\underline{q}}$.*

Proof. Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\sqrt{u_{n+1}} \leq u_n+1 \leq u_{n+1}$. We prove more precisely that for any sequence $\underline{q}$ such that $q_{n+1} > q_n^{u_n}$ for all $n \geq 1$, the sequence $\underline{q}' = (q_{2n})_{n\geq 1}$ has the property that $\mathsf{S}_{\underline{q}',\underline{u}} \neq \mathsf{S}_{\underline{q},\underline{u}}$. This proves Theorem 2.6.3, since any strictly increasing sequence has a subsequence satisfying $q_{n+1} > q_n^{u_n}$.

The method of the proof is very similar to Example 2.6.2. Assuming $q_{n+1} > q_n^{u_n}$ for all $n \geq 1$, we define $d_n = q_{2n-1}^{\lfloor\sqrt{u_n}\rfloor}$. We check that the number

$$\xi = \sum_{n\geq 1} \frac{1}{d_n}$$

satisfies $\xi \in \mathsf{S}_{\underline{q}',\underline{u}}$ and $\xi \notin \mathsf{S}_{\underline{q},\underline{u}}$.

Set $b_n = d_1 d_2 \cdots d_n$ and

$$a_n = \sum_{m=1}^{n} \frac{b_n}{d_m} = \sum_{m=1}^{n} \prod_{1\leq i\leq n, i\neq m} d_i,$$

so that

$$\xi - \frac{a_n}{b_n} = \sum_{m \geq n+1} \frac{1}{d_m}\cdot$$

It is easy to check from the definition of d_n and q_n that, for sufficiently large n,

$$b_n \leq q_1^{\lfloor\sqrt{u_1}\rfloor} q_3^{\lfloor\sqrt{u_2}\rfloor} \dots q_{2n-1}^{\lfloor\sqrt{u_n}\rfloor} \leq q_{2n-3}^{n-2+u_{n-1}} q_{2n-1}^{\lfloor\sqrt{u_n}\rfloor} \leq q_{2n-1}^{1+\lfloor\sqrt{u_n}\rfloor} \leq q_{2n-1}^{u_{2n-1}} \leq q_{2n},$$

and

$$\frac{1}{d_{n+1}} \leq \xi - \frac{a_n}{b_n} \leq \frac{2}{d_{n+1}}\cdot$$

For each $n \geq 1$, since $d_{n+1} = q_{2n+1}^{\lfloor\sqrt{u_{n+1}}\rfloor} \geq q_{2n}^{\lfloor\sqrt{u_{n+1}}\rfloor u_{2n}} > q_{2n}^{u_n}$, we deduce

$$\left|\xi - \frac{a_n}{b_n}\right| \leq \frac{2}{q_{2n}^{u_n}},$$

hence $\xi \in \mathsf{S}_{\underline{q}',\underline{u}}$.

Now the only task left is to prove that $\xi \notin \mathsf{S}_{\underline{q},\underline{u}}$. For, let κ_1 and κ_2 be two positive real numbers, and let n be sufficiently large positive integer. Let s be a positive integer with $q_{2n+1} \leq s \leq q_{2n+1}^{\kappa_1}$ and let r be an integer. If $r/s = a_n/b_n$, then

$$\left|\xi - \frac{r}{s}\right| = \left|\xi - \frac{a_n}{b_n}\right| > \frac{1}{q_{2n}^{u_n}} > \frac{1}{q_{2n+1}^{\kappa_2 u_{2n+1}}}\cdot$$

Assume now $r/s \neq a_n/b_n$. From the inequality

$$\left|\xi - \frac{a_n}{b_n}\right| \leq \frac{2}{q_{2n+1}^{\lfloor\sqrt{u_{n+1}}\rfloor}} \leq \frac{1}{2q_{2n+1}^{\kappa_1+2}},$$

we deduce that

$$\begin{aligned}
\left|\xi - \frac{r}{s}\right| &\geq \left|\frac{r}{s} - \frac{a_n}{b_n}\right| - \left|\xi - \frac{a_n}{b_n}\right| \\
&\geq \frac{1}{sb_n} - \frac{1}{2q_{2n+1}^{\kappa_1+2}} \\
&\geq \frac{1}{2q_{2n+1}^{\kappa_1+2}} \geq \frac{1}{q_{2n+1}^{\kappa_2 u_{2n+1}}},
\end{aligned}$$

for sufficiently large n. This completes the proof that $\xi \notin \mathsf{S}_{\underline{q},\underline{u}}$. □

2.7 Liouville sets and union of two sequences

Let $\underline{x}$ and $\underline{y}$ be two strictly increasing sequence of positive integers. Then, the **union** of $\underline{x}$ and $\underline{y}$ written as $\underline{\mathrm{x}}\vee\underline{\mathrm{y}}$, is the unique increasing sequence of positive integers that belong to either $\underline{x}$ or $\underline{y}$.

For example, the sequence $\underline{n}$ is the union of the sequence $\{1, 3, 5, \dots\}$ and $\{2, 4, 6, \dots\}$.

Proposition 2.7.1. *For $0 < \tau < 1$, let $\underline{q}^{(\tau)}$ denote the sequence $(q_n^{(\tau)})_{n\geq 1}$ where*

$$q_n^{(\tau)} = 2^{n!\lfloor n^\tau \rfloor} \qquad (n \geq 1).$$

Then the sets $\mathsf{S}_{\underline{q}^{(\tau)}}$, $0 < \tau < 1$ are nonempty, and for $\tau \neq \tau'$, the Liouville sets corresponding to the union $\underline{q}^{(\tau)} \vee \underline{q}^{(\tau')}$ is empty.

Proof. For a fixed τ in the open interval $(0, 1)$, and for each $n \geq 1$,

$$(1 + 1/n) \leq \frac{(n+1)\lfloor (n+1)^\tau \rfloor}{n\lfloor n^\tau \rfloor} = \frac{\log q_{n+1}^{(\tau)}}{n \log q_n^{(\tau)}} \leq \frac{(n+1)(n+1)^\tau}{n\lfloor n^{\tau+1} \rfloor} \leq \frac{(1+1/n)(n+1)^\tau}{n^\tau - 1}.$$

Thus

$$\lim_{n\to\infty} \frac{\log q_{n+1}^{(\tau)}}{n \log q_n^{(\tau)}} = 1.$$

And therefore, by Theorem 2.5.1 the sets $\mathsf{S}_{\underline{q}^{(\tau)}}$ are nonempty for each τ in the interval $(0, 1)$. In fact, if $(e_n)_{n\geq 1}$ is a bounded sequence of integers with infinitely many nonzero terms, then

$$\sum_{n\geq 1} \frac{e_n}{q_n^{(\tau)}} \in \mathsf{S}_{\underline{q}^{(\tau)}}.$$

Let $0 < \tau_1 < \tau_2 < 1$. For $n \geq 1$, define

$$q_{2n} = q_n^{(\tau_1)} = 2^{n!\lfloor n^{\tau_1} \rfloor} \quad \text{and} \quad q_{2n+1} = q_n^{(\tau_2)} = 2^{n!\lfloor n^{\tau_2} \rfloor}.$$

Note that $\underline{q} = (q_m)_{m\geq 1}$ is the union of the sequence $\underline{q}^{(\tau_1)}$ and $\underline{q}^{(\tau_2)}$. One can easily checks that $\underline{q}$ is an increasing sequence with

$$\frac{\log q_{2n+1}}{n \log q_{2n}} \to 0 \quad \text{and} \quad \frac{\log q_{2n+2}}{n \log q_{2n+1}} \to 0.$$

From Theorem 2.5.1, we can deduce that $\mathsf{S}_{\underline{q}} = \emptyset$. □

2(14) If $\underline{q} = \underline{q}' \vee \underline{q}''$, then by Proposition 2.6.1, we see that $\mathsf{S}_{\underline{q}} \subseteq \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. Proposition 2.7.1 gives an example, where the set $\mathsf{S}_{\underline{q}}$ is empty. We shall show that this is not the case always. More precisely, we have the following:

Theorem 2.7.2. *There exists two increasing sequences $\underline{q}'$ and $\underline{q}''$ of positive integers such that $\mathsf{S}_{\underline{q}' \vee \underline{q}''}$ is a strict nonempty subset of $\mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$.*

The construction takes several steps.

(a) Let $(\lambda_s)_{s\geq 0}$ be a strictly increasing sequence of positive integers with $\lambda_0 = 1$. We define two sequences $(n'_k)_{k\geq 1}$ and $(n''_h)_{h\geq 1}$ of positive integers as follows. The sequence $(n'_k)_{k\geq 1}$ is the strictly increasing sequence formed by those integers n, for which there exists an integer $s \geq 0$ with $\lambda_{2s} \leq n < \lambda_{2s+1}$, while $(n''_h)_{h\geq 1}$ is the strictly increasing sequence formed by those integers n for which there exists $s \geq 0$ with $\lambda_{2s+1} \leq n < \lambda_{2s+2}$.

(b) Note that the sequences $(n'_k)_{k\geq 1}$ and $(n''_h)_{h\geq 1}$ completely covers $\mathbb{N}$ in the sense that, each $n \in \mathbb{N}$ is exactly one of the form: $n = n'_k$ for some $k \geq 1$, or $n = n''_h$ for some $h \geq 1$. Indeed, if $n \in \mathbb{N}$ with $\lambda_{2s} \leq n < \lambda_{2s+1}$ for some $s \geq 0$, then $n = n'_k$, where

$$k = n - \lambda_{2s} + \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1 - \lambda_0;$$

while if $\lambda_{2s+1} \leq n < \lambda_{2s+2}$ $(s \geq 0)$, then $n = n''_h$. with

$$h = n - \lambda_{2s+1} + \lambda_{2s} - \lambda_{2s-1} + \cdots + \lambda_2 - \lambda_1.$$

For instance, when $\lambda_s = s + 1$ $(s \geq 0)$, the sequence $(n'_k)_{k\geq 1}$ is the sequence $(1, 3, 5, \dots)$ of odd positive integers, while $(n''_h)_{h\geq 1}$ is the sequence $(2, 4, 6, \dots)$ of even positive integers. Another example is $\lambda_s = s!$, which occurs in the Erdős paper [11].

(c) When, $n = \lambda_{2s}$, we write $n = n'_{k(s)}$ where

$$k(s) = n - \lambda_{2s} + \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1 - \lambda_0 = \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1 - \lambda_0 < \lambda_{2s-1}.$$

Notice that $\lambda_{2s} - 1 = n''_h$ with $h = \lambda_{2s} - k(s)$.

(d) Next, we define two increasing sequences $(d_n)_{n\geq 1}$ and $\underline{q} = (q_n)_{n\geq 1}$ of positive integers by induction as follows: set $d_1 = 2$, and for each $n \geq 1$, let

$$d_{n+1} = \begin{cases} kd_n & \text{if } n = n'_k, \\ hd_n & \text{if } n = n''_h, \end{cases}$$

and $q_n = 2^{d_n}$.

(e) For example, when $\lambda_s = s+1$ ($s \geq 0$), the sequence $(d_n)_{n\geq 1}$ is given by the following formula:

$$d_{n+1} = \begin{cases} \frac{n(n-2)\cdots 2}{2^{n/2}} d_1 & \text{if } n \text{ is even}, \\ \frac{(n-1)(n-3)\cdots 2}{2^{(n-1)/2}} d_1 & \text{if } n \text{ is odd}. \end{cases}$$

(f) Finally, let $\underline{q}' = (q'_k)_{k\geq 1}$ and $\underline{q}'' = (q''_h)_{h\geq 1}$ be two subsequences of $\underline{q}$ defined by

$$q'_k = q_{n'_k} \quad (k \geq 1),$$

$$q''_h = q_{n''_h} \quad (h \geq 1).$$

Hence $\underline{q}$ is the union of the two sequences $\underline{q}'$ and $\underline{q}''$.

(g) Now we check that the number

$$\xi = \sum_{n\geq 1} \frac{1}{q_n}$$

belongs to $\mathsf{S}_{\underline{q}'} \bigcap \mathsf{S}_{\underline{q}''}$ (and hence the sets $\mathsf{S}_{\underline{q}'}$ and $\mathsf{S}_{\underline{q}''}$ are uncountable by Theorem 2.5.1).

For each $n \geq 1$, let

$$a_n = \sum_{m=1}^{n} 2^{d_n - d_m}.$$

Then

$$\frac{1}{q_{n+1}} < \xi - \frac{a_n}{q_n} = \sum_{m\geq n+1} \frac{1}{q_m} < \frac{2}{q_{n+1}}.$$

If $n = n'_k$, then

$$\left| \xi - \frac{a_{n'_k}}{q'_k} \right| < \frac{2}{(q'_k)^k}$$

while if $n = n''_h$, then

$$\left|\xi - \frac{a_{n''_h}}{q''_h}\right| < \frac{2}{(q''_h)^h}.$$

This proves $\xi \in \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$.

(h) Now, we choose $\lambda_s = 2^{2^s}$ for $s \geq 2$ and we prove that ξ does not belong to $\mathsf{S}_{\underline{q}}$. Notice that $\lambda_{2s-1} = 2^{2^{2s-1}} = \sqrt{\lambda_{2s}}$. Let $n = \lambda_{2s} = n'_{k(s)}$. We have $k(s) < \sqrt{\lambda_{2s}}$ and

$$\left|\xi - \frac{a_n}{q_n}\right| > \frac{1}{q_{n+1}} = \frac{1}{q_n^{k(s)}} > \frac{1}{q_n^{\sqrt{n}}}.$$

Let κ_1 and κ_2 be two positive real numbers and assume s is sufficiently large. Further, let $u/v \in \mathbb{Q}$ with $v \leq q_n^{\kappa_1}$. If $u/v = a_n/q_n$, then

$$\left|\xi - \frac{u}{v}\right| = \left|\xi - \frac{a_n}{q_n}\right| > \frac{1}{q_n^{\sqrt{n}}} > \frac{1}{q_n^{\kappa_2 n}}$$

for sufficiently large n. On the other hand, if $u/v \neq a_n/q_n$, then

$$\left|\xi - \frac{u}{v}\right| \geq \left|\frac{u}{v} - \frac{a_n}{q_n}\right| - \left|\xi - \frac{a_n}{q_n}\right|$$

with

$$\left|\frac{u}{v} - \frac{a_n}{q_n}\right| \geq \frac{1}{vq_n} \geq \frac{1}{q_n^{\kappa_1+1}} > \frac{2}{q_n^{\sqrt{n}}}$$

and

$$\left|\xi - \frac{a_n}{q_n}\right| > \frac{1}{q_n^{\sqrt{n}}} \geq \frac{1}{q_n^n}.$$

Hence

$$\left|\xi - \frac{u}{v}\right| > \frac{2}{q_n^{\sqrt{n}}} - \frac{1}{q_n^n} = \frac{2q_n^n - q_n^{\sqrt{n}}}{q_n^n} \geq \frac{1}{q_n^n}$$

for sufficiently large n. This completes the proof of Theorem 2.7.2.

2.8 Equivalence relation induced by Liouville sets

2(15) We define a binary relation $\sim$ on $\mathbf{L}$ as follows: *for* $\xi, \eta \in \mathbf{L}$, *we say* $\xi \sim \eta$ *if* $\{\xi, \eta\}$ *is a Liouville set.* The relation $\sim$ is clearly both reflexive and symmetric. We shall show that $\sim$ is not transitive.

Theorem 2.8.1. *Let ξ and η be Liouville numbers. Then there exists a subset ϑ of* $\mathbf{L}$ *having the power of continuum such that for each $\varrho \in \vartheta$, the sets $\{\xi, \varrho\}$ and $\{\varrho, \eta\}$ are Liouville sets.*

2(16) By Theorem 2.8.1, the relation $\sim$ is not transitive. For example, let t be an irrational real number which is not a Liouville number. Then by a result due to Erdős [11], we can write $t = \xi + \eta$ with two Liouville numbers ξ and η. Since any irrational number in a Liouville fields are Liouville numbers, the set $\{\xi, \eta\}$ is not a Liouville Set, that is $\xi \nsim \eta$. But by Theorem 2.8.1, there exists a (in fact uncountably many) Liouville number ϱ such that $\xi \sim \varrho$ and $\varrho \sim \eta$. Thus, the relation $\sim$ is not transitive. Note that (again by Theorem 2.8.1), the equivalence relation induced by $\sim$ is trivial.

The proof of Theorem 2.8.1 as a consequence of Theorem 2.5.1 relies on the following elementary lemma.

Lemma 2.8.2. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two strictly increasing sequences of positive integers. Then there exists a strictly increasing sequence of positive integers $(q_n)_{n\geq 1}$ satisfying the following properties:*
(i) The sequence $(q_{2n})_{n\geq 1}$ is a subsequence of the sequence $(a_n)_{n\geq 1}$.
(ii) The sequence $(q_{2n+1})_{n\geq 0}$ is a subsequence of the sequence $(b_n)_{n\geq 1}$.
(iii) For $n \geq 1$, $q_{n+1} \geq q_n^n$.

Proof of Lemma 2.8.2. We construct the sequence $(q_n)_{n\geq 1}$ inductively as follows. Set $q_1 = b_1$ and let q_2 be the least integer a_i satisfying $a_i \geq b_1$. Once q_n is known for some $n \geq 2$, we define q_{n+1} to be the least integer satisfying the following properties:
- $q_{n+1} \in \{a_1, a_2, \dots\}$ if n is odd, $q_{n+1} \in \{b_1, b_2, \dots\}$ if n is even.
- $q_{n+1} \geq q_n^n$. □

Proof of Theorem 2.8.1. Let ξ and η be Liouville numbers. There exist increasing sequences of positive integers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that

$$\|a_n\xi\| \leq a_n^{-n} \quad \text{and} \quad \|b_n\eta\| \leq b_n^{-n}$$

for sufficiently large n. Let $\underline{q} = (q_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers satisfying the conclusion of Lemma 2.8.2. Since $\lim_{n\to\infty} \frac{\log q_{n+1}}{n \log q_n} \geq 1$, by Theorem 2.5.1, the Liouville set $\mathsf{S}_{\underline{q}}$ is nonempty (and hence uncountable). Let $\varrho \in \mathsf{S}_{\underline{q}}$. We denote by $\underline{q}'$

the subsequence $(q_2, q_4, \dots, q_{2n}, \dots)$ of $\underline{q}$ and by $\underline{q}''$ the subsequence $(q_1, q_3, \dots, q_{2n+1}, \dots)$. We have $\varrho \in \mathsf{S}_{\underline{q}} \subseteq \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. Since the sequence $(a_n)_{n\geq 1}$ is increasing, we have $q_{2n} \geq a_n$, and hence $\xi \in \mathsf{S}_{\underline{q}'}$. Also, since the sequence $(b_n)_{n\geq 1}$ is increasing, we have $q_{2n+1} \geq b_n$, hence $\eta \in \mathsf{S}_{\underline{q}''}$. Finally, ξ and ϱ belong to the Liouville set $\mathsf{S}_{\underline{q}'}$, while η and ϱ belong to the Liouville set $\mathsf{S}_{\underline{q}''}$. $\square$

2.9 A topological property of the sets $\mathsf{S}_{\underline{q},\underline{u}}$

Proposition 2.9.1. *The sets $\mathsf{S}_{\underline{q},\underline{u}}$ are not G_δ subsets of $\mathbb{R}$. If they are nonempty, then they are dense in $\mathbb{R}$.*

Proof. Suppose that $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty, say, $\gamma \in \mathsf{S}_{\underline{q},\underline{u}}$. By Theorem 2.4.1, $\gamma + \mathbb{Q}$ is contained in $\mathsf{S}_{\underline{q},\underline{u}}$, hence $\mathsf{S}_{\underline{q},\underline{u}}$ is dense in $\mathbb{R}$.

Let t be an irrational real number which is not Liouville. Hence $t \notin \mathbb{Q}_{\underline{q},\underline{u}}$, and therefore, by Theorem 2.4.1, $\mathsf{S}_{\underline{q},\underline{u}} \cap (t + \mathsf{S}_{\underline{q},\underline{u}}) = \emptyset$. This implies that $\mathsf{S}_{\underline{q},\underline{u}}$ is not a G_δ dense subset of $\mathbb{R}$. $\square$

Chapter 3

Mahler Sets and Mahler Fields

3.1 Introduction

In [18, 1932], K. Mahler using the construction similar to Liouville numbers, classified the set of complex numbers into four disjoint classes; namely, A-numbers, S-numbers, T-numbers and U-numbers.[1] His idea was to classify complex numbers ξ according to how the values $|P(\xi)|$, for nonzero integer polynomials P, approach zero. The class of A-number is precisely, $\mathbb{A}$, the set of all algebraic numbers. The class of U-numbers are further divided into classes of type U_m-numbers for each $m \in \mathbb{N}$. We denote the set of U-numbers (respectively, of U_m-numbers) by $\mathbf{U}$ (respectively, by $\mathbf{U}_m$)$\cdot$ The class of U_1-numbers is precisely the set of all Liouville numbers[2] and therefore coincide with $\mathbf{L}$.

It was an open problem at that time, whether all the classes S-, T- and U- are nonempty. Mahler himself proved that, with respect to Lebesgue measure, almost all numbers are S-numbers, and hence the class of S-numbers is nonempty. In fact, he constructed many examples of S-numbers. For example, he showed that the Champernowne number 0.123456789101112..., is an S-number. In [16], W. J. LeVeque proved that the sets $\mathbf{U}_m$ are nonempty for each positive integer m. In [22, 1968], W. M. Schmidt proved the existence of T-numbers, almost thirty five years after the work of Mahler.

In [14], J. F. Koksma using the construction closely analogous to Mahler, classified the set of transcendental numbers into three disjoint classes; namely, S^*-numbers, T^*-numbers and U^*-numbers. His idea was to classify complex numbers ξ according to how

[1] See the Appendix for the definition of Mahler's and Koksma's classifications of complex numbers.
[2] While collaborating with Professor M. Waldschmidt on Liouville numbers, he suggested to work on algebraic approximation of complex numbers. This led to the work on this chapter.

the difference $|\xi - \alpha|$, for $\alpha \in \mathbb{A}$ with $\alpha \neq \xi$, approach zero. In the same paper [14], J. F. Koksma proved that these two classifications are same; that is, the S^*-, T^*- and U^*-classes being in fact identical with the S-, T- and U-classes respectively. Later [31], E. Wirsing gave a simple proof of these two classifications are same. Here we use the Koksma definition of U-numbers.

In this chapter we introduce the notions of a Mahler set and of a Mahler field. They generalize the notions of Liouville sets and of Liouville fields. Mahler sets are subsets of $\mathbf{U}$, and they have many properties similar to Liouville sets. Like Chapter 2, we give a family of Mahler sets and Mahler fields (see Section 3.4 and Section 3.5). In Section 3.6, we prove that the sets $\mathbf{U}_m$ are nonempty (this was already confirmed by W. J. LeVeque [16]). In Section 3.7, we classify all the finite extensions of Mahler fields. In Section 3.7.2, we study quadratic extensions of certain Liouville fields. Section 3.8 is devoted to study the image of the Mahler fields under the classical exponential function. In Section 3.9, we discuss the image of those U-numbers lies in Mahler fields under some special power series with algebraic coefficients.

3.2 Mahler's U-numbers

The U-numbers are generalization of Liouville numbers in the following sense: *A transcendental number ξ is said to be a U-number if there exists a positive integer m with the property that, to every positive integer n there corresponds to an algebraic number α_n of degree m with height*[3] *$H(\alpha_n) > 1$ such that*

$$|\xi - \alpha_n| < H(\alpha_n)^{-n}. \tag{3.1}$$

The least such integer m is called the *type* of the U-number ξ, and in this case ξ is called a U_m*-number.* More precisely, we have the following:

Definition 3.2.1. *Let m be a positive integer. A complex number ξ is called a* **U_m-number** *or a* **U-number of type** *m, if for every real number $\omega > 0$ there exists a sequence $(\alpha_n)_{n\geq 1}$ of distinct algebraic numbers of degree m such that*

$$0 < |\xi - \alpha_n| < H(\alpha_n)^{-\omega} \tag{3.2}$$

[3] See the Appendix for the definition and some properties of the height $H(\alpha)$ of an algebraic number α.

and if there exist positive real numbers c and κ depending only on ξ and m such that the relation

$$|\xi - \beta| > cH(\beta)^{-\kappa} \tag{3.3}$$

holds for every algebraic number β of degree strictly less than m.

We denote the set of U_m-numbers by $\mathbf{U}_m$, and we denote the union $\bigcup_{m\geq 1} \mathbf{U}_m$ by $\mathbf{U}$. The elements of $\mathbf{U}$ are called U-numbers.

Example 3.2.2. *Let m be a positive integer and let α be a m-th root of unity. Then $\xi = \sum_{n=1}^{\infty}(\frac{\alpha}{2})^{n!}$ is a U-number of type at most $\varphi(m)$, where φ is the Euler's totient function.*

Proof. This can be easily verified. Write $\xi = \sum_{n=1}^{m-1}(\frac{\alpha}{2})^{n!} + \sum_{n=m}^{\infty} 2^{-n!}$. Since $\sum_{n=m}^{\infty} 2^{-n!}$ is a Liouville number (and hence a U-number) and since algebraic dependent numbers belongs to the same Mahler class [6, p. 86], ξ is a U-number. Since ξ can be approximated by algebraic numbers of degree at most $\varphi(m)$, ξ is a U-number of type at most $\varphi(m)$. □

The following (which is a modification of the example given in [7, p. 153]) is an example of U_m-number for each positive integer m. In fact, it proves that the set of U_m-numbers is uncountable for each positive integer m.

Example 3.2.3. *Let m be a positive integer, and let a_j be an element of $\{-1, +1\}$. Let ξ be the real positive real m-th root of $\left(1 + \sum_{j\geq 1} a_j 10^{-j!}\right)/2$. Then ξ is a U-number type m.*

Proof. It is clear that ξ is a U-number. Indeed, since $\eta = (1 + \sum_{j\geq 1} a_j 10^{-j!})/2$ is a Liouville number, and $\{\xi, \eta\}$ is algebraically dependent, (by [6, p. 86]) ξ is a U-number. Moreover, its type does not exceed m. This can be easily verified. For, set

$$a_k = 10^{k!}\left(1 + \sum_{j=1}^{k} a_j 10^{-j!}\right), \ b_k = 2.10^{k!}, \text{ and } \alpha_k = \text{positive real } m\text{-th root of } \frac{a_k}{b_k}$$

for all $k \geq 1$. Then, since $|\frac{a_k}{b_k}| < 1$, we have $H(\alpha_k) = 2.10^{k!}$ and

$$|\xi - \alpha_k| \leq 2\,|\xi^m - \alpha_k^m| \leq 2.10^{-(k+1)!} \leq 2^{k+2} H(\alpha_k)^{-k-1}. \tag{3.4}$$

This proves that, type of ξ does not exceed m.

Now we prove that type of ξ is not less than m. For, let β be a nonzero real algebraic number of degree n strictly less than m and of height greater than $H(\alpha_1)$. Since the sequence $(H(\alpha_k))_{k\geq 1}$ is strictly increasing, there exists a positive integer k such that

$$H(\alpha_k) \leq H(\beta)^{2m} \leq H(\alpha_{k+1}) \leq H(\alpha_k)^{k+1}. \tag{3.5}$$

By Lemma *A*.12,

$$|\alpha_k - \beta| > cH(\alpha_k)^{-n} H(\beta)^{-m} \geq cH(\alpha_k)^{-m-(k+1)/2} \tag{3.6}$$

for some constant $c = c(m, n)$. By taking $H(\beta)$ large enough, the index $k = k(m)$ satisfies

$$cH(\alpha_k)^{-m+(k+1)/2} > 2^{k+3}$$

and it follows from (3.4) and (3.6) that $|\alpha_k - \beta| > 2\,|\xi - \alpha_k|$. Thus, except for finitely many algebraic numbers β of degree strictly less than m, we have

$$|\xi - \beta| \geq |\alpha_k - \beta| - |\xi - \alpha_k| > |\alpha_k - \beta|/2 > (C/2)H(\beta)^{-m-2m^2} \tag{3.7}$$

where $C = \min_{1\leq n\leq m} c(n, m)$. Hence ξ is not a U-number of type strictly less than m. $\square$

3.3 Mahler sets and Mahler fields

An algebraic number field (or simply **number field**) K is a field extension of finite degree over $\mathbb{Q}$.

3(1) Note that in general, all the $\alpha_n{}'s$ in Definition 3.2.1 of U-numbers need not lie in a fixed number field. We are interested to study about those U-numbers for which all the $\alpha_n{}'s$ are in some fixed number field K.

Definition 3.3.1. *Let K be a number field of degree m over $\mathbb{Q}$. A subset* S *of* **U** *is called a* **Mahler set** *over K, if there exists an increasing sequence $\underline{q} = (q_n)_{n\geq 1}$ of positive integers such that:* for each element $\xi \in \mathsf{S}$, there exists a sequence $\left(\alpha_n\right)_{n\geq 1}$ of distinct elements of K all having same degree over $\mathbb{Q}$, and there exist two positive real numbers $\kappa_1 = \kappa_1(\xi)$

and $\kappa_2 = \kappa_2(\xi)$ such that, for each sufficiently large n,

$$H(\alpha_n) \leq q_n^{\kappa_1} \text{ and } |\xi - \alpha_n| \leq \frac{1}{q_n^{\kappa_2 n}}. \tag{3.8}$$

Definition 3.3.2. *Let K be a number field. A field extension $\mathcal{K}$ of K is called a* **Mahler field** *over K if $\mathcal{K} = K(\mathsf{S})$ for some Mahler set S over K.*

Thus, Liouville sets are Mahler sets over $\mathbb{Q}$, and Liouville fields are Mahler fields over $\mathbb{Q}$.

3.4 The family of Mahler sets $(\mathsf{S}_{K,\underline{q},\underline{u}})_{\underline{q},\underline{u}}$

3.4.1 The sets $\mathsf{S}_{K,\underline{q},\underline{u}}$

Definition 3.4.1. *Let K be a number field of degree m over $\mathbb{Q}$, and let $(\underline{q}, \underline{u})$ be an admissible pair. Let t be a positive integer such that $1 \leq t \leq m$. An* **$\mathbb{A}_t$-sequence** *for a U-number ξ over K with respect to $(\underline{q}, \underline{u})$ (if it exists) is a sequence $(\alpha_n)_{n\geq 1}$ of distinct elements of K all having same degree t over $\mathbb{Q}$, with two positive constants $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ such that for sufficiently large n, we have*

$$H(\alpha_n) \leq q_n^{\kappa_1} \text{ and } 0 < |\xi - \alpha_n| \leq \frac{1}{q_n^{\kappa_2 u_n}}. \tag{3.9}$$

3(2) Since any infinite subsequence of an $\mathbb{A}_t$-sequence is also an $\mathbb{A}_t$-sequence, an $\mathbb{A}_t$-sequence for a U-number ξ is not unique. It is clear that, if a U-number ξ has an $\mathbb{A}_t$-sequence, then $\xi \in \bigcup_{k=1}^{t} \mathbf{U}_k$.

Definition 3.4.2. *Let K be a number field of degree m over $\mathbb{Q}$. For any admissible pair $(\underline{q}, \underline{u})$, let $\mathsf{S}_{K,\underline{q},\underline{u}}$ denotes the set of all U-numbers having an $\mathbb{A}_t$-sequence (for some positive integer t) over K with respect to the admissible pair $(\underline{q}, \underline{u})$. The situation when $\underline{u} = \underline{n}$, we denote $\mathsf{S}_{K,\underline{q},\underline{u}}$ by $\mathsf{S}_{K,\underline{q}}$.*

Note that, when $K = \mathbb{Q}$, the sets $\mathsf{S}_{K,\underline{q},\underline{u}}$ coincide with the sets $\mathsf{S}_{\underline{q},\underline{u}}$ defined in Chapter 2, and therefore we denote $\mathsf{S}_{\mathbb{Q},\underline{q},\underline{u}}$ by $\mathsf{S}_{\underline{q},\underline{u}}$.

The proof of the following lemma is similar to the proof of Lemma 2.3.2 and we left to the interesting reader to make necessary changes.

Lemma 3.4.3. *Let K be a number field. Then, for each admissible pair $(\underline{q},\underline{u})$, the set $\mathsf{S}_{K,\underline{q},\underline{u}}$ is a Mahler set over K.*

Example 3.4.4. *Let K be a number field of degree m over $\mathbb{Q}$. Let α be a nonzero element of K of absolute value less than 1. Let $\underline{q}=(q_n)_{n\geq 1}$ be the sequence of positive integers where $q_n=2^{n!}$ for each $n\geq 1$. Then $\xi=\sum_{k=1}^{\infty}\frac{\alpha^{k!}}{2^{k!}}$ is an element of $\mathsf{S}_{K,\underline{q}}$.*

For each positive integer n, let $\alpha_n=\sum_{k=1}^{n}\frac{\alpha^{k!}}{2^{k!}}$, and let $P_n(X)=\sum_{k=1}^{n}2^{n!-k!}X^{k!}$. Note that, for each $n\geq 1$, $P_n(\alpha)=2^{n!}\alpha_n$. Moreover, since $P_n(\alpha)\in K$ for each $n\geq 1$, $P_n(\alpha)$ has degree at most m over $\mathbb{Q}$. Also, the length $L(P_n)=\sum_{k=1}^{n}2^{n!-k!}$ is bounded above by $2^{2(n!)}$. Thus, by Lemma *A.2* and by Theorem *A.10*,

$$\begin{aligned}H(\alpha_n) &= H(P_n(\alpha)/2^{n!})\\ &\leq 2^{m+1}H(P_n(\alpha))2^{m(n!)}\\ &\leq 2^{m(n!)}2^{m+1}2^{m}L(P_n)H(\alpha)^{m}(m+1)^{m/2}\end{aligned}$$

for each $n\geq 1$. Hence there exist a positive real number $\kappa_1>0$ depends only on m such that, for sufficiently large n,

$$H(\alpha_n)\leq 2^{(n!)\kappa_1}. \tag{3.10}$$

Finally, the inequality $|\alpha-\alpha_n|<\frac{1}{2^{n(n!)}}$ (which can be easily verified) together with (3.10), we have $\xi=\sum_{k=1}^{\infty}\frac{\alpha^{k!}}{2^{k!}}$ is an element of $\mathsf{S}_{K,\underline{q}}$.

3.5 The family of Mahler fields $(K_{\underline{q},\underline{u}})_{\underline{q},\underline{u}}$

Here after, let K denotes a fixed number field of degree m over $\mathbb{Q}$.

3.5.1 The fields $K_{\underline{q},\underline{u}}$

For any admissible pair $(\underline{q},\underline{u})$, we let $K_{\underline{q},\underline{u}}:=K\cup\mathsf{S}_{K,\underline{q},\underline{u}}$. The situation when $\underline{u}=\underline{n}$, we denote $K_{\underline{q},\underline{u}}$ by $K_{\underline{q}}$. The following theorem generalize Theorem 2.4.1, and provides a large supply of Mahler fields.

Theorem 3.5.1. *For each admissible pair $(\underline{q},\underline{u})$, the set $K_{\underline{q},\underline{u}}$ is a field.*

3(3) By Theorem 3.5.1, $K_{\underline{q},\underline{u}}$ is a Mahler field over K, and any Mahler field over K is a subfield of $K_{\underline{q}}$ for some $\underline{q}$. Another consequence from Theorem 3.5.1 is that any

Mahler field over K will be of the form $K\bigcup \mathsf{S}$ for some Mahler set S over K. It follows that, once if we assume the empty set is a Mahler set, then any subset of a Mahler set is a Mahler set and any subfield of a Mahler field is a Mahler field.

3(4) Suppose that $u_{n+1} \geq u_n + 1$ for sufficiently large n. Then, by Theorem 2.5.1, the set $\mathsf{S}_{\underline{q},\underline{u}}$ is not empty if and only if $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0$. Moreover, if the set $\mathsf{S}_{\underline{q},\underline{u}} \neq \emptyset$ then it has the power of continuum. Since $\mathsf{S}_{\underline{q},\underline{u}} \subseteq \mathsf{S}_{K,\underline{q},\underline{u}}$ we see that $\mathsf{S}_{K,\underline{q},\underline{u}}$ has the power of continuum, if $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n}$ is strictly positive.

The proof of Theorem 3.5.1 is very similar to Theorem 2.4.1. For the sake of completeness we shall give the proof of Theorem 3.5.1 here. It is enough to prove the following:

(a) $\xi + \eta \in K_{\underline{q},\underline{u}}$ for all ξ, η in $K_{\underline{q},\underline{u}}$,

(b) $\xi\eta \in K_{\underline{q},\underline{u}}$ for ξ, η in $K_{\underline{q},\underline{u}}$, and

(c) $\xi^{-1} \in K_{\underline{q},\underline{u}}$ for each nonzero element ξ in $K_{\underline{q},\underline{u}}$.

3.5.2 Proof of (a) and (b).

Let ξ and η be elements of $K_{\underline{q},\underline{u}}$. If both ξ and η are elements of K, then (a) and (b) follows trivially. Assume that at least one of ξ and η is not an element of K. Suppose that ξ is not an element of K. Then, there exist a positive integer t and a sequence $(\alpha_n)_{n\geq 1}$ of distinct elements of K all are of degree t over $\mathbb{Q}$, with two positive constants $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ such that

$$H(\alpha_n) \leq q_n^{\kappa_1} \quad \text{and} \quad 0 < |\xi - \alpha_n| \leq \frac{1}{q_n^{\kappa_2 u_n}}\cdot$$

holds for all sufficiently large n.

The case where η is an element of K. Let s_1 (respectively, s_2) be the smallest positive integer s for which infinitely many $\eta\alpha_n$ (respectively, infinitely many $\eta + \alpha_n$) has degree s over $\mathbb{Q}$.

If $\eta = 0$, then both $\xi\eta$ and $\xi + \eta$ are elements of $K_{\underline{q},\underline{u}}$. So, we assume that $\eta \neq 0$; and in this case we shall show that both $\xi\eta$ and $\xi + \eta$ are elements of $K_{\underline{q},\underline{u}}$. The idea is to show that:

- there exists a subsequence $(\alpha_{n_k})_{k\geq 1}$ of $(\alpha_n)_{n\geq 1}$,

- and a subsequence $(\alpha_{m_k})_{k\geq 1}$ of $(\alpha_n)_{n\geq 1}$,

such that $(\eta\alpha_{n_k})_{k\geq 1}$ is an $\mathbb{A}_{s_1}$-sequence for $\xi\eta$ and $(\eta+\alpha_{m_k})_{k\geq 1}$ is an $\mathbb{A}_{s_2}$-sequence for $\xi+\eta$.

By Lemma A.8, and since $\underline{q}=(q_n)_{n\geq 1}$ is strictly increasing, there exist a constant $\kappa_3=\kappa_3(\xi,\eta,m)$ (recall that m is the degree of K) such that

$$H(\eta\alpha_n)\leq q_n^{\kappa_3} \text{ and } H(\eta+\alpha_n)\leq q_n^{\kappa_3}, \tag{3.11}$$

holds for all sufficiently large n. Moreover,

$$0<|\eta\xi-\eta\alpha_n|=|\eta||\xi-\alpha_n|\leq\frac{|\eta|}{q_n^{\kappa_2 u_n}},$$

and since $\underline{q}=(q_n)_{n\geq 1}$ is strictly increasing, there exists a constant $\kappa_4=\kappa_4(\xi,\eta)$ such that

$$0<|\eta\xi-\eta\alpha_n|\leq\frac{1}{q_n^{\kappa_4 u_n}}, \tag{3.12}$$

holds for all sufficiently large n. From equation (3.11) and (3.12), we see that $\eta\xi$ is an element of $K_{\underline{q},\underline{u}}$.

Finally, the inequality

$$|\eta+\xi-(\eta+\alpha_n)|=|\xi-\alpha_n|\leq\frac{1}{q_n^{\kappa_2 u_n}}$$

together with (3.11) shows that $\eta+\xi$ is an element of $K_{\underline{q},\underline{u}}$.

The case where η is an element of $\mathsf{S}_{K,\underline{q},\underline{u}}$. There exists a positive integer t' and a sequence $(\beta_n)_{n\geq 1}$ of distinct elements of K all having same degree t' over $\mathbb{Q}$ with two positive constants κ_5,κ_6 depends on η such that

$$H(\beta_n)\leq q_n^{\kappa_5} \quad\text{and}\quad 0<\big|\eta-\beta_n\big|\leq\frac{1}{q_n^{\kappa_6 u_n}}$$

for all sufficiently large n.

We shall prove here that $\xi\eta$ is an element of $K_{\underline{q},\underline{u}}$, and the proof of $\xi+\eta\in K_{\underline{q},\underline{u}}$ is very similar and therefore we omit here.

If $\xi\eta$ is an element of K, then there is nothing to prove. Suppose that $\xi\eta\notin K$. Then we shall show that $\xi\eta\in\mathsf{S}_{K,\underline{q},\underline{u}}$. The idea is to show that there exists a subsequence $(\alpha_{n_k})_{k\geq 1}$ of

$(\alpha_n)_{n\geq 1}$ and a subsequence $(\beta_{m_k})_{k\geq 1}$ of $(\beta_n)_{n\geq 1}$ such that $(\alpha_{n_k}\beta_{m_k})_{k\geq 1}$ is an $\mathbb{A}_d$-sequence for $\xi\eta$, where d is the smallest positive integer r for which infinitely many $\alpha_{n_k}\beta_{n_k}$ has degree r over $\mathbb{Q}$.

By the triangle inequality, we have

$$0 < |\xi\eta - \alpha_n\beta_n| \leq |\xi|\big|\eta - \beta_n\big| + |\beta_n|\big|\xi - \alpha_n\big|.$$

Since the sequence $(\beta_n)_{n\geq 1}$ is bounded, there exists a positive constant $\kappa_7 = \kappa_7(\xi, \eta, m)$ such that

$$\big|\xi\eta - \alpha_n\beta_n\big| \leq \frac{1}{{q_n}^{\kappa_7 u_n}}, \tag{3.13}$$

holds for all sufficiently large n.

By Lemma A.8, and since $q_n \geq 2$ for all sufficiently large n,

$$\begin{aligned} H(\alpha_n\beta_n) &\leq 2^{3m} H(\alpha_n)^m H(\beta_n)^m \\ &\leq 2^{3m} q_n^{(\kappa_1+\kappa_5)m} \\ &\leq q_n^{3m+(\kappa_1+\kappa_5)m}. \end{aligned}$$

Hence, there exists a positive constant $\kappa_8 = \kappa_8(\xi, \eta, m)$ depends only on m such that

$$H(\alpha_n\beta_n) \leq q_n^{\kappa_8}, \tag{3.14}$$

holds for all sufficiently large n.

By (3.14) and (3.13), we see that $\xi\eta$ is an element of $K_{\underline{q},\underline{u}}$.

3.5.3 Proof of (c).

Let ξ be a nonzero element of $K_{\underline{q},\underline{u}}$. If $\xi \in K$, then (c) follows trivially (since K is a field). So, we assume that $\xi \in \mathsf{S}_{K,\underline{q},\underline{u}}$. Then, there exist a positive integer t'' and a sequence $(\alpha_n)_{n\geq 1}$ of distinct elements of K all having degree t'' over $\mathbb{Q}$, with two positive constants $\kappa = \kappa(\xi)$ and $\kappa' = \kappa'(\xi)$ such that

$$H(\alpha_n) \leq q_n^{\kappa} \quad \text{and} \quad 0 < \big|\xi - \alpha_n\big| \leq \frac{1}{q_n^{u_n\kappa'}}.$$

Now, $\xi^{-1} \in K_{\underline{q},\underline{u}}$ follows from the following equality

$$\left|\frac{1}{\xi} - \frac{1}{\alpha_n}\right| = \left|\frac{1}{\xi\alpha_n}\right| |\xi - \alpha_n|$$

and $H({\alpha_n}^{-1}) = H(\alpha_n)$ for all $n \geq 1$. This completes the proof of Theorem 3.5.1. □

The following corollary gives an upper bound for the type of the image of a U-number ξ, under a polynomial $P(X)$ with algebraic coefficients.

Corollary 3.5.2. *Let ξ be a U-number and let L be a field extension of K of degree n over K. Suppose that there exists a positive integer t such that ξ has an $\mathbb{A}_t$-sequence over K with respect to some admissible pair $(\underline{q}, \underline{u})$. Then, for any non-constant polynomial $P \in L[X]$, we have $P(\xi) \in \bigcup_{k=1}^{mn} U_k$.*

Proof. First note that, by the assumption, $\xi \in L_{\underline{q},\underline{u}}$. Moreover, since $P(\xi)$ is a transcendental number for any non-constant polynomial $P \in L[X]$, and $L_{\underline{q},\underline{u}}$ is a field (thanks to Theorem 3.5.1), $P(\xi) \in \mathsf{S}_{L,\underline{q},\underline{u}} \subseteq \bigcup_{k=1}^{mn} U_k$. □

3.6 Product of U-numbers with algebraic numbers

3(5) Let β be a nonzero element of K and let ξ be a U-number. Since two algebraically dependent complex numbers belong to the same Mahler class [6, p. 86], $\beta\xi$ is a U-number. *What is the type of the U-number $\beta\xi$?* We answer this question for a class of U-numbers which we termed as *special U-numbers.*

Definition 3.6.1. *Let m be a positive integer. A complex number ξ is called a* **special U-number of type** m *if there exists a sequence $(\alpha_n)_{n\geq 1}$ of distinct algebraic numbers of degree m such that*

$$0 < |\xi - \alpha_n| < H(\alpha_n)^{-\omega_n} \tag{3.15}$$

with $\omega_n \to \infty$ as $n \to \infty$, and, for some $r \geq 1$, and for all sufficiently large n, we have

$$H(\alpha_n) < H(\alpha_{n+1}) < H(\alpha_n)^{r\omega_n}. \tag{3.16}$$

By a result in [6, pp. 90-92], a special U-number of type m is indeed a U-number of type m. Special U-numbers of type 1 are called *special Liouville numbers.*

Theorem 3.6.2. *Let $(\underline{q}, \underline{u})$ be an admissible pair and let β be a nonzero element of K. Let ξ be a special U-number of type t such that $\xi \in \mathsf{S}_{K,\underline{q},\underline{u}}$. Suppose that there exists an $\mathbb{A}_t$-sequence $(\alpha_n)_{n\geq 1}$ of ξ (over K with respect to $(\underline{q}, \underline{u})$) which satisfies (3.15), (3.16); and for some $\kappa > 0$, $H(\beta\alpha_n) \geq H(\alpha_n)^\kappa$ for all sufficiently large n. Then $\beta\xi \in \mathbf{U}_D$, where D is the smallest integer d for which infinitely many $\beta\alpha_n$ have degree d.*

Proof. By the assumption on ξ, the sequence $(H(\beta\alpha_n))_{n\geq 1}$ is unbounded; and therefore, we can assume that

$$H(\beta\alpha_n) < H(\beta\alpha_{n+1}) \tag{3.17}$$

for each $n \geq 1$. By Lemma *A.8*, we have

$$H(\beta\alpha_n) < 2^{3m} H(\beta)^m H(\alpha_n)^m$$

for each $n \geq 1$. Moreover, the sequence $(H(\alpha_n))_{n\geq 1}$ is strictly increasing,

$$2^{3m} H(\beta)^m < H(\alpha_n)$$

for sufficiently large n. Thus,

$$H(\beta\alpha_n) < H(\alpha_n)^{m+1}. \tag{3.18}$$

for sufficiently large n.

Now the inequality (3.18) together with (3.17), we have

$$\begin{aligned} H(\beta\alpha_n) &< H(\beta\alpha_{n+1}) \\ &\leq H(\alpha_{n+1})^{m+1} \\ &< H(\alpha_n)^{r\omega_n(m+1)} \\ &\leq H(\beta\alpha_n)^{r(m+1)\omega_n/\kappa} \end{aligned} \tag{3.19}$$

for sufficiently large n. By (3.15) and (3.18),

$$|\beta\xi - \beta\alpha_n| = |\beta||\xi - \alpha_n| < |\beta| H(\beta\alpha_n)^{-\omega_n/m+1}. \tag{3.20}$$

By (3.20) and (3.19), we see that $\beta\xi$ is a $\mathbf{U}_D$-number, where D is the smallest integer d for which infinitely many $\beta\alpha_n$ have degree d. $\square$

Corollary 3.6.3. *Let ξ be a special Liouville number and let α be a nonzero element of K of degree d over $\mathbb{Q}$. If there exists a sequence $(\frac{a_n}{b_n})_{n\geq 1}$ of rational numbers which satisfies (3.15) and (3.16) such that for some $\kappa > 0$, $H(\alpha a_n/b_n) \geq a_n^{\kappa}$ for all sufficiently large n, then $\alpha\xi \in \mathbf{U}_d$.*

Proof. By Theorem 3.6.2, it suffices to show that, for sufficiently large n, $H(a_n/b_n) \leq a_n^{\kappa'}$ for some $\kappa' > 0$. But this can be easily verified. For example, since division of a U-number by a non-zero integer will not change the type of a U-number, we can assume that $0 < \xi < 1$. By this assumption, $H(a_n/b_n) = b_n$ for each $n \geq 1$. Moreover, since $\xi^{-1} \in \mathbf{L}$, there exist a real number $\kappa' > 0$ such that $b_n < a_n^{\kappa'}$ holds for all sufficiently large n. We thus have, for sufficiently large n, $H(a_n/b_n) \leq a_n^{\kappa'}$ for some $\kappa' > 0$. □

Corollary 3.6.3 can be applied to construct explicit examples of U_m-numbers for each positive integer m.

Example 3.6.4. *Let α be an mth root of 2 and let $\xi = 1 + \sum_{k=1}^{\infty} \frac{1}{3^{k!}}$. Then, $\alpha\xi \in \mathbf{U}_m$.*

Proof. For each $n \geq 1$, let $a_n = 3^{n!} + \sum_{k=1}^{n} 3^{n!-k!}$ and $b_n = 3^{n!}$. Since $b_n < a_n$, $H(a_n/b_n) = a_n$ for $n \geq 1$. Moreover,

$$\begin{aligned} 3^{(n+1)!} + \sum_{k=1}^{n+1} 3^{(n+1)!-k!} &< (3+1)^{(n+1)!} \\ &< 3^{2(n+1)!}. \end{aligned}$$

Hence,

$$H(\frac{a_n}{b_n}) < H(\frac{a_{n+1}}{b_{n+1}}) < H(\frac{a_n}{b_n})^{2(n+1)} \tag{3.21}$$

for $n \geq 1$. Now

$$\begin{aligned} \left|\xi - \frac{a_n}{b_n}\right| &= \frac{1}{3^{(n+1)!}} + \frac{1}{3^{(n+2)!}} + \cdots \\ &< \frac{1}{3^{(n+1)!-1}} < \frac{1}{3^{n(n!)}} < \frac{1}{\left(3^{n!} + \sum_{k=1}^{n} 3^{n!-k!}\right)^{n/2}}. \end{aligned}$$

Thus, for $n \geq 1$,

$$\left|\xi - \frac{a_n}{b_n}\right| < \frac{1}{H(\frac{a_n}{b_n})^{n/2}}. \tag{3.22}$$

For $n \geq 1$, the minimal polynomial of $\alpha a_n/b_n$ over $\mathbf{Z}$ is given by $b_n^m X^m - 2a_n^m$, and hence, $H(\frac{\alpha a_n}{b_n}) = 2a_n^m$. Thus, for all $n \geq 1$,

$$a_n^m < 2a_n^m = H(\frac{\alpha a_n}{b_n}). \tag{3.23}$$

By (3.21), (3.22), (3.23) and by Theorem 3.6.2, we see that $\alpha\xi \in \mathbf{U}_m$. □

Note. It is clear that, for any admissible pair $(\underline{q}, \underline{u})$, $\mathsf{S}_{K,\underline{q},\underline{u}} \subseteq \bigcup_{k=1}^{m} \mathbf{U}_k$. A natural question is: *when* $\mathsf{S}_{K,\underline{q},\underline{u}} \bigcap \mathbf{U}_m$ *is nonempty ?* By Corollary 3.6.3, $S_{K,\underline{q},\underline{u}} \bigcap \mathbf{U}_m \neq \emptyset$ if there exists an element $\alpha \in K$ of degree m over $\mathbb{Q}$, together with an element $\xi \in S_{\underline{q},\underline{u}}$ satisfies Corollary 3.6.3.

3.7 Finite extensions of Mahler fields

3.7.1 Numbers algebraic over Mahler fields

Let $\mathcal{K}$ be a Mahler field over K, and let $\mathcal{L}$ be a finite extension of $\mathcal{K}$. Since $\mathcal{L}$ is a finite separable extension of $\mathcal{K}$, $\mathcal{L} = \mathcal{K}(\eta)$ for some $\eta \in \mathcal{L}$. In this section, we prove that such a η is either an algebraic number or a U-number. More precisely, we prove the following:

Theorem 3.7.1. *A complex number η is algebraic over a Mahler field $\mathcal{K}$ if and only if it is either*

(i) an algebraic number, or

(ii) a U-number for which there exists a sequence $(P_i(X))_{i\geq 1}$ of polynomials which are of same degree over K, and converges coefficient-wise to a polynomial $P(X) \in \mathcal{K}[X]$ such that (for each $i \geq 1$) there exists a zero β_i of $P_i(X)$ satisfies:

$$|\eta - \beta_i| \leq \frac{1}{H(\beta_i)^{\omega_i}}, \tag{3.24}$$

with $\lim_{i\to\infty} \omega_i = \infty$.

3(6) By Theorem 3.7.1, we see that the algebraic closure of any Mahler field over K is a subset of $\mathbb{A} \cup \mathbf{U}$.

To prove Theorem 3.7.1, we need the following:

Proposition 3.7.2. *Let $(P_n)_{n\geq 1}$ be a sequence of polynomials of degree m over $\mathbb{C}$. For each $n \geq 1$, write $P_n(X) = a_0^{(n)} + a_1^{(n)}X + \cdots + a_m^{(n)}X^m$; let $\alpha_1^{(n)}, \ldots, \alpha_m^{(n)}$ be the set of zeros of P_n (counted with multiplicity). Suppose that for each k with $0 \leq k \leq m$, $\lim_{n\to\infty} a_k^{(n)} = \xi_k$ exists. Finally set $P(X) = \xi_0 + \xi_1 X + \cdots + \xi_m X^m$. Then*

(a) P_n converges uniformly on every compact subsets of $\mathbb{C}$ to the polynomial $P(X)$.

(b) If $\alpha_1, \ldots, \alpha_m$ are the zeros of $P(X)$, then the zeros of P_n's can be arranged so that for each k with $1 \leq k \leq m$, $\alpha_k^{(n)} \to \alpha_k$ as $n \to \infty$.

Proof. For $z \in \mathbb{C}$,

$$\begin{aligned} |P_n(z) - P(z)| &= \left|(a_0^{(n)} - \xi_0) + (a_1^{(n)} - \xi_1)z + \cdots + (a_m^{(n)} - \xi_m)z^m\right| \\ &\leq \left|a_0^{(n)} - \xi_0\right| + \left|a_1^{(n)} - \xi_1\right| |z| + \cdots + \left|a_m^{(n)} - \xi_m\right| |z|^m. \end{aligned}$$

Let $\varepsilon > 0$ be given, and let k be an integer with $0 \leq k \leq m$. Since $\lim_{n\to\infty} a_k^{(n)} = \xi_k$, there exist a positive integer N_k such that $|a_k^{(n)} - \xi_k| < \varepsilon$ for all $n \geq N_k$. Let $N = \max_{0\leq k\leq m} N_k$. We then have,

$$|P_n(z) - P(z)| < \varepsilon(1 + |z| + \cdots + |z|^m)$$

for all $n \geq N$. This shows that the convergence is uniform on every compact subset of $\mathbb{C}$. Hence (a) follows. Now (b) follows from the theorem of Hurwitz [27, p. 119]. □

Proof of Theorem 3.7.1. Let $(\underline{q}, \underline{u})$ be an admissible pair such that $\mathcal{K} \subseteq K_{\underline{q},\underline{u}}$ (such an admissible pair exists!). Let $\eta \in \mathbb{C}$ be algebraic of degree d over $\mathcal{K}$· Let $P(X)$ be the minimal polynomial of η over $\mathcal{K}$ of degree d. If $P \in K[X]$, then $\eta \in \mathbb{A}$. Suppose that $P(X)$ does not belong to $K[X]$. This means that some of the coefficients of P is in $\mathsf{S}_{K,\underline{q},\underline{u}}$. So, we can write

$$P(X) = f(X) + \xi_1 X^{r_1} + \cdots + \xi_k X^{r_k},$$

where $f \in K[X]$, $\xi_1, \xi_2, \ldots, \xi_k$ are elements of $\mathsf{S}_{K,\underline{q},\underline{u}}$ and $r_1, r_2, \ldots, r_k$ are non-negative distinct integers with $k \geq 1$.

For each i with $1 \leq i \leq k$, let $(\alpha_n^{(i)})_{n\geq 1}$ be a sequence of elements of K all having same degree over $\mathbb{Q}$, with two positive constants κ_{1i} and κ_{2i} (depends only on ξ_i) such that

$$H(\alpha_n^{(i)}) \leq {q_n}^{\kappa_{1i}} \text{ and } |\xi_i - \alpha_n^{(i)}| \leq \frac{1}{q_n^{\kappa_{2i} u_n}}. \tag{3.25}$$

Now for each $n \geq 1$, let $P_n(X) = f(X) + \alpha_n^{(1)} X^{r_1} + \cdots + \alpha_n^{(k)} X^{r_k}$ and let $\beta_1^{(n)}, \ldots, \beta_d^{(n)}$ be the roots of P_n (written with multiplicity). Clearly, the sequence $(P_n(X))_{n\geq 1}$ of polynomials converges to $P(X)$ coefficient-wise. And therefore, by Proposition 3.7.2, there exists an integer j with $1 \leq j \leq d$ such that $\beta_j^{(n)} \to \eta$ as $n \to \infty$. Without loss of generality, we assume that $j = 1$.

Let $\mathbb{B}(\eta, R)$ be an open ball for which P has no zeros other than η. By Proposition 3.7.2, the sequence $(P_n(X))_{n\geq 1}$ converges to $P(X)$ uniformly on compact subsets of $\mathbb{C}$. Therefore by Rouché's theorem, P and P_n have the same number of zeros inside $\mathbb{B}(\eta, R)$ for all sufficiently large n. Hence, for $1 < i \leq d$, $\left|\eta - \beta_i^{(n)}\right| > R$ for all sufficiently large n. Moreover, for each positive integer n,

$$\begin{aligned} |P_n(\eta)| &= |P_n(\eta) - P(\eta)| \leq \left|\alpha_n^{(1)} - \xi_1\right| |\eta|^{r_1} + \cdots + \left|\alpha_n^{(k)} - \xi_k\right| |\eta|^{r_k} \\ &\leq \frac{|\eta|^{r_1} + \cdots + |\eta|^{r_k}}{q_n^{\kappa u_n}}, \end{aligned}$$

where $\kappa = \min\limits_{1\leq i\leq k} \kappa_{2i}$. Thus,

$$\left|\eta - \beta_1^{(n)}\right| \prod_{1<i\leq d} \left|\eta - \beta_i^{(n)}\right| = |P_n(\eta)| \leq \frac{|\eta|^{r_1} + \cdots + |\eta|^{r_k}}{q_n^{\kappa u_n}}.$$

It follows that, for all sufficiently large n,

$$\left|\eta - \beta_1^{(n)}\right| \leq \frac{|\eta|^{r_1} + \cdots + |\eta|^{r_k}}{R^{d-1} q_n^{\kappa u_n}} \leq \frac{c}{q_n^{\kappa u_n}}, \tag{3.26}$$

for some constant $c = c(\eta) > 0$.

Write $f(X) = k_1 X^{s_1} + k_2 X^{s_2} + \cdots + k_l X^{s_l}$, where $s_1, s_2, \ldots, s_l$ are distinct non-negative integers. For $n \geq 1$, let

$$F_n(Y, X_1, \ldots, X_l, Z_1, \ldots, Z_k) = X_1 Y^{s_1} + \cdots + X_l Y^{s_l} + Z_1 Y^{r_1} + \cdots + Z_k Y^{r_k}.$$

Then,

$$F_n(\beta_1^{(n)}, k_1, k_2, \ldots, k_l, \alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(k)}) = P_n(\beta_1^{(n)}) = 0,$$

and hence, by Lemma *A*.11 we have

$$H(\beta_1^{(n)}) \leq 3^{3dm} \left(H(k_1) \ldots H(k_l) H(\alpha_n^{(1)}) \ldots H(\alpha_n^{(k)})\right)^m \qquad (n \geq 1).$$

Thus, for all sufficiently large n, $H(\beta_1^{(n)}) < q_n^{\kappa'}$ for some $\kappa' > 0$. Using this inequality in (3.26),

$$\left|\eta - \beta_1^{(n)}\right| \leq \frac{1}{H(\beta_1^{(n)})^{\kappa'' u_n}} \tag{3.27}$$

for some $\kappa'' > 0$, holds for all sufficiently large n.

Since the polynomials in the sequence $(P_n(X))_{n\geq 1}$ have bounded degree (namely d), there exists a subsequence $(\beta_1^{(n_k)})_{k\geq 1}$ of $(\beta_1^{(n)})_{n\geq 1}$ whose terms have same degree over K. This proves that η is a U-number for which the sequence $(\beta_1^{(n_k)})_{k\geq 1}$ satisfies the condition (ii) of Theorem 3.7.1.

Conversely, let $\eta \in \mathbb{C}$. If η is an algebraic number, then clearly η is algebraic over $\mathcal{K}$. So, assume that η is a U-number for which there exists a sequence $(\beta_i)_{i\geq 1}$ of complex numbers which are algebraic of same degree over K which satisfies Theorem 3.7.1. Then the sequence $(\beta_i)_{i\geq 1}$ converges to η and hence by the theorem of Hurwitz [27, p. 119], $P(\eta) = 0$. Since $P \in \mathcal{K}[X]$, η is algebraic over $\mathcal{K}$, and this completes the proof of Theorem 3.7.1.

3.7.2 Quadratic extensions of the Liouville field $\mathbb{Q}_{\underline{q},\underline{u}}$

Let $(\underline{q},\underline{u})$ be an admissible pair. Let d be a negative square-free integer, and let $K = \mathbb{Q}(\sqrt{d})$ be the associated imaginary quadratic field. In this section, we prove that $K_{\underline{q},\underline{u}}$ is a quadratic extension of $\mathbb{Q}_{\underline{q},\underline{u}}$. More precisely, we have the following:

Theorem 3.7.3. *Let K be an imaginary quadratic field. Then for any admissible pair $(\underline{q},\underline{u})$, the field $K_{\underline{q},\underline{u}}$ is a quadratic extension of $\mathbb{Q}_{\underline{q},\underline{u}}$. In other words, for each negative square-free integer d, $\mathbb{Q}(\sqrt{d})_{\underline{q},\underline{u}} = \mathbb{Q}_{\underline{q},\underline{u}}(\sqrt{d})$.*

Proof. We shall give the proof when $d = -1$. The proof for the general case is very similar, and we left to the interesting reader to make necessary changes.

Thus we have $K = \mathbb{Q}(\mathfrak{i})$, and we shall show that $[K_{\underline{q},\underline{u}} : \mathbb{Q}_{\underline{q},\underline{u}}] = 2$. To show this, we prove that $K_{\underline{q},\underline{u}} = \mathbb{Q}_{\underline{q},\underline{u}}(\mathfrak{i})$ and hence the result follows, since $[\mathbb{Q}_{\underline{q},\underline{u}}(\mathfrak{i}) : \mathbb{Q}_{\underline{q},\underline{u}}] = 2$. One way is easy, namely $\mathbb{Q}_{\underline{q},\underline{u}}(\mathfrak{i}) \subseteq K_{\underline{q},\underline{u}}$. This follows from the fact that $\mathbb{Q}_{\underline{q},\underline{u}} \subseteq K_{\underline{q},\underline{u}}$ and $\mathfrak{i} \in K$. Conversely, let $\xi \in K_{\underline{q},\underline{u}}$. If $\xi \in K$, then clearly $\xi \in \mathbb{Q}_{\underline{q},\underline{u}}(\mathfrak{i})$ since $K \subseteq \mathbb{Q}_{\underline{q},\underline{u}}(\mathfrak{i})$. So assume that, $\xi \in \mathsf{S}_{K,\underline{q},\underline{u}}$. Then by definition, there exists a sequence $(\alpha_n)_{n\geq 1}$ of elements of K of same degree over $\mathbb{Q}$ and two positive constants κ_1 and κ_2 (depends on ξ) such that

$$H(\alpha_n) \leq q_n^{\kappa_1} \quad \text{and} \quad 0 < \left|\xi - \alpha_n\right| \leq \frac{1}{q_n^{\kappa_2 u_n}}.$$

For each $n \geq 1$, write $\alpha_n = \frac{a_n}{b_n} + \mathrm{i}\frac{c_n}{d_n}$ and $\xi = \mathrm{Re}(\xi) + \mathrm{i}\,\mathrm{Im}(\xi)$. It is well known that $\lim_{n\to\infty} a_n/b_n = \mathrm{Re}(\xi)$ and $\lim_{n\to\infty} c_n/d_n = \mathrm{Im}(\xi)$. We shall show that $\mathrm{Re}(\xi), \mathrm{Im}(\xi) \in \mathbb{Q}_{\underline{q},\underline{u}}$, and hence the theorem follows.

By Lemma *A.2* and Lemma *A.8*, we have

$$H(a_n/b_n) = H\left(\frac{\alpha_n + \overline{\alpha_n}}{2}\right) \leq 2^3 H(\alpha_n + \overline{\alpha_n}) \leq 2^9 H(\alpha_n)^2 H(\overline{\alpha_n})^2 = 2^9 H(\alpha_n)^4,$$

since $H(\alpha_n) = H(\overline{\alpha_n})$. Moreover, the sequence $\underline{q}$ is increasing, we can find a real number $\kappa_3 > 0$ such that

$$H(a_n/b_n) \leq q_n^{\kappa_3}. \tag{3.28}$$

Using the inequality $\mathrm{Re}(z) \leq |z|$ (which holds for all nonzero complex numbers z), we have

$$\left|\mathrm{Re}(\xi) - \frac{a_n}{b_n}\right| \leq |\xi - \alpha_n| \leq \frac{1}{q_n^{\kappa_2 u_n}}. \tag{3.29}$$

By (3.28) and (3.29), we see that $\mathrm{Re}(\xi) \in \mathbb{Q}_{\underline{q},\underline{u}}$. The proof of $\mathrm{Im}(\xi) \in \mathbb{Q}_{\underline{q},\underline{u}}$, is very similar and we left to the reader. □

Note. We are not able to prove Theorem 3.7.3 for real quadratic fields. In the proof of Theorem 3.7.3 we mainly use the following inequality: *if $\alpha = a+b\sqrt{d}$ is a nonzero element of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, with $a, b \in \mathbb{Q}$, then $|a| \leq |\alpha|$ and $|b\sqrt{d}| \leq |\alpha|$*, which is in general not true for real quadratic fields.

3.8 The exponential function and Mahler fields

The classic Lindemann-Weierstrass theorem states that: *for any $n \geq 1$ number of $\mathbb{Q}$-linearly independent algebraic numbers $\alpha_1, \ldots, \alpha_n$, the transcendence degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ over $\mathbb{Q}$ is at least n.* As a simple corollary of this result we see that, for any nonzero algebraic number α, e^α is a transcendental number. It follows that $e^\alpha \notin K$ for any nonzero element $\alpha \in K$. We interested to ask whether a similar result hold for Mahler fields over K. We shall prove that under some conditions on the admissible pair $(\underline{q}, \underline{u})$, the Mahler fields $K_{\underline{q},\underline{u}}$ has the required property, namely: $e^\xi \notin K_{\underline{q},\underline{u}}$ for all nonzero element $\xi \in K_{\underline{q},\underline{u}}$. More precisely, we prove the following:

Theorem 3.8.1. *Let $(\underline{q}, \underline{u})$ be an admissible pair. Suppose that the sequences $\underline{q} = (q_n)_{n\geq 1}$ and $\underline{u} = (u_n)_{n\geq 1}$ satisfying the property that, the sequence $\underline{v} = (v_n)_{n\geq 1}$ defined by $v_n =$*

$\frac{u_n}{\log q_n}$, *is unbounded. Then for all nonzero element* $\xi \in K_{\underline{q},\underline{u}}$, *we have* $e^{\xi} \notin K_{\underline{q},\underline{u}}$.

For the proof of Theorem 3.8.1, we need the following result due to P. L. Cijsouw [10].

Theorem 3.8.2. *Let* $\xi \in \mathbb{C}, \xi \neq 0$, *and* α, β *be algebraic numbers of height at most* $H_1 \geq 3, H_2 \geq 3$ *and of degree* N *and* M *respectively. Then there exists an effectively computable positive real number* $C = C(\xi, N, M)$ *such that the following inequality holds:*

$$\max\{|\xi - \alpha|, |e^{\xi} - \beta|\} \geq \exp(-C \log H_1 \log H_2). \tag{3.30}$$

In fact, P. L. Cijsouw computed C explicitly. Indeed one can take C to be $C_1 e^{2\log|\xi|} N_1^3$, where C_1 is an absolute constant and $N_1 = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$.

3.8.1 Proof of Theorem 3.8.1.

Let $(\underline{q}, \underline{u})$ be an admissible pair which satisfies the assumptions of Theorem 3.8.1. First we show that these assumptions implies that e^{ξ} does not belong to $\mathsf{S}_{K,\underline{q},\underline{u}}$ for all $\xi \in \mathsf{S}_{K,\underline{q},\underline{u}}$.

Let ξ be any element of $\mathsf{S}_{K,\underline{q},\underline{u}}$. Then there exists a sequence $(\alpha_n)_{n\geq 1}$ of distinct elements of K of fixed degree N over $\mathbb{Q}$ and two positive constants κ_1 and κ_2 (depends only on ξ) such that for all sufficiently large n,

$$H(\alpha_n) \leq q_n^{\kappa_1} \text{ and } 0 < |\xi - \alpha_n| \leq \frac{1}{{q_n}^{\kappa_2 u_n}}. \tag{3.31}$$

Suppose for the sake of contradiction that $e^{\xi} \in \mathsf{S}_{K,\underline{q},\underline{u}}$. This assures the existence of a sequence $(\beta_n)_{n\geq 1}$ of distinct elements of K of fixed degree M over $\mathbb{Q}$ and there are positive constants κ_3, κ_4 (depends on ξ) such that

$$H(\beta_n) \leq {q_n}^{\kappa_3} \text{ and } |e^{\xi} - \beta_n| \leq \frac{1}{{q_n}^{\kappa_4 u_n}}. \tag{3.32}$$

It is easily seen that for any pair of positive integers s and H, there are only finitely many algebraic numbers of height at most H, and of degree at most s. Since the terms of the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ respectively are distinct, one can assume that $H(\alpha_n) \geq 3$ and $H(\beta_n) \geq 3$ for all sufficiently large n.

Set $\kappa_5 = \min\{\kappa_2, \kappa_4\}$ and let $\kappa_6 = \max\{\kappa_1, \kappa_3\}$. Now applying Theorem 3.8.2 to ξ, we deduce that

$$\frac{1}{{q_n}^{\kappa_5 u_n}} > e^{-D\kappa_6^2 (\log q_n)^2}$$

holds for all sufficiently large n; here D is a constant depends only on ξ and m. (Recall here that m is the degree of K over $\mathbb{Q}$.) From this inequality, we can deduce that $u_n < \frac{D\kappa_6^2}{\kappa_5} \log q_n$ for all sufficiently large n. Hence the sequence $(\frac{u_n}{\log q_n})_{n\geq 1}$ is bounded. This contradiction establish that $e^{\xi} \notin \mathsf{S}_{K,\underline{q},\underline{u}}$ for all $\xi \in \mathsf{S}_{K,\underline{q},\underline{u}}$.

To complete the proof of Theorem 3.8.1 (by the above discussion and since e^{α} is transcendental for each nonzero algebraic number α), it suffices to prove the following two statements.

(i) For any nonzero algebraic number α, e^{α} is not a U-number.

(ii) For any U-number ξ, e^{ξ} is a transcendental number.

Proof of the statement (i). Let α be a nonzero algebraic number. Taking $\xi = \alpha$ in Theorem 3.8.2, we see that for any algebraic number β with height $H \geq 3$ and degree d,

$$\begin{aligned} |e^{\alpha} - \beta| &\geq e^{-C \log H} \\ &= H^{-C} \end{aligned}$$

for some absolute constant $C > 0$. By this inequality, one can easily see that, e^{α} is not a U-number.

Proof of the statement (ii). Let ξ be a U-number of type t. Then to every positive integer n, there corresponds to an algebraic number α_n of degree t with $H(\alpha_n) > 1$ such that

$$|\xi - \alpha_n| < H(\alpha_n)^{-n}. \tag{3.33}$$

Suppose that e^{ξ} is an algebraic number, say β. Let n be a positive integer. Replacing e^{ξ} by β and α by α_n in (3.30), we see that

$$\begin{aligned} H(\alpha_n)^{-n} > |\xi - \alpha_n| &\geq e^{-C_1 \log H(\alpha_n)} \\ &= H(\alpha_n)^{-C_1} \end{aligned}$$

for some absolute constant $C_1 > 0$ and for each sufficiently large n. From this inequality, we deduce that $n < C_1$. This contradiction proves that e^{ξ} is transcendental for each U-number ξ. This proves (ii), and hence the proof of Theorem 3.8.1 is complete.

3.9 Formal power series and Mahler fields

Let $\underline{c} = (c_n)_{n\geq 1}$ be a sequence of nonzero elements of K, and let $F(z) = \sum_{n=1}^{\infty} c_n z^{n-1}$ be a formal power series associated to the sequence $\underline{c} = (c_n)_{n\geq 1}$. In this section, we study the transcendence of the values of $F(z)$ for U-numbers lies in Mahler fields over K. The result we prove here is a generalization of Theorem 5 in [13].

Theorem 3.9.1. *Let $(\underline{q}, \underline{u})$ be an admissible pair, and let $\underline{c} = (c_n)_{n\geq 1}$ be a sequence of non-zero elements of K. Let $G(z) = \sum_{n=1}^{\infty} \left(\frac{c_n}{q_n}\right) z^{n-1}$ be a power series over K. Suppose that the sequences $\underline{q}, \underline{u}$ and $\underline{c}$ satisfies the following conditions:*

(i) $\liminf_{n\to\infty} \dfrac{\log q_{n+1}}{\log q_n} = \lambda > 1,$

(ii) $\limsup_{n\to\infty} \dfrac{\log q_{n+1}}{\log q_n} = \infty,$

(iii) $\lim_{n\to\infty} \dfrac{\log q_n}{n} = \infty.$

(iv) For all sufficiently large n, $H(c_n) \leq q_n^{\kappa}$ for some real number κ with $0 < \kappa < 1$.

Then, for all $\xi \in S_{K,\underline{q},\underline{u}}$, we have either $G(\xi) \in K$ or $G(\xi) \in \bigcup_{i=1}^{m} \mathbf{U}_i$.

The proof of Theorem 3.9.1 is more technical, and the method of the proof is very similar to the proof of Theorem 5 in [13].

Proof. Let $\xi \in S_{K,\underline{q},\underline{u}}$. Then there exists a sequence $(\alpha_n)_{n\geq 1}$ of elements of K of same degree over $\mathbb{Q}$ and with two positive constants $\kappa_1 = \kappa_1(\xi)$ and $\kappa_2 = \kappa_2(\xi)$ such that for all sufficiently large n,

$$H(\alpha_n) \leq q_n^{\kappa_1} \text{ and } 0 < |\xi - \alpha_n| \leq \frac{1}{{q_n}^{\kappa_2 u_n}}\cdot \tag{3.34}$$

From the assumption (ii), one can deduce that, for each positive integer n, there exists an integer m_n such that $q_{m_n+1} > q_{m_n}^{u_n}$. We define $\underline{q}' := (q'_n)_{n\geq 1}$ to be the sequence of positive integers where $q'_n = q_{m_n}$ for each $n \geq 1$. The idea of the proof is to show that $G(\xi) \in K_{\underline{q}',\underline{u}}$. If $G(\xi) \in K$, then there is nothing to prove. So, we assume that $G(\xi) \notin K$, and in this case we prove that $G(\xi) \in S_{K,\underline{q}',\underline{u}}$. To prove $G(\xi) \in S_{K,\underline{q}',\underline{u}}$, it suffices to show that for some positive integer t, $G(\xi)$ has an $\mathbb{A}_t$-sequence over K with respect $(\underline{q}', \underline{u})$.

For each $n \geq 1$, let $G_n(z) = \sum_{k=1}^{n+1} \left(\frac{c_k}{q_k}\right) z^{k-1}$ and let $\beta_n = G_{m_n}(\alpha_n)$. Then clearly $\beta_n \in K$ for each $n \geq 1$, and has degree at most m over $\mathbb{Q}$. Let d be the smallest positive integer s such that infinitely many β_n has degree s. We will show that the sequence $(\beta_n)_{n\geq 1}$ is an $\mathbb{A}_d$-sequence for $G(\xi)$ over K with respect to $(\underline{q}', \underline{u})$.

There are two conditions needed to verify that $(\beta_n)_{n\geq 1}$ is an $\mathbb{A}_d$-sequence for $G(\xi)$ over K with respect $(\underline{q}', \underline{u})$. They are the following: *there exist two positive constants $\kappa_3 = \kappa_3(\xi)$ and $\kappa_4 = \kappa_4(\xi)$ such that, for sufficiently large n, we have*

(a) $H(\beta_n) \leq (q_n')^{\kappa_3}$, *and*

(b) $0 < |G(\xi) - \beta_n| \leq \frac{1}{(q_n')^{\kappa_4 u_n}}$.

Proof of (a). Let $Q_n = [q_1, \ldots, q_{n+1}]$ be the least common multiple of $q_1, \ldots, q_{n+1}$. From the condition (i), we can find a positive integer N such that for all $n \geq N+1$, $q_n < q_{n+1}^{\lambda_1}$ where $\lambda_1 = \frac{2}{\lambda+1}$. By induction, one can easily see that for each integer i with $N+1 \leq i \leq n$,

$$q_i < q_n^{\lambda_1^{n-i}}. \tag{3.35}$$

Hence for $n \geq N+1$,

$$\begin{aligned} Q_n &< [q_1, \ldots, q_{N+1}] q_n^{\lambda_1^{n-N-2}} q_n^{\lambda_1^{n-N-3}} \cdots q_n^{\lambda_1} q_n \\ &= [q_1, \ldots, q_{N+1}] q_n^{1+\lambda_1+\lambda_1^2+\cdots+\lambda_1^{n-N-2}} \leq [q_1, \ldots, q_{N+1}] q_n^{\frac{1}{1-\lambda_1}}. \end{aligned}$$

Moreover, the sequence $(q_n)_{n\geq 1}$ is strictly increasing, $[q_1, \ldots, q_{N+1}] \leq q_n$, for sufficiently large n, and therefore,

$$Q_n \leq q_n^{1+\frac{1}{1-\lambda_1}} \tag{3.36}$$

for all sufficiently large n.

By definition, $\beta_n = G_{m_n}(\alpha_n) = \displaystyle\sum_{k=1}^{m_n+1} \left(\frac{c_k}{q_k}\right) \alpha_n^{k-1}$, and hence,

$$\beta_n Q_{m_n+1} = a_1 c_1 + a_2(c_2 \alpha_n) + a_3(c_3 \alpha_n^2) + \cdots + a_{m_n+1}(c_{m_n+1} \alpha_n^{m_n}),$$

where $a_i = \frac{Q_{m_n+1}}{q_i}$ for $i = 1, 2, \ldots, m_n+1$. Thus, we have a polynomial

$$F(Y, X_1, X_2, \ldots, X_{m_n+1}) = Q_{m_n+1} Y - (a_1 X_1 + a_2 X_2 + \cdots + a_{m_n+1} X_{m_n+1})$$

with integer coefficients such that $F(\beta_n, c_1, c_2\alpha_n, \ldots, c_{m_n+1}\alpha_n^{m_n}) = 0$. Now by Lemma A.11 and since $a_i \leq Q_{m_n+1}$ for $i = 1, 2, \ldots, m_n + 1$, we have

$$H(\beta_n) \leq 3^{2m+(m_n+1)m} Q_{m_n+1}^m H(c_0)^m H(c_1\alpha_n)^m \ldots H(c_{m_n}\alpha_n m_n)^m.$$

By condition (iii), one can deduce that $q_n \geq 3^n$ for all sufficiently large n. Moreover, since $H(c_n) < q_n^\kappa$ and $H(\alpha_n) < q_n^{\kappa_1}$ for each n, by (3.36) and by Lemma A.8, we can find a positive constant $\kappa_3 > 0$ such that

$$H(\beta_n) \leq q_{m_n}^{\kappa_3} \tag{3.37}$$

for all sufficiently large n. This completes the proof of (a).

Proof of (b). Set $R = \sup_{z \in \bar{\mathbb{B}}(\xi,1)} \left|G'_{m_n}(z)\right|$, where $\bar{\mathbb{B}}(\xi, 1) = \{z : |\xi - z| \leq 1\}$ and G'_{m_n} is the derivative of G_{m_n}. Then, by the mean value theorem,

$$|G_{m_n}(\xi) - \beta_n| = |G_{m_n}(\xi) - G_{m_n}(\alpha_n)| < \frac{R}{q_{m_n}^{\kappa_2 u_{m_n}}}. \tag{3.38}$$

Moreover,

$$|G(\xi) - G_n(\xi)| = \left| \sum_{k \geq n+2} \left(\frac{c_k}{q_k}\right) \xi^{k-1} \right| \leq |\xi|^{n+1} \sum_{i=2}^{\infty} \left(\frac{|c_{n+i}|}{q_{n+i}}\right) |\xi|^{i-2}. \tag{3.39}$$

By Lemma A.3, $|c_n| \leq 2H(c_n)$ for each $n \geq 1$; further $H(c_n) \leq q_n^\kappa$ (thanks to condition (iv)) for sufficiently large n. Hence,

$$\begin{aligned} |G(\xi) - G_n(\xi)| &\leq 2\,|\xi|^{n+1} \sum_{i=2}^{\infty} \left(\frac{q_{n+i}^\kappa}{q_{n+i}}\right) |\xi|^{i-2} \\ &= 2\,|\xi|^{n+1} \sum_{i=2}^{\infty} \left(\frac{1}{q_{n+i}^{1-\kappa}}\right) |\xi|^{i-2} \\ &= \left(\frac{2\,|\xi|^{n+1}}{q_{n+2}^{1-\kappa}}\right) \sum_{i=2}^{\infty} \left(\frac{q_{n+2}}{q_{n+i}}\right)^{1-\kappa} |\xi|^{i-2} \end{aligned} \tag{3.40}$$

holds for all sufficiently large n.

Since $\lim_{n\to\infty} \frac{q_{n+1}}{q_{n+2}} = 0$, there exist a natural number M such that both $\left|\frac{q_{n+1}}{q_{n+2}}\right|$ and

$\left|\frac{\xi q_{n+1}}{q_{n+2}}\right|$ are strictly less than $\frac{1}{2}$ for all $n \geq M$. Hence by induction, one can show that,

$$|\xi^{i-2}|\left(\frac{q_{n+1}}{q_{n+i}}\right) < \frac{1}{2^{i-2}} \tag{3.41}$$

for a fixed integer $n \geq M-1$, and for all $i \geq 2$. Using the inequality (3.41) in (3.40), we have

$$|G(\xi) - G_n(\xi)| \leq \frac{4|\xi|^{n+1}}{q_{n+2}^{1-\kappa}}, \tag{3.42}$$

holds for all sufficiently large n. Thus, for all sufficiently large n, we have

$$|G(\xi) - G_{m_n}(\xi)| \leq \frac{4|\xi|^{m_n+1}}{q_{m_n+2}^{1-\kappa}} \leq \frac{4|\xi|^{m_n+1}}{q_{m_n}^{u_n(1-\kappa)}}.$$

From the condition (iii), one can deduce that $4|\xi|^{n+1} \leq q_n^{u_n(\frac{1-\kappa}{2})}$ for all sufficiently large n, and hence by (3.38) and (3.42),

$$\begin{aligned} |G(\xi) - \beta_n| &= |G(\xi) - G_{m_n}(\alpha_n)| \\ &\leq |G(\xi) - G_{m_n}(\xi)| + |G_{m_n}(\xi) - G_{m_n}(\alpha_n)| \\ &\leq \frac{1}{q_{m_n}^{u_n(\frac{1-\kappa}{2})}} + \frac{R}{q_{m_n}^{\kappa_2 u_{m_n}}} \leq \frac{1}{q_{m_n}^{u_n(\frac{1-\kappa}{2})}} + \frac{R}{q_{m_n}^{\kappa_2 u_n}}. \end{aligned}$$

Finally, we thus have, for all sufficiently large $n \geq M$,

$$|G(\xi) - \beta_n| \leq \frac{1}{(q_n')^{\kappa_4 u_n}} \tag{3.43}$$

for some $\kappa_4 > 0$. This completes the proof of (b). By (3.37) and (3.43), we see that $G(\xi) \in K_{\underline{q}',\underline{u}}$, and hence the theorem follows. □

3.10 Concluding remarks

The U-numbers are complex analog of Liouville numbers. They share many properties in common: for example

(i) both are uncountable sets of Lebesgue measure zero,

(ii) both are G_δ-subsets, since

$$\mathbf{L} = \bigcap_{n\geq 1} V_n \quad \text{with} \quad V_n = \bigcup_{q\geq 2}\bigcup_{p\in\mathbf{Z}} \left(\frac{p}{q} - \frac{1}{q^n}\,,\ \frac{p}{q} + \frac{1}{q^n}\right) \setminus \left\{\frac{p}{q}\right\},$$

and

$$\mathbf{U} = \bigcap_{n\geq 1} W_n \quad \text{with} \quad W_n = \bigcup_{m\geq 1}\ \bigcup_{\alpha\in\mathbb{A}_m^*} \mathbb{B}(\alpha, H(\alpha)^{-n}) \setminus \{\alpha\}\,,$$

where $\mathbb{A}_m^*$ denotes the set of nonzero algebraic numbers of degree m over $\mathbb{Q}$, and $\mathbb{B}(\alpha, r)$ denotes a open ball of radius r with center α in $\mathbb{C}$.

(iii) both sets have Hausdorff dimension zero.

Clearly $\mathbf{U}_m \subseteq \bigcap_{n\geq 1}\bigcup_{\alpha\in\mathbb{A}_m^*} \mathbb{B}(\alpha, H(\alpha)^{-n}) \setminus \{\alpha\}$. Since $\mathbf{U}_1 \cap \mathbf{U}_m = \emptyset$ for $m > 1$, by Baire's theorem, $\mathbf{U}_m \cap \mathbf{R} \subsetneq \bigcap_{n\geq 1}\bigcup_{\alpha\in\mathbb{A}_m^*\cap\mathbf{R}} (\alpha - H(\alpha)^{-n}, \alpha + H(\alpha)^{-n}) \setminus \{\alpha\}$. From this observation it follows that, there are real U-numbers ξ of type t with $1 < t < m$, for which there exists a sequence $(\alpha_n)_{n\geq 1}$ of algebraic numbers of degree m such that

$$|\xi - \alpha_n| \leq H(\alpha_n)^{-n}$$

for all $n \geq 1$. This shows that a U-number ξ of type t may be well approximated by algebraic numbers of degree $m > t$.

We complete this chapter with two interesting questions for which we do not know any answers at present.

Problem 1. Given any U-number ξ, can we find an admissible pair $(\underline{q}, \underline{u})$ such that $\xi \in K_{\underline{q},\underline{u}}$?

Problem 2. Let K be a number field and let L be a finite extension of K. Is it true that for any admissible pair $(\underline{q}, \underline{u})$, $L_{\underline{q},\underline{u}}$ is an algebraic extension of $K_{\underline{q},\underline{u}}$? If this is the case, what can we say about $[L_{\underline{q},\underline{u}} : K_{\underline{q},\underline{u}}]$?

Chapter 4

Some partial results towards Schanuel's conjecture for U-numbers

4.1 Introduction

A set of complex numbers $\{x_1, \ldots, x_n\}$ $(n \geq 1)$ is said to be **algebraically dependent** if there exist a nonzero polynomial $P \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$. A subset S of $\mathbb{C}$ is said to be **algebraically independent** if no nonempty finite subset of S is algebraically dependent. For any subfield L of $\mathbb{C}$, the **transcendence degree of** L (over $\mathbb{Q}$) is defined to be the cardinality of a maximal algebraically independent subset of L. We denote the transcendence degree of the field L by $\mathrm{trdeg}_{\mathbb{Q}}\mathsf{L}$.

In 1960, S. Schanuel made a famous conjecture concerning transcendence degree of certain field extensions over the field of rational numbers. The conjecture reads as follows:

Conjecture (Schanuel)**.** *Let n be a positive integer. Then, given any n number of $\mathbb{Q}$–linearly independent complex numbers $x_1, \ldots, x_n$, the transcendence degree of the field $\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ over $\mathbb{Q}$ is at least n.*

This conjecture was first stated in a course given by S. Lang [15, pp. 30-31] at Columbia in the 1960s. This conjecture is one of the major open problems in transcendental number theory, and if proved, then it includes most of the known results about the algebraic independence of the values of the exponential function. We are very far from the proof of this conjecture. For example, we do not know the truth of this conjecture in the case $n = 2$. There are very few cases where the conjecture is proved to be true. The special case where, $x_1, x_2, \ldots, x_n$ are all algebraic numbers is completely solved, and is now known as, Lindemann-Weierstrass theorem. For $n = 1$, this is just Hermite-Lindemann Theorem: *For any nonzero complex number ξ, at least one of the two numbers ξ, e^{ξ} is transcendental.*

There are other variations of Schanuel Conjecture. Schanuel himself made a similar conjecture for power series over $\mathbb{C}$, which reads as follows:

Conjecture (Schanuel, power series version)**.** *Given any $n \geq 1$ $\mathbb{Q}$-linearly independent formal power series $f_1(t), f_2(t), \ldots, f_n(t)$ over $\mathbb{C}$, the field $\mathbb{C}(t, f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n})$ has transcendence degree at least n over the field $\mathbb{C}(t)$.*

This conjecture was completely solved by J. Ax in [5].

4.2 Schanuel's conjecture and Liouville numbers

4(1) Schanuel's conjecture implies that for any $\mathbb{Q}$-linearly independent complex numbers $\xi_1, \ldots, \xi_n$ $(n \geq 1)$, the transcendence degree of the field

$$\mathbb{Q}(\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}) \tag{4.1}$$

is at least n. One may ask whether the transcendence degree is at least $n+1$ when the following additional assumption is made: for each $i = 1, \ldots, n$, one at least of the two numbers ξ_i, e^{ξ_i} is a U-number.

We first proceed the case where: for each $i = 1, \ldots, n$, one at least of the two numbers ξ_i, e^{ξ_i} is a Liouville number. We show that for each pair of integers (n, m) with $n \geq m \geq 1$, there exist uncountably many tuples $\xi_1, \ldots, \xi_n$ consisting of $\mathbb{Q}$–linearly independent real numbers, such that the numbers $\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}$ are all Liouville numbers, and the transcendence degree of the field $\mathbb{Q}(\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n})$ is $n + m$. More precisely, we prove the following:

Theorem 4.2.1. *Let $n \geq 1$ and $1 \leq m \leq n$ be given integers. Then there exist uncountably many n-tuples $(\xi_1, \ldots, \xi_n) \in \mathbf{L}^n$ such that $\xi_1, \ldots, \xi_n$ are linearly independent over $\mathbb{Q}$, $e^{\xi_i} \in \mathbf{L}$ for all $i = 1, 2, \ldots, n$ and*

$$\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}) = n + m.$$

4(2) Theorem 4.2.1 is tight when $n = 1$: the result does not hold for $m = 0$. Indeed, since the set of α in $\mathbf{L}$ such that α and e^α are algebraically dependent over $\mathbb{Q}$ is countable, one cannot get uncountably many $\alpha \in \mathbf{L}$ such that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha, e^\alpha) = 1$.

We need the following propositions and a corollary for the proof of Theorem 4.2.1.

Proposition 4.2.2.
Let $g_1, g_2, \ldots, g_n$ be polynomials in $\mathbb{C}[z]$. Then the following two conditions are equivalent.

(i) For $1 \le i < j \le n$, the function $g_i - g_j$ is not constant.

(ii) The functions $e^{g_1}, \ldots, e^{g_n}$ are linearly independent over $\mathbb{C}(z)$.

Proposition 4.2.3. *Let $f_1, f_2, \ldots, f_m$ be polynomials in $\mathbb{C}[z]$. Then the following two conditions are equivalent.*

(i) For any nonzero tuple $(a_1, \ldots, a_m) \in \mathbb{Z}^m$, the function $a_1 f_1 + \cdots + a_m f_m$ is not constant.

(ii) The functions $e^{f_1}, \ldots, e^{f_m}$ are algebraically independent over $\mathbb{C}(z)$.

Since the functions $1, z, z^2, \ldots, z^m, \ldots$ are linearly independent over $\mathbb{C}$, we deduce from Proposition 4.2.3 the following:

Corollary 4.2.4. *The functions*

$$z,\ e^z,\ e^{z^2},\ \ldots,\ e^{z^m}, \ldots$$

are algebraically independent over $\mathbb{C}$.

4(3) For the proof of Proposition 4.2.2 and Proposition 4.2.3, we introduce the quotient vector space $\mathcal{V} = \mathbb{C}[z]/\mathbb{C}$ and the canonical surjective linear map $s : \mathbb{C}[z] \to \mathcal{V}$ with kernel $\mathbb{C}$. Assertion (i) in Proposition 4.2.2 means that $s(g_1), \ldots, s(g_n)$ are pairwise distinct, while assertion (i) in Proposition 4.2.3 means that $s(f_1), \ldots, s(f_m)$ are linearly independent over $\mathbb{Q}$.

4.2.1 Proof of Proposition 4.2.2

(i) ⇒ (ii). We prove this result by induction on n. For $n = 1$, there is no condition on g_1, the function e^{g_1} is not zero, hence the result is true. Assume $n \geq 2$ and assume that, for any $n' < n$, the result holds for n'. Let $A_1, \dots, A_n$ be polynomials in $\mathbb{C}[z]$, not all of which are zero; consider the function

$$G(z) = A_1(z)e^{g_1(z)} + \cdots + A_n(z)e^{g_n(z)}.$$

The goal is to check that G is not the zero function. The idea is to show that the associated function $H = e^{-g_n}G$ is not the zero function (and hence G is not the zero function).

Using the induction hypothesis, we may assume $A_i \neq 0$ for $1 \leq i \leq n$. Define $h_i = g_i - g_n$ $(1 \leq i \leq n-1)$. From $h_i - h_j = g_i - g_j$, we deduce that $s(h_1), s(h_2), \dots, s(h_{n-1})$ are distinct in $\mathcal{V}$. By definition,

$$H(z) = A_1(z)e^{h_1(z)} + \cdots + A_{n-1}(z)e^{h_{n-1}(z)} + A_n(z).$$

Write $D = d/dz$ and let N be a positive integer with $N > \deg A_n$, so that $D^N A_n = 0$. One can easily see that, for any nonzero polynomial $A \in \mathbb{C}[z]$ and any non-constant polynomial $h \in \mathbb{C}[z]$, we can write $D(Ae^h) = Be^h$ for some nonzero polynomial $B \in \mathbb{C}[z]$. Therefore, for $i = 1, \dots, n-1$ and for any integer $t \geq 0$, we can write

$$D^t\bigl(A_i(z)e^{h_i(z)}\bigr) = A_{i,t}(z)e^{h_i(z)}$$

where $A_{i,t}$ is a nonzero polynomial in $\mathbb{C}[z]$. By the induction hypothesis, applied to the functions $h_1, \dots, h_{n-1}$, we have

$$D^N H(z) = A_{1,N}(z)e^{h_1(z)} + \cdots + A_{n-1,N}(z)e^{h_{n-1}(z)},$$

is not the zero function, hence $H \neq 0$. This proves that G is not the zero function.

(ii) ⇒ (i). If $g_1 - g_2$ is a constant c, then the pair (e^{g_1}, e^{g_2}) is a zero of the linear form $X_1 - e^c X_2$ over $\mathbb{C}$.

4.2.2 Proof of Proposition 4.2.3

(i) ⇒ (ii). Assume that for any nonzero tuple $(a_1, \dots, a_m) \in \mathbb{Z}^m$, $a_1 f_1 + \cdots + a_m f_m$ is not constant. Consider a nonzero polynomial

$$P(X_1,\ldots,X_m)=\sum_{\lambda_1=0}^{d_1}\cdots\sum_{\lambda_m=0}^{d_m}p_{\lambda_1,\ldots,\lambda_m}(z)X_1^{\lambda_1}\ldots X_m^{\lambda_m}\in\mathbb{C}[z,X_1,\ldots,X_m]$$

and let F be the entire function $F=P(e^{f_1},\ldots,e^{f_m})$. Let $\{g_1,\ldots,g_n\}$ be the set of functions $\lambda_1 f_1+\cdots+\lambda_m f_m$ where $p_{\lambda_1,\ldots,\lambda_m}(z)\neq 0$. For $1\leq i\leq n$, we set

$$A_i(z)=p_{\lambda_1,\ldots,\lambda_m}(z)\in\mathbb{C}[z]$$

where $(\lambda_1,\ldots,\lambda_m)$ is defined by $g_i=\lambda_1 f_1+\cdots+\lambda_m f_m$, so that

$$F(z)=A_1(z)e^{g_1(z)}+\cdots+A_n(z)e^{g_n(z)}.$$

The assumption on $f_1,\ldots,f_m$ implies that the functions $g_1,\ldots,g_n$ satisfy the assumption (i) of Proposition 4.2.2, hence the function F is not the zero function. This proves that the functions e^{f_1}, …, e^{f_m} are algebraically independent over $\mathbb{C}(z)$.
(ii) $\Rightarrow$ (i). If there exists $(a_1,\ldots,a_m)\in\mathbb{Z}^m\setminus\{(0,\ldots,0)\}$ such that the function $a_1f_1+\cdots+a_mf_m$ is a constant c, then for the polynomial

$$P(X_1,\ldots,X_m)=\prod_{a_i>0}X_i^{a_i}-e^c\prod_{a_i<0}X_i^{|a_i|}$$

we have $P(e^{f_1},\ldots,e^{f_m})=0$, therefore the functions $e^{f_1},\ldots,e^{f_m}$ are algebraically dependent over $\mathbb{C}$ (hence over $\mathbb{C}(z)$).

4.2.3 Proof of Theorem 4.2.1

We are now in position to prove Theorem 4.2.1, which states that *for integers n and m with $1\leq m\leq n$, there are uncountably many n-tuples $(\alpha_1,\ldots,\alpha_n)\in\mathbf{L}^n$ such that $\alpha_1,\ldots,\alpha_n$ are linearly independent over $\mathbb{Q}$, $e^{\alpha_i}\in\mathbf{L}$ for all $i=1,2,\ldots,n$ and*

$$\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha_1,\ldots,\alpha_n,e^{\alpha_1},\ldots,e^{\alpha_n})=n+m.$$

Let us start the proof. Let n and m be integers such that $1\leq m\leq n$. We shall prove the assertion by induction on $m \geq 1$. Assume $m = 1$. We prove the result for all $n \geq 1$. For each nonzero polynomial $P(X_0, X_1, \ldots, X_n) \in \mathbb{Q}[X_0, \ldots, X_n]$ in $n+1$ variables with rational coefficients, we define a function

$$f_P : \mathbb{C} \to \mathbb{C} \text{ by } f_P(z) = P(z, e^z, \ldots, e^{z^n}).$$

Using Corollary 4.2.4, we deduce that the set $Z(f_P)$, of all zeros of f_P in $\mathbb{C}$, as the zero locus of a non-zero complex analytic map f_P, is discrete in $\mathbb{C}$. Hence its intersection with $\mathbb{R}$, say Z_P, is discrete in $\mathbb{R}$. Therefore, $\mathbb{R} \setminus Z_P$ is both open and dense in $\mathbb{R}$. From Proposition 1.2.2 and Baire's theorem, it follows that the set

$$E = \left\{ \alpha \in \mathbf{L} \mid \quad e^{\alpha^j} \in \mathbf{L} \text{ for } j = 1, \ldots, n \right\} \cap \bigcap_{P \in \mathbb{Q}[X_0, \ldots, X_n] \setminus \{0\}} (\mathbb{R} \setminus Z_P)$$

is a G_δ–subset of $\mathbb{R}$. Therefore, by Corollary 1.2.5, E is uncountable. For any $\alpha \in E$, the numbers $\alpha, e^\alpha, e^{\alpha^2}, \ldots, e^{\alpha^n}$ are in $\mathbf{L}$ and are algebraically independent over $\mathbb{Q}$. Since α is a Liouville number, $\alpha^2, \ldots, \alpha^n$ are also Liouville numbers and $\alpha, \alpha^2, \ldots, \alpha^n$ are linearly independent over $\mathbb{Q}$ (and they are trivially algebraically dependent over $\mathbb{Q}$). We then have

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha, \alpha^2, \ldots, \alpha^n, e^\alpha, \ldots, e^{\alpha^n}) = n + 1$$

and we conclude that the assertion is true for $m = 1$ and for all $n \geq 1$.

Assume that $1 < m \leq n$. Also, suppose the assertion is true for $m - 1$ and for all $n \geq m - 1$. In particular, the assertion is true for $m - 1$ and $n - 1$. Hence, there are uncountably many $n - 1$ tuples $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbf{L}^{n-1}$ such that $\alpha_1, \ldots, \alpha_{n-1}$ are linearly independent over $\mathbb{Q}$; $e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}$ are Liouville numbers and

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}) = n + m - 2. \tag{4.2}$$

Choose such an $(n-1)$-tuple $(\alpha_1, \ldots, \alpha_{n-1})$. Consider the following subset of $\mathbb{R}$:

$$E' = \{\alpha \in \mathbb{R} \mid \quad \alpha, e^\alpha \text{ are algebraically independent over } \mathbb{Q}(\alpha_1, \ldots \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})\} \tag{4.3}$$

If $P(X, Y) \in \mathbb{Q}(\alpha_1, \ldots, \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})[X, Y]$ is a polynomial, we define an analytic function $f(z) = P(z, e^z)$ in $\mathbb{C}$. Since z, e^z are algebraically independent functions over $\mathbb{C}$ (by Corollary 4.2.4), f is a nonzero function when P is a nonzero polynomial. Therefore, the set of zeros of f in $\mathbb{C}$ is countable when $P(X,Y)$ is nonzero. Since there are only countably many polynomials $P(X,Y)$ with coefficients in the field $\mathbb{Q}(\alpha_1,\dots\alpha_{n-1},e^{\alpha_1},\dots,e^{\alpha_{n-1}})$, we conclude that $\mathbb{R}\backslash E'$ is countable. Therefore $F = E'\cap \mathbf{L}$ is uncountable. For each $\alpha\in F$, the two numbers α, e^{α} are algebraically independent over $\mathbb{Q}(\alpha_1,\dots,\alpha_{n-1},e^{\alpha_1},\dots,e^{\alpha_{n-1}})$. From (4.2) we deduce

$$\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha_1,\dots,\alpha_{n-1},\alpha,e^{\alpha_1},\dots,e^{\alpha_{n-1}},e^{\alpha}) = n+m.$$

This completes the proof of Theorem 4.2.1. □

4(4) The proof of Theorem 4.2.1 shows that, it can be extended to any $G_\delta-$subset of $\mathbb{C}$. Thus we have the following:

Theorem 4.2.5. *Let $n\geq 1$ and $1\leq m\leq n$ be given integers. Let E be a $G_\delta-$subset of $\mathbb{C}$. Then there exist uncountably many n-tuples $(\xi_1,\dots,\xi_n)\in \mathsf{E}^n$ such that $\xi_1,\dots,\xi_n$ are linearly independent over $\mathbb{Q}$, $e^{\xi_i}\in\mathsf{E}$ for all $i=1,2,\dots,n$ and*

$$\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(\xi_1,\dots,\xi_n,e^{\xi_1},\dots,e^{\xi_n}) = n+m.$$

4.3 Schanuel's conjecture and U-numbers

Proposition 4.3.1. *Let G be a $G_\delta-$subset of $\mathbb{C}$. Then for any line L in $\mathbb{C}$, $G\setminus L$ is a $G_\delta-$subset of $\mathbb{C}$.*

Proof. By Baire's theorem it is enough to prove that, for every open dense subset E of $\mathbb{C}$, $E\setminus L$ is a dense open subset of $\mathbb{C}$. So, let E be an open dense subset of $\mathbb{C}$. Since L is closed and E is open, $E\setminus L$ is open in $\mathbb{C}$. Let F be any open subset of $\mathbb{C}$. Then $F\setminus L$ is open, and hence, $E\cap(F\setminus L)$ is nonempty. But this implies $F\cap(E\setminus L)$ is nonempty. This proves the proposition. □

It is clear that $\mathbf{U}$ is a G_δ–subset of $\mathbb{C}$ (see Section 3.10). Combining Proposition 4.3.1 (where we take $L=\mathbb{R}$) with Theorem 4.2.5, we have the following:

Theorem 4.3.2. *Let $n\geq 1$ and $1\leq m\leq n$ be given integers. Then there exist uncountably many n-tuples $(\xi_1,\dots,\xi_n)$ of U-numbers of degree >1 such that $e^{\xi_i}\in U\setminus U_1$ for all $i=1,2,\dots,n$ and*

$$\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(\xi_1,\dots,\xi_n,e^{\xi_1},\dots,e^{\xi_n}) = n+m.$$

4.4 Schanuel's conjecture and G_δ-subsets

4(5) Let n be a positive integer, and let $\xi_1, \dots, \xi_n$ be $\mathbb{Q}$-linearly independent complex numbers. To each such an n-tuple $(\xi_1, \dots, \xi_n)$, we associate the following set:

$$\mathsf{E}_{(\xi_1,\dots,\xi_n)} := \{c \in \mathbb{C} \mid \quad \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(c\xi_1, c\xi_2, \dots, c\xi_n, e^{c\xi_1}, e^{c\xi_2}, \dots, e^{c\xi_n}) \geq n+1\}.$$

In this section, we prove that for any n-tuple $(\xi_1, \dots, \xi_n)$ of $\mathbb{Q}$-linearly independent complex numbers, $\mathsf{E}_{(\xi_1,\dots,\xi_n)}$ is a $G_\delta-$subset of $\mathbb{C}$. In fact, we prove a more general result, which was suggested by Professor M. Waldschmidt in a personal communication.

Theorem 4.4.1. *Let $\xi_1, \dots, \xi_n, \eta_0, \eta_1, \dots, \eta_t$ be $n+t+1$ complex numbers, with $n \geq 1$ and $t \geq 0$. Assume $\xi_1, \dots, \xi_n$ are $\mathbb{Q}$-linearly independent, $\eta_0 \neq 0$ and $\eta_1/\eta_0, \dots, \eta_t/\eta_0$ are algebraically independent over $\mathbb{Q}$. Then there exists a G_δ-subset E of $\mathbb{C}$ such that for all $c \in \mathsf{E}$, the numbers*

$$c\eta_0, c\eta_1, \dots, c\eta_t, e^{c\xi_1}, \dots, e^{c\xi_n} \tag{4.4}$$

are algebraically independent over $\mathbb{Q}$; in other terms

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(c\eta_0, c\eta_1, \dots, c\eta_t, e^{c\xi_1}, \dots, e^{c\xi_n}) = n+t+1. \tag{4.5}$$

Proof. From the assumption on the ξ_i's, and since $\eta_0 \neq 0$, the numbers $\xi_1/\eta_0, \dots, \xi_n/\eta_0$ are linearly independent over $\mathbb{Q}$. Hence, the polynomials $f_i(Z) = \xi_i Z/\eta_0$, for $i = 1, 2, \dots, n$ satisfies the condition (i) of Proposition 4.2.3. Therefore, the functions $e^{\xi_1 Z/\eta_0}, \dots, e^{\xi_n Z/\eta_0}$ are algebraically independent over $\mathbb{C}(Z)$.

To each nonzero polynomial $P \in \mathbb{Z}[Y_1 \dots, Y_t, Z, X_1, \dots, X_n]$, we associate an entire function F_P from $\mathbb{C}$ to $\mathbb{C}$ by the formula:

$$F_P(z) = P(z\eta_1/\eta_0, \dots, z\eta_t/\eta_0, z, e^{z\xi_1/\eta_0}, \dots, e^{z\xi_n/\eta_0}).$$

Since $\eta_1/\eta_0, \dots, \eta_t/\eta_0$ are algebraically independent over $\mathbb{Q}$, the polynomial

$$G_p(Z, X_1, \dots, X_n) = P(Z\eta_1/\eta_0, \dots, Z\eta_t/\eta_0, Z, X_1, \dots, X_n) \tag{4.6}$$

is a nonzero polynomial in $\mathbb{Q}(\eta_1/\eta_0, \dots, \eta_t/\eta_0)[Z, X_1, \dots, X_n]$. Moreover, the functions $e^{\xi_1 Z/\eta_0}$, $e^{\xi_2 Z/\eta_0}, \dots, e^{\xi_n Z/\eta_0}$ are algebraically independent over $\mathbb{C}(Z)$; the same is true over

$\mathbb{Q}(\eta_1/\eta_0, \dots, \eta_t/\eta_0)(Z)$. Hence, $F_P(z)$ is not the zero function, and therefore the function $F_P(\eta_0 z)$ is also not the zero function. It follows that the set $Z(F_P(\eta_0 z))$ of zeros of the nonzero analytic map $F_P(\eta_0 z)$, is discrete in $\mathbb{C}$. Hence its complement is a dense open subset of $\mathbb{C}$. Thus by Baire's theorem, the set

$$E = \bigcap_{P \in \mathbb{Z}[Y_1,\dots,Y_t,Z,X_1,\dots,X_n]\setminus\{0\}} \mathbb{C} \setminus Z(F_P(\eta_0 z))$$

is a G_δ subset of $\mathbb{C}$.

If $c \in E$, then for any nonzero polynomial $P \in \mathbb{Z}[Y_1 \dots, Y_t, Z, X_1, \dots, X_n]$ we have $F_P(c\eta_0) \neq 0$. That is, $P(c\eta_0, c\eta_1, \dots, c\eta_t, e^{c\xi_1}, \dots, e^{c\xi_n}) \neq 0$. In other words,

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(c\eta_0, c\eta_1, \dots, c\eta_t, e^{c\xi_1}, \dots, e^{c\xi_n}) = n + t + 1.$$

This completes the proof of Theorem 4.4.1. □

By taking $t = 0$, and $\eta_0 = \xi_1$ in Theorem 4.4.1 we get the following:

Theorem 4.4.2. *Let $\xi_1, \xi_2, \dots, \xi_n$ be n complex numbers with $n \geq 1$, and they are linearly independent over $\mathbb{Q}$. Then, there exists a G_δ-subset E of $\mathbb{C}$ such that for all $c \in \mathsf{E}$, we have*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(c\xi_1, c\xi_2, \dots, c\xi_n, e^{c\xi_1}, e^{c\xi_2}, \dots, e^{c\xi_n}) \geq n + 1.$$

Appendix A

Heights and Mahler's classification

A.1 Height of an algebraic number

Here we recall some of the properties of absolute logarithmic Weil height and of (usual) height of an algebraic number which will be needed for our thesis. We also prove some new results here.

Definition A.1. *Let α be an algebraic number with the minimal polynomial $f(X) = a_0X^m + \cdots + a_m$ over $\mathbb{Z}$, and let $\{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_m\}$ be the set of all Galois conjugates of α. Then, the* **absolute logarithmic Weil height** *$h(\alpha)$, of α is given by*

$$h(\alpha) = \frac{1}{m}\left(\log a_0 + \sum_{i=1}^{m} \log\max\{1, |\alpha_i|\}\right).$$

while

$$H(\alpha) = H(f) = \max_{0\le i\le m}|a_i|$$

is called the **usual height**, *or simply* **height**, *of α (and of the polynomial f).*

Throughout this section, let α be denote an algebraic number of degree m over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$. It is known that every element of K is of the form $P(\alpha)$ for some polynomials $P(x) \in \mathbb{Q}[x]$. We are interested to study the relationship between $H(\alpha)$ and $H(P(\alpha))$. First, we recall some of the results in this direction.

Lemma A.2. *[7, p.222] Let a, b, and c be integers with $c \neq 0$. Then we have*

$$H\left(\frac{a\alpha + b}{c}\right) \le 2^{m+1}H(\alpha)\max\{|a|, |b|, |c|\}^m.$$

Lemma A.3. *[16, p.224] Let $\phi(X)$ and $\phi_1(X)$ be two polynomials with integer coefficients of degree d and d_1 respectively such that $\phi_1(X)$ divides $\phi(X)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the zeros of $\phi(X)$ (counted with multiplicity). Then*

$$\max_{1\le k\le d}|\alpha_k| \le 2H(\phi)$$

and there is a constant $c > 0$ that depends only on d_1, such that

$$H(\phi_1) < cH(\phi)^{d_1+1}.$$

The following Lemma is due to C. L. Siegel.

Lemma A.4. *Following the notations of Lemma A.3, we have*

$$H(\phi_1) \leq \frac{d!}{d_1!}H(\phi)$$

The following lemma gives an upper bound for the absolute logarithmic Weil height of sum and product of elements of K.

Lemma A.5. *[30, p.75] For any algebraic numbers γ and β we have*

(i) $h(\gamma\beta) \leq h(\gamma) + h(\beta)$

(ii) $h(\gamma + \beta) \leq \log 2 + h(\gamma) + h(\beta)$.

Lemma A.6. *[30, p.80] For any algebraic number γ of degree d over $\mathbb{Q}$, we have*

$$\frac{1}{d}\log H(\gamma) - \log 2 \leq h(\gamma) \leq \frac{1}{d}\log H(\gamma) + \frac{1}{2d}\log(d+1).$$

Corollary A.7. *For any algebraic number γ, we have*

$$e^{h(\gamma)} \leq 2H(\gamma).$$

Proof. If γ is a rational number, say $\gamma = \frac{a}{b}$, then

$$h(\gamma) = \begin{cases} \log|b|, & \text{if } a \leq b; \\ \log|a|, & \text{if } a > b, \end{cases}$$

and hence the result follows trivially.

For otherwise, by Lemma A.6, we have

$$h(\gamma) \leq \frac{1}{d}\log\left(\sqrt{d+1}H(\gamma)\right).$$

Therefore,

$$\begin{aligned} e^{h(\gamma)} &< ((d+1)H(\gamma))^{1/d} \\ &< \sqrt[d]{d+1}H(\gamma), \end{aligned}$$

where d is the degree of γ over $\mathbf{Q}$. Since $\sqrt[d]{d+1} \leq 2$ for $d \geq 2$, the result follows. □

The following lemma gives an upper bound for the height of sum and product of elements of K.

Lemma A.8. *Let γ, β be elements of K. Then we have*

(i) $H(\gamma\beta) \leq 2^{3m} H(\gamma)^m H(\beta)^m$.

(ii) $H(\gamma+\beta) \leq 2^{4m} H(\gamma)^m H(\beta)^m$.

Proof. Let N be the degree of $\gamma\beta$ over $\mathbb{Q}$. Then, by Lemma A.5 and Lemma A.6, we have

$$\begin{aligned} \frac{1}{N}\log H(\gamma\beta) &\leq \log 2 + h(\gamma\beta) \\ &\leq \log 2 + h(\gamma) + h(\beta). \end{aligned}$$

By Corollary A.7, and since $N \leq m$, we have

$$\begin{aligned} H(\gamma\beta) &\leq \left(2e^{h(\gamma)}e^{h(\beta)}\right)^m \\ &\leq 2^{3m} H(\gamma)^m H(\beta)^m. \end{aligned}$$

This proves (i).

Now we prove the result for the sum $\gamma+\beta$. Since the degree of $\gamma+\beta$ (over $\mathbb{Q}$) is at most m, by Lemma A.5 and Lemma A.6, we have

$$\begin{aligned} \log H(\gamma+\beta) &\leq m(\log 2 + h(\gamma+\beta)) \\ &\leq m(2\log 2 + h(\gamma) + h(\beta)). \end{aligned}$$

Using Corollary A.7, we have

$$\begin{aligned} H(\gamma+\beta) &\leq 2^{2m} e^{mh(\gamma)} e^{mh(\beta)} \\ &\leq 2^{4m} H(\gamma)^m H(\beta)^m, \end{aligned}$$

which proves (ii). □

In the following, let $P(X) = a_0X^r + \cdots + a_r$ denote a non-constant polynomial of degree $r \leq m$ over $\mathbf{Z}$ and let s be the degree of $P(\alpha)$ over $\mathbb{Q}$. Let $\Phi_P : \mathbf{Z}[X] \to \mathbf{Z}[X]$ be the ring homomorphism defined by $X \mapsto P(X)$.

Proposition A.9. *For all $Q(X) \in \mathbf{Z}[X]$, we have*

$$H(\Phi_P(Q)) \leq (\deg(Q)+1)H(Q)[(r+1)H(P)]^{\deg(Q)}.$$

Proof. Let $Q(x) = c_0x^d + \cdots + c_d$ be a polynomial of degree d over $\mathbf{Z}$. Then, by definition

$$\Phi_P(Q(x)) = \sum_{n=0}^{d} c_n(a_0x^r + \cdots + a_r)^n.$$

By the multinomial theorem, we have

$$\Phi_P(Q(x)) = \sum_{n=0}^{d} c_n \sum_{n_0+n_1+\cdots+n_r=n} \binom{n}{n_0, n_1, \ldots, n_r} a_0^{n_0} a_1^{n_1} \ldots a_r^{n_r} x^{rn_0+\cdots+2n_{r-2}+n_{r-1}},$$

where $\binom{n}{n_0, n_1, \ldots, n_r} = \frac{n!}{n_0!n_1!\cdots n_r!}$, and the inner sum is over all n-tuples $(n_0, n_1, \ldots, n_r)$ of non-negative integers.

By interchanging the sum, one can write $\Phi_P(Q(x)) = \sum_{i=0}^{dr} b_i x^i$, where

$$b_i = \sum_{n=0}^{d_i} c_n \sum_{\substack{n_0,n_1,\ldots,n_r \geq 0, \\ n_0+n_1+\cdots+n_r=n, \\ rn_0+\cdots+2n_{r-2}+n_{r-1}=i}} \binom{n}{n_0, n_1, \ldots, n_r} a_0^{n_0} a_1^{n_1} \ldots a_r^{n_r}.$$

Hence by definition,

$$H(\Phi_P(Q)) = \max_{0\le i\le dr} |b_i| \le H(Q) \sum_{n=0}^{d} \sum_{rn_0+\cdots+n_{r-1}=i} \binom{n}{n_0, n_1, \ldots, n_r} H(P)^n \tag{4.7}$$

where the inner sum is over all n-tuples $(n_0, \ldots, n_r)$ of non-negative integers that add up to n.

From (4.7), one can easily deduce that, $H(\Phi_P(Q)) \le (d+1)H(Q)H(P)^d(r+1)^d$. □

The following theorem relates the height of $P(\alpha)$ with the height of α.

Theorem A.10. *Let Q be the minimal polynomial of $P(\alpha)$ of degree d over* $\mathbf{Z}$*. Then,*

$$\frac{H(P(\alpha))^{1/d}}{2L(P)\sqrt{m+1}} \le H(\alpha) \le \frac{(dr)!}{m!}(d+1)H(P(\alpha))\left[(r+1)H(P)\right]^d. \tag{4.8}$$

Proof. Let Q be the minimal polynomial of $P(\alpha)$ of degree d over $\mathbf{Z}$. Then α is a root of $\Phi_P(Q)$, and hence, the minimal polynomial of α divides $\Phi_P(Q)$ over $\mathbf{Z}$. By Lemma *A*.4, we have $H(\alpha) < \frac{(dr)!}{m!}H(\Phi_P(Q))$. By using Proposition *A*.9, we have

$$H(\alpha) \le \frac{(dr)!}{m!}(d+1)H(P(\alpha))\left[(r+1)H(P)\right]^d. \tag{4.9}$$

By Lemma *A*.6, we have

$$\begin{aligned}
\frac{1}{d}\log H(P(\alpha)) - \log 2 &\le h(P(\alpha)) \\
&\le \log L(P) + rh(\alpha) \\
&\le \log L(P) + m\left(\frac{1}{m}\log H(\alpha) + \frac{1}{2m}\log(m+1)\right) \\
&\le \log L(P) + \log H(\alpha) + \log(m+1)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$H(P(\alpha))^{\frac{1}{d}} \le 2L(P)H(\alpha)\sqrt{m+1}. \tag{4.10}$$

From (4.9) and (4.10), the result follows. □

Suppose that β is algebraic over K. Then the following lemma gives an upper bound for the height of β in terms of the heights and degrees of the coefficients occur in the minimal polynomial of β over K.

Lemma A.11 ([12]). *Let n be a positive integer, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be algebraic numbers belong to K. Let $F(Y, X_1, \ldots, X_n)$ be a nonconstant polynomial in the variables $Y, X_1, \ldots, X_n$ over $\mathbb{Z}$, and with degree at least one in Y. Suppose that γ is an algebraic number with $F(\gamma, \alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. Then,*

$$H(\gamma) \leq 3^{2dm+(d_1+\cdots+d_n)m}(H_F H(\alpha_1)^{d_1} \cdots H(\alpha_n)^{d_n})^m \tag{4.11}$$

where H_F is the maximum of the absolute values of the coefficients of F, d_i is the degree of X_i $(i = 1, 2, \ldots, n)$ in F and d is the degree of Y in F.

The following lemma gives the lower bound for the difference between two distinct nonzero algebraic numbers (see [7, pp. 153-154] for more details).

Lemma A.12. *Let α and β be two distinct nonzero algebraic numbers of degree n and m, respectively. Then there exists a constant $c = c(n, m)$ such that*

$$|\alpha - \beta| > cH(\alpha)^{-m}H(\beta)^{-n}. \tag{4.12}$$

A.2 Mahler's and Koksma's classification of complex numbers

In this section, we define the Mahler's and Koksma's classifications of complex numbers. The materials here are taken from [7]. For more details on both the classifications and also for a literature survey on these topics, we refer to Chapter 3 of [7].

The **Mahler classification of complex numbers** can be obtained by the following algorithm.

(i) For any complex number ξ, we put

$$\omega_m(\xi, N) = \inf |P(\xi)|,$$

where the infimum is taken over all polynomials $P(X)$ with rational integer coefficients such that $\deg(P) \leq m, H(P) \leq N$, and $P(\xi) \neq 0$.

(ii) Then, we put

$$\omega_m(\xi) = \limsup_{N\to\infty} \frac{-\log \omega_m(\xi, N)}{\log N}, \qquad \omega(\xi) = \limsup_{m\to\infty} \frac{\omega_m(\xi)}{m}.$$

(iii) **(Mahler's classification).** Finally, ξ is called an

- A-number, if $\omega(\xi) = 0$;
- S-number, if $0 < \omega(\xi) < \infty$;
- T-number, if $\omega(\xi) = \infty$ and $\omega_m(\xi) < \infty$ for all $m \geq 1$;
- U-number, if $\omega(\xi) = \infty$ and there exists positive integer m_0 such that $\omega_m(\xi) = \infty$ for all $m \geq m_0$.

The **Koksma's classification of complex numbers** can be obtained in a similar way as we obtained the Mahler's classification. The algorithm is given below:

(i) For any complex number ξ, we first put

$$\omega_m^*(\xi, N) = \inf |\xi - \alpha|,$$

where the infimum is taken over all algebraic numbers α of degree at most m and height at most N such that $\alpha \neq \xi$.

(ii) Then, we put

$$\omega_m^*(\xi) = \limsup_{N\to\infty} \frac{-\log (N\omega_m^*(\xi, N))}{\log N}, \qquad \omega^*(\xi) = \limsup_{m\to\infty} \frac{\omega_m^*(\xi)}{m}.$$

(iii) **(Koksma's classification)** Koksma classified the complex numbers as A^*-, S^*-, T^*-, or U^*-numbers in the same way as Mahler did, but with ω^* in place of ω. Thus, ξ is called an

- A^*-number, if $\omega^*(\xi) = 0$;
- S^*-number, if $0 < \omega^*(\xi) < \infty$;
- T^*-number, if $\omega^*(\xi) = \infty$ and $\omega_m^*(\xi) < \infty$ for all $m \geq 1$;
- U^*-number, if $\omega^*(\xi) = \infty$ and there exists positive integer m_0 such that $\omega_m^*(\xi) = \infty$ for all $m \geq m_0$.

Bibliography

[1] W. W. Adams, *On the algebraic independence of certain Liouville numbers*, J. Pure Appl. Algebra **13** (1978), no. 1, 41–47.

[2] K. Alniaçik, *On the subclasses $\mathbf{U}_m$ in Mahler's classification of the transcendental numbers*, St. niv. Fen Fak. Mec. Seri A **44** (1979), 39–82. (Doktora tezi, stanbul niversitesi, 1978).

[3] K. Alniaçik, *Representation of real numbers as sum of U_2 numbers*, Acta Arith. **55** (1990), 301–310.

[4] K. Alniaçik and É. Saias, *Une remarque sur les G_δ–denses.* Arch. Math. (Basel) **62** (1994), no. 5, 425–426.

[5] J. Ax, *On Schanuel's Conjecture*, Annals of Mathematics **93** (2)(1971), 252–268.

[6] A. Baker, *Transcendental Number Theory*, Cambridge University Press, 1975.

[7] Y. Bugeaud, *Approximation by Algebraic Numbers, Cambridge Tracts in Mathematics*, **160**, Cambridge University Press, Cambridge, 2004.

[8] E. B. Burger, *On Liouville decompositions in local fields*, Proc. Amer. Math. Soc, **124** (11) (1996), 3305–3310.

[9] E. B. Burger, *Diophantine inequalities and irrationality measures for certain transcendental numbers.* Indian J. Pure Appl. Math. **32** (2001), no. 10, 1591–1599.

[10] P.L. Cijsouw, *On the simultaneous approximation of certain numbers*, Duke Mathematical Journal **42** (1975), 249–257.

[11] P. Erdős, *Representation of real numbers as sums and products of Liouville numbers*, Michigan Math. J, **9** (1) (1962) 59–60.

[12] O.S. İçen, *Anhang zu den Arbeiten "Über die Funkionswerte der p-adischen elliptischen Funkionen I und II"*, İst. Üniv. Fen Fak. Mec. Serie A, **38** (1973), 25–35.

[13] G. Karadeniz Gözeri, *On some power series with algebraic coefficients and Liouville numbers*, J. Inequal. Appl. **178** (2013), 9 pp.

[14] J. F. Koksma, *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen,* Monatsh. Math. Phys. **48** (1939) 176–189.

[15] S. Lang, *Introduction to Transcendental Numbers*, Addison-Wesley, 1966.

[16] W. J. LeVeque, *On Mahler's U-numbers*, J. London Math. Soc., **28** (1953), 220–229.

[17] J. Liouville, *Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible des irrationnelles algébriques*, J. Math. Pures et Appl. **18** (1844) 883–885, and 910–911.

[18] K. Mahler, *Zur Approximation der Exponentialfunktion und des Logarithmus, I, II*, J. reine u. angew. Math., **166** (1932) 118–136.

[19] É. Maillet, *Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions*, Paris, 1906.

[20] É. Maillet, *Sur quelques propriétés des nombres transcendants de Liouville*, Bull. S.M.F **50** (1922), 74–99.

[21] G. J. Rieger, *Über die Lösbarkeit von Gleichungssystemen durch Liouville–Zahlen.* Arch. Math. (Basel) **26** (1975), 40–43.

[22] W. M. Schmidt, *Mahler's T-numbers,* Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., (1971), 275–286.

[23] W. Schwarz, *Liouville–Zahlen und der Satz von Baire.* Math.-Phys. Semesterber. **24** (1977), no. 1, 84–87.

[24] K. Senthil Kumar, R. Thangadurai and M. Waldschmidt, *Liouville numbers, Liouville sets and Liouville fields*, To appear in Proc. Amer. Math. Soc.

[25] K. Senthil Kumar, R. Thangadurai and M. Waldschmidt, *Liouville numbers and Schanuel's conjecture*, Arch. Math. (Basel) **102** (2014), no. 1, 59–70.

[26] K. Senthil Kumar, *Fields of Mahler's U-numbers*, preprint.

[27] E. C. Titchmarsh, *The Theory of Functions*, Second Edition, Oxford University Press, 1939.

[28] M. Waldschmidt, *Algebraic independence of transcendental numbers.* In: Gel'fond's method and its developments. *Perspectives in mathematics,* 551–571, Birkhuser, Basel-Boston, Mass., 1984.

[29] M. Waldschmidt, *Indépendance algébrique de nombres de Liouville*, Cinquante ans de polynômes, Lecture Notes in Math. (Springer Verlag), **1415** (1990), 225–235.

[30] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Grundlehren der Mathematischen Wissenschaften **326,** Springer-Verlag, Berlin-Heidelberg, 2000.

[31] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades*, J. reine angew. Math., **206** (1961), 67–77.

Index of special symbols

$\mathbb{A}$ is the set of algebraic numbers.

$\underline{q}$ is a strictly increasing sequence of positive integers.

$\underline{u}$ is an unbounded sequence of positive real numbers.

$\lceil x \rceil$ is the smallest integer greater than x.

$\mathbb{C}$ is the set of complex numbers.

$\lfloor x \rfloor$ is the largest integer less than x.

$H(\alpha)$ is the height of the algebraic number α.

$\mathrm{Im}(\xi)$ is the imaginary part of the complex number ξ.

$\mathbb{Z}$ is the set of integers.

$\mathbf{L}$ is the set of Liouville numbers.

$\mathbb{N}$ is the set of positive integers.

K is a number field.

$\mathbb{Q}$ is the set of rational numbers.

$\mathbb{R}$ is the set of real numbers.

$\mathrm{Re}(\xi)$ is the real part of the complex number ξ.

i is a square root of -1.

$\mathrm{trdeg}_{\mathbb{Q}}\mathsf{L}$ is the transcendence degree of the field L over $\mathbb{Q}$.

$\mathbf{U}_m$ is the set of U_m-numbers.

$\underline{x} \vee \underline{y}$ is the union of the sequences $\underline{x}$ and $\underline{y}$.

$\mathbf{U}$ is the set of U-numbers.

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