On the Serre-Swan Theorem, and on Vector Bundles over Real Abelian Varieties

By

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Certificate

This is to certify that the Ph.D. thesis titled "**On the Serre-Swan Theorem, and on Vector Bundles over Real Abelian Varieties**" by **Archana Subhash Morye** is a record of bonafide research work done under my supervision. It is further certified that the thesis represents independent and original work by the candidate.

Thesis Supervisor:

Prof. N. Raghavendra

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Declaration

The author hereby declares that the work in the thesis titled "**On the Serre-Swan Theorem, and on Vector Bundles over Real Abelian Varieties**," submitted for Ph.D. degree to the **Homi Bhabha National Institute** has been carried out at Harish-Chandra Research Institute under the supervision of Professor N. Raghavendra. Wherever contributions of others are involved, every effort has been made to indicate that clearly, with due reference to the literature. The author attests that the work is original and has not been submitted in part or full for any degree or diploma to any other institute or university.

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Abstract

This thesis is divided into two parts.

In Chapter 1 we prove a generalization of two classical results of Serre and Swan on the relation between locally free sheaves and projective modules, by emphasizing the axiomatic aspect of the problem. We determine a class of ringed spaces $(X, \mathcal{O}_X)$ for which the category of locally free sheaves of bounded rank over X is equivalent to the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules. The well-known Serre-Swan theorems for affine schemes, differentiable manifolds, Stein spaces, etc., are then derived.

In Chapter 2 we study real algebraic vector bundles over a real abelian variety. The main theorem in this part gives various equivalent criteria for a real algebraic vector bundle over a real abelian variety to admit a flat holomorphic connection. In the course of the proof of the main theorem we also derive a version of a result of Simpson for real abelian varieties.

SYNOPSIS

1 Introduction

This thesis is divided into two parts. The first part is a generalization of two classical results of Serre and Swan on the relation between locally free sheaves and projective modules, by emphasizing the axiomatic aspect of the problem. We determine a class of ringed spaces, $(X, \mathcal{O}_X)$ for which the category of locally free sheaves of bounded rank over X is equivalent to the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules. The well-known Serre-Swan theorems for affine schemes, differentiable manifolds, Stein spaces, etc., are then derived. In the second part we study real algebraic vector bundles over a real abelian variety. The main theorem in this part gives various equivalent conditions for a real algebraic vector bundle over a real abelian variety to admit a flat holomorphic connection. In the course of proof of the main theorem we also derive a version of a result of Simpson for real abelian varieties.

The summary of my thesis work is given in Section 2 and Section 3. Section 2 is based on the paper [Mor2009]. Section 3 is based on unpublished work of mine, which is being prepared into a paper for publication.

In the following sections we will assume that all rings are commutative unless otherwise mentioned. All compact and paracompact topological spaces are assumed to be Hausdorff.

2 The Serre-Swan Theorem for Ringed Spaces

It is a well-known theorem due to Serre, that for an affine scheme $(X, \mathcal{O}_X)$, there is an equivalence of categories between locally free $\mathcal{O}_X$-modules of finite rank, and finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules [Ser55, Section 50, Corollaire to Proposition 4, p. 242]. Later Swan proved the same equivalence when X is a paracompact topological space of finite covering dimension, and $\mathcal{O}_X$ is the sheaf of continuous real-valued functions on X [Swa62, Theorem 2 and p. 277]. The same equivalence is true for finite dimensional connected Stein spaces, [For67, Satz 6.7 and Satz 6.8].

2.1 The Serre-Swan Theorem Let $(X, \mathcal{O}_X)$ be a ringed space, and A denote the ring $\Gamma(X, \mathcal{O}_X)$. We will denote by $\mathcal{O}_X$-**mod** the category of $\mathcal{O}_X$-modules, and **Lfb**(X) the full subcategory of $\mathcal{O}_X$-**mod** consisting of locally free $\mathcal{O}_X$-modules of bounded rank. Let A-**mod** denote the category of A-modules, and **Fgp**(A) the full subcategory of A-**mod** consisting of finitely generated projective A-modules. We have the canonical functor, $\Gamma(X, \bullet) : \mathcal{O}_X\textbf{-mod} \to A\textbf{-mod}$, given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. We say that *the Serre-Swan Theorem holds for a ringed space* $(X, \mathcal{O}_X)$ if $\Gamma(X, \mathcal{F})$ is a finitely generated projective module for every sheaf $\mathcal{F}$ in **Lfb**(X), and the functor $\Gamma(X, \bullet) : \textbf{Lfb}(X) \to \textbf{Fgp}(\Gamma(X, \mathcal{O}_X))$ is an equivalence of categories.

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *generated by global sections* if there is a family of sections $(s_i)_{i \in I}$ in $\Gamma(X, \mathcal{F})$ such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. We will say that $\mathcal{F}$ is *finitely generated by global sections* if a finite family of global sections $(s_i)_{i \in I}$ exists with the above property.

Definition 2.1 Let $(X, \mathcal{O}_X)$ be a locally ringed space. Then, a subcategory $\mathcal{C}$ of $\mathcal{O}_X$-**mod** is called an *admissible subcategory* if it satisfies the following conditions:

C1. $\mathcal{C}$ is a full abelian subcategory of $\mathcal{O}_X$-**mod**, and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ belongs to $\mathcal{C}$ for every pair of sheaves $\mathcal{F}$ in **Lfb**(X) and $\mathcal{G}$ in $\mathcal{C}$, where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$.

C2. Every sheaf in $\mathcal{C}$ is acyclic, and generated by global sections.

C3. **Lfb**(X) is a subcategory of $\mathcal{C}$.

Admissible subcategories naturally arise in many ringed spaces. In the case of an affine scheme, **Qcoh**(X), the subcategory of quasi-coherent $\mathcal{O}_X$-module, in the case of differentiable manifold $\mathcal{O}_X$-**mod** itself are examples of admissible subcategories. Moreover, if $(X, \mathcal{O}_X)$ is a connected finite-dimensional Stein space, **Coh**(X) the category of coherent $\mathcal{O}_X$-modules is an admissible subcategory of $\mathcal{O}_X$-**mod**.

The main theorem of this section is the following:

Theorem 2.2 *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $A = \Gamma(X, \mathcal{O}_X)$. Assume that $\mathcal{O}_X$-**mod** contains an admissible subcategory $\mathcal{C}$, and that every sheaf in **Lfb**(X) is finitely generated by global sections. Then,* $\Gamma(X, \bullet) : \textbf{Lfb}(X) \to$

$\mathbf{Fgp}(A)$ *is an equivalence of categories, i.e., the Serre-Swan Theorem holds for* $(X, \mathcal{O}_X)$.

Given an A-module M, we get a presheaf $\mathcal{P}(M)$ on X by defining $\mathcal{P}(M)(U) = M \otimes_A \mathcal{O}_X(U)$ for every open set U of X. We will denote by $\mathcal{S}(M)$ the sheaf associated to the presheaf $\mathcal{P}(M)$. Similarly, for a homomorphism $u : M \to N$ of A-modules, we get a homomorphism $\mathcal{S}(u) : \mathcal{S}(M) \to \mathcal{S}(N)$ of $\mathcal{O}_X$-modules. Thus, we get a functor $\mathcal{S} : A\text{-}\mathbf{mod} \to \mathcal{O}_X\text{-}\mathbf{mod}$. We proved that, if $(X, \mathcal{O}_X)$ is as in Theorem 2.2, then $\mathcal{S} : \mathbf{Fgp}(A) \to \mathbf{Lfb}(X)$ is a quasi-inverse of $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$.

2.2 Some Special Cases Following are some important examples of locally ringed spaces for which the Serre-Swan Theorem holds, and can be derived from Theorem 2.2.

Corollary 2.3 (Serre's Theorem)[Ser55, Section 50, Corollaire to Proposition 4, p. 242] *Let* $(X, \mathcal{O}_X)$ *be an affine scheme, and let* A *denote its coordinate ring* $\Gamma(X, \mathcal{O}_X)$*. Then, a quasicoherent* $\mathcal{O}_X$*-mdoule* $\mathcal{F}$ *is locally free* $\mathcal{O}_X$*-module of finite rank if and only if* $\Gamma(X, \mathcal{F})$ *is a finitely generated projective* A*-module. The functor* $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ *is an equivalence of categories.*

Corollary 2.4 *Let* $(X, \mathcal{O}_X)$ *be a ringed space such that,* X *is a paracompact topological space of bounded topological dimension, and* $\mathcal{O}_X$ *is a fine sheaf. Then, the Serre-Swan Theorem holds for* $(X, \mathcal{O}_X)$.

Corollary 2.5 (Swan's Theorem)[Swa62, Theorem 2 and p. 277] *Let* X *be a paracompact topological space of bounded topological dimension, and let* $\mathcal{C}_X$ *denote the sheaf of continuous real-valued functions on* X*. Let* $C(X)$ *denote the* $\mathbf{R}$*-algebra* $\Gamma(X, \mathcal{C}_X)$*. Then, the functor* $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(C(X))$ *is an equivalence of categories.*

Another interesting examples are affine differentiable spaces [GS2003, pp. 22, 30, 44], locally ringed spaces $(X, \mathcal{O}_X)$ whose center is compact [Mul76, Section 3, definition, p. 63], regular ringed spaces [Pie67, p. 8, Definition 10.2], for which the Serre-Swan Theorem holds.

Let $(X, \mathcal{O}_X)$ be a Stein space, and let $A = \Gamma(X, \mathcal{O}_X)$. Recall that, a topological module M over the algebra A is called a *Stein module* if there exists a coherent $\mathcal{O}_X$-module $\mathcal{M}$, such that $\Gamma(X, \mathcal{M})$ is isomorphic to M [For67, Section 2, p. 383]. Let $\mathfrak{S}$-$\mathbf{mod}$ denote the category of Stein modules over A.

Corollary 2.6 [For67, Satz 6.7 and Satz 6.8] *Let $(X, \mathcal{O}_X)$ be a finite-dimensional connected Stein space. Then $\Gamma(X, \bullet) : \mathbf{Coh}(X) \to \mathfrak{S}$-**mod** is an equivalence of category with quasi-inverse $\mathcal{S} : \mathfrak{S}$-**mod** $\to \mathbf{Coh}(X)$. Moreover, the category of locally free sheaves of finite rank is equivalent to the category of finitely generated projective A-modules.*

3 Real Vector Bundles over Real Abelian Varieties

In this part of the thesis, we study various equivalent conditions for the presence of real holomorphic connections in a real holomorphic vector bundle over a real abelian variety. Holomorphic connections in holomorphic bundles over a complex abelian variety were studied by Balaji and Biswas [BB2009], Biswas [Bis2004], Biswas and Iyer [BI2007], Biswas and Gómez [BG2008, Theorem 4.1], Biswas and Subramanian [BS2004]. In the part of this thesis we prove analogues, for real abelian varieties, of some of the results in the above papers.

3.1 Real Structures on a Ringed Space over C Let $(X, \mathcal{O}_X)$ be a ringed space over $\mathbf{C}$. A *real structure on* $(X, \mathcal{O}_X)$ is a pair $(\sigma, \tilde{\sigma})$ consisting of a continuous map $\sigma : X \to X$ such that $\sigma^2 = \mathbf{1}_X$, and for every $U \subset X$ open, $\tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$, a $\mathbf{C}$-antilinear ring homomorphism, which is compatible with restrictions, i.e., for every $V \subset U$ open in X,

$$\begin{array}{ccc}
\mathcal{O}_X(U) & \xrightarrow{\tilde{\sigma}_U} & \mathcal{O}_X(\sigma(U)) \\
\downarrow{\scriptstyle \rho_{V,U}} & & \downarrow{\scriptstyle \rho_{\sigma(V),\sigma(U)}} \\
\mathcal{O}_X(V) & \xrightarrow[\tilde{\sigma}_V]{} & \mathcal{O}_X(\sigma(V))
\end{array}$$

commutes, and such that, $\tilde{\sigma}_{\sigma(U)} \circ \tilde{\sigma}_U = \mathbf{1}_{\mathcal{O}_X(U)}$, for every $U \subset X$ open.

Let X be a complex manifold and $\sigma : X \to X$ be an anti-holomorphic involution. Let $\mathcal{O}_X$ denote the structure sheaf of holomorphic functions on X. Define $\tilde{\sigma} : \mathcal{O}_X \to \mathcal{O}_X$ as follows: For every $U \subset X$ open, $\tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$, by $f \mapsto \overline{f \circ \sigma}$. Then, $(\sigma, \tilde{\sigma})$ is a real structure on $(X, \mathcal{O}_X)$. Moreover all real structures on complex manifolds are of this form.

Proposition 3.1 *If $(X, \mathcal{O}_X)$ is a complex manifold, and $(\sigma, \tilde{\sigma})$ is a real structure on $(X, \mathcal{O}_X)$, then the following are true:*

1. *The involution σ is antiholomorphic.*

2. *For any open $U \subset X$, the $\mathbf{C}$-antilinear ring homomorphism $\tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$ is given by $s \mapsto \overline{s \circ \sigma}$.*

In particular the sheaf morphism $\tilde{\sigma}$ is uniquely determined by the map σ.

We will call a complex manifold with an antiholomorphic involution a *real holomorphic manifold*.

Definition 3.2 Let $(X, \mathcal{O}_X)$ be a ringed space over $\mathbf{C}$, with a real structure $(\sigma, \tilde{\sigma})$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *real structure on* $\mathcal{F}$ is a family $\alpha^{\mathcal{F}} = (\alpha^{\mathcal{F}}_U)_{U \in \mathrm{op}(X)}$ of morphisms of abelian groups $\alpha^{\mathcal{F}}_U : \mathcal{F}(U) \to \mathcal{F}(\sigma(U))$, such that:

1. The abelian group homomorphisms are compatible with restriction morphisms, that is, for every $V \subset U$ open in X the following diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha^{\mathcal{F}}_U"] \arrow[d, "\rho_{V,U}"'] & \mathcal{F}(\sigma(U)) \arrow[d, "\rho_{\sigma(V),\sigma(U)}"] \\ \mathcal{F}(V) \arrow[r, "\alpha^{\mathcal{F}}_V"'] & \mathcal{F}(\sigma(V)) \end{tikzcd}$$

   commutes.

2. For every open subset U of X, $\alpha^{\mathcal{F}}_{\sigma(U)} \circ \alpha^{\mathcal{F}}_U = \mathbf{1}_{\mathcal{F}(U)}$.

3. $\alpha^{\mathcal{F}}_U(fs) = \tilde{\sigma}_U(f)\alpha^{\mathcal{F}}_U(s)$, for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.

A *real $\mathcal{O}_X$-module* is a pair $(\mathcal{F}, \alpha^{\mathcal{F}})$ such that $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\alpha^{\mathcal{F}}$ is a real structure on $\mathcal{F}$. A *real vector bundle over* X is a real $\mathcal{O}_X$-module $(\mathcal{F}, \alpha^{\mathcal{F}})$ such that the $\mathcal{O}_X$-module $\mathcal{F}$ is locally free. We sometimes denote by a real sheaf $(\mathcal{F}, \alpha^{\mathcal{F}})$ by just $\mathcal{F}$ when no confusion is likely to occur.

Remark 3.3 Let $(X, \mathcal{O}_X)$ be a ringed space with a real structure $(\sigma, \tilde{\sigma})$, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We will define an $\mathcal{O}_X$-module $\mathcal{F}^\sigma$ as follows: For an open set U of X, $\mathcal{F}^\sigma(U) = \mathcal{F}(\sigma(U))$ as an abelian group, and for every $f \in \mathcal{O}_X(U)$, $s \in \mathcal{F}^\sigma(U)$, $f \cdot s = \tilde{\sigma}_U(f)s$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Define, $\varphi^\sigma : \mathcal{F}^\sigma \to \mathcal{G}^\sigma$ as follows: For every $U \subset X$ open, $\varphi^\sigma_U : \mathcal{F}^\sigma(U) \to \mathcal{G}^\sigma(U)$, by $\varphi^\sigma_U = \varphi_{\sigma(U)}$. Then, it is easy to check that $\mathcal{F}^\sigma$ is an $\mathcal{O}_X$-module, and φ^σ is a homomorphism of $\mathcal{O}_X$-modules. Hence, we get a functor,

$$\bullet^\sigma : \mathcal{O}_X\text{-}\mathbf{mod} \to \mathcal{O}_X\text{-}\mathbf{mod}.$$

We can rephrase the definition of a real structure on an $\mathcal{O}_X$-module $\mathcal{F}$ using the functor $\bullet^\sigma$. A real structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is an $\mathcal{O}_X$-modules homomorphism $\alpha^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\sigma$ such that, $(\alpha^{\mathcal{F}})^\sigma \circ \alpha^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}}$.

Let $(X, \mathcal{O}_X)$ be a ringed space with a real structure $(\sigma, \tilde{\sigma})$. Let $\mathcal{O}_X\text{-}\mathbf{mod}^{\mathbf{real}}$ be the category defined as follows:

$$\begin{aligned}\mathrm{Ob}(\mathcal{O}_X\text{-}\mathbf{mod}^{\mathbf{real}}) &= \text{real } \mathcal{O}_X\text{-modules}\\ &= \left\{ (\mathcal{F}, \alpha^{\mathcal{F}}) \,\middle|\, \begin{array}{l} \mathcal{F} \text{ is an } \mathcal{O}_X\text{-module, and} \\ \alpha^{\mathcal{F}} \text{ is a real structure on } \mathcal{F} \end{array} \right\}.\end{aligned}$$

Let $(\mathcal{F}, \alpha^{\mathcal{F}})$, $(\mathcal{G}, \alpha^{\mathcal{G}})$ be $\mathcal{O}_X$-modules with a real structures. Then define,

$$\mathrm{Hom}_{\mathbf{real}}((\mathcal{F}, \alpha^{\mathcal{F}}), (\mathcal{G}, \alpha^{\mathcal{G}})) = \{\varphi \in \mathrm{Hom}_{\mathcal{O}_X\text{-}\mathbf{mod}}(\mathcal{F}, \mathcal{G}) \,|\, \varphi^\sigma \circ \alpha^{\mathcal{F}} = \alpha^{\mathcal{G}} \circ \varphi\}$$

If $(\mathcal{F}, \alpha^{\mathcal{F}})$ is a real $\mathcal{O}_X$-module, then a real sheaf $(\mathcal{G}, \alpha^{\mathcal{G}})$ is called a *real subsheaf of* $\mathcal{F}$ if $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, and the inclusion morphism $i : \mathcal{G} \to \mathcal{F}$ is a real morphism. If X is a real holomorphic manifold with an anti-holomorphic involution σ, then the tangent bundle TX, the co-tangent bundle T^*X, exterior algebra bundles are canonically real holomorphic vector bundles. Moreover, if E is a real vector bundle then the exterior algebra bundles with values in E are also real. If α^E is a real structure on E, then we will denote by the same α^E real structures on all the exterior algebra bundles with values in E.

A *real abelian variety* X is a complex abelian variety which is a real holomorphic manifold such that the corresponding antiholomorphic involution σ is compatible with the group operation, i.e., $\sigma(x+y) = \sigma(x)+\sigma(y)$ for all $x, y \in X$. We will denote by $\mathcal{R}(X)$ the set of fixed points of σ, i.e., $\{x \in X \,|\, \sigma(x) = x\}$. Points in $\mathcal{R}(X)$ are called *real points.* We will denote by τ_x the translation by x, for all $x \in X$. If $x \in \mathcal{R}(X)$, we get $\sigma \circ \tau_x(y) = \tau_x \circ \sigma(y)$. Then it is easy to see that the pullback of a real vector bundle E over X, by τ_x, ${\tau_x}^*(E)$ is also real. A real vector bundle E is called *real homogeneous*, if $\tau_x^*(E)$ is isomorphic to E in the category of $\mathcal{O}_X\text{-}\mathbf{mod}^{\mathbf{real}}$, for all $x \in \mathcal{R}(X)$.

3.2 Real Flat Connections Let X be a real holomorphic manifold with antiholomorphic involution σ. Let E be a holomorphic vector bundle on X with a holomorphic connection D. Define, $D^\sigma : E^\sigma \to \Omega^1_X(E^\sigma)$, as follows, if $s \in E^\sigma(U)$, $D^\sigma(s) = D_{\sigma(U)}(s)$. This defines a holomorphic connection in E^σ. (Note that

$\Omega^1_X(E^\sigma)$ is canonically isomorphic to $\Omega^1_X(E)^\sigma$.)

Definition 3.4 Let D be a holomorphic connection in a real holomorphic vector bundle (E, α^E). Then, D is called *real* if the following diagram commutes,

$$\begin{tikzcd} E \arrow[r, "D"] \arrow[d, "\alpha^E"'] & \Omega^1_X(E) \arrow[d, "\alpha^E"] \\ E^\sigma \arrow[r, "D^\sigma"'] & \Omega^1_X(E^\sigma). \end{tikzcd}$$

Let (X, g) be a real compact Kähler manifold with antiholomorphic involution, σ. We will denote by $\mu(E)$ the *slope* of E, i.e., $\mu(E) = \frac{\text{degree}(E)}{\text{rank}(E)}$. (The *degree* of a torsion-free coherent sheaf $\mathcal{F}$ over a compact Kähler manifold X of dimension n, with the Kähler form Φ is $\deg(\mathcal{F}) = \int_X c_1(\mathcal{F}) \wedge \Phi^{n-1}$.)

Definition 3.5 A real holomorphic vector bundle E is called *real stable* (respectively *real semistable*) if for every proper real holomorphic coherent subsheaf $\mathcal{F}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(E)$ we have

$$\mu(\mathcal{F}) < \mu(E), \quad (\text{respectively } \mu(\mathcal{F}) \leq \mu(E)).$$

The main results in this part of the thesis are as follows.

Theorem 3.6 *Let (X, σ) be a real abelian variety, and let (E, α^E) be a real holomorphic vector bundle over X. Then the following are equivalent:*

1. *The real holomorphic vector bundle (E, α^E) admits a real holomorphic connection.*

2. *The real holomorphic vector bundle (E, α^E) is real homogeneous.*

3. *The real holomorphic vector bundle (E, α^E) is real semistable with $c_1(E) = c_2(E) = 0$.*

4. *The real holomorphic vector bundle (E, α^E) admits a filtration*

$$E^\bullet : \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that E_i is real sub-bundle of (E, α^E), $c_j(E_i) = 0$, for $j = 1, 2$ and $i = 1, \ldots, n$, and E_i/E_{i-1} is real polystable.

5. *The real holomorphic vector bundle* (E, α^E) *admits a real flat holomorphic connection.*

The following is the real analogue of a special case of a result of Simpson [Sim92, Theorem, p. 39].

Corollary 3.7 *Let* X *be a real abelian variety, and* E *a real semistable holomorphic vector bundle over* X*, such that* $c_1(E) = c_2(E) = 0$*. Then,* E *is obtained by successive extensions of real stable holomorphic vector bundles with vanishing Chern classes.*

List of Publications and Preprints:

1. A. S. Morye, *Note on the Serre-Swan Theorem*, accepted for publication in Math. Nachr.
2. A. S. Morye, *Real vector bundles over a real abelian variety*, Preprint.

Dedicated to my family

Contents

Conventions and Notations

If $\mathcal{C}$ is a category, and if X and Y are objects in $\mathcal{C}$, then the set of morphisms from X into Y will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, or by $\mathrm{Hom}(X, Y)$ when no confusion is likely to occur. If $X = Y$, then the set is denoted by $\mathrm{End}_{\mathcal{C}}(X)$ or $\mathrm{End}(X)$. Furthermore, $X \cong_{\mathcal{C}} Y$ or $X \cong Y$ means that X is isomorphic to Y. If $\mathcal{C}$ is a category, then $\mathcal{C}^{\mathrm{op}}$ will denote the *opposite category of* $\mathcal{C}$, i.e., $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, for every X, Y in $\mathcal{C}^{\mathrm{op}}$.

We will assume that all rings are commutative with identity unless otherwise mentioned. We do not exclude the possibility that the identity element 1 equals the zero element 0 of A. In that case, A equals the singleton $\{0\}$, and is called a zero ring. However, when we say that A is an integral domain or a field, we assume that A is not a zero ring. Every homomorphism $\varphi : A \to B$ is assumed to be unital, i.e., $\varphi(1) = 1$. By an A-module, we means a left A-module. All A-modules M are assumed to be unital, i.e., $1 \cdot x = x$, for all $x \in M$. For any A-module M, and for every set I, we denote by $E^{(I)}$ the direct sum $\oplus_{i \in I} E_i$ where $E_i = E$ for all $i \in I$. Similarly, E^I denotes the direct product $\prod_{i \in I} E_i$, where $E_i = E$ for all $i \in I$.

Let X be a topological space, and let $\mathcal{B}$ be a base for the topology on X. If $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$, then we will denote by the same symbol $\mathcal{F}$ the canonical presheaf on X associated to $\mathcal{F}$. Recall that, $\mathcal{F}$ is a sheaf on X if and only if $\mathcal{F}$ is a sheaf on $\mathcal{B}$. For a presheaf $\mathcal{F}$ and for every pair (U, V) of open subsets of X with $U \supset V$, we will denote the sheaf restriction map by $\rho^{\mathcal{F}}_{V,U}$ or $\rho_{V,U}$ when no confusion is likely to occur.

Let X be a topological space, and $\mathcal{F}$ a sheaf on X. The sheaf cohomology group of the space X of degree q and with coefficients in $\mathcal{F}$ is denoted by $H^q(X, \mathcal{F})$ [Har77, Chaper III, §. 2, Definition 2.2, p. 207].

All compact and paracompact topological spaces are assumed to be Hausdorff. Topological manifolds are assumed to be Hausdorff topological spaces with a countable basis. In particular they are paracompact.

We will denote by $\mathbf{C}$ (respectively $\mathbf{R}$) the field of complex numbers (respectively real numbers). A fixed square root of -1 is denoted by ι. The ring of integers is denoted by $\mathbf{Z}$, and a set of nonnegative integers is denoted by $\mathbf{N}$.

Chapter 1

Introduction

In this thesis we study two problems on vector bundles. The first problem is regarding the equivalence between the category of vector bundles over a ringed space $(X, \mathcal{O}_X)$, and the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules. The other problem concerns, the various equivalent criteria for a real algebraic vector bundle over a real abelian variety to admit a real flat holomorphic connection.

The Serre-Swan theorem for ringed spaces

In Chapter 2 we generalize two classical results of Serre and Swan on the relation between locally free sheaves and projective modules by emphasizing the axiomatic aspect of the problem.

For any ringed space $(X, \mathcal{O}_X)$, let $\mathcal{O}_X$-**mod** denote the category of $\mathcal{O}_X$-modules, and $\mathbf{Lfb}(X)$ the full subcategory of $\mathcal{O}_X$-**mod** consisting of locally free $\mathcal{O}_X$-modules of bounded rank. Let $A = \Gamma(X, \mathcal{O}_X)$, and $\mathbf{Fgp}(A)$ the category of finitely generated projective A-modules. We will say that *the Serre-Swan Theorem holds for a ringed space* $(X, \mathcal{O}_X)$ if for every $\mathcal{F}$ in $\mathbf{Lfb}(X)$, $\Gamma(X, \mathcal{F})$ is in $\mathbf{Fgp}(A)$, and the canonical functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(\Gamma(X, \mathcal{O}_X))$ is an equivalence of categories. A full abelian subcategory $\mathcal{C}$ of $\mathcal{O}_X$-**mod** is called an *admissible subcategory* if $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ belongs to $\mathcal{C}$ for every pair of sheaves $\mathcal{F}$ in $\mathbf{Lfb}(X)$ and $\mathcal{G}$ in $\mathcal{C}$, every sheaf in $\mathcal{C}$ is acyclic, and generated by global sections, and $\mathbf{Lfb}(X)$ is a subcategory of $\mathcal{C}$.

In Section 2.1, we recall various known facts about sheaves which we will be using in this chapter. In Section 2.2, we prove that the Serre-Swan Theorem holds for a locally ringed space $(X, \mathcal{O}_X)$ if every locally free $\mathcal{O}_X$-module of bounded rank is finitely generated by global sections, and if the category $\mathcal{O}_X$-**mod** contains an admissible subcategory. The wellknown Serre-Swan Theorems for affine schemes, differentiable manifolds, Stein spaces etc. are then derived in the last section.

Real vector bundles over a real abelian variety

Holomorphic connections in holomorphic bundles over a complex abelian variety were studied by Balaji and Biswas [BB2009], Biswas [Bis2004], Biswas and Iyer [BI2007], Biswas and Gómez [BG2008, Theorem 4.1], Biswas and Subramanian [BS2004]. In the third chapter we prove analogues, for real abelian varieties, of some of the results in the above papers.

A *real abelain variety* is a pair (X, σ), where X is a complex abelian variety, and σ is an antiholomorphic involution on X which is compatible with the group operation in X. A *real holomorphic vector bundle* over (X, σ) is a pair (E, α^E), where E is a holomorphic vector bundle over X, and α^E is a family $\left(\alpha_U^E\right)_{U \in \mathrm{op}(X)}$ of morphisms of abelian groups $\alpha_U^{\mathcal{F}} : \mathcal{F}(U) \to \mathcal{F}(\sigma(U))$, which are compatible with restriction morphisms, such that for every open subset U of X, $\alpha_{\sigma(U)}^{\mathcal{F}} \circ \alpha_U^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}(U)}$, and $\alpha_U^{\mathcal{F}}(fs) = \overline{f \circ \sigma} \alpha_U^{\mathcal{F}}(s)$, for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$. Let $\tau_x : X \to X$ be the translation by x, for all $x \in X$. A real vector bundle E is called *real homogeneous*, if $\tau_x^*(E)$ is isomorphic to E in the category of $\mathcal{O}_X$-$\mathbf{mod}^{\mathbf{real}}$ (Subsection 3.1.3), for all real point x in X (that is, $\sigma(x) = x$). A *real holomorphic connection* in (E, α^E) is a holomorphic connection which is compatible with real structure in E.

The main theorem in this chapter asserts that the following five conditions are equivalent: (1) a real holomorphic vector bundle (E, α^E) over a real abelian variety (X, σ) admits a real holomorphic connection. (2) (E, α^E) is real homogeneous. (3) (E, α^E) is real semistable (Definition 3.4.8), and $c_1(E) = c_2(E) = 0$. (4) (E, α^E) admits a filtration $E^\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, such that E_i is a real sub-bundle of (E, α^E), $c_j(E_i) = 0$, for $j = 1, 2$ and $i = 1, \ldots, n$, and E_i/E_{i-1} is real polystable. (5) (E, α^E) admits a real flat holomorphic connection.

In Section 3.1, we discuss real structure on a ringed space over $\mathbf{C}$. In the second section, we show how real structures on a holomorphic manifold come from antiholomorphic involutions. Also we recall various notions about real abelian varieties. In some subsections of Section 3.3 and 3.4 we develop real analogues for geometric notions such as Hermitian metrics, connections, stability, Kähler manifolds etc. In the last subsection, we prove the main theorem of this chapter.

A more detailed introduction to the thesis, including precise definitions, results and statements of the theorems, is given in the synopsis (p. **i**).

Chapter 2

The Serre-Swan Theorem for Ringed Spaces

It is a wellknown theorem due to Serre that for an affine scheme $(X, \mathcal{O}_X)$, there is a categorical equivalence between locally free $\mathcal{O}_X$-modules of finite rank, and finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules [Ser55, Section 50, Corollaire to Proposition 4, p. 242]. Later Swan proved the same equivalence when X is a paracompact topological space of finite covering dimension, and $\mathcal{O}_X$ is the sheaf of continuous real-valued functions on X [Swa62, Theorem 2 and p. 277]. In this chapter we will generalize this result to a large class of locally ringed spaces.

In Section 2.1 we recall some background material about sheaves. All facts are known, and proofs of some facts are given just for the sake of completeness. In Section 2.2, we will prove that for a locally ringed space $(X, \mathcal{O}_X)$, the category of locally free sheaves of bounded rank is equivalent to the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules if every locally free $\mathcal{O}_X$-module of bounded rank is finitely generated by global sections, and if the category $\mathcal{O}_X$-**mod** contains an admissible subcategory (Definition 2.2.6). In Section 2.3, we will show that certain wellknown classes of locally ringed spaces satisfy the above conditions, and hence the Serre-Swan Theorem holds for them. Appendix A is the list of definitions and results from category theory which we will be using in this chapter.

2.1 Theory of Sheaves

We begin this section by giving some examples of locally ringed spaces, and then we discuss two canonical functors $\Gamma(X, \bullet)$ and $\mathcal{S}$, which eventually give us the required equivalence of categories. In this section we collect general facts about sheaves to the extent necessary for this chapter. In Subsections 2.1.4, 2.1.5, 2.1.6 and 2.1.7, we briefly recall definitions and properties of sheaves of finite type, sheaves of finite presentation, coherent sheaves, and locally free sheaves. Results and proofs in this section are mainly taken from [Gro60].

2.1.1 Ringed Spaces

Recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings on X.

Definition 2.1.1 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A *morphism of ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ is a pair $(u, \tilde{u})$, where

1. $u : X \to Y$ is a continuous map.

2. $\tilde{u}$ is an assignment which attaches to each open subset V of Y, a homomorphism of rings $\tilde{u}_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(u^{-1}(V))$, such that for every pair (V, V') of open subsets of Y with $V \supset V'$, the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "\tilde{u}_V"] \arrow[d, "\rho^{\mathcal{O}_Y}_{V',V}"'] & \mathcal{O}_X(u^{-1}(V)) \arrow[d, "\rho^{\mathcal{O}_X}_{u^{-1}(V'),u^{-1}(V)}"] \\
\mathcal{O}_Y(V') \arrow[r, "\tilde{u}_{V'}"'] & \mathcal{O}_X(u^{-1}(V'))
\end{tikzcd}$$

commutes.

We then write that $(u, \tilde{u}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces.

Let $(X, \mathcal{O}_X)$ be a ringed space. Then, the *identity morphism of* $(X, \mathcal{O}_X)$, which is denoted by $\mathbf{1}_{(X,\mathcal{O}_X)}$, is the morphism of ringed spaces $(u, \tilde{u}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ defined as follows:

1. $u = \mathbf{1}_X : X \to X$.

2. $\tilde{u}$ is the assignment which attaches to each open subset V of X, the identity homomorphism $\mathbf{1}_{\mathcal{O}_X(V)} : \mathcal{O}_X(V) \to \mathcal{O}_X(V)$.

The presheaf axioms imply that $(u, \tilde{u})$ is indeed a morphism of ringed spaces from $(X, \mathcal{O}_X)$ to itself.

Let $(u, \tilde{u}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(v, \tilde{v}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be two morphism of ringed spaces. Then, the *composition of* $(u, \tilde{u})$ *and* $(v, \tilde{v})$, which is denoted by $(v, \tilde{v}) \circ (u, \tilde{u})$, is the morphism of ringed spaces $(w, \tilde{w}) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ defined as follows:

1. $w = v \circ u : X \to Z$.

2. $\tilde{w}$ is the assignment which attaches to each open subset W of Z, the homomorphism $w_W : \mathcal{O}_Z(W) \to \mathcal{O}_X(w^{-1}(W))$ which is the composite of the ring homomorphisms

$$\mathcal{O}_Z(W) \xrightarrow{\tilde{v}_W} \mathcal{O}_Y(v^{-1}(W)) \xrightarrow{\tilde{u}_{v^{-1}(W)}} \mathcal{O}_X(w^{-1}(W))$$

   where one is using the equality $w^{-1}(W) = u^{-1}(v^{-1}(W))$.

Then, it is easy to verify that $(w, \tilde{w})$ is indeed a morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Z, \mathcal{O}_Z)$.

Thus, we get a category **Rsp**, whose objects are ringed spaces, and whose morphisms are morphisms of ringed spaces.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, and $(u, \tilde{u})$ a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$. Then for every $x \in X$, we have a canonical homomorphism of rings,

$$u_x : \mathcal{O}_{Y,u(x)} \to \mathcal{O}_{X,x} \tag{2.1}$$

defined as follows: Let $\theta \in \mathcal{O}_{Y,u(x)}$. Then there exist an open neighborhood U of $u(x)$ in Y, and $s \in \mathcal{O}_Y(U)$ such that, $(s)_{u(x)} = \theta$. Since $s \in \mathcal{O}_Y(U)$, $\tilde{u}_U(s)$ is a section of $\mathcal{O}_X$ on the open neighborhood $u^{-1}(U)$ of x. Therefore, $(\tilde{u}_U(s))_x$ is an element of $\mathcal{O}_{X,x}$. Define $u_x(\theta) = (\tilde{u}_U(s))_x$. It follows from this definition that, u_x is independent of the choice of a pair (U, s).

Definition 2.1.2 A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. In that case, the maximal ideal in $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$. The *residue field* $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ of $\mathcal{O}_{X,x}$ is denoted by $k(x)$.

Definition 2.1.3 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. A *morphism of locally ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(u, \tilde{u}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that, for all $x \in X$, the canonical homomorphism (2.1) $u_x : \mathcal{O}_{Y,u(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, that is, $u_x(\mathfrak{m}_{Y,u(x)}) \subset \mathfrak{m}_{X,x}$.

Locally ringed spaces form a subcategory of the category **Rsp** . We will denote it by **LRsp**.

Example 2.1.4 Let X be a topological space, and let $\mathcal{C}_X$ be the sheaf of continuous real valued functions on X. Then, $(X, \mathcal{C}_X)$ is a locally ringed space. For $x \in X$, $\mathcal{C}_{X,x}$ is the ring of germs of continuous functions at x, and

$$\mathfrak{m}_{X,x} = \{(f)_x \in \mathcal{C}_{X,x} \mid f(x) = 0\}.$$

If $u : X \to Y$ is a continuous map, define $\tilde{u}_V : \mathcal{C}_Y(V) \to \mathcal{C}_X(u^{-1}(V))$, by $g \mapsto g \circ (u|_{u^{-1}(V)})$. Then, $(u, \tilde{u})$ is a morphism of locally ringed spaces. In fact $\tilde{u}_x((g)_{u(x)}) = (g \circ u)_x \in \mathfrak{m}_{X,x}$, if $(g)_{u(x)} \in \mathfrak{m}_{Y,u(x)}$.

Similarly if X is a differentiable manifold, and $\mathcal{C}_X^\infty$ is the sheaf of $\mathcal{C}^\infty$ real valued functions of X, then $(X, \mathcal{C}_X^\infty)$ is a locally ringed space.

Example 2.1.5 (Affine Schemes) Let A be a ring. The set $\operatorname{Spec}(A)$ of all prime ideals in A is called the *prime spectrum*, or just *spectrum* of A. We will denote by $\mathrm{D}(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid f \notin \mathfrak{p}\}$. Recall that the family $\mathcal{B} = \{\mathrm{D}(f)\}_{f \in A}$ is a base for the topology of $\operatorname{Spec}(A)$. If S is a multiplicative subset of A, $S^{-1}A$ denote the ring of fraction of A with denominators in S. If $S = \{f^n \mid f \in A, n \in \mathbf{N}\}$, then $S^{-1}A$ we will denote by A_f. For every $f \in A$, define $\mathcal{F}(\mathrm{D}(f)) = A_f$. Then $\mathcal{F}$ forms a welldefined sheaf of ring on $\mathcal{B}$ [Liu2002, pp. 42–45, 2.3.1]. Let $\tilde{A}$ be the sheaf of ring on $\operatorname{Spec}(A)$ associated to the sheaf of ring $\mathcal{F}$ on $\mathcal{B}$. For every $\mathfrak{p} \in \operatorname{Spec}(A)$, the stalk of $\tilde{A}$ at $\mathfrak{p}$ is $\tilde{A}_\mathfrak{p} = A_\mathfrak{p}$, where $A_\mathfrak{p}$ denotes the ring $S^{-1}A$, for $S = \{f \in A \mid f \notin \mathfrak{p}\}$. Note that, $A_\mathfrak{p}$ is a local ring with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$. Therefore, $(\operatorname{Spec}(A), \tilde{A})$ is a locally ringed space. Let $\varphi : B \to A$ be a homomorphism of rings. Let

$${}^a\varphi : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$$

be the *continuous map associated to* φ, that is , ${}^a\varphi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$, for all $\mathfrak{p}$ in $\operatorname{Spec}(A)$. Let $g \in B$, and $f = \varphi(g) \in A$. Then ${}^a\varphi^{-1}(\mathrm{D}(g)) = \mathrm{D}(f)$, and we also get a canonical morphism of rings,

$$\begin{aligned} \varphi_g : B_g &\longrightarrow A_f \\ \frac{b}{g^n} &\longmapsto \frac{\varphi(b)}{f^n}. \end{aligned}$$

This morphism of sheaves on the base $\mathcal{B}$ induces a morphism

$$\tilde{\varphi} : \tilde{B}(U) \to \tilde{A}(\varphi^{-1}(U))$$

for all open sets U in $\mathrm{Spec}(B)$ such that $\tilde{\varphi}(\mathrm{D}(g)) = \varphi_g$ for all $g \in B$. This gives a morphism of ringed spaces

$$({}^a\varphi, \tilde{\varphi}) : (\mathrm{Spec}(A), \tilde{A}) \to (\mathrm{Spec}(B), \tilde{B}).$$

In fact, it is easy to see that $({}^a\varphi, \tilde{\varphi})$ is a morphism of locally ringed spaces.

Definition 2.1.6 An *affine scheme* is a ringed space isomorphic to $(\mathrm{Spec}(A), \tilde{A})$ for some ring A.

Definition 2.1.7 Let k be a field. A ringed space $(X, \mathcal{O}_X)$ is called a *ringed space over* k if for every $U \subset X$ open, $\mathcal{O}_X(U)$ is a k-algebra, and for each pair (U, V) of open subsets of X such that $U \supset V$ the restriction homomorphism

$$\rho_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

is a homomorphism of k-algebras. A *morphism of k-ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces

$$(u, \tilde{u}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that for all $V \subset Y$ open the ring homomorphism

$$\tilde{u}_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(u^{-1}(V))$$

is a homomorphism of k-algebras.

A *locally ringed space over* k is a ringed space over k which is a locally ringed space.

2.1.2 The Functors $\Gamma(X, \bullet)$ and $\mathcal{S}$

For any ringed space $(X, \mathcal{O}_X)$, let $\mathcal{O}_X$-**mod** denote the category of $\mathcal{O}_X$-modules. For any ring A, let A-**mod** denote the category of A-modules. We will denote the $\Gamma(X, \mathcal{O}_X)$-module of homomorphism of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. If $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, and U is an open subset of X, then $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ denotes the homomorphism of $\mathcal{O}_X(U)$-modules induced by φ. Let A denote the ring $\Gamma(X, \mathcal{O}_X)$ of global sections of $\mathcal{O}_X$.

We have a canonical functor

$$\Gamma(X, \bullet) : \mathcal{O}_X\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod},$$

defined by, $\Gamma(X, \bullet)(\mathcal{F}) = \Gamma(X, \mathcal{F})$, and for $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$,

$$\Gamma(X, \bullet)(\varphi) = \varphi_X : \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}).$$

Recall that $\Gamma(X, \bullet)$ is a left exact functor.

Let M be an A-module. Define a presheaf $\mathcal{P}(M)$ on X by $\mathcal{P}(M)(U) = M \otimes_A \mathcal{O}_X(U)$ for every open set U of X. Note that, $\mathcal{O}_X(U)$ is canonically an A-module. Indeed if $f \in A$, and $s \in \mathcal{O}_X(U)$, then $f \cdot s = f|_U s \in \mathcal{O}_X(U)$. Therefore, $\mathcal{P}(M)(U)$ is an $\mathcal{O}_X(U)$-module. We will denote by $\mathcal{S}(M)$ the sheaf associated to the presheaf $\mathcal{P}(M)$. Similarly for a homomorphism $u : M \to N$, and for every $U \subset X$ open, define $(\mathcal{P}(M)(u))_U = u \otimes \mathbf{1}_{\mathcal{O}_X(U)}$. This morphism of presheaves induces a morphism of sheaves $\mathcal{S}(u) : \mathcal{S}(M) \to \mathcal{S}(N)$. Thus, we get a functor

$$\mathcal{S} : A\text{-}\mathbf{mod} \longrightarrow \mathcal{O}_X\text{-}\mathbf{mod}. \tag{2.2}$$

Since for every x in X, the functor $\bullet \otimes_A \mathcal{O}_{X,x}$ is right exact, the functor $\mathcal{S}$ is right exact.

Remark 2.1.8 Let $(X, \mathcal{O}_X)$ be a ringed space. For every $\mathcal{O}_X$-module $\mathcal{F}$, define a homomorphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$\varphi_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F})$$

by $\varphi_{\mathcal{F}}(u) = u_X(1)$, where $1 \in \Gamma(X, \mathcal{O}_X)$ is the identity section of $\mathcal{O}_X$ on X.

Then, $\varphi_{\mathcal{F}}$ is an isomorphism, whose inverse

$$\psi_{\mathcal{F}} : \Gamma(X, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$$

is given by, $\psi_{\mathcal{F}}(s)_U(f) = f \cdot (s|_U)$ for $s \in \Gamma(X, \mathcal{F})$, $U \subset X$ open, and $f \in \mathcal{O}_X(U)$. We say that the morphism of $\mathcal{O}_X$-module $u = \psi_{\mathcal{F}}(s) : \mathcal{O}_X \to \mathcal{F}$ is *defined by the section* s of $\mathcal{F}$ on X. Thus, we get an isomorphism of functors

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \bullet) \cong \Gamma(X, \bullet).$$

That is, the $\mathcal{O}_X$-module $\mathcal{O}_X$ represents $\Gamma(X, \bullet)$, and the identity section $1 \in \Gamma(X, \mathcal{O}_X)$ is a universal element for $\Gamma(X, \bullet)$ (**A:10** and **A:11**). Thus, the pair $(\mathcal{O}_X, 1)$ is a representation of $\Gamma(X, \bullet)$.

More generally, if I is an arbitrary index set, consider the direct sum $\mathcal{O}_X^{(I)} = \oplus_{i \in I} \mathcal{O}_X$, and for each $i \in I$, let $h_i : \mathcal{O}_X \to \mathcal{O}_X^{(I)}$ denote the canonical injective morphism from the i^{th} factor $\mathcal{O}_X$ to $\mathcal{O}_X^{(I)}$. Then, for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a homomorphism of $\Gamma(X, \mathcal{O}_X)$-modules,

$$\begin{aligned} \varphi_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F}) &\longrightarrow \Gamma(X, \mathcal{F})^I = \prod_{i \in I} \Gamma(X, \mathcal{F}), \\ u &\longmapsto ((u \circ h_i)_X(1))_{i \in I}. \end{aligned}$$

The map $\varphi_{\mathcal{F}}$ is an isomorphism, whose inverse

$$\psi_{\mathcal{F}} : \Gamma(X, \mathcal{F})^I \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F})$$

is given by $\psi_{\mathcal{F}}(s)_U(f) = \sum_{i \in I} f_i\big(s_i|_U\big)$ for $s = (s_i)_{i \in I} \in \Gamma(X, \mathcal{F})^I$, $U \subset X$ open, and $f = (f_i)_{i \in I} \in \mathcal{O}_X^{(I)}(U)$. (Since $f \in \mathcal{O}_X^{(I)}(U)$, the above sum is a finite sum, hence welldefined.) Thus, we get

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \bullet) \cong \Gamma(X, \bullet)^I. \tag{2.3}$$

We say that the morphism of $\mathcal{O}_X$-modules $u = \psi_{\mathcal{F}}(s) : \mathcal{O}_X^{(I)} \to \mathcal{F}$ is *defined by the family of sections* $s = (s_i)_{i \in I}$ *of* $\mathcal{F}$ on X. Let

$$F_I = \Gamma(X, \bullet)^I : \mathcal{O}_X\textbf{-mod} \to \Gamma(X, \mathcal{O}_X)\textbf{-mod},$$

be the functor defined by $F_I(\mathcal{F}) = \Gamma(X, \mathcal{F})^I$, and $F_I(u) = \prod_{i\in I} u_X$, for $u : \mathcal{F}' \to \mathcal{F}$ a morphism of $\mathcal{O}_X$-module. Then, the $\mathcal{O}_X$-module, $\mathcal{O}_X^I$ represents F_I, and the family of sections $e = (1)_{i\in I} \in F_I(\mathcal{O}_X)$ is a universal element for F_I. Thus, the pair $(\mathcal{O}_X^{(I)}, e)$ is a representation of F_I.

Proposition 2.1.9 *For any ringed space $(X, \mathcal{O}_X)$, the functor $\mathcal{S}$ is a left adjoint of $\Gamma(X, \bullet)$.*

Proof. Let $\Gamma(X, \mathcal{O}_X) = A$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and M an A-module. Let $u : M \to \Gamma(X, \mathcal{F})$ be a homomorphism of A-modules. Consider a homomorphism of presheaves, such that for every open subset U of X,

$$\lambda_u : M \otimes_A \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$$

is given by $\lambda_u(m \otimes_A f) = f \cdot u(m)|_U$, for $m \in M$, and $f \in \mathcal{O}_X(U)$. Let $\lambda_{\mathcal{F},M}(u)$ be the morphism of sheaves associated to λ_u. Thus, we get a map,

$$\lambda_{\mathcal{F},M} : \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(M), \mathcal{F}). \tag{2.4}$$

We will prove that $\lambda : A\text{-}\mathbf{mod}^{\mathrm{op}} \times \mathcal{O}_X\text{-}\mathbf{mod} \to \mathbf{Set}$, given by $M \times \mathcal{F} \mapsto \lambda_{\mathcal{F},M}$ is an adjunction between $\Gamma(X, \bullet)$ and $\mathcal{S}$ (**A:3**). So we have to check λ is functorial in $\mathcal{F}$ and M, and $\lambda_{\mathcal{F},M}$ is a bijection. We must show that for every $g : M' \to M$ and $\psi : \mathcal{F} \to \mathcal{F}'$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \arrow[r, "\lambda_{\mathcal{F},M}"] \arrow[d, "\mathrm{Hom}(g,\Gamma(X,\psi))"'] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(M), \mathcal{F}) \arrow[d, "\mathrm{Hom}(\mathcal{S}(g),\psi)"] \\ \mathrm{Hom}_A(M', \Gamma(X, \mathcal{F}')) \arrow[r, "\lambda_{\mathcal{F}',M'}"'] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(M'), \mathcal{F}') \end{tikzcd} \tag{2.5}$$

commutes. Let $u : M \to \Gamma(X, \mathcal{F})$, then

$$\mathrm{Hom}(\mathcal{S}(g), \psi) \circ \lambda_{\mathcal{F},M}(u) = \psi \circ \lambda_{\mathcal{F},M}(u) \circ \mathcal{S}(g) : \mathcal{S}(M') \to \mathcal{F}', \tag{2.6}$$

and

$$\lambda_{\mathcal{F}',M'} \circ \mathrm{Hom}(g, \psi)(u) = \lambda_{\mathcal{F}',M'}(\Gamma(X, \psi) \circ u \circ g) : \mathcal{S}(M') \to \mathcal{F}'. \tag{2.7}$$

It is enough to check that, this two morphism agrees on the presheaf $\mathcal{P}(M')$. Let

U be an open subset of X, and $m' \otimes s \in \mathcal{P}(M')(U) = M' \otimes \mathcal{O}_X(U)$. Then,

$$\begin{aligned}\psi_U \circ \lambda_{\mathcal{F}',M'}(u) \circ \mathcal{S}(g)(m' \otimes s) = \psi_U \circ \lambda_u(g(m') \otimes s) = \psi_U\Big(s \cdot u(g(m'))|_U\Big) \\ = s \cdot \Big(\Gamma(X, \psi)(u \circ g(m'))\Big)|_U\end{aligned}$$

since ψ is a sheaf morphism. Hence, $\psi_U \circ \lambda_{\mathcal{F}',M'}(u) \circ \mathcal{S}(g)(m' \otimes s) = \lambda_{\mathcal{F}',M'}(\Gamma(X, \psi) \circ u \circ g)(m' \otimes s)$. Thus, the two morphisms (2.6) and (2.7) agree, that is, the diagram (2.5) commutes. Hence, λ is functorial in $\mathcal{F}$ and M. It remains to check that $\lambda_{\mathcal{F},M}$ is a bijection. Define

$$\varphi_{\mathcal{F},M} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(M), \mathcal{F}) \longrightarrow \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \tag{2.8}$$

by $\varphi_{\mathcal{F},M}(u)(m) = u_X(m \otimes 1)$ for $u : \mathcal{S}(M) \to \mathcal{F}$, and $m \in M$. Now it is easy to show that $\lambda_{\mathcal{F},M} \circ \varphi_{\mathcal{F},M} = \mathbf{1}_{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(M),\mathcal{F})}$, and $\varphi_{\mathcal{F},M} \circ \lambda_{\mathcal{F},M} = \mathbf{1}_{\mathrm{Hom}_A(M,\Gamma(X,\mathcal{F}))}$. Hence, $\lambda_{\mathcal{F},M}$ is a bijection. $\square$

2.1.3 The Sheaf of Morphisms from $\mathcal{F}$ to $\mathcal{G}$

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. For every open subset U of X, define

$$\big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big)(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Recall that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the set of $\mathcal{O}_X$-module homomorphisms from $\mathcal{F}$ to $\mathcal{G}$, and it is canonically an $\Gamma(X, \mathcal{O}_X)$-module. If $U \subset V$, we have a canonical restriction map,

$$\big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big)(V) \longrightarrow \big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big)(U).$$

This defines a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ called the *sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$*. There is a canonical structure of an $\mathcal{O}_X$-module on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ defined as follows. Let U be an open subset of X, and let $v : \mathcal{F}|_U \to \mathcal{G}|_U$ be a morphism of $\mathcal{O}_X|_U$-modules. Let $f \in \mathcal{O}_X(U)$. Define $fu : \mathcal{F}|_U \to \mathcal{G}|_U$ by setting, for every open $V \subset U$,

$$(fu)_V : \mathcal{F}(V) \to \mathcal{G}(V),$$

to be $(fv)_V(s) = f|_V \cdot u_V(s)$, $s \in \mathcal{F}(V)$.

Definition 2.1.10 For every point $x \in X$, there exists a canonical homomorphism of $\mathcal{O}_{X,x}$-modules

$$\varphi_x : (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \tag{2.9}$$

defined as follows. Let $\alpha \in (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x$, choose an open neighborhood U of x, and a morphism of $\mathcal{O}_X|_U$-modules, $u : \mathcal{F}|_U \to \mathcal{G}|_U$, such that α equals the germ of u at x. For each point $y \in U$, let $u_y : \mathcal{F}_y \to \mathcal{G}_y$ denote the $\mathcal{O}_{X,y}$-module homomorphism induced by u (stalk homomorphism). Define

$$\varphi_x(\alpha) = u_x.$$

It follows from this definition that $\varphi_x(\alpha)$ is independent of the choices of U and u.

Note that the canonical homomorphism φ_x is functorial. The canonical homomorphism φ_x is in general neither injective nor surjective.

Remark 2.1.11 Let $\mathcal{F}$ be a presheaf, and $\mathcal{G}$ a sheaf on a topological space X. For each point $x \in X$, let $\alpha^x : \mathcal{F}_x \to \mathcal{G}_x$ be a map. Suppose that for every $x \in X$, every open neighborhood U of x, and for every section $s \in \Gamma(U, \mathcal{F})$, there exist an open neighborhood V of x in U, and a section $t \in \Gamma(V, \mathcal{G})$, such that $\alpha^y((s)_y) = (t)_y$ for all $y \in V$. Then, there exists a unique morphism of presheaves $u : \mathcal{F} \to \mathcal{G}$ such that $(u)_x = \alpha^x$ for all $x \in X$.

Proposition 2.1.12 *Let $(X, \mathcal{O}_X)$ be a ringed space. For every $\mathcal{O}_X$-module $\mathcal{G}$, the functor*

$$\mathrm{Hom}_{\mathcal{O}_X}(\bullet, \mathcal{G}) : \mathcal{O}_X\textbf{-mod}^{\mathrm{op}} \longrightarrow \Gamma(X, \mathcal{O}_X)\textbf{-mod}$$

is left exact. In particular, the functor

$$\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{G}) : \mathcal{O}_X\textbf{-mod}^{\mathrm{op}} \longrightarrow \mathcal{O}_X\textbf{-mod}$$

is left exact.

Proof. We have to show that if $\mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \to 0$ is an exact sequence of

$\mathcal{O}_X$-modules, then the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \xrightarrow{\mathrm{Hom}_{\mathcal{O}_X}(v,\mathcal{G})} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\mathrm{Hom}_{\mathcal{O}_X}(u,\mathcal{G})} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G})$$

is exact. Let $w \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G})$ be such that $\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G})(w) = 0$, that is, for every $x \in X$, and $\theta \in \mathcal{F}_x$, $w_x \circ v_x(\theta) = 0$. Let $\theta'' \in \mathcal{F}''_x$. Then, there exists $\theta \in \mathcal{F}_x$ such that $v_x(\theta) = \theta''$. Therefore, $w_x(\theta'') = w_x(v_x(\theta)) = 0$. Hence, $w_x = 0$ for all $x \in X$, that is, $w = 0$. Hence, $\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G})$ is injective. Since $\mathrm{Hom}_{\mathcal{O}_X}(\bullet, \mathcal{G})$ is a functor,

$$\mathrm{Hom}_{\mathcal{O}_X}(u, \mathcal{G}) \circ \mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(v \circ u, \mathcal{G}) = 0.$$

This implies that $\mathrm{Im}(\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G})) \subset \mathrm{Ker}(\mathrm{Hom}_{\mathcal{O}_X}(u, \mathcal{G}))$.

Now let $g \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{O}_X}(u, \mathcal{G}))$. Define $f^x : \mathcal{F}''_x \to \mathcal{G}_x$ as follows. For every $\theta'' \in \mathcal{F}''_x$, there exists $\theta \in \mathcal{F}_x$ such that $v_x(\theta) = \theta''$. Define, $f^x(\theta'') = g_x(\theta)$. Since $g \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{O}_X}(u, \mathcal{G}))$, f^x is well defined. Now it is easy to check the condition in Remark 2.1.11, which gives a morphism $f : \mathcal{F}'' \to \mathcal{G}$. For all $x \in X$, and $\theta \in \mathcal{F}'_x$,

$$(\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G})(f))_x(\theta) = f_x \circ v_x(\theta) = f^x \circ v_x(\theta) = g_x(\theta),$$

hence, $\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G})(f) = g$. Thus, $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{O}_X}(u, \mathcal{G})) \subset \mathrm{Im}(\mathrm{Hom}_{\mathcal{O}_X}(v, \mathcal{G}))$. This proves that, $\mathrm{Hom}_{\mathcal{O}_X}(\bullet, \mathcal{G})$ is left exact. The second statement follows from the first by putting an arbitrary open subset of X in the place of X. □

Remark 2.1.13 Recall that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{G}) \cong \Gamma(X, \mathcal{G})^I$ (2.3). Therefore, if I is a finite set then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^I, \mathcal{G}) \cong \Gamma(X, \mathcal{G})^I.$$

Let U be an open subset of X. Then, the above isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules gives an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^I, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X^I|_U, \mathcal{G}|_U) \cong \Gamma(U, \mathcal{G})^I = \mathcal{G}(U)^I.$$

Thus, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^I, \mathcal{G}) \cong \mathcal{G}^I$ for every finite set I.

2.1.4 Sheaves of Finite type

Definition 2.1.14 An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be of *finite type* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is generated by a finite family of sections of $\mathcal{F}$ on U.

The following properties of sheaves of finite type are easy to verify.

Proposition 2.1.15 *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *If $u : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of $\mathcal{O}_X$-modules, and if $\mathcal{F}$ is of finite type, then so is $\mathcal{G}$. Thus, every quotient sheaf of a sheaf of finite type is also of finite type.*

2. *The direct sum and the tensor product (over $\mathcal{O}_X$) of a finite family of sheaves of finite type are also of finite type.*

Proposition 2.1.16 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$, and let $s_1, s_2, \dots, s_n$ be sections of $\mathcal{F}$ on an open neighborhood U of x, such that the family $\left((s_i)_x\right)_{i=1}^n$ generates $\mathcal{F}_x$. Then, there exists an open neighborhood V of x in U, such that the family $\left((s_i)_y\right)_{i=1}^n$ generates $\mathcal{F}_y$ for all $y \in V$.*

Proof. Since $\mathcal{F}$ is of finite type, there exist an open neighborhood U_1 of x in U, and sections $t_1, \dots, t_p \in \Gamma(U_1, \mathcal{F})$ such that the family $\left((t_i)_y\right)_{i=1}^p$ generates $\mathcal{F}_y$ for all $y \in U_1$. Since the family $\left((s_j)_x\right)_{j=1}^n$ also generate $\mathcal{F}_x$, there exist germs $\theta_{ij} \in \mathcal{O}_{X,x}$ such that

$$(t_i)_x = \sum_{j=1}^{n} \theta_{ij} (s_j)_x \quad (i = 1, \dots, p).$$

As the family $(\theta_{ij})_{i,j}$ is finite, there exist an open neighborhood U_2 of x in U_1, and sections $f_{ij} \in \Gamma(U_2, \mathcal{O}_X)$ such that $\theta_{ij} = (f_{ij})_x$ for all i, j, and

$$(t_i)_x = \sum_{j=1}^{n} (f_{ij})_x (s_j)_x \quad (i = 1, \dots, p).$$

Therefore, there exists an open neighborhood V of x in U such that

$$(t_i)|_V = \sum_{j=1}^{n} (f_{ij})|_V (s_j)|_V \quad (i = 1, \dots, p),$$

hence, $(t_i)_y = \sum_j (f_{ij})_y (s_j)_y$ for all $y \in V$. Therefore, $(s_j)_y$ generates $\mathcal{F}_y$ for all $y \in V$. □

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Recall that the *support of* $\mathcal{F}$ is the set

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Note that the support of a sheaf need not be a closed subset of X.

Corollary 2.1.17 *If $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then* $\mathrm{Supp}(\mathcal{F})$ *is a closed subset of X.*

Proof. Let $x \in X \backslash \mathrm{Supp}(\mathcal{F})$. Then, $\mathcal{F}_x = 0$, hence the germ 0_x of the zero section $0 \in \Gamma(X, \mathcal{F})$ generates $\mathcal{F}_x$. Thus, by Proposition 2.1.16, there exists an open neighborhood V of x such that 0_y generates $\mathcal{F}_y$, for all $y \in V$, that is, $\mathcal{F}_y = 0$ for all $y \in V$. Therefore, V is an open neighborhood of x in $X \backslash \mathrm{Supp}(\mathcal{F})$. □

Corollary 2.1.18 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type, and let $u : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $x \in X$ and suppose that the stalk homomorphism $u_x : \mathcal{F}_x \to \mathcal{G}_x$ equals* 0*. Then, there exists an open neighborhood U of x such that $u_y = 0$ for all $y \in V$.*

Proof. Let $\mathcal{H} = \mathrm{Im}(u)$. Then $\mathcal{H}$ is an $\mathcal{O}_X$-submodule of $\mathcal{G}$, and u induces an exact sequence of $\mathcal{O}_X$-modules $\mathcal{F} \xrightarrow{u'} \mathcal{H} \longrightarrow 0$. Since u' is surjective, and $\mathcal{F}$ is of finite type, by (1) of Proposition 2.1.15, $\mathcal{H}$ is of finite type. Since

$$u'_y = u_y : \mathcal{F}_y \to \mathcal{H}_y = \mathrm{Im}(u_y)$$

for all $y \in X$, we have $\mathcal{H}_y = 0$ if and only if $u_y = 0$. Thus,

$$\mathrm{Supp}(\mathcal{H}) = \{y \in X \mid u_y \neq 0\}.$$

By Corollary 2.1.17, since $x \in X \backslash \mathrm{Supp}(\mathcal{H})$, there exists an open neighborhood U of x in $X \backslash \mathrm{Supp}(\mathcal{H})$. We have $u_y = 0$ for all $y \in U$. □

Corollary 2.1.19 *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules of finite type. Let $u : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules, and $x \in X$. If $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective, then there exists an open neighborhood U of x, such that $u|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ is surjective.*

Proof. Since $\mathcal{G}$ is of finite type, $\mathrm{Coker}(u) = \mathcal{G}/\mathrm{Im}(u)$ is also of finite type (Proposition 2.1.15, (1)). Hence, by Corollary 2.1.17, $F = \mathrm{Supp}(\mathrm{Coker}(u))$ is a closed subset of X. If u_x is surjective, the $(\mathrm{Coker}(u))_x = \mathrm{Coker}(u_x) = 0$, hence $x \in X \backslash F$. Therefore, $U = X \backslash F$ is an open neighborhood of x such that $u|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ is surjective. □

Corollary 2.1.20 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then, for every $\mathcal{O}_X$-module $\mathcal{G}$, and for every point $x \in X$, the canonical homomorphism of $\mathcal{O}_{X,x}$-modules (2.9)*

$$\varphi_x : (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is injective.

Proof. Let $\alpha \in (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x$, and suppose that $\varphi_x(\alpha) = 0$. Let U and u be as in Definition 2.1.10. Then $u_x = \varphi_x(\alpha) = 0$, so by corollary 2.1.18, there exists an open neighborhood V of x in U such that $u_y = 0$ for all $y \in V$. We must show that $u|_V : \mathcal{F}|_V \to \mathcal{G}|_V$ is equals 0, which will imply that $\alpha = 0$. Let W be an open subset of V. Consider the homomorphism

$$\left(u|_V\right)_W = u_W : \mathcal{F}(W) \to \mathcal{G}(W).$$

Let $s \in \mathcal{F}(W)$. Then, $(u_W(s))_y = u_y((s)_y) = 0$ for all $y \in W$. Since $\mathcal{G}$ is a sheaf, this implies that $u_W(s) = 0$. Therefore, $u_W = 0$ for all W open in V, hence $u|_V = 0$. □

Corollary 2.1.21 *Let $(X, \mathcal{O}_X)$ be a ringed space such that, X is quasicompact. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ is finitely generated by global sections if it is generated by global sections.*

Proof. Let $(s_i)_{i \in I}$ be a family of global sections of $\mathcal{F}$ such that the family $\left((s_i)_x\right)_{i \in I}$ generates $\mathcal{F}_x$ for all $x \in X$. Let $x \in X$. Since $\mathcal{F}$ is of finite type, there exists a finite family $(\theta_j)_{j \in J_x}$ of generators of $\mathcal{F}_x$. As $\left((s_i)_x\right)_{i \in I}$ also generates $\mathcal{F}_x$, there exist a finite subset I_x if I, and $a_{ij} \in \mathcal{O}_{X,x}(i \in I_x, j \in J_x)$ such that

$$\theta_j = \sum_{i \in I_x} a_{ij} (s_i)_x.$$

Thus, the family $\left((s_i)_x\right)_{i \in I_x}$ generates $\mathcal{F}_x$. By Proportion 2.1.16 there exists an open neighborhood U_x of x, such that the family $\left((s_i)_y\right)_{i \in I_x}$ generates F_y for all

$y \in U_x$. Since X is quasicompact there exists $x_1, \ldots, x_n$ such that $\cup_{i=1}^n U_{x_i} = X$. This implies that, the family

$$\left((s_i)_{i \in I_{x_\alpha}}\right)_{1 \leq \alpha \leq n}$$

of global sections of $\mathcal{F}$ is such that for all $x \in X$ their stalk generate $\mathcal{F}_x$. Thus, $\mathcal{F}$ is finitely generated by global sections. □

2.1.5 Sheaves of Finite Presentation

Definition 2.1.22 An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be is of *finite presentation* if for every point x in X, there exist an open neighborhood U of x, and an exact sequence of $\mathcal{O}_X|_U$-modules

$$\mathcal{O}_X^p|_U \xrightarrow{u} \mathcal{O}_X^q|_U \xrightarrow{v} \mathcal{F}|_U \to 0$$

Proposition 2.1.23 *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, and let $x \in X$. Then, for every $\mathcal{O}_X$-module $\mathcal{G}$, the canonical homomorphism of $\mathcal{O}_{X,x}$-modules (2.9)*

$$\varphi_x : \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is an isomorphism.

Proof. Let U be an open neighborhood of x, on which there exists an exact sequence of $\mathcal{O}_X|_U$-modules

$$\mathcal{O}_X^p|_U \xrightarrow{u} \mathcal{O}_X^q|_U \xrightarrow{v} \mathcal{F}|_U \to 0.$$

Since the proposition is local with respect to x, we will replace X by U, so $X = U$. By Proposition 2.1.12, $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{G})$ is a left exact functor, so we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{G}).$$

This induces an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)_x \to \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{G})\right)_x \to \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{G})\right)_x.$$

Since $\mathcal{O}_{X,x}^p \xrightarrow{u_x} \mathcal{O}_{X,x}^q \xrightarrow{v_x} \mathcal{F}_x \longrightarrow 0$ is an exact sequence of $\mathcal{O}_{X,x}$-modules, and since $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\bullet, \mathcal{G}_x)$ is a left exact functor, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^q, \mathcal{G}_x) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^p, \mathcal{G}_x).$$

Since the canonical homomorphism φ_x is functorial, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_x \arrow[r] \arrow[d, "\varphi_x"] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{G}))_x \arrow[r] \arrow[d, "\varphi_x"] & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{G}))_x \arrow[d, "\varphi_x"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^q, \mathcal{G}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^p, \mathcal{G}_x)
\end{tikzcd}$$

By Remark 2.1.13 there exists a canonical isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{G}) \cong \mathcal{G}^q$. Thus, the modules in the second column of the above diagram are identifies with $\mathcal{G}_x^q$ and $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^q, \mathcal{G}_x) \cong \mathcal{G}_x^q$ respectively, and the second vertical arrow is an isomorphism. For the same reasons, the third vertical arrow is also an isomorphism. Since the rows of the above diagram are exact, it follows that the first vertical arrows also is an isomorphism. □

Corollary 2.1.24 *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules of finite presentation, and let $x \in X$. Suppose that $f : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism of $\mathcal{O}_{X,x}$-modules. Then there exist an open neighborhood V of x, and an isomorphism of $\mathcal{O}_X|_V$-modules $u : \mathcal{F}|_V \to \mathcal{G}|_V$ such that $u_x = f$.*

Proof. Let $g : \mathcal{G}_x \to \mathcal{F}_x$ be the inverse of f. By Proposition 2.1.23, there exist an open neighborhood V of x, and sections $u \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$ and $v \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}))$ such that $u_x = f$ and $v_x = g$. Because,

$$(u \circ v)_x = u_x \circ v_x = f \circ g = \mathbf{1}_{\mathcal{G}_x} \text{ and } (v \circ u)_x = v_x \circ u_x = g \circ f = \mathbf{1}_{\mathcal{F}_x}$$

the Proposition 2.1.23 again implies that the germs, at x, of $u \circ v$ and $v \circ u$ equal those of the corresponding identity morphisms. Therefore, there exist an open neighborhood U of x in V such that

$$(u \circ v)|_U = \mathbf{1}_{\mathcal{G}|_U} \text{ and } (v \circ u)|_U = \mathbf{1}_{\mathcal{F}|_U}.$$

Thus, $u|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ is an isomorphism. □

2.1.6 Coherent Sheaves

In this subsection we collect some general facts about coherent sheaves which we will be using in further sections. We use [Ser55] as a reference for coherent sheaves.

Definition 2.1.25 Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if it satisfies the following conditions:

1. $\mathcal{F}$ is of finite type.
2. For every open subset U of X, for every integer $p \in \mathbf{N}$, and for every morphism of $\mathcal{O}_X|_U$-modules
$$u : \mathcal{O}_X^p|_U \longrightarrow \mathcal{F}|_U,$$
the $\mathcal{O}_X|_U$-submodule $\ker(u)$ of $\mathcal{O}_X^p|_U$ is of finite type.

Let $(X, \mathcal{O}_X)$ be a ringed space. *A sheaf of ideals $\mathcal{I}$ in $\mathcal{O}_X$ is said to be coherent* if it is coherent as an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

Theorem 2.1.26 (Oka's Coherence Theorem) [GR84, Chapter 2, §5, 2 Theorem, p. 59] *For every positive integer n, the structure sheaf $\mathcal{O}_{\mathbf{C}^n}$ (sheaf of holomorphic functions on $\mathbf{C}^n$) is a coherent sheaf of rings.*

Since coherence is a local condition Oka's Theorem implies that for every complex manifold X, the structure sheaf $\mathcal{O}_X$ is coherent.

Proposition 2.1.27 *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, every coherent $\mathcal{O}_X$-module is of finite presentation.*

Proof. Let $x \in X$. Since $\mathcal{F}$ is of finite type, there is an open neighborhood U of x, and an exact sequence of $\mathcal{O}_X|_U$-modules
$$\mathcal{O}_X^p|_U \xrightarrow{u} \mathcal{F}|_U \longrightarrow 0 \quad (p \in \mathbf{N}).$$
Since $\mathcal{F}$ is coherent, $\mathcal{K} = \ker(u)$ is an $\mathcal{O}_X|_U$-module of finite type, therefore, there exist an open neighborhood V of x in U, and an exact sequence
$$\mathcal{O}_X^q|_V \xrightarrow{v} \mathcal{K}|_V \longrightarrow 0 \quad (q \in \mathbf{N}).$$

Let $i : \mathcal{K} \to \mathcal{O}_X^p|_V$ be the inclusion morphism, let $w = i|_V \circ v : \mathcal{O}_X^q|_V \to \mathcal{O}_X^p|_V$. Then

$$\mathcal{O}_X^q|_V \xrightarrow{w} \mathcal{O}_X^p|_V \xrightarrow{u|_V} \mathcal{F}|_V \longrightarrow 0$$

is exact. □

Remark 2.1.28 The converse of the above proposition is in general not true. It is however true if $\mathcal{O}_X$ is coherent (see Corollary 2.1.31 and Corollary 2.1.32).

Proposition 2.1.29 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then every $\mathcal{O}_X$-submodule of finite type in $\mathcal{F}$ is also coherent.*

Proof. Let $\mathcal{G}$ be an $\mathcal{O}_X$-submodule of $\mathcal{F}$ of finite type. We only have to check the second condition in the definition of coherence. Let $x \in X$ and let U be an open neighborhood of x, and let

$$u : \mathcal{O}_X^p|_U \longrightarrow \mathcal{G}|_U$$

be a morphism on $\mathcal{O}_X|_U$-modules. Then

$$\ker(u) = \ker(i|_u \circ u : \mathcal{O}_X^p|_U \to \mathcal{F}|_U),$$

where $i : \mathcal{G} \to \mathcal{F}$ is the inclusion morphism. Since $\mathcal{F}$ is coherent, it follows that $\ker(u)$ is of finite type. Therefore, $\mathcal{G}$ is coherent. □

Theorem 2.1.30 (Three Lemma) *Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. If two of the $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are coherent, then so is the third.

Following are some consequences of Three Lemma.

Corollary 2.1.31 (Corollary to Theorem 2.1.30) *The direct sum of a finite family of coherent $\mathcal{O}_X$-modules is coherent.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves, there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow 0,$$

hence by Theorem 2.1.30 $\mathcal{F} \oplus \mathcal{G}$ is coherent. The corollary follows by induction. $\square$

Corollary 2.1.32 (Corollary to Theorem 2.1.30) *If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, and if $u : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then* $\mathrm{Im}(u)$, $\ker(u)$, *and* $\mathrm{Coker}(u)$ *are coherent $\mathcal{O}_X$-modules.*

Proof. Since $\mathcal{F}$ is of finite type, and since $u : \mathcal{F} \to \mathrm{Im}(u)$ is a surjective morphism, by (1) of Proposition 2.1.15, $\mathrm{Im}(u)$ is of finite type. Now by Proposition 2.1.29, since $\mathcal{G}$ is coherent, $\mathrm{Im}(u)$ is coherent. There are exact sequences

$$0 \longrightarrow \ker(u) \longrightarrow \mathcal{F} \xrightarrow{u} \mathrm{Im}(u) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}(u) \xrightarrow{i} \mathcal{G} \longrightarrow \mathrm{Coker}(u) \longrightarrow 0,$$

where $i : \mathrm{Im}(u) \to \mathcal{G}$ is inclusion morphism. Theorem 2.1.30 now implies the coherence of $\ker(u)$ and $\mathrm{Coker}(u)$. $\square$

Remark 2.1.33 Let $\mathbf{Coh}(X)$ be the full subcategory of $\mathcal{O}_X$-**mod** consists of coherent $\mathcal{O}_X$-modules. A subcategory of an abelian category is abelian if contains a zero object, and is closed with respect to finite coproducts, kernels, and cokernels. Now since $\mathcal{O}_X$-**mod** is an abelian category, by Corollary 2.1.31 and Corollary 2.1.32, the category $\mathbf{Coh}(X)$ is an abelian category.

Corollary 2.1.34 (Corollary to Theorem 2.1.30) *If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.*

Proof. Since the question is local, we may assume that there is an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^p \xrightarrow{u} \mathcal{O}_X^q \xrightarrow{v} \mathcal{F} \longrightarrow 0.$$

Since $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{G}) : \mathcal{O}_X\text{-}\mathbf{mod}^{\mathrm{op}} \to \mathcal{O}_X\text{-}\mathbf{mod}$ is a left exact functor (Proposition 2.1.12), we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{G}).$$

Now, by Remark 2.1.13, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{G}) \cong \mathcal{G}^n$, and is hence coherent (Corollary 2.1.31) for all $n \in N$. Thus, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the kernel of a morphism between coherent sheaves, and is hence coherent. □

2.1.7 Locally Free Sheaves

Definition 2.1.35 Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if for every point $x \in X$, there exist an open neighborhood U of x, and a set I, such that $\mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$ as an $\mathcal{O}_X|_U$-module.

If A is a nonzero ring, and M a free A-module, any two bases have the same cardinality [Bou89, Corollary to Proposition 3, p. 294], and this common cardinality is called the *rank of* M.

Remark 2.1.36 Let X be a ringed space such that $\mathrm{Supp}(\mathcal{O}_X) = X$. Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module, and let $x \in X$. Then, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module. Since $\mathcal{O}_{X,x} \neq 0$, the above paragraph implies that the *rank of* $\mathcal{F}_x$ (as a free $\mathcal{O}_{X,x}$-module) is well defined, we denote it by $\mathrm{rk}_x(\mathcal{F})$. The definition implies that there exist an open neighborhood U of x, such that

$$\mathrm{rk}_y(\mathcal{F}) = \mathrm{rk}_x(\mathcal{F}), \text{ for all } y \in U.$$

Proposition 2.1.37 *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, and let $x \in X$. Suppose that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank n. Then, n is finite, and there exists an open neighborhood U of x, such that $\mathcal{F}|_U$ is a locally free $(\mathcal{O}_X|_U)$-module of rank n.*

Proof. Since $\mathcal{F}_x$ is a finitely generated free $\mathcal{O}_{X,x}$-module, the rank n of $\mathcal{F}_x$ is finite. Let $u : \mathcal{F}_x \to \mathcal{O}_{X,x}^n$ be an isomorphism of $\mathcal{O}_{X,x}$-modules. Since $\mathcal{F}$ and $\mathcal{O}_X^n$ are of finite presentation, Corollary 2.1.24 implies that there exist an open neighborhood U of x, and an isomorphism of $(\mathcal{O}_X|_U)$-modules $\varphi : \mathcal{F}|_U \to \mathcal{O}_X^n|_U$ such that $\varphi_x = u$. □

Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $x \in X$. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we define,

$$\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) = \mathcal{F}_x / \mathfrak{m}_{X,x}\mathcal{F}_x,$$

where, $\mathfrak{m}_{X,x}$ is the maximal ideal in $\mathcal{O}_{X,x}$, and $k(x)$ the residue field of X at x. We call $\mathcal{F}(x)$ the *fibre* of $\mathcal{F}$ at x, it is a $k(x)$-vector space. The *rank of* $\mathcal{F}$ *at* x is defined by

$$\mathrm{rk}_x(\mathcal{F}) = \dim_{k(x)} \mathcal{F}(x).$$

If $\mathrm{rk}_x(\mathcal{F}) = n$ is independent of x, we say that $\mathcal{F}$ *is of rank* n. If $\mathrm{rk}_x(\mathcal{F})$ is finite for all $x \in X$, we say that $\mathcal{F}$ *is of finite rank.* If $\{\mathrm{rk}_x(\mathcal{F}) \,|\, x \in X\} < \infty$, then we call $\mathcal{F}$ *is of bounded rank.*

Remark 2.1.38 Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. For $x \in X$, then there exists a set I such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{(I)}$. Hence, we get,

$$\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) \cong \mathcal{O}_{X,x}^{(I)} \otimes_{\mathcal{O}_{X,x}} k(x) = k(x)^{(I)}.$$

This implies that, $\dim_{k(x)} \mathcal{F}(x)$ equals the rank of the free $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. Hence the definition of the rank of $\mathcal{F}$ as defined in Remark 2.1.36 is the same as that of the definition of the rank of $\mathcal{F}$ as an $\mathcal{O}_X$-module over a locally ringed space $(X, \mathcal{O}_X)$.

Let $\mathbf{Lfb}(X)$ denotes the full subcategory of $\mathcal{O}_X$-**mod** consisting of locally free $\mathcal{O}_X$-modules of bounded rank.

Proposition 2.1.39 *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then, the following are equivalent:*

1. *$\mathcal{F}$ is locally free.*
2. *$\mathcal{F}$ is of finite presentation, and for every point $x \in X$, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module.*
3. *For every point $x \in X$, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function $X \to \mathbf{N}$, $x \mapsto \mathrm{rk}_x(\mathcal{F})$ is locally constant.*

Proof. For any ringed space (1) implies (2) is clearly true. By Proposition 2.1.37, (2) implies (1) for any ringed space.

By Remark 2.1.36 and Remark 2.1.38 (1) implies (3). Therefore, it remains to prove (3) implies (1). Let $z \in X$, and let $n = \mathrm{rk}_z(\mathcal{F})$. By hypothesis, there exists an open neighborhood U of z, such that $\mathrm{rk}_z(\mathcal{F}) = n$ for all $x \in U$. Let $\{\theta_1, \ldots, \theta_n\}$ be a basis of the free $\mathcal{O}_{X,z}$-module $\mathcal{F}_z$. Replacing U by a smaller set, we may assume there exist sections $s_1, \ldots, s_n \in \Gamma(U, \mathcal{F})$ such that $(s_i)_z = \theta_i$.

Let $u : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ be the morphism of $\mathcal{O}_X|_U$-modules defined by the s_i. Then, $u_z : \mathcal{O}_{X,z}^n \to \mathcal{F}_z$ is an isomorphism. By Corollary 2.1.19, there exists an open neighborhood V of z in U such that $u|_U : \mathcal{O}_X^n|_U \to \mathcal{G}|_U$ is a surjective morphism. Therefore, for all $x \in V$, $u_x : \mathcal{O}_{X,x}^n \to \mathcal{F}_x$ is a surjective. Since $\mathcal{F}_x$ is free of rank n, there exists an isomorphism $\varphi^x : \mathcal{F}_x \to \mathcal{O}_{X,x}^n$. The map $\varphi^x \circ u_x : \mathcal{O}_{X,x}^n \to \mathcal{O}_{X,x}^n$ is a surjective endomorphism of $\mathcal{O}_{X,x}^n$. By [Eis95, Corollary 4.4, p. 120], $\varphi^x \circ u_x$ is an isomorphism. Therefore, u_x is an isomorphism for all $x \in V$, hence $u|_U : \mathcal{O}_X^n|_U \to \mathcal{G}|_U$ is an isomorphism. This shows that $\mathcal{F}$ is locally free. □

2.2 The Serre-Swan Theorem

In this section we will prove the main theorem of this chapter. We will use results from Section 2.1 to prove the theorem. We will define a notion of an admissible subcategory in Subsection 2.2.1. The main theorem and it proof are given in Subsection 2.2.2.

The functor $\Gamma(X, \bullet)$ defined in Subsection 2.1.2 is in general not fully faithful. Let X be a compact Riemann surface, and let $\mathcal{O}_X$ be a sheaf of holomorphic functions over X. Then $\Gamma(X, \mathcal{O}_X) = \mathbf{C}$. Let L be a line bundle on X, with negative degree. Then $\Gamma(X, L) = 0$, so is $\mathrm{End}(\Gamma(X, L))$. On the other hand $\mathbf{1}_L : L \to L$ and $0 : L \to L$ are two distinct morphism of L, hence, $\mathrm{End}_{\mathcal{O}_X}(L) \to \mathrm{End}_{\mathbf{C}}(\Gamma(X, L))$ is not injective, and $\Gamma(X, \bullet)$ is not fully faithful. But under certain conditions restriction of $\Gamma(X, \bullet)$ to a subcategory of $\mathcal{O}_X$-**mod** is fully faithful.

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *generated by global sections* if there is a family of sections $(s_i)_{i \in I}$ in $\Gamma(X, \mathcal{F})$ such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. We will say that $\mathcal{F}$ is *finitely generated by global sections* if a finite family of global sections $(s_i)_{i \in I}$ exists with the above property.

Proposition 2.2.1 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{C}$ be a full abelian subcategory of $\mathcal{O}_X$-**mod**, such that $\mathcal{O}_X$ belongs to $\mathcal{C}$. Suppose that every sheaf in $\mathcal{C}$ is generated by global sections. Then $\Gamma(X, \bullet) : \mathcal{C} \to \Gamma(X, \mathcal{O}_X)$-**mod** is fully faithful.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(\mathcal{C})$. Then, we have to show that the homomorphism

$$\varphi_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_A(\Gamma(X,\mathcal{F}),\Gamma(X,\mathcal{G})), \quad u \mapsto u_X \tag{2.10}$$

is a bijection. First we will prove that $\varphi_{\mathcal{F},\mathcal{G}}$ is injective. Let $u \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ be such that $u_X = 0$, we will then show that $u = 0$. To show this it suffices to show that $u_x(w) = 0$ for all $x \in X$, and $w \in \mathcal{F}_x$. Since $\mathcal{F}$ is generated by global sections, $w = \sum_{i=1}^n a_i \cdot (s_i)_x$, where $a_i \in \mathcal{O}_{X,x}$, and $s_i \in \Gamma(X,\mathcal{F})$, $i = 1,\ldots,n$. Thus,

$$\begin{aligned} u_x(w) = u_x\left(\sum_{i=1}^n a_i \cdot (s_i)_x\right) &= \sum_{i=1}^n a_i \cdot u_x\left((s_i)_x\right) \\ = \sum_{i=1}^n a_i \cdot \left(u_X(s_i)\right)_x &= \sum_{i=1}^n a_i \cdot 0 = 0. \end{aligned}$$

Hence we are through. Now we will show that $\varphi_{\mathcal{F},\mathcal{G}}$ is surjective. Let $\alpha : \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{G})$ be a homomorphism of A-modules. For every point x in X, let us define $\alpha^x : \mathcal{F}_x \to \mathcal{G}_x$ as follows. Let $w = \sum_{i=1}^n a_i \cdot (s_i)_x$ in $\mathcal{F}_x$ defined as above. Define

$$\alpha^x(w) = \sum_{i=1}^n a_i \cdot (\alpha(s_i))_x.$$

To prove α^x is welldefined it is enough to check that if $\sum_{i=1}^n a_i \cdot (s_i)_x = 0$, then $\sum_{i=1}^n a_i \cdot (\alpha(s_i))_x = 0$. Consider a homomorphism of $\mathcal{O}_X$-modules

$$\varphi : \mathcal{O}_X^n \to \mathcal{F}$$

defined by the family $(s_i)_{i=1}^n$. Then, $\mathcal{K} = \mathrm{Ker}(\varphi) \in \mathrm{Ob}(\mathcal{C})$, therefore $\mathcal{K}$ is also generated by global sections. As $(a_1,\ldots,a_n) \in \mathcal{K}_x$, there exist $g_1,\ldots,g_p \in \Gamma(X,\mathcal{K}) \subseteq \mathcal{O}_X^n(X) = A^n$, and $z_1,\ldots,z_p \in \mathcal{O}_{X,x}$ such that $(a_1,\ldots,a_n) = \sum_{j=1}^p z_j \cdot (g_j)_x$. Let $g_j = (g_{j1},\ldots,g_{jn})$, for $g_{ji} \in A$, $i = 1,\ldots,n$, and $j = 1,\ldots,p$. Then $a_i = \sum_{j=1}^p z_j \cdot (g_{ji})_x$, for $i = 1,\ldots,n$, and $\sum_{i=1}^n g_{ji}s_i = 0$, for $j = 1,\ldots,p$. Consider,

$$\begin{aligned} \sum_{i=1}^n a_i \cdot \alpha(s_i)_x = \sum_{i,j=1}^{n,p} z_j \cdot (g_{ji})_x \alpha(s_i)_x &= \sum_{j=1}^p z_j \cdot \left(\sum_{i=1}^n g_{ji}\alpha(s_i)\right)_x \\ = \sum_{j=1}^p z_j \cdot \left(\alpha\left(\sum_{i=1}^n g_{ji}s_i\right)\right)_x &= 0. \end{aligned}$$

Therefore, α^x is a welldefined map for every x in X. Now, we will check that the $\alpha^x, (x \in X)$ give rise to a homomorphism of $\mathcal{O}_X$-module $u : \mathcal{F} \to \mathcal{G}$. So we have to check condition of Remark 2.1.11. Let U be an open neighborhood of x, and let $s \in \mathcal{F}(U)$. We have as before, $s_x = \sum_{i=1}^n a_i \cdot (s_i)_x$ for some $s_i \in \Gamma(X, \mathcal{F})$, and for some $a_i \in \mathcal{O}_{X,x}$, $i = 1, \ldots, n$. Thus, there exist an open neighborhood V of x in U, and $f_i \in \mathcal{O}_X(V)$, such that $a_i = (f_i)_x$, $(i = 1, \ldots, n)$, and $s|_V = \sum_{i=1}^n f_i \cdot s_i|_V$. Define $t = \sum_{i=1}^n f_i \cdot \alpha(s_i)|_V \in \mathcal{G}(V)$. Then, $\alpha^y(s_y) = \alpha^y\left(\sum_{i=1}^n (f_i)_y (s_i)_y\right) = \sum_{i=1}^n (f_i)_y (\alpha(s_i))_y = t_y$, for all y in V. This proves that, there exists a unique homomorphism $u : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules such that $u_x = \alpha^x$ for all x in X. Also it is easy to check that $u_X = \alpha$. □

Recall that a sheaf of abelian groups $\mathcal{F}$ over a paracompact space X is *fine* if for any locally finite open cover $(U_i)_{i\in I}$ of X there exists a family $(\eta_i)_{i\in I}$ of morphisms $\eta_i : \mathcal{F} \to \mathcal{F}(i \in I)$ such that

1. For every $i \in I$, there exists an open neighborhood V_i of $X \backslash U_i$ such that $(\eta_i)_x = 0 : \mathcal{F}_x \to \mathcal{F}_x$ for all $x \in V_i$.

2. $\sum \eta_i = 1$.

The family $(\eta_i)_{i\in I}$ is called a *partition of unity* of $\mathcal{F}$ subordinate to the covering $(U_i)_{i=I}$. Recall that the *support of a sheaf of morphism* of an $\mathcal{O}_X$-modules $\eta : \mathcal{F} \to \mathcal{G}$ is defined as the closure of the set $\{y \in X \mid (\eta)_y = 0\}$, and it is denoted by $\mathrm{supp}(\eta)$. By the first condition of the definition of the fine sheaf, $\mathrm{supp}(\eta_i) \subset U_i$ for all $i \in I$.

Proposition 2.2.2 *Let $(X, \mathcal{O}_X)$ be a ringed space, such that X is a paracompact topological space, and $\mathcal{O}_X$ is a fine sheaf. Consider an $\mathcal{O}_X$-module $\mathcal{F}$. Let x be a point in X, U an open neighborhood of x, and $s' \in \mathcal{F}(U)$. Then, there exist an open neighborhood V of x in U, and a global section s of $\mathcal{F}$, such that $s|_V = s'|_V$. In particular, the canonical homomorphism $\rho_x : \Gamma(X, \mathcal{F}) \to \mathcal{F}_x$ is surjective, and hence $\mathcal{F}$ is generated by global sections.*

Proof. Since paracompact topological spaces are normal, so is X. Therefore, there exists a closed neighborhood N of x such that $N \subset U$. Let the interior of N be V and $U_1 = U$, $U_2 = X \backslash N$. Then $X = U_1 \cup U_2$. Thus, there exists a partition of unity subordinate to $\{U_1, U_2\}$ of $\mathcal{O}_X$, say $\{\eta_1, \eta_2\}$. Let $M = \mathrm{supp}(\eta_1)$. Define $s_1 = (\eta_1)_U(1)s' \in \mathcal{F}(U)$, and $s_2 = 0 \in \mathcal{F}(W)$, where $W = X \backslash M$. For $z \in U \cap W$,

$z \notin \mathrm{supp}(\eta_1)$, hence $(s_1)_z = (\eta_1)_z(1)(s')_z = 0$. Thus, $s_1|_{U\cap W} = s_2|_{U\cap W} = 0$. Note that, $X = U \cup W$. Indeed, if $z \notin U$, then $z \notin \mathrm{supp}(\eta_1)$, that is, $z \in W$. Since $\mathcal{F}$ is a sheaf, there exists a global section s of $\mathcal{F}$ such that $s|_U = s_1$, and $s|_W = s_2$. Since $V \subset X \backslash U_2 \subset X \backslash \mathrm{supp}(\eta_2)$, we get $(\eta_1)_z = \mathbf{1}_{\mathcal{O}_{X,z}}$ for every $z \in V$. Thus, for every $z \in V$,

$$\left(s|_V\right)_z = (\eta_1)_z(1)(s')_z = (s')_z.$$

Hence $s|_V = s'|_V$. For the second statement, let $\theta \in \mathcal{F}_x$. Then there exist an open neighborhood U of x, and $s' \in \mathcal{F}(U)$ such that $s'_x = \theta$. By the first part, there exist $s \in \Gamma(X, \mathcal{F})$, and an neighborhood V of x in U such that $s|_V = s'|_V$. Therefore,

$$(s'|_V)_x = (s|_V)_x = (s')_x = \theta.$$

Hence, ρ_x is surjective. □

It follows from Proposition 2.2.1 that for a ringed space X satisfying the conditions of Proposition 2.2.2, the functor $\Gamma(X, \bullet) : \mathcal{O}_X$-**mod** $\to A$-**mod** is fully faithful. In particular, if $(X, \mathcal{O}_X)$ is a differentiable manifold then $\Gamma(X, \bullet)$ is fully faithful.

Remark 2.2.3 Let $(X, \mathcal{O}_X)$ be a ringed space, and let A denote the ring $\Gamma(X, \mathcal{O}_X)$. If $\mathcal{C}$ is a subcategory of $\mathcal{O}_X$-**mod**, define A-$\mathbf{mod}_{\mathcal{C}}$ to be the full subcategory of A-**mod** consisting of A-modules M such that $M \cong \Gamma(X, \mathcal{F})$ for some $\mathcal{F}$ in $\mathcal{C}$. Let $\mathcal{C}$ be a subcategory of $\mathcal{O}_X$-**mod** as in Proposition 2.2.1, then $\Gamma(X, \bullet) : \mathcal{C} \to A$-$\mathbf{mod}_{\mathcal{C}}$ is an equivalence of categories. Indeed, it follows from Proposition 2.2.1 that $\Gamma(X, \bullet)$ is fully faithful, and by the definition of A-$\mathbf{mod}_{\mathcal{C}}$, it is essentially surjective.

Proposition 2.2.4 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let A denote the ring $\Gamma(X, \mathcal{O}_X)$. Suppose $\mathcal{C}$ is as in Proposition 2.2.1. Then, the canonical homomorphism $\mathcal{S}(\Gamma(X, \mathcal{F})) \to \mathcal{F}$ is an isomorphism for every sheaf $\mathcal{F}$ in $\mathcal{C}$.*

Proof. Let $M = \Gamma(X, \mathcal{F})$. Recall that $\mathcal{P}(M)$ denotes the presheaf tensor product of M and $\mathcal{O}_X$ over A. Let $u' : \mathcal{P}(M) \to \mathcal{F}$ be the morphism of presheaves such that for every open subset U of X, $u'_U : M \otimes_A \mathcal{O}_X(U) \to \mathcal{F}(U)$ is given by $u'_U(m \otimes_A f) = f \cdot m|_U$ for $m \in M$ and $f \in \mathcal{O}_X(U)$. Let $u : \mathcal{S}(M) \to \mathcal{F}$ be the morphism of sheaves associated to u'. (Note that, $u = \lambda_{\mathcal{F},M}(\mathbf{1}_M)$, where $\lambda_{\mathcal{F},M}$

is (2.4), i.e., u is the counit morphism of $\mathcal{F}$ with respect to the adjunction λ, (**A:5**).) Thus, $u_x : M \otimes_A \mathcal{O}_{X,x} \to \mathcal{F}_x$ is such that

$$u_x\left(\sum_{i=1}^{n} s_i \otimes_A a_i\right) = \sum_{i=1}^{n} a_i \cdot (s_i)_x,$$

for $x \in X$, $a_i \in \mathcal{O}_{X,x}$, and $s_i \in M$, $i = 1, \ldots, n$. Since u is a morphism of sheaves to prove that u is an isomorphism, it is enough to prove that u_x is an isomorphism for every x in X. The sheaf $\mathcal{F}$ is generated by global sections, therefore any germ w belongs to $\mathcal{F}_x$ can be written as

$$w = \sum_{i=1}^{n} a_i \cdot (s_i)_x, \text{ for some } a_i \in \mathcal{O}_{X,x}, \text{ and } s_i \in M, \ i = 1, \ldots, n.$$

Define $\alpha = \sum_{i=1}^{n} s_i \otimes_A a_i$. Then

$$u_x(\alpha) = u_x\left(\sum_{i=1}^{n} s_i \otimes_A a_i\right) = \sum_{i=1}^{n} a_i \cdot (s_i)_x = w.$$

Therefore, u_x is surjective. Now we will show that u_x is injective. Let $w = \sum_{i=1}^{n} s_i \otimes_A a_i$, where $s_i \in M$, and $a_i \in \mathcal{O}_{X,x}$, $i = 1, \ldots, n$, and suppose that $u_x(w) = \sum_{i=1}^{n} a_i \cdot (s_i)_x = 0$. Consider the morphism $\varphi : \mathcal{O}_X^n \to \mathcal{F}$ of $\mathcal{O}_X$-modules defined by the family $(s_i)_{i=1}^n$. Then, $\mathcal{K} = \mathrm{Ker}(\varphi)$ belongs to $\mathcal{C}$, and $(a_1, \ldots, a_n)$ belongs to $\mathcal{K}_x$. Therefore, there exist $g_1, \ldots, g_p \in \Gamma(X, \mathcal{K}) \subseteq \mathcal{O}_X^n(X) = A^n$, and $z_1, \ldots, z_p \in \mathcal{O}_{X,x}$ such that

$$(a_1, \ldots, a_n) = \sum_{j=1}^{p} z_j \cdot (g_j)_x.$$

Let $g_j = (g_{j1}, \ldots, g_{jn})$, where $g_{ji} \in A$, $i = 1, \ldots, n$, and $j = 1, \ldots, p$. Then, $a_i = \sum_{j=1}^{p} z_j \cdot (g_{ji})_x$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} g_{ji} s_i = 0$, for $j = 1, \ldots, p$. Thus,

$$\begin{aligned} w = \sum_{i=1}^{n} s_i \otimes_A a_i &= \sum_{i,j=1}^{n,p} s_i \otimes_A z_j \cdot (g_{ji})_x = \sum_{i,j=1}^{n,p} (g_{ji} s_i) \otimes_A z_j \\ &= \sum_{j=1}^{p}\left(\sum_{i=1}^{n} g_{ji} s_i\right) \otimes_A z_j = 0. \end{aligned}$$

Therefore, u_x is injective. □

Remark 2.2.5 One can also prove Proposition 2.2.1 using Proposition 2.2.4. Indeed we can define the inverse of a morphism $\varphi_{\mathcal{F},\mathcal{G}}$ (2.10). If v is the inverse of u in Proposition 2.2.4, then for all x in X,

$$\begin{aligned} v_x : \mathcal{F}_x &\longrightarrow \Gamma(X,\mathcal{F}) \otimes_A \mathcal{O}_{X,x}, \\ w &\longmapsto \sum_{i=1}^{n} s_i \otimes_A a_i, \end{aligned}$$

where $w = \sum_{i=1}^{n} a_i \cdot (s_i)_x$, $a_i \in \mathcal{O}_{X,x}$, $s_i \in \Gamma(X,\mathcal{F})$, $i = 1, \ldots, n$. Let $u_{\mathcal{G}}$ be the unit morphism of $\mathcal{G}$ with respect to the adjunction λ, (2.4), i.e., $u_{\mathcal{G}} = \varphi_{\mathcal{G},\Gamma(X,\mathcal{G})}(\mathbf{1}_{\mathcal{S}(M)})$ (2.8). Then, the inverse of $\varphi_{\mathcal{F},\mathcal{G}}$ is given by

$$\begin{aligned} \psi : \mathrm{Hom}_A(\Gamma(X,\mathcal{F}),\Gamma(X,\mathcal{G})) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}), \\ \alpha &\longmapsto u_{\mathcal{G}} \circ \mathcal{S}(\alpha) \circ v. \end{aligned}$$

2.2.1 Admissible Subcategories

Definition 2.2.6 Let $(X,\mathcal{O}_X)$ be a locally ringed space. Then, a subcategory $\mathcal{C}$ of $\mathcal{O}_X$-**mod** is called an *admissible subcategory* if it satisfies the following conditions:

C1. $\mathcal{C}$ is a full abelian subcategory of $\mathcal{O}_X$-**mod**, and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ belongs to $\mathcal{C}$ for every pair of sheaves $\mathcal{F}$ in $\mathbf{Lfb}(X)$ and $\mathcal{G}$ in $\mathcal{C}$, where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ denotes the sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$.

C2. Every sheaf in $\mathcal{C}$ is acyclic, and generated by global sections.

C3. $\mathbf{Lfb}(X)$ is a subcategory of $\mathcal{C}$.

Admissible subcategories naturally arise in many ringed spaces. In Section 2.3 we will see that, in the case of an affine scheme, $\mathbf{Qcoh}(X)$, the subcategory of quasi-coherent $\mathcal{O}_X$-module, in the case of differentiable manifold $\mathcal{O}_X$-**mod** itself are examples of admissible subcategories. We will also see some more examples of an admissible subcategories.

Corollary 2.2.7 [Corollary to Proposition 2.2.4] *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let A denote its ring of global sections. Assume that $\mathcal{O}_X$-**mod** contains an admissible subcategory $\mathcal{C}$. Then, every finitely generated projective module P is isomorphic to $\Gamma(X, \mathcal{S}(P))$.*

Proof. Every finitely generated projective module is finitely presented, therefore we get an exact sequence,

$$A^p \xrightarrow{\alpha} A^q \to P \to 0, \text{ for some } p, q \in \mathbf{N}.$$

Applying the functor $\mathcal{S}$, and by Proposition 2.2.4 we get an exact sequence

$$\mathcal{O}_X^p \xrightarrow{\psi(\alpha)} \mathcal{O}_X^q \to \mathcal{S}(P) \to 0,$$

where ψ is as in Remark 2.2.5. Since $\mathcal{C}$ is an admissible subcategory by **C3** sheaf $\mathcal{O}_X^m$ is in $\mathcal{C}$ for every integer m, and so by **C1** the sheaf $\mathcal{S}(P) \in \mathrm{Ob}(\mathcal{C})$. Also $\Gamma(X, \bullet)$ is an exact functor restricted to the category of acyclic $\mathcal{O}_X$-modules, it follows that

$$A^p \xrightarrow{\psi(\alpha)_X} A^q \to \Gamma(X, \mathcal{S}(P)) \to 0$$

is an exact sequence. By Remark 2.2.5, $\psi(\alpha)_X = \alpha$. Therefore, $\Gamma(X, \mathcal{S}(P)) \cong P$, being cokernels of the same map. □

2.2.2 Main Theorem

Let $(X, \mathcal{O}_X)$ be a ringed space, and A denote the ring $\Gamma(X, \mathcal{O}_X)$. Let $\mathbf{Fgp}(A)$ denote the full subcategory of A-**mod** consisting of finitely generated projective A-modules. We say that *the Serre-Swan Theorem holds for a ringed space* $(X, \mathcal{O}_X)$ if $\Gamma(X, \mathcal{F})$ is a finitely generated projective module for every sheaf $\mathcal{F}$ in $\mathbf{Lfb}(X)$, and the functor

$$\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(\Gamma(X, \mathcal{O}_X))$$

is an equivalence of categories. In this subsection we will prove that if $\mathcal{O}_X$-**mod** contains an admissible subcategory, and if all sheaves in $\mathbf{Lfb}(X)$ are finitely generated by global sections then the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$. To prove the theorem we will need some preliminary results.

Proposition 2.2.8 *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let A denote the ring $\Gamma(X, \mathcal{O}_X)$. Assume that $\mathcal{O}_X$-**mod** contains an admissible subcategory $\mathcal{C}$. If a locally free sheaf of bounded rank $\mathcal{F}$ is finitely generated by global sections, then $\Gamma(X, \mathcal{F})$ is a finitely generated projective A-module.*

Proof. By the hypothesis, there exists a surjective morphism $u : \mathcal{O}_X^n \to \mathcal{F}$ for some $n \in \mathbf{N}$. Consider the exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^n \xrightarrow{u} \mathcal{F} \to 0 \tag{2.11}$$

where $\mathcal{K} = \ker u$. Since $\mathcal{F}$ is locally free of bounded rank, by [Gro57, Corollaire to Proposition 4.2.3, p. 189]

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}) \cong H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})).$$

As $\mathcal{C}$ is an admissible subcategory $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}) \in \mathrm{Ob}(\mathcal{C})$. Hence $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$ is acyclic. Therefore, $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}) = 0$, and consequently the exact sequence (2.11) splits. This implies that $\mathcal{O}_X^n \cong \mathcal{F} \oplus \mathcal{K}$. Thus

$$A^n \cong \Gamma(X, \mathcal{O}_X^n) \cong \Gamma(X, \mathcal{F}) \oplus \Gamma(X, \mathcal{K}).$$

This proves that $\Gamma(X, \mathcal{F})$ is a finitely generated projective A-module. □

Remark 2.2.9 Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules of finite presentation. Let $x \in X$, and suppose that the $\mathcal{O}_{X,x}$-modules $\mathcal{F}_x$ and $\mathcal{G}_x$ are isomorphic. Then, there exists an open neighborhood U of x such that the $\mathcal{O}_X|_U$-modules $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic [Gro60, Chap. 0, 5.2.7, pp. 46-47]. This is immediately follows from Corollary 2.1.24.

Lemma 2.2.10 *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $x \in X$, and suppose that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module. Then there exist an open neighborhood U of x, and $n \in \mathbf{N}$, such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_X^n|_U$.*

Proof. Since $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, there exists an integer $n \in \mathbf{N}$ such that $\mathcal{O}_{X,x}$-modules $\mathcal{O}_{X,x}^n$ and $\mathcal{F}_x$ are isomorphic. All locally free sheaves of finite type are of finite presentation, hence $\mathcal{O}_X^n$ is of finite presentation. From Remark 2.2.9 there exists an open neighborhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_X^n|_U$.

Remark 2.2.11 One can prove Lemma 2.2.10 using Fitting ideals. Indeed, let M be a finitely generated A-module. For $r \in \mathbf{N}$ we denote the r-th Fitting ideal of M by $\mathfrak{F}_r(M)$. Let $(X, \mathcal{O}_X)$ be a ringed space, and consider an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation. We denote the r-th Fitting ideal sheaf of $\mathcal{G}$ by $\mathcal{F}_r(\mathcal{G})$, for $r \geq 0$. For every $x \in X$, $(\mathcal{F}_r(\mathcal{G}))_x = \mathfrak{F}_r(\mathcal{G}_x)$, and the ideal sheaf $\mathcal{F}_r(\mathcal{G})$ is of finite type as an $\mathcal{O}_X$-module. Since $\mathcal{F}_r(\mathcal{G})$ is of finite type, if $(\mathcal{F}_r(\mathcal{G}))_x = \mathcal{O}_{X,x}$, then there exists an open neighborhood U of x such that $(\mathcal{F}_r(\mathcal{G}))_y = \mathcal{O}_{X,y}$, for all y in U (This follows from Proposition 2.1.16). Also, by Corollary 2.1.17 the support of $\mathcal{F}_r(\mathcal{G})$, that is the set $\{x \in X \mid (\mathcal{F}_r(\mathcal{G}))_x \neq 0\}$ is closed in X. Further let $x \in X$, and an $\mathcal{O}_{X,x}$-module $\mathcal{G}_x$ is free of rank n. Then there exists an open neighborhood U of x such that for all $y \in U$, $\mathcal{G}_y$ is a free $\mathcal{O}_{X,y}$-module of rank n. (This follows from the fact that, if M is a free A-module of rank q, then $\mathfrak{F}_r(M) = 0$ for $0 \leq r < q$, and $\mathfrak{F}_r(M) = A$ for $r \geq q$ [Nor76, Exercise 1, p. 90], and converse is true if A is a local ring [Eis95, Propostion 20.8, p. 500].) Now since $\mathcal{G}$ is of finite type, by Proposition 2.1.39, $\mathcal{G}|_U \cong \mathcal{O}_X^n|_U$.

Lemma 2.2.12 *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let A denote the ring $\Gamma(X, \mathcal{O}_X)$. Then for every finitely generated projective A-module P, the sheaf $\mathcal{S}(P)$ is a locally free $\mathcal{O}_X$-module of bounded rank.*

Proof. It is given that, an A-module P is finitely generated and projective, hence it is of finite presentation. Therefore, we get an exact sequence of A-modules

$$A^p \to A^q \to P \to 0, \text{ for some } p, q \in \mathbf{N}.$$

Since the functor $\mathcal{S}$ is right exact, and $\mathcal{S}(A^m) \cong \mathcal{O}_X^m$ for all $m \in \mathbf{N}$, we get an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^p \to \mathcal{O}_X^q \to \mathcal{S}(P) \to 0.$$

This shows that $\mathcal{S}(P)$ is of finite presentation. Since for every $x \in X$, $\mathcal{O}_{X,x}$ is a local ring, and P is a finitely generated projective module $P \otimes_A \mathcal{O}_{X,x}$ is a free $\mathcal{O}_{X,x}$-module of finite rank. We denote the rank $\mathrm{rk}_x(\mathcal{S}(P))$ of the sheaf $\mathcal{S}(P)$ at every point x in X by n_x. Therefore, $P \otimes_A \mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{X,x}^{n_x}$. Now by Lemma 2.2.10 $\mathcal{S}(P)$ is a locally free $\mathcal{O}_X$-module. Also the family of integers $(n_x)_{x \in X}$ is bounded above by q, so the sheaf $\mathcal{S}(P)$ is of bounded rank.

Now we are ready for giving proof of the main theorem.

Theorem 2.2.13 *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $A = \Gamma(X, \mathcal{O}_X)$. Assume that $\mathcal{O}_X$-**mod** contains an admissible subcategory $\mathcal{C}$, and that every sheaf in $\mathbf{Lfb}(X)$ is finitely generated by global sections. Then, $\Gamma(X, \mathcal{F})$ is a finitely generated projective module for every sheaf $\mathcal{F}$ in $\mathbf{Lfb}(X)$, and $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ is an equivalence of categories, i.e., the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$.*

Proof. It follows from Proposition 2.2.8 that, if an $\mathcal{O}_X$-module $\mathcal{F}$ is locally free of bounded rank then $\Gamma(X, \mathcal{F})$ is finitely generated projective A-module, hence the restriction of the functor $\Gamma(X, \bullet)$ from the subcategory $\mathbf{Lfb}(X)$ to the subcategory $\mathbf{Fgp}(A)$ is welldefined. Since $\mathbf{Lfb}(X)$ is a subcategory of the admissible subcategory $\mathcal{C}$, Proposition 2.2.1 implies that, $\Gamma(X, \bullet)$ is fully faithful. By Corollary 2.2.7, if P is finitely generated projective module, then P is isomorphic to $\Gamma(X, \mathcal{S}(P))$. The sheaf $\mathcal{S}(P)$ is locally free of bounded rank is follows from Lemma 2.2.12. Hence the functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ is essentially surjective. Therefore, $\Gamma(X, \bullet)$ is an equivalence of categories. Moreover, $\mathcal{S}$ is a quasi-inverse of $\Gamma(X, \bullet)$ is a consequence of Proposition 2.2.4 and Lemma 2.2.12.

2.3 Some Special Cases

In this section we will discuss some important examples of locally ringed spaces for which the Serre-Swan Theorem holds. In Subsections 2.3.8, 2.3.10 we discussed examples of affine scheme and topological space respectively. Subsections 2.3.3 and 2.3.5 we showed the Serre-Swan Theorem holds for $\mathcal{C}^\infty$-differentiable spaces and Stein spaces.

2.3.1 Serre's Theorem

Let A be a ring, and let $(X, \mathcal{O}_X)$ denote $(\mathrm{Spec}(A), \tilde{A})$ the corresponding affine scheme. Recall that for every A-module M, we can define an $\mathcal{O}_X$-module $\tilde{M}$. This is defined by

$$\mathcal{F}_M(\mathrm{D}(f)) = M_f, \quad f \in A.$$

If $\mathrm{D}(f) \supset \mathrm{D}(g)$, then $S'_f \subset S'_g$, where for any multiplicative subset S of A

$$S' = \{s' \in A \,|\, as' \in S \text{ for some } a \in A\}$$

is the *saturation of* S, and $S_f = \{f^k \mid k \in \mathbf{N}\}$. Identifying $S^{-1}M$ with $S'^{-1}M$, we define

$$\rho_{f,g} : S_f'^{-1}M \longrightarrow S_f'^{-1}M$$
$$\rho_{f,g}\left(\frac{m}{s}\right) = \frac{m}{s} \in S_g'^{-1}M, \qquad \text{for } m \in M, s \in S'_f.$$

This define a sheaf $\mathcal{F}_M$ on the base $\mathcal{B} = \{\mathrm{D}(f) \mid f \in A\}$ of X. The associated sheaf on X is denoted by $\tilde{M}$, it is called the *sheaf on* X *associated to the* A*-module* M, and is an $\mathcal{O}_X$-module. If $u : M \to N$ is a homomorphism of A-modules, then for every $f \in A$, there exist a homomorphism of A_f-modules

$$u_f : M_f \longrightarrow N_f$$
$$u_f\left(\frac{m}{s}\right) = \frac{u(m)}{s}, \qquad \text{for } m \in M \text{ and } s \in S_f.$$

This induces a homomorphism of $\mathcal{O}_X$-modules

$$\tilde{u} : \tilde{M} \longrightarrow \tilde{N}$$

such that $\tilde{u}_{\mathrm{D}(f)} = u$ for all $f \in A$. Then this defines a fully faithful, faithfully exact and additive functor

$$\tilde{\bullet} : A\textbf{-mod} \longrightarrow \mathcal{O}_X\textbf{-mod}$$
$$M \longrightarrow \tilde{M}.$$

Remark 2.3.1 Let $(X, \mathcal{O}_X)$ is an affine space, and let $A = \Gamma(X, \mathcal{O}_X)$. Hence $(X, \mathcal{O}_X) \cong (\mathrm{Spec}(A), \tilde{A})$, so we identify $(X, \mathcal{O}_X)$ with $(\mathrm{Spec}(A), \tilde{A})$. Let M be an A-module. Recall that the canonical functor (2.2) $\mathcal{S} : A\textbf{-mod} \to \mathcal{O}_X\textbf{-mod}$ is such that $\mathcal{S}(M)_{\mathfrak{p}} = \mathcal{P}(M)_{\mathfrak{p}}$, for all $\mathfrak{p} \in X$. Hence we get,

$$\mathcal{S}(M)_{\mathfrak{p}} \cong \mathcal{P}(M)_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}} = M_{\mathfrak{p}} \cong (\tilde{M})_{\mathfrak{p}}.$$

Moreover these isomorphisms are natural, that is, the functors $\mathcal{S}$ and $\tilde{\bullet}$ are isomorphic.

Definition 2.3.2 Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* if for every point $x \in X$, there exist an open neighborhood U of

x such that $\mathcal{F}|_U$ is the cokernel of a morphism of $\mathcal{O}_X|_U$-modules,

$$u : \mathcal{O}_X^{(I)}|_U \longrightarrow \mathcal{O}_X^{(J)}|_U,$$

where I and J are arbitrary index sets, that is, if there exists an exact sequence of $\mathcal{O}_X|_U$-modules.

$$\mathcal{O}_X^{(I)}|_U \xrightarrow{u} \mathcal{O}_X^{(J)}|_U \xrightarrow{v} \mathcal{F}|_U \longrightarrow 0$$

for some open neighborhood U of every point $x \in X$.

Let $\mathbf{Qcoh}(X)$ denote the full subcategory of $\mathcal{O}_X$-**mod** consists of quasicoherent $\mathcal{O}_X$-modules. If M is an A-module, then since a free resolution exists for every module, and the functor $\tilde{\bullet}$ is exact, $\tilde{M}$ is quasicoherent. Moreover the functor

$$\tilde{\bullet} : A\text{-}\mathbf{mod} \longrightarrow \mathbf{Qcoh}(X)$$

is an equivalence of categories with quasi-inverse

$$\Gamma(X, \bullet) : \mathbf{Qcoh}(X) \longrightarrow A\text{-}\mathbf{mod}$$

[Gro60, Théorème (I.4.1), p. 90].

Remark 2.3.3 Recall that, an A-module M is said to be *finitely presented* if it is the cokernel of an A-module homomorphism $A^p \to A^q$ for some $p, q \in \mathbf{N}$ or, equivalently, if there exists an exact sequence of A-modules of the form $A^p \to A^q \to M \to 0$ with $p, q \in \mathbf{N}$. Following are two facts about finitely presented modules which we will use in later propositions.

1. Let $(f_i)_{i \in I}$ be a finite family of elements of a ring A, generating the ideal A of A. For an A-module M to be finitely presented, it is necessary and sufficient that, for every index i, the A_{f_i}-module M_{f_i} be finitely presented [Bou98, Chapter II, §5.1, Corollary to Proposition 3, p. 109].

2. Let S be a multiplicative subset of A, and let N, M be A-modules. If M is finitely presented, then the natural $S^{-1}A$-module homomorphism

$$\varphi : S^{-1}(\mathrm{Hom}_A(M, N)) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$
$$\varphi\left(\left(\frac{u}{s}\right)\right)\left(\frac{m}{t}\right) = \frac{u(m)}{st},$$

where $m \in M$, $s, t \in S$ and $u \in \mathrm{Hom}_A(M, N)$, is an isomorphism. [Eis95, Chapter 2, Proposition 2:10, p. 68]

Remark 2.3.4 Let X be a topological space, and $\mathcal{B}$ a base for the topology on X. For any presheaf $\mathcal{F}$ on X, let $\mathcal{F}_{\mathcal{B}}$ denote the presheaf on $\mathcal{B}$ induced by $\mathcal{F}$. If $u : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves in X, let $u_{\mathcal{B}} : \mathcal{F}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$ denote the induced morphism of presheaves on $\mathcal{B}$.

Let $\mathcal{F}$ be a presheaf on X, and $\mathcal{F}'$ the associated sheaf on X. Let $\mathcal{G}$ be another sheaf on X. Suppose that $u : \mathcal{F}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$ is a morphism of presheaves on $\mathcal{B}$. Then, there exists a unique morphism of sheaves on X, $u' : \mathcal{F}' \to \mathcal{G}$, such that $(u' \circ i)_{\mathcal{B}} = u$, where $i : \mathcal{F} \to \mathcal{F}'$ is the canonical morphism of presheaves on X. If $u_{\mathcal{B}}$ is an isomorphism of presheaves on $\mathcal{B}$ then u is also an isomorphism of sheaves on X. This can be proved using Remark 2.1.11.

Proposition 2.3.5 [Gro60, Chapter I, (ii) of Corollaire (I.3.12), p. 88] *If M and N are A-modules, there exists a canonical morphism of $\mathcal{O}_X$-modules*

$$v : (\mathrm{Hom}_A(M, N))\tilde{} \longrightarrow \mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N}).$$

If M is finitely presented, then v is an isomorphism.

Proof. Let $\mathcal{F} = (\mathrm{Hom}_A(M, N))\tilde{}$ and $\mathcal{G} = \mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N})$. Let $\mathcal{B} = \{\mathrm{D}(f) \mid f \in A\}$. Then, for all $B = \mathrm{D}(f) \in \mathcal{B}$

$$\mathcal{F}(B) = \Gamma(\mathrm{D}(f), (\mathrm{Hom}_A(M, N))\tilde{}) = (\mathrm{Hom}_A(M, N))_f = {S_f}^{-1}(\mathrm{Hom}_A(M, N))$$

where $S_f = \{f^n \mid n \geq 0\}$ and $\mathcal{G}(B) = \mathrm{Hom}_{\tilde{A}|_B}(\tilde{M}|_B, \tilde{N}|_B)$. Recall that there exist a canonical isomorphism of ringed spaces $(\mathrm{D}(f), \tilde{A}|_{\mathrm{D}(f)}) \cong (\mathrm{Spec}(A_f), \tilde{A}_f)$, and if we identified these two ringed spaces, there exists a canonical isomorphism of $\tilde{A}_f$-modules $\tilde{M}|_{\mathrm{D}(f)} = \tilde{M}_f$. Therefore, $\mathcal{G}(B) = \mathrm{Hom}_{\tilde{A}_f}(\tilde{M}_f, \tilde{N}_f)$. Since $\tilde{\bullet}$ is fully faithful functor, there exists a canonical isomorphism

$$\mathrm{Hom}_{\tilde{A}_f}(\tilde{M}_f, \tilde{N}_f) \cong \mathrm{Hom}_{A_f}(M_f, N_f).$$

By (2) of Remark 2.3.3, we have a unique homomorphism of ${S_f}^{-1}A$-modules

$$\varphi : {S_f}^{-1}(\mathrm{Hom}_A(M, N)) \longrightarrow \mathrm{Hom}_{{S_f}^{-1}A}({S_f}^{-1}M, {S_f}^{-1}N)$$

such that

$$\varphi\left(\left(\frac{u}{s}\right)\right)\left(\frac{m}{t}\right) = \frac{u(m)}{st},$$

where $u \in \mathrm{Hom}_A(M, N)$, $m \in M$ and $s, t \in S_f$. Thus, we get a homomorphism of A_f-modules

$$u_B = \varphi : \mathcal{F}(B) \longrightarrow \mathcal{G}(B).$$

If $B \supset B'$ are two members of $\mathcal{B}$, then

$$u'_B \circ \rho_{B',B} = \rho_{B',B} \circ u_B.$$

Therefore, the $u_{\mathcal{B}}$ define a morphism of presheaves on $\mathcal{B}$, $u : \mathcal{F}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$. By Remark 2.3.4, there exists a unique morphism of $\mathcal{O}_X$-modules $v = u' : \mathcal{F} \to \mathcal{G}$ such that $v_{\mathcal{B}} = u_{\mathcal{B}}$ for all $B \in \mathcal{B}$ (since $\mathcal{F}$ is a sheaf on X). If M is finitely presented, then by (2) of Remark 2.3.3 each $v_{\mathcal{B}} = u_{\mathcal{B}} = \varphi$ is an isomorphism. Since $\mathcal{F}$ and $\mathcal{G}$ are sheaves, this implies that $v = u'$ is an isomorphism of sheaves on X (by Remark 2.3.4). $\square$

Proposition 2.3.6 *Let A be a ring, and $(X, \mathcal{O}_X)$ the associated affine scheme. Let M be an A-module. Then, $\tilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is a finitely presented A-module.*

Proof. If M is finitely presented, then $\tilde{M}$ is of finite presentation is follows from the fact that $\tilde{\bullet}$ is an exact functor. Suppose $\tilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. As the principal open subsets $\{\mathrm{D}(f)\}_{f\in A}$ form a basis of the topology of X, and as X is quasi-compact, we may assume that there exists a finite open covering $X = \cup_i \mathrm{D}(f_i)$ such that $\tilde{M}|_{\mathrm{D}(f_i)}$ admits a finite presentation, that is there exists an exact sequence of $\mathcal{O}_X|_{\mathrm{D}(f_i)}$-modules

$$\left(\mathcal{O}_X|_{\mathrm{D}(f_i)}\right)^{p_i} \to \left(\mathcal{O}_X|_{\mathrm{D}(f_i)}\right)^{q_i} \to \tilde{M}|_{\mathrm{D}(f_i)} \to 0, \text{ for some } p_i, q_i \in \mathbf{N}$$

for all i. Since $X|_{\mathrm{D}(f_i)} = (\tilde{A}_{f_i})$ and $\tilde{M}|_{\mathrm{D}(f_i)} = (\tilde{M}_{f_i})$, this gives an exact sequence of A_{f_i}-modules,

$$A_{f_i}^{p_i} \to A_{f_i}^{q_i} \to M_{f_i} \to 0, \text{ for some } p_i, q_i \in \mathbf{N}$$

since $\tilde{\bullet}$ is faithfully exact. By (1) Remark 2.3.3, M is a finitely presented A-module. $\square$

Lemma 2.3.7 *Let A be an affine scheme, and $(X, \mathcal{O}_X)$ the associated affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite presentation then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is also quasicoherent. In particular, if $\mathcal{F}$ is locally free $\mathcal{O}_X$-module then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasicoherent.*

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent $\mathcal{F} \cong \tilde{M}$, and $\mathcal{G} \cong \tilde{N}$ where $\Gamma(X, \mathcal{F}) = M$, and $\Gamma(X, \mathcal{G}) = N$. By Proposition 2.3.6, since $\mathcal{F}$ is of finite presentation M is finitely presented A-module. Hence by Proposition 2.3.5,

$$(\mathrm{Hom}_A(M, N))\tilde{} \cong \mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N})$$

(note that $\tilde{A} = \mathcal{O}_X$). Since $(\mathrm{Hom}_A(M, N))\tilde{}$ is quasicoherent so is $\mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N})$. For any ringed space locally free sheaves are finitely presentation, hence second statement is true. $\square$

Corollary 2.3.8 (To Theorem 2.2.13) [Ser55, Section 50, Corollaire to Proposition 4, p. 242] *Let $(X, \mathcal{O}_X)$ be an affine scheme, and let A denote its coordinate ring $\Gamma(X, \mathcal{O}_X)$. Then, a quasicoherent sheaf $\mathcal{F}$ is locally free $\mathcal{O}_X$-module of finite rank if and only if $\Gamma(X, \mathcal{F})$ is a finitely generated projective A-module. The functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ is an equivalence of categories, with a quasi-inverse $\tilde{\bullet} : \mathbf{Fgp}(A) \to \mathbf{Lfb}(X)$.*

Proof. The functor $\tilde{\bullet}$ is an equivalence of categories from A-**mod** to $\mathbf{Qcoh}(X)$, with quasi-inverse $\Gamma(X, \bullet) : \mathbf{Qcoh}(X) \to A$-**mod** [Gro60, Chap. I, Théorème (1.4.1), p. 90]. Since A-**mod** is an abelian category so is $\mathbf{Qcoh}(X)$. Now by Lemma 2.3.7 the category $\mathbf{Qcoh}(X)$ satisfies condition **C1** of Definition 2.2.6. Quasicoherent $\mathcal{O}_X$-modules over an affine scheme are acyclic [Liu2002, Theorem 2.18, p. 186]. Let sheaf $\mathcal{F}$ be an quasicoherent sheaf, and $M = \Gamma(X, \mathcal{F})$. Then $\mathcal{F} \cong \tilde{M}$, and $\tilde{M}$ is clearly generated by global sections. Obvious $\mathbf{Lfb}(X)$ is a subcategory of $\mathbf{Qcoh}(X)$, hence $\mathbf{Qcoh}(X)$ is an admissible subcategory. Since X is quasicompact, by Corollary 2.1.21 locally free sheaves of finite rank are finitely generated by global sections. Therefore, the corollary will follows from Theorem 2.2.13.

2.3.2 Swan's Theorem

Let $\mathcal{F}$ be a sheaf over a topological space X, and S a closed subset of X. Then, $\mathcal{F}(S)$ is defined as

$$\mathcal{F}(S) = \varinjlim_{U \supset X} \mathcal{F}(U).$$

Recall that a sheaf $\mathcal{F}$ over a space X is *soft* if for any closed subset $S \subset X$ the restriction mapping

$$\begin{aligned} \Gamma(X, \mathcal{F}) &\longrightarrow \Gamma(S, \mathcal{F}) \\ s &\longmapsto \big((s)_x\big)_{x \in S} \end{aligned}$$

is surjective. Every soft sheaf S over a paracompact space is acyclic [Wel80, Chapter II, Theorem 3.11, (a) 2, p. 56].

Consider a ringed space $(X, \mathcal{O}_X)$ as in Proposition 2.2.2. Further, assume that X is of bounded topological dimension. Then, locally free sheaves of bounded rank over X are finitely generated by global sections. This fact is standard when $(X, \mathcal{O}_X)$ is a differential manifold [Wel80, Chap. III, Proposition 4.1], and the proof in the general case is similar.

Corollary 2.3.9 (To Theorem 2.2.13) *Let $(X, \mathcal{O}_X)$ be a ringed space such that, X is a paracompact topological space of bounded topological dimension, and $\mathcal{O}_X$ is a fine sheaf. Then, the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$.*

Proof. The category $\mathcal{O}_X$-**mod** clearly satisfies **C1** and **C3** of Definition 2.2.6. Since X is a paracompact topological space, fine sheaves on X are soft ([Wel80, Chapter II, Proposition 3.5, p. 53]) and hence acyclic. Since $\mathcal{O}_X$ is a fine sheaf, every $\mathcal{O}_X$-module is fine ([Wel80, Chapter II, Example 3.4(e), p. 53] and hence acyclic. Also by Proposition 2.2.2 $\mathcal{F}$ is generated by global sections, hence $\mathcal{O}_X$-**mod** satisfies **C2**. From the previous paragraph every sheaf $\mathcal{F}$ in $\mathbf{Lfb}(X)$ is finitely generated by global sections. Now the corollary follows applying Theorem 2.2.13, with $\mathcal{C} = \mathcal{O}_X$-**mod**. □

The sheaf of continuous real-valued functions on a paracompact topological space is a fine sheaf. Hence, the following is an immediate consequence of Corollary 2.3.9.

Corollary 2.3.10 [Swa62, Theorem 2 and p. 277] *Let X be a paracompact topological space of bounded topological dimension, and let $\mathcal{C}_X$ denote the sheaf of continuous real-valued functions on X. Let $C(X)$ denote the $\mathbf{R}$-algebra $\Gamma(X, \mathcal{C}_X)$.*

Then, the functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(C(X))$ *is an equivalence of categories.*

It follows from Corollary 2.3.9 that if X is a differentiable manifold of bounded dimension, and $\mathcal{C}^\infty(\mathbf{R})$ (respectively $\mathcal{C}^\infty(\mathbf{C})$) is the sheaf of differentiable real-valued (respectively, complex-valued) functions on X, then the category of real (respectively, complex) differentiable vector bundles on a manifold X is equivalent to the category of finitely generated projective $\Gamma(X, \mathcal{C}^\infty(\mathbf{R}))$-modules (respectively $\Gamma(X, \mathcal{C}^\infty(\mathbf{C}))$-modules).

2.3.3 $\mathcal{C}^\infty$-Differentiable Spaces

An ideal $\mathfrak{m} \subset A$ is said to be *real* if the natural map $\mathbf{R} \to A/\mathfrak{m}$ ($\mathbf{R} \to A \to A/\mathfrak{m}$) is an isomorphism (in particular $\mathfrak{m}$ is a maximal ideal of A). The kernel of any morphism of $\mathbf{R}$-algebras $A \to \mathbf{R}$ is a real ideal, so we get a natural bijection between the set of all real ideals of A and the set of all morphisms of $\mathbf{R}$-algebras $A \to \mathbf{R}$.

Definition 2.3.11 The *real spectrum* of a $\mathbf{R}$-algebra A is the set

$$\mathrm{Spec}_{\mathrm{r}}(A) = \mathrm{Hom}_{\mathbf{R}\text{-alg}}(A, \mathbf{R}) = \{\text{real ideals of } A\}.$$

If M is a differentiable manifold, then

$$\mathcal{C}^\infty(M) = \{f : M \to \mathbf{R} \mid f \text{ is } \mathcal{C}^\infty \text{ function}\}$$

is a Fréchet algebra [GS2003, p. 28]. An ideal $\mathfrak{a}$ is called a *closed ideal* if it is closed with respect to the Fréchet topology of uniform convergence on compact sets of functions and their derivatives.

Definition 2.3.12 We say that a $\mathbf{R}$-algebra is a *differentiable algebra* if it is (algebraically) isomorphic to a quotient

$$A \cong \mathcal{C}^\infty(\mathbf{R}^n)/\mathfrak{a}$$

for some natural number n, and some closed ideal $\mathfrak{a}$ of $\mathcal{C}^\infty(\mathbf{R}^n)$. A map $A \to B$ between differentiable algebras is said to be a *morphism of differentiable algebras* when it is a morphism of $\mathbf{R}$-algebras.

For any quotient algebra $A = \mathcal{C}^\infty(\mathbf{R})/\mathfrak{a}$, $\operatorname{Spec_r}(A)$ is homeomorphic to

$$(\mathfrak{a})_0 := \{x \in \mathbf{R}^n \mid f(x) = 0, \text{ for all } f \in \mathfrak{a}\}.$$

Hence $\operatorname{Spec_r}(A)$ is a closed set in $\mathbf{R}^n$, in particular it is paracompact [GS2003, Proposition 2.13, p. 30]. Let A be a differentiable algebra. Let $U \subset \operatorname{Spec_r}(A)$ open, define

$$S_U = \{s \in A \mid s(x) \neq 0, \text{for all } x \in U\}$$

(note that $x \in U \subset \operatorname{Spec_r}(A) = \operatorname{Hom}_{\mathbf{R}\text{-alg}}(A, \mathbf{R})$, then $s(x)$ is define as $s(x) = x(s) \in \mathbf{R}$. Then S_U is a multiplicative subset of A. Define a presheaf $\mathcal{F}$ by $\mathcal{F}(U) = {S_U}^{-1}A$. The sheaf associated to presheaf $\mathcal{F}$ is called the *structural sheaf on* $\operatorname{Spec_r}(A)$. This define $(\operatorname{Spec_r}(A), \tilde{A})$ a locally ringed space over $\mathbf{R}$. Let M be an A-module. If $U \subset \operatorname{Spec_r}(A)$ is open, then

$$\mathcal{G}(U) = {S_U}^{-1}M = M \otimes_A {S_U}^{-1}A$$

is a presheaf. We will denote by $\tilde{M}$ the sheaf associated to the presheaf $\mathcal{G}$, we will call $\tilde{M}$ *the sheaf associated to* M. This gives us a functor

$$\tilde{\bullet} : A\text{-}\mathbf{mod} \longrightarrow \tilde{A}\text{-}\mathbf{mod}.$$

It is easy to see that the functor $\mathcal{S}$ (2.2) is isomorphic to the functor $\tilde{\bullet}$.

Definition 2.3.13 The *real spectrum* of a differentiable algebra is defined to be the locally ringed space over $\mathbf{R}$, $(\operatorname{Spec_r}(A), \tilde{A})$.

Definition 2.3.14 A locally ringed space over $\mathbf{R}$, $(X, \mathcal{O}_X)$ is said to be an *affine differentiable space* (of finite type) if it is isomorphic to the real spectrum of some differentiable algebra. A locally ringed space over $\mathbf{R}$ is said to be a *differentiable space* (locally of finite type) if any point $x \in X$ has an open neighborhood U in X such that $(U, \mathcal{O}_X|_U)$ is an affine differentiable space.

Corollary 2.3.15 (To Theorem 2.2.13) [GS2003, Theorem 3.11, 4.16] *Let A be a differentiable algebra, and let $(X, \mathcal{O}_X)$ denote its real spectrum* $(\operatorname{Spec_r}(A), \tilde{A})$. *Then, the functor, $\Gamma(X, \bullet) : \mathcal{O}_X\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}_{\mathcal{O}_X\text{-}\mathbf{mod}}$ is an equivalence of categories. Moreover, the Serre-Swan Theorem holds for X.*

Proof. The topological space X is homeomorphic to a closed subset of $\mathbf{R}^n$ for some finite n, therefore X is a paracompact topological space of bounded topological dimension. Since the sheaf $\mathcal{O}_X$ admits partition of unity [GS2003, Theorem of partition of unity, p. 52] it a fine sheaf. Now the corollary follows from Remark 2.2.3 and Corollary 2.3.9.

2.3.4 Some Other Examples

In the paper [Mul76], Mulvey proved that for a locally ringed space $(X, \mathcal{O}_X)$ (not necessarily commutative) whose center is compact, the Serre-Swan theorem holds. A ringed space $(X, \mathcal{O}_X)$ is said to be compact provided that, the topological space X is compact, and that for every $x, x' \in X$, there exists an element $a \in \Gamma(X, \mathcal{O}_X)$ satisfying $a(x) = 1$ and $a(x') = 0$, [Mul76, Section 3, definition, p. 63]. If $(X, \mathcal{O}_X)$ is commutative (i.e., $\Gamma(X, \mathcal{O}_X)$ is a commutative ring) then the above result follows from Theorem 2.2.13.

Corollary 2.3.16 [Mul76, Theorem 4.1] *If a locally ringed space $(X, \mathcal{O}_X)$ is compact, then the category of locally free $\mathcal{O}_X$-modules of bounded rank is equivalent to the category of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules.*

Proof. The structure sheaf $\mathcal{O}_X$ is a fine sheaf (this is follows from [Mul78, Corollary 1.3]), and X is a compact topological space. Therefore, by Proposition 2.2.2 every sheaf in $\mathcal{O}_X$-**mod** is generated by global sections. Also, since X is compact, locally free sheaves of bounded rank over X are finitely generated by global sections (by Corollary 2.1.21). On a paracompact space fine sheaves are acyclic, hence all $\mathcal{O}_X$-modules are acyclic. Therefore, the category $\mathcal{O}_X$-**mod** is admissible. Now, the corollary follows from Theorem 2.2.13 by taking category $\mathcal{C}$ to be the category $\mathcal{O}_X$-**mod**. □

Recall that [Pie67, p. 8, Definition 10.2], a ringed space $(X, \mathcal{O}_X)$ is called *regular ringed space* if X is a profinite space, i.e., a compact totally disconnected space, and $\mathcal{O}_{X,x}$ is a field for every $x \in X$.

Corollary 2.3.17 [Pie67, Theorem 15.3] *Let $(X, \mathcal{O}_X)$ be a regular ringed space, and let A denote $\Gamma(X, \mathcal{O}_X)$. Then Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$. Moreover, coherent sheaves over X are locally free $\mathcal{O}_X$-modules of bounded rank. Hence,* $\mathbf{Coh}(X)$ *and* $\mathbf{Fgp}(A)$ *are equivalent.*

Proof. Note that regular ringed spaces are commutative and compact ringed spaces [Mul76, pp. 65-66]. Therefore, the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$ by Corollary 2.3.16. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Since X is a regular ringed space, $\mathcal{O}_{X,x}$ is a field for every $x \in X$. Thus, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$. Since by Proposition 2.1.27 coherent sheaves are of finitely presentation, Lemma 2.2.10 implies that $\mathcal{F}$ is locally free. Also X is compact, therefore $\mathcal{F}$ is of bounded rank. Hence, $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of bounded rank. □

2.3.5 Stein Spaces

Let $D \subset \mathbf{C}^m$ be open connected, and let $\mathcal{O}_D$ be the sheaf of holomorphic function on D. Let $\mathcal{I}$ be a coherent ideal in $\mathcal{O}_D$. Then the cokernel of an inclusion morphism $i : \mathfrak{I} \to \mathcal{O}_D$, $\mathcal{O}_D/\mathcal{I}$ is coherent (by Corollary 2.1.32). In particular, it is of finite type. Let $A = \mathrm{Supp}(\mathcal{O}_D/\mathcal{I})$. By Corollary 2.1.17, A is a closed subset of D. Let $\mathcal{O}_A = \mathcal{O}_D/\mathfrak{I}$. Then $(A, \mathcal{O}_A)$ is locally ringed space over $\mathbf{C}$. The ringed space $(A, \mathcal{O}_A)$ is called a *closed complex subspace* of D. A *complex space* $(X, \mathcal{O}_X)$ is a locally ringed space over $\mathbf{C}$ in which every point has a neighborhood U so that $(U, \mathcal{O}_X|_U)$ is isomorphic to a closed complex subspace $(A, \mathcal{O}_A)$ of an open connected subset in some $\mathbf{C}^n$. Since coherence is a local condition $\mathcal{O}_X$ is coherent.

Definition 2.3.18 (Stein Sets) [GR79, Chapter IV, §1, p. 100, Definition 1] A closed subset P of a complex space $(X, \mathcal{O}_X)$ is called a *Stein set* in X if Theorem B is valid on P (that is, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, $H^q(P, \mathcal{F}) = 0$ for all $q \geq 1$). A complex space which is itself is a Stein set is called a *Stein space*.

Theorem 2.3.19 (Theorem A for Stein Sets) [GR79, Chapter IV, §1, p. 101, Theorem 2] *Let P be a Stein set in X and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module on P. Then $\Gamma(P, \mathcal{F})$ generates every stalk $\mathcal{F}_x$ for all $x \in P$.*

For equivalent criteria for Stein spaces see [GR79, Chapter IV, Section 1, Theorem 2, p. 101 and Chapter V, Section 4, Theorem 3, p. 152].

Let $(X, \mathcal{O}_X)$ be a Stein space, and let $A = \Gamma(X, \mathcal{O}_X)$. Recall that a topological module M over the algebra A is called a *Stein module* if there exists a coherent $\mathcal{O}_X$-module $\mathcal{M}$, such that $\Gamma(X, \mathcal{M})$ is isomorphic to M [For67, Section 2, p. 383]. Let $\mathfrak{S}$**-mod** denote the category of Stein modules over A. Since $\mathcal{O}_X$ is coherent, Corollary 2.1.31 implies that every sheaf in $\mathbf{Lfb}(X)$ is also coherent.

Thus, $\mathbf{Coh}(X)$ satisfy **C1** and **C3** of Definition 2.2.6 (see Remark 2.1.33). Since THEOREM A and THEOREM B are valid for Stein spaces, the category $\mathbf{Coh}(X)$ satisfies **C2**. Thus, $\mathbf{Coh}(X)$ is an admissible subcategory of $\mathcal{O}_X$-**mod**. Note that the full subcategory $A\text{-}\mathbf{mod}_{\mathbf{Coh}(X)}$ of category A-**mod** is precisely $\mathfrak{S}$-**mod**.

Corollary 2.3.20 [For67, Satz 6.7 and Satz 6.8] *Let $(X, \mathcal{O}_X)$ be a finite-dimensional connected Stein space. Then $\Gamma(X, \bullet) : \mathbf{Coh}(X) \to \mathfrak{S}\text{-}\mathbf{mod}$ is an equivalence of category with quasi-inverse $\mathcal{S} : \mathfrak{S}\text{-}\mathbf{mod} \to \mathbf{Coh}(X)$. Moreover, the category of locally free sheaves of finite rank is equivalent to the category of finitely generated projective A-modules.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. Since $\mathbf{Coh}(X)$ is an admissible subcategory, by Theorem 2.2.13 and Remark 2.2.3 to prove the corollary it is enough to prove that, every locally free sheaf of bounded rank is finitely generated by global sections. Let $\mathcal{M}$ be locally free sheaf on X of bounded rank, and $M = \Gamma(X, \mathcal{M})$ be the corresponding Stein module over A. For a Stein module M let $d_x(M)$ denote the rank of $\mathcal{M}$ at a point x. Let $d = \sup\{d_x(M) \,|\, \forall x \in X\}$. Since $\mathcal{M}$ is of finite rank $d < \infty$. Since X is connected and finite dimensional, A is indecomposable and finite dimensional [For67, Section 1, Subsection 3, p. 382]. Therefore, M is a finitely generated A-module [For67, Corollary 4.7]. Now since $\mathcal{M}$ is generated by global sections, and $\Gamma(X, \mathcal{M})$ is a finitely generated, $\mathcal{M}$ is finitely generated by global sections.

Note that the spectrum of A, denoted by $S(A)$ is homeomorphic to X [For67, Section 1, Satz 1, Beweis c, p. 380]. Therefore, the definition of d in [For67, Corollary 4.7] is same as that in the Corollary 2.3.20.

Remark 2.3.21 Every non-compact connected Riemann surface is a Stein space [GR79, Chapter V, §5, p. 134], hence by the above corollary Serre-Swan Theorem holds for non-compact Riemann surfaces. Note that every vector bundle over a non-compact connected Riemann surface $(X, \mathcal{O}_X)$ is trivial [For91, Chapter 3, §30, Theorem 30.4, p. 229]. Let $A = \Gamma(X, \mathcal{O}_X)$. Therefore, if $\mathcal{F}$ is in $\mathbf{Lfb}(X)$, then $\Gamma(X, \mathcal{F}) \cong A^n$ for some $n \in N \cup \{0\}$. Conversely, let P be a finitely generated projective A-module. Then, $\mathcal{S}(P)$ is in $\mathbf{Lfb}(X)$, and hence, $\Gamma(X, \mathcal{S}(P)) \cong P \cong A^n$ for some n. This implies that every projective A-module is free.

On the other hand, for compact Riemann surfaces the Serre-Swan theorem does not hold. Let $(X, \mathcal{O}_X)$ be a compact Riemann surface, and let L be a line bundle over X of negative degree. Then the zero $\mathcal{O}_X$-module 0 and L are non-isomorphic vector bundles over X. But, $\Gamma(X, 0)$ and $\Gamma(X, L)$ are 0, that is isomorphic $\Gamma(X, \mathcal{O}_X) = \mathbf{C}$-modules. Hence, the functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to A$-**mod** is not faithful.

Chapter 3

Real Vector Bundles over a Real Abelian Variety

In this chapter, we study various equivalent conditions for the presence of real holomorphic connections in a real holomorphic vector bundle over a real abelian variety. Holomorphic connections in holomorphic bundles over a complex abelian variety were studied by Balaji and Biswas [BB2009], Biswas [Bis2004] Biswas and Iyer [BI2007], Biswas and Gómez [BG2008],and Biswas and Subramanian [BS2004]. In this chapter we prove analogues, for real abelian varieties, of some of the results in the above papers.

This chapter is divided into four sections. In the first section, we develop some preliminaries regarding real structures on a ringed space over $\mathbf{C}$. We show in Section 3.2 that real structures on a complex manifolds are completely determined by antiholomorphic involutions. We also recall the notions of real abelian varieties, and of real homogenous vector bundles over real abelian variety. Section 3.3 discusses the concept of real holomorphic and real $\mathcal{C}^\infty$ connections. We show that if a real vector bundle over a real abelian variety admit a real connection, then it is real homogenous. In the last section we gives various equivalent conditions for a real algebraic vector bundle over a real abelian variety to admit a real flat holomorphic connection.

We refer to the books [BCR98] and [Sil89] for real algebraic geometry.

3.1 Real Structures on a Ringed Space over C

Let $(X, \mathcal{O}_X)$ be a ringed space over $\mathbf{C}$. A *real structure on* $(X, \mathcal{O}_X)$ is a pair $(\sigma, \tilde{\sigma})$ consisting of a continuous map $\sigma : X \to X$ such that $\sigma^2 = \mathbf{1}_X$, and for every $U \subset X$ open, a $\mathbf{C}$-antilinear ring homomorphism

$$\tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$$

which is compatible with restrictions, that is, for every $V \subset U$ open in X,

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "\tilde{\sigma}_U"] \arrow[d, "\rho_{V,U}"'] & \mathcal{O}_X(\sigma(U)) \arrow[d, "\rho_{\sigma(V),\sigma(U)}"] \\
\mathcal{O}_X(V) \arrow[r, "\tilde{\sigma}_V"'] & \mathcal{O}_X(\sigma(V))
\end{tikzcd}$$

commutes, and such that, $\tilde{\sigma}_{\sigma(U)} \circ \tilde{\sigma}_U = \mathbf{1}_{\mathcal{O}_X(U)}$, for every $U \subset X$ open.

A *real ringed space* is a pair $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$, where $(X, \mathcal{O}_X)$ is a ringed space, and $(\sigma, \tilde{\sigma})$ is a real structure on $(X, \mathcal{O}_X)$.

Definition 3.1.1 Let $\big((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)\big)$, $\big((Y, \mathcal{O}_Y), (\sigma_Y, \tilde{\sigma}_Y)\big)$ be two real ringed spaces. A *morphism of real ringed spaces* is a morphism of ringed spaces

$$(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that

1. $\sigma_Y \circ \varphi = \varphi \circ \sigma_X$.

2. For every $U \subset Y$ open, the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\tilde{\varphi}_U"] \arrow[d, "(\tilde{\sigma}_Y)_U"'] & \mathcal{O}_X(\varphi^{-1}(U)) \arrow[d, "(\tilde{\sigma}_X)_{(\varphi^{-1}(U))}"] \\
\mathcal{O}_Y(\sigma_Y(U)) \arrow[r, "\tilde{\varphi}_{\sigma_Y(U)}"'] & \mathcal{O}_X(\varphi^{-1}(\sigma_Y(U)))
\end{tikzcd}$$

commutes. Note that by (1) $\varphi^{-1}(\sigma_Y(U)) = \sigma_X(\varphi^{-1}(U))$, so

$$\mathcal{O}_X(\varphi^{-1}(\sigma_Y(U))) = \mathcal{O}_X(\sigma_X(\varphi^{-1}(U))).$$

We denote by **Rsp/C** the category of ringed spaces over **C**, and $(\mathbf{Rsp}/\mathbf{C})^{\mathbf{real}}$ the category of real ringed spaces, with real morphisms.

Remark 3.1.2 Let $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a real morphism of real ringed spaces $\big((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)\big)$ and $\big((Y, \mathcal{O}_Y), (\sigma_Y, \tilde{\sigma}_Y)\big)$. Then,

$$\big((X, \varphi^{-1}(\mathcal{O}_Y)), (\sigma_X, \varphi^{-1}(\tilde{\sigma}_Y))\big)$$

is a real ringed space. Moreover, $\mathcal{O}_X$ is a real $\varphi^{-1}(\mathcal{O}_Y)$-module.

Remark 3.1.3 Let X be a complex manifold. Suppose that, $\sigma : X \to X$ is an antiholomorphic involution, that is, $\sigma^2 = \mathbf{1}_X$. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Define $\tilde{\sigma} : \mathcal{O}_X \to \mathcal{O}_X$ as follows. For every $U \subset X$ open,

$$\begin{aligned} \tilde{\sigma}_U : \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(\sigma(U)), \\ f &\longmapsto \overline{f \circ \sigma}. \end{aligned}$$

Since σ is antiholomorphic, $\overline{f \circ \sigma}$ is holomorphic, and hence belongs to $\mathcal{O}_X(\sigma(U))$. For $a \in \mathbf{C}$,

$$\tilde{\sigma}_U(af) = \overline{(af) \circ \sigma} = \bar{a}(\overline{f \circ \sigma}) = \bar{a}\tilde{\sigma}_U(f),$$

that is, $\tilde{\sigma}_U$ is $\mathbf{C}$-antilinear. For every open subset U of X,

$$\tilde{\sigma}_{(\sigma(U))} \circ \tilde{\sigma}_U(f) = \tilde{\sigma}_{(\sigma(U))}(\overline{f \circ \sigma}) = \overline{\overline{f \circ \sigma} \circ \sigma} = f = \mathbf{1}_{\mathcal{O}_X(U)}(f).$$

Clearly, the family $(\tilde{\sigma}_U)_{U \in \mathrm{op}(X)}$ commutes with restrictions. Therefore, $(\sigma, \tilde{\sigma})$ is a real structure on $(X, \mathcal{O}_X)$. We will denote by i_σ the above real structure $(\sigma, \tilde{\sigma})$ on $(X, \mathcal{O}_X)$ induced by σ.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex manifolds with antiholomorphic involutions σ_X and σ_Y, respectively. Let $\varphi : X \to Y$ be a holomorphic map from X to Y such that

$$\sigma_Y \circ \varphi = \varphi \circ \sigma_X.$$

Then, $(\varphi, \tilde{\varphi})$ is real, where $\tilde{\varphi}$ is given by

$$\tilde{\varphi}_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U)), \quad \tilde{\varphi}_U(f) = f \circ \varphi$$

for every open subset U of Y. Let i_φ denote the above real morphism $(\varphi, \tilde{\varphi})$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ induced by φ.

We will later show that all real structures and real morphisms of complex manifolds are of this form (Proposition 3.2.7).

3.1.1 Real $\mathcal{O}_X$-modules

Definition 3.1.4 Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *real structure on* $\mathcal{F}$ is a family $\alpha^{\mathcal{F}} = (\alpha^{\mathcal{F}}_U)_{U\in \mathrm{op}(X)}$ of morphisms of abelian groups $\alpha^{\mathcal{F}}_U : \mathcal{F}(U) \to \mathcal{F}(\sigma(U))$ such that:

1. The abelian group homomorphisms are compatible with restriction morphisms, that is, for every $V \subset U$ open in X, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha^{\mathcal{F}}_U"] \arrow[d, "\rho_{V,U}"'] & \mathcal{F}(\sigma(U)) \arrow[d, "\rho_{\sigma(V),\sigma(U)}"] \\ \mathcal{F}(V) \arrow[r, "\alpha^{\mathcal{F}}_V"'] & \mathcal{F}(\sigma(V)) \end{tikzcd}$$

   commutes.

2. For every open subset U of X, $\alpha^{\mathcal{F}}_{\sigma(U)} \circ \alpha^{\mathcal{F}}_U = \mathbf{1}_{\mathcal{F}(U)}$.

3. $\alpha^{\mathcal{F}}_U(fs) = \tilde{\sigma}_U(f)\alpha^{\mathcal{F}}_U(s)$ for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.

A *real* $\mathcal{O}_X$*-module* is a pair $(\mathcal{F}, \alpha^{\mathcal{F}})$, where $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\alpha^{\mathcal{F}}$ is a real structure on $\mathcal{F}$. We sometimes denote by a real sheaf $(\mathcal{F}, \alpha^{\mathcal{F}})$ by just $\mathcal{F}$ when no confusion is likely to occur. A *real vector bundle over* X is a real $\mathcal{O}_X$-module $(\mathcal{F}, \alpha^{\mathcal{F}})$ such that the $\mathcal{O}_X$-module $\mathcal{F}$ is locally free.

A global section s of a real $\mathcal{O}_X$-module $(\mathcal{F}, \alpha^{\mathcal{F}})$ is called *real* if for all $x \in X$, $(\alpha^{\mathcal{F}}(s))_x = (s)_{\sigma(x)}$.

3.1.2 The Functor $\bullet^{\sigma}$

Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define an $\mathcal{O}_X$-module $\mathcal{F}^{\sigma}$ as follows. For any open subset U of X, $\mathcal{F}^{\sigma}(U) = \mathcal{F}(\sigma(U))$, and for every $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}^{\sigma}(U)$, $f \cdot s = \tilde{\sigma}_U(f)s$. Note that, $\tilde{\sigma}_U(f) \in \mathcal{O}_X(\sigma(U))$, and $s \in \mathcal{F}(\sigma(U))$, therefore,

$$f \cdot s = \tilde{\sigma}_U(f)s \in \mathcal{F}(\sigma(U)) = \mathcal{F}^{\sigma}(U).$$

If V, U are open subset of X with $V \subset U$, and $s \in \mathcal{F}^\sigma(U)$, then define the restriction of s to V by

$$\rho^{\mathcal{F}^\sigma}_{V,U}(s) = \rho^{\mathcal{F}}_{\sigma(V),\sigma(U)}(s) \in \mathcal{F}(\sigma(V)) = \mathcal{F}^\sigma(V).$$

It is easy to check that $\mathcal{F}^\sigma$ is an $\mathcal{O}_X$-module.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Define $\varphi^\sigma : \mathcal{F}^\sigma \to \mathcal{G}^\sigma$ by

$$\varphi^\sigma_U : \mathcal{F}^\sigma(U) \to \mathcal{G}^\sigma(U), \quad \varphi^\sigma_U = \varphi_{\sigma(U)}$$

for every open subset U of X. If $f \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}^\sigma(U)$ then

$$\varphi^\sigma_U(f \cdot s) = \varphi_{\sigma(U)}\big(\tilde{\sigma}_U(f)(s)\big) = \tilde{\sigma}_U(f)\varphi_{\sigma(U)}(s),$$

since $\varphi_{\sigma(U)}$ is an $\mathcal{O}_X(\sigma(U))$-linear. Therefore, $\varphi^\sigma(f \cdot s) = f \cdot \varphi^\sigma_U(s)$. It follows that φ^σ is a homomorphism of $\mathcal{O}_X$-modules.

This gives a functor

$$\bullet^\sigma : \mathcal{O}_X\text{-}\mathbf{mod} \to \mathcal{O}_X\text{-}\mathbf{mod}. \tag{3.1}$$

Remark 3.1.5 Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then, $\sigma_*(\mathcal{F}) = \mathcal{F}^\sigma$ as an $\mathcal{O}_X$-module. Hence, for all $x \in X$, $\sigma_*(\mathcal{F})_x = \mathcal{F}^\sigma_x$.

The following are some properties of the functor $\bullet^\sigma$.

1. For every $\mathcal{O}_X$-module $\mathcal{F}$, $(\mathcal{F}^\sigma)^\sigma = \mathcal{F}$.

2. We can rephrase the definition of a real structure on an $\mathcal{O}_X$-module using the functor $\bullet^\sigma$. Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space. A real structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is given by any of the following:

   (a) For each open set U of X, an abelian group homomorphism $\alpha^{\mathcal{F}}_U : \mathcal{F}(U) \to \mathcal{F}(\sigma(U))$, compatible with restrictions, such that $\alpha^{\mathcal{F}}_{\sigma(U)} \circ \alpha^{\mathcal{F}}_U = \mathbf{1}_{\mathcal{F}(U)}$ for all $U \subset X$ open, and $\alpha^{\mathcal{F}}_U(fs) = \tilde{\sigma}_U(f)\alpha^{\mathcal{F}}_U(s)$ for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.

   (b) An $\mathcal{O}_X$-module homomorphism $\alpha^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\sigma$ such that $(\alpha^{\mathcal{F}})^\sigma \circ \alpha^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}}$.

3. $\bullet^\sigma$ is an exact functor.

   This property can be verified directly. Alternatively, recall the following fact. Let $\psi : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of sets in X. We will denote by $\psi_*(\mathcal{F})$ the sheaf of direct image of $\mathcal{F}$ by ψ. For all $x \in X$ we have a canonical function

$$\psi_x : \psi_*(\mathcal{F})_{\psi(x)} \to \mathcal{F}_x$$

   define as follows. Let $\theta \in \psi_*(\mathcal{F})_{\psi(x)}$. Let U be an open neighborhood of $\psi(x)$ in Y, and $s \in \psi_*(\mathcal{F})(U)$ such that $s_{\psi(x)} = \theta$. Since $s \in \mathcal{F}(\psi^{-1}(U))$, s is also a section of $\mathcal{F}$ on the open neighborhood $\psi^{-1}(U)$ of x. It follows from this definition that $\psi_x(\theta)$ is independent of the choice of (U, s). In general ψ_x is neither injective or surjective. But, if $\psi : X \to Y$ is a homeomorphism from X onto the subspace $\psi(X)$ of Y, then for all $x \in X$ the above canonical map ψ_x is a bijection. Since $\sigma^2 = \mathbf{1}_X$, σ is a homeomorphism. Thus, for every $\mathcal{O}_X$-module $\mathcal{F}$,

$$\sigma_x : \sigma_*(\mathcal{F})_{\sigma(x)} = \mathcal{F}^\sigma_{\sigma(x)} \to \mathcal{F}_x \tag{3.2}$$

   is a bijection.

   Let $\alpha : 0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. Exactness of the functor $\bullet^\sigma$ follows from the facts that, for every $x \in X$ the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\sigma(x)} \arrow[r, "f_{\sigma(x)}"] & \mathcal{G}_{\sigma(x)} \arrow[r, "g_{\sigma(x)}"] & \mathcal{H}_{\sigma(x)} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F}^\sigma_x \arrow[r, "f^\sigma_x"'] \arrow[u, "\sigma_{\sigma(x)}"'] & \mathcal{G}^\sigma_x \arrow[r, "g^\sigma_x"'] \arrow[u, "\sigma_{\sigma(x)}"'] & \mathcal{H}^\sigma_x \arrow[r] \arrow[u, "\sigma_{\sigma(x)}"'] & 0,
\end{tikzcd}$$

   commutes, (3.2), and $0 \to \mathcal{F}_{\sigma(x)} \xrightarrow{f_{\sigma(x)}} \mathcal{G}_{\sigma(x)} \xrightarrow{g_{\sigma(x)}} \mathcal{H}_{\sigma(x)} \to 0$ is exact.

Remark 3.1.6 Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space, and let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Define

$$\varphi : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^\sigma \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\sigma, \mathcal{G}^\sigma)$$

as follows. Let $t \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^\sigma(U) = \mathrm{Hom}_{\mathcal{O}_X|_{\sigma(U)}}(\mathcal{F}|_{\sigma(U)}, \mathcal{G}|_{\sigma(U)})$, $U \subset X$

open. Then $\varphi_U(t) : \mathcal{F}^\sigma|_U \to \mathcal{G}^\sigma|_U$ is define by

$$\big(\varphi_U(t)\big)_V : \mathcal{F}^\sigma(V) \to \mathcal{G}^\sigma(V), \quad \big(\varphi_U(t)\big)_V = t_{\sigma(V)}$$

for $V \subset U$ open. Note that $\sigma(V) \subset \sigma(U)$ is open, therefore

$$t_{\sigma(V)} : \mathcal{F}(\sigma(V)) = \mathcal{F}^\sigma(V) \to \mathcal{G}(\sigma(V)) = \mathcal{G}^\sigma(V).$$

Then φ is an isomorphism of $\mathcal{O}_X$-modules. We will identify $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^\sigma$ with $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\sigma, \mathcal{G}^\sigma)$ by this isomorphism.

Remark 3.1.7 Let $\big((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)\big)$ be a real ringed space, and let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. By (3.2) for every $x \in X$, $\mathcal{F}_x^{\sigma_X}$ (respectively $\mathcal{G}_x^{\sigma_X}$) is canonically isomorphic to $\mathcal{F}_{\sigma(x)}$ (respectively $\mathcal{G}_{\sigma(x)}$), and we can identify $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\sigma_X}$ with $\mathcal{F}^{\sigma_X} \otimes_{\mathcal{O}_X} \mathcal{G}^{\sigma_X}$.

Suppose $\big((Y, \mathcal{O}_Y), (\sigma_Y, \tilde{\sigma}_Y)\big)$ be a real ringed space, and $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a real morphism. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module then $\varphi^{-1}(\mathcal{F}^{\sigma_Y})$ is canonically isomorphic to $\big(\varphi^{-1}(\mathcal{F})\big)^{\sigma_Y}$ as an $\varphi^{-1}(\mathcal{O}_Y)$-module.

3.1.3 The Category $\mathcal{O}_X$-mod$^{\mathbf{real}}$

Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space. Let $\mathcal{O}_X$-$\mathbf{mod}^{\mathbf{real}}$ be a category defined as follows.

$$\begin{aligned} \mathrm{Ob}(\mathcal{O}_X\text{-}\mathbf{mod}^{\mathbf{real}}) &= \text{real } \mathcal{O}_X\text{-modules} \\ &= \left\{ (\mathcal{F}, \alpha^{\mathcal{F}}) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is an } \mathcal{O}_X\text{-module, and} \\ \alpha^{\mathcal{F}} \text{ is a real structure on } \mathcal{F} \end{array} \right\}. \end{aligned} \tag{3.3}$$

If $(\mathcal{F}, \alpha^{\mathcal{F}})$, $(\mathcal{G}, \alpha^{\mathcal{G}})$ are real $\mathcal{O}_X$-modules, then, by definition,

$$\mathrm{Hom}_{\mathbf{real}}((\mathcal{F}, \alpha^{\mathcal{F}}), (\mathcal{G}, \alpha^{\mathcal{G}})) = \{\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \,|\, \varphi^\sigma \circ \alpha^{\mathcal{F}} = \alpha^{\mathcal{G}} \circ \varphi\}. \tag{3.4}$$

3.1.4 Real Structures in Associated $\mathcal{O}_X$-modules

Let $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ be a real ringed space. If $(\mathcal{F}, \alpha^{\mathcal{F}})$ is a real $\mathcal{O}_X$-module, then a subsheaf $(\mathcal{G}, \alpha^{\mathcal{G}})$ is called a *real subsheaf* of $(\mathcal{F}, \alpha^{\mathcal{F}})$ if $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, and

$i : \mathcal{G} \to \mathcal{F}$ a real morphism of $\mathcal{O}_X$-modules. In this case, $\alpha^{\mathcal{G}} = \alpha^{\mathcal{F}}|_{\mathcal{G}}$, that is, for every open subset U of X,

$$\alpha_U^{\mathcal{F}}\big(\mathcal{G}(U)\big) = \mathcal{G}\big(\sigma(U)\big).$$

Let $\mathcal{G}$ be a subsheaf of a real subsheaf $(\mathcal{F}, \alpha^{\mathcal{F}})$. Then $(\mathcal{G}, \alpha^{\mathcal{F}}|_{\mathcal{G}})$ is a real subsheaf of $(\mathcal{F}, \alpha^{\mathcal{F}})$ if $\alpha_U^{\mathcal{F}}\big(\mathcal{G}(U)\big) = \mathcal{G}\big(\sigma(U)\big)$ for all open subset $U \subset X$. We denote the real subseaf $(\mathcal{G}, \alpha^{\mathcal{G}})$ of $(\mathcal{F}, \alpha^{\mathcal{F}})$ by $\mathcal{G}$, since $\alpha^{\mathcal{G}}$ is determined by $\alpha^{\mathcal{F}}$.

Let $(\mathcal{F}, \alpha^{\mathcal{F}})$, $(\mathcal{G}, \alpha^{\mathcal{G}})$ be real $\mathcal{O}_X$-modules. Then $\alpha^{\mathcal{F}}$ and $\alpha^{\mathcal{G}}$ induce a real structure on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Explicitly, define

$$\alpha : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^{\sigma} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^{\sigma}, \mathcal{G}^{\sigma}) \tag{3.5}$$

as follows. For $U \subset X$ open,

$$\alpha_U : \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}^{\sigma}|_U, \mathcal{G}^{\sigma}|_U)$$

is given by

$$\alpha_U(t)_V = \alpha_V^{\mathcal{G}} \circ t_V \circ (\alpha^{\mathcal{F}})_V^{\sigma}$$

for $V \subset U$ open, $t \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. Then, it is easy to see that the family $(\alpha_U)_{U \in \mathrm{op}(X)}$ is a real structure on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Hence, $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \alpha)$ is a real $\mathcal{O}_X$-module.

Note that, $\tilde{\sigma} : \mathcal{O}_X \to \mathcal{O}_X$ is a real structure on $\mathcal{O}_X$ as an $\mathcal{O}_X$-module. If $(\mathcal{F}, \alpha^{\mathcal{F}})$ is a real $\mathcal{O}_X$-module, then $\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ has a canonical real structure given by (3.5). We will denote it by $\big(\alpha^{\mathcal{F}}\big)^*$. Hence, $\big(\mathcal{F}^*, \big(\alpha^{\mathcal{F}}\big)^*\big)$ is a real $\mathcal{O}_X$-module.

Let $(\mathcal{F}, \alpha^{\mathcal{F}})$, $(\mathcal{G}, \alpha^{\mathcal{G}})$ be real $\mathcal{O}_X$-modules, then $(\mathcal{F} \otimes \mathcal{G}, \alpha^{\mathcal{F} \otimes \mathcal{G}})$ is a real $\mathcal{O}_X$-module, where $\alpha^{\mathcal{F} \otimes \mathcal{G}} = \alpha^{\mathcal{F}} \otimes \alpha^{\mathcal{G}}$, that is, for every $U \subset X$ open

$$\alpha_U^{\mathcal{F} \otimes \mathcal{G}} : (\mathcal{F} \otimes \mathcal{G})(U) \longrightarrow (\mathcal{F} \otimes \mathcal{G})^{\sigma}(U)$$

is such that, for every $s \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$,

$$\alpha_U^{\mathcal{F} \otimes \mathcal{G}}(s \otimes t) = \alpha_U^{\mathcal{F}}(s) \otimes \alpha_U^{\mathcal{G}}(t).$$

Remark 3.1.8 Let $\big((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)\big)$ and $\big((Y, \mathcal{O}_Y), (\sigma_Y, \tilde{\sigma}_Y)\big)$ be real ringed spaces, and $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a real morphism. If $(\mathcal{F}, \alpha^{\mathcal{F}})$ is a real $\mathcal{O}_Y$-module, then the pull back

$$\varphi^*(\mathcal{F}) = \varphi^{-1}(\mathcal{F}) \otimes_{\varphi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

has a real structure induced from $\alpha^{\mathcal{F}}$ and $(\varphi, \tilde{\varphi})$. This follows from Remark 3.1.2 and Remark 3.1.7. Indeed, the real structure $\alpha^{\varphi^*(\mathcal{F})}$ is

$$\varphi^{-1}(\alpha^{\mathcal{F}}) \otimes_{\varphi^{-1}(\sigma_Y)} \tilde{\sigma}_X.$$

Hence, $(\varphi^*(\mathcal{F}), \alpha^{\varphi^*(\mathcal{F})})$ is a real $\mathcal{O}_X$-module.

3.2 Real Holomorphic Spaces

In this section we study real ringed spaces whose underlying ringed spaces are complex spaces. In particular we will study real holomorphic manifolds, real abelian varieties, and real holomorphic vector bundles over them. At the end of this section we discuss the relation between real abelian varieties and abelian varieties over $\mathbf{R}$. More details about this can be found in [Hui92].

Definition 3.2.1 A *real holomorphic space* is a real ringed space $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$, where a ringed space $(X, \mathcal{O}_X)$ is a complex space. A *morphism of real holomorphic spaces* is a morphism of complex spaces which is real (that is, a morphism of real ringed space, Definition 3.1.1).

Recall that a complex space is a locally ringed space (Section 2.3.5). Since morphisms of complex spaces are automatically local [GR84, §4, pp. 6-7], the category of complex spaces is a full subcategory of $\mathbf{Rsp}/\mathbf{C}$. Therefore, the category of real holomorphic spaces is a full subcategory of $(\mathbf{Rsp}/\mathbf{C})^{\mathbf{real}}$.

Let $(X, \mathcal{O}_X)$ be a complex space. We will denote by $\mathfrak{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$. Recall that $\mathcal{O}_{X,x} = \mathbf{C} \oplus \mathfrak{m}_x$, and that for every section $s \in \mathcal{O}_X(U)$, $U \subset X$ open, there exists a unique continuous $\mathbf{C}$-valued function $[s] : U \to \mathbf{C}$, given by $[s](x) = c_x$, where $s_x = c_x + t_x$, $c_x \in \mathbf{C}$ and $t_x \in \mathfrak{m}_x$. Let $\mathcal{C}_X$ denotes the sheaf of $\mathbf{C}$-valued continuous functions of X. Then, we get a morphism of sheaves $\mathcal{O}_X \to \mathcal{C}_X$, given for every open $U \subset X$, by setting, $\mathcal{O}_X(U) \to \mathcal{C}_X(U)$, $s \mapsto [s]$.

If $(X, \mathcal{O}_X)$ is reduced complex space, then by Rückert's Nullstellensatz this morphism of sheaves is injective, hence one can consider $\mathcal{O}_X$ as a subsheaf of $\mathcal{C}_X$ [Rem94, p. 27, (3.2)]. Therefore, using this canonical morphism, we will identify sections of $\mathcal{O}_X$ with $\mathbf{C}$-valued continuous maps.

Remark 3.2.2 If $(X, \mathcal{O}_X)$ is a reduced complex space, then, for any complex space $(Y, \mathcal{O}_Y)$, every morphism

$$(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is determined by the map $\varphi : X \to Y$. In fact, for any open subset V of Y, the $\mathbf{C}$-algebra homomorphism is given by

$$\begin{aligned} \tilde{\varphi}_V : \mathcal{O}_Y(V) &\longrightarrow \mathcal{O}_X(\varphi^{-1}(V)) \\ g &\longmapsto g \circ \varphi \end{aligned}$$

[Rem94, §3, 3, p. 28].

3.2.1 Real Holomorphic Manifolds

Remark 3.2.3 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex manifolds. Let

$$(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of complex spaces. Then by Remark 3.2.2, the following are true:

1. The continuous map φ is holomorphic, that is, for every holomorphic chart (V, β) on Y, and for every holomorphic chart (U, α) on X such that $\varphi(U) \subset V$, the map $\beta \circ \varphi \circ \alpha^{-1} : \alpha(U) \subset \mathbf{C}^m \to \beta(V) \subset \mathbf{C}^n$ is holomorphic.

2. For every open set $V \subset Y$, the $\mathbf{C}$-algebra homomorphism $\tilde{\varphi}_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V))$ is given by $g \mapsto g \circ \varphi$.

Remark 3.2.4 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex manifolds, and let $\varphi : X \to Y$ be a continuous map. Then, φ is antiholomorphic (that is, for every holomorphic chart (V, β) on Y, and for every holomorphic chart (U, α) on X such that $\varphi(U) \subset V$, the map $\beta \circ \varphi \circ \alpha^{-1} : \alpha(U) \subset \mathbf{C}^m \to \beta(V) \subset \mathbf{C}^n$ is anti-holomorphic) if and only if for every open set V in Y, and every holomorphic function $g : V \to \mathbf{C}$, the function $\overline{g \circ \varphi} : \varphi^{-1}(V) \to \mathbf{C}$ is holomorphic, that is, $\overline{g \circ \varphi} \in \mathcal{O}_X(\varphi^{-1}(V))$.

Proposition 3.2.5 *If $(X, \mathcal{O}_X)$ is a complex manifold, and $(\sigma, \tilde{\sigma})$ is a real structure on $(X, \mathcal{O}_X)$, then the following are true:*

1. *The involution σ is antiholomorphic.*
2. *For any open $U \subset X$, the $\mathbf{C}$-antilinear ring homomorphism $\tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$ is given by $s \mapsto \overline{s \circ \sigma}$.*

In particular the sheaf morphism $\tilde{\sigma}$ is uniquely determined by the map σ.

Proof. Define $\tilde{\sigma}' : \mathcal{O}_X \to \mathcal{O}_X$ by, $\tilde{\sigma}'_U(s) = \overline{\tilde{\sigma}(s)}$ for $U \subset X$ open and $s \in \mathcal{O}_X(U)$. Then, $\tilde{\sigma}'$ is a morphism of $\mathbf{C}$-algebras. Indeed, $\tilde{\sigma}'_U(as) = \overline{\tilde{\sigma}_U(as)} = \overline{\bar{a}\tilde{\sigma}_U(s)} = a\tilde{\sigma}'_U(s)$, for all $a \in \mathbf{C}$ and $s \in \mathcal{O}_X(U)$. Hence, $(\sigma, \tilde{\sigma}')$ is a morphism of reduced complex spaces. Therefore, by Remark 3.2.2, $\tilde{\sigma}'_U(s) = s \circ \sigma$, that is, $\tilde{\sigma}_U(s) = \overline{s \circ \sigma}$. This proves 2. Moreover, since $\tilde{\sigma}_U(s) \in \mathcal{O}_X(\sigma(U)) = \mathcal{O}_X(\sigma^{-1}(U))$, by Remark 3.2.4, σ is antiholomorphic. Hence, 1 is true. $\square$

Definition 3.2.6 A real holomorphic space $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$ is called a *real holomorphic manifold* if the underlying complex space $(X, \mathcal{O}_X)$ is a complex manifold. Let $\mathbf{C}\text{-Mfld}^{\mathbf{real}}$ be the category of real holomorphic manifolds with real morphisms.

Let $\mathcal{C}$ be a category defined by setting,

$$\mathrm{Ob}(\mathcal{C}) = \left\{ (X, \sigma_X) \,\middle|\, \begin{array}{l} X \text{ is a complex manifold with} \\ \text{an antiholomorphic involution } \sigma_X \end{array} \right\},$$

and for $(X, \sigma_X), (Y, \sigma_Y) \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}((X, \sigma_X), (Y, \sigma_Y)) = \{\varphi : X \to Y \mid \varphi \text{ is holomorphic, and } \sigma_Y \circ \varphi = \varphi \circ \sigma_X\}.$$

Proposition 3.2.7 *The Category $\mathcal{C}$ defined as above and the category* $\mathbf{C}\text{-Mfld}^{\mathbf{real}}$ *are equivalent.*

Proof. By the example in Remark 3.1.3, there is a functor $i : \mathcal{C} \to \mathbf{C}\text{-Mfld}^{\mathbf{real}}$ through the following assignments

1. $\mathrm{Ob}(C) \to \mathrm{Ob}(\mathbf{C}\text{-Mfld}^{\mathbf{real}})$, $(X, \sigma_X) \mapsto ((X, \mathcal{O}_X), i_{\sigma_X})$ and

2. $\mathrm{Hom}_{\mathcal{C}}((X, \sigma_X), (Y, \sigma_Y)) \to \mathrm{Hom}_{\mathbf{C}\text{-Mfld}^{\mathbf{real}}}(((X, \mathcal{O}_X), i_{\sigma_X}), ((Y, \mathcal{O}_Y), i_{\sigma_Y}))$, $\varphi \mapsto i_\varphi$.

Define a functor $G : \mathbf{C}\text{-Mfld}^{\mathbf{real}} \to \mathcal{C}$ by defining

$$\mathrm{Ob}(\mathbf{C}\text{-Mfld}^{\mathbf{real}}) \to \mathrm{Ob}(\mathcal{C}), \quad ((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)) \mapsto (X, \sigma_X)$$

and

$$\mathrm{Hom}_{\mathbf{C}\text{-Mfld}^{\mathbf{real}}}(((X, \mathcal{O}_X), (\sigma_X, \tilde{\sigma}_X)), ((Y, \mathcal{O}_Y), (\sigma_Y, \tilde{\sigma}_Y))) \to \mathrm{Hom}_{\mathcal{C}}((X, \sigma_X), (Y, \sigma_Y)),$$
$$(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \mapsto \varphi : X \to Y.$$

Note that by (1) of Proposition 3.2.5, σ_X is antiholomorphic involution, hence, (X, σ_X) is in $\mathcal{C}$. And by (1) of Definition 3.1.1, $\sigma_Y \circ \varphi = \varphi \circ \sigma_X$, hence $\varphi \in \mathrm{Hom}_{\mathcal{C}}((X, \sigma_X), (Y, \sigma_Y))$. This shows that the functor G is welldefined. It is clear from definitions of i and G that $G \circ i = \mathbf{1}_{\mathcal{C}}$. Now by Proposition 3.2.5, the real structure $(\sigma, \tilde{\sigma})$ on $\big((X, \mathcal{O}_X), (\sigma, \tilde{\sigma})\big)$, is i_σ. And by Remark 3.2.2, the real morphism $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is i_φ. This implies that $i \circ G = \mathbf{1}_{\mathbf{C}\text{-Mfld}^{\mathbf{real}}}$. It follows that $\mathcal{C}$ and $\mathbf{C}$-Mfld$^{\mathbf{real}}$ are equivalent categories. □

By the above proposition, we can identify a real holomorphic manifold to a complex manifold together with an antiholomorphic involution. Hence now onwards we will denote real holomorphic manifold by (X, σ_X), where X is a complex manifold and σ_X is an antiholomorphic involution on X.

A real vector bundle over a real complex manifold (X, σ) is called a *real holomorphic vector bundle.*

Remark 3.2.8 Let $(\mathcal{F}, \alpha^{\mathcal{F}})$ be a real holomorphic vector bundle over a real holomorphic manifold (X, σ). Then, $\alpha^{\mathcal{F}}$ induces an antiholomorphic involution α^E on the total space E of $\mathcal{F}$. Moreover, $\alpha^E(E(x)) \subset E(\sigma(x))$, and $\alpha^E : E(x) \to E(\sigma(x))$ is $\mathbf{C}$-antilinear (where $E(y)$ denotes the fibre of the vector bundle E over y for any $y \in X$). The map α^E is defined as follows. Let $e \in E$, and let $x = \pi(e)$, where $\pi : E \to X$ denotes the bundle projection. Then, there exist an open neighborhood U of x in X, and a section $s \in \Gamma(U, \mathcal{F})$ such that $s(x) = e$. Define $\alpha^E(e) = \big(\alpha^{\mathcal{F}}_U(s)\big)(\sigma(x))$. Then, $\alpha^E(e)$ is independent of the choice of s. As e varies over E, the $\alpha^E(e)$ defines the above map $\alpha^E : E \to E$. Using local holomorphic frames of E, one can check that α^E is antiholomorphic, and has the required properties.

3.2.2 Real Abelian Varieties

A *real abelian variety* is a real holomorphic manifold (X, σ), where the underlying complex manifold is an abelian variety, and the antiholomorphic involution σ is compatible with the group operation, that is,

$$\sigma(x+y) = \sigma(x) + \sigma(y) \quad \text{for all } x, y \in X.$$

A real morphisms of real abelian varieties are just real morphisms of underlying real holomorphic manifolds. We will denote by $\mathbf{Ab}^{\mathbf{real}}$ the full subcategory of $\mathbf{C}\text{-Mfld}^{\mathbf{real}}$ consisting of real abelian varieties.

The set of fixed points of σ, that is the set

$$\{x \in X \mid \sigma(x) = x\},$$

is denoted by $\mathcal{R}(X)$, and is called *the set of real points of* X. For any point $x \in X$, let $\tau_x : X \to X$, $y \mapsto y + x$ be the *translation of* X *by* x. If $x \in \mathcal{R}(X)$, then τ_x is a real morphism, since

$$\sigma \circ \tau_x(y) = \sigma(y+x) = \sigma(x) + \sigma(y) = x + \sigma(y) = \tau_x \circ \sigma(y).$$

Recall that a vector bundle over an abelian variety is called *homogeneous* if it is invariant under all translations.

Definition 3.2.9 Let (E, α^E) be a real vector bundle over a real abelian variety (X, σ). If for all $x \in \mathcal{R}(X)$, $(\tau_x^*(E), \alpha^{\tau_x^*(E)})$ is isomorphic to (E, α^E) in the category of $\mathcal{O}_X\text{-}\mathbf{mod}^{\mathbf{real}}$, then (E, α^E) is said to be *real homogeneous*.

Note that, since τ_x and E are real, $(\tau_x^*(E), \alpha^{\tau_x^*(E)})$ is real (Remark 3.1.8).

3.2.3 Abelian Varieties over R

Recall that an *algebraic variety over a field* K is a geometrically reduced, separated scheme over K, which is of finite type over K. Let L/K be a finite Galois extension with the Galois group G. Suppose Y is an algebraic variety over L, and G acts on Y, that is, for very $g \in G$, we are given a morphism of schemes, $\psi_g : Y \to Y$, such that $\psi_1 = \mathbf{1}_Y$ and $\psi_g \circ \psi_h = \psi_{gh}$ for $g, h \in G$. Such an action is called a *descent datum for* Y with respect to the field extension L/K if the diagram

$$\begin{tikzcd} Y \arrow[r, "\psi_g"] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(L) \arrow[r, "\mathrm{Spec}(g^{-1})"] & \mathrm{Spec}(L) \end{tikzcd}$$

commutes for every $g \in G$.

If X is an algebraic variety over K, then $X_{(L)} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is an algebraic variety over L. Moreover $\{1 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(g^{-1})\}_{g \in G}$ is a descent datum for $X_{(L)}$.

Remark 3.2.10 Let $L = \mathbf{C}$ and $K = \mathbf{R}$ in the previous paragraph. Note that the Galois group of $\mathbf{C}/\mathbf{R}$ is $\{1, c\}$, where $c : \mathbf{C} \to \mathbf{C}$ denotes the conjugation. Let $(Y, (\sigma, \tilde{\sigma}))$ be a real ringed space, with Y an algebraic variety over $\mathbf{C}$. Let $\psi_\sigma = (\sigma, \tilde{\sigma})$. Since $(\sigma, \tilde{\sigma})$ is a real structure, $\{\psi_1 = \mathbf{1}_Y, \psi_\sigma\}$ is a descent datum for Y.

Moreover, if Y is quasi-projective then there exists a unique (up to $\mathbf{R}$-isomorphism) algebraic variety X over $\mathbf{R}$ such that $X_{(\mathbf{C})}$ and Y are real isomorphic [Hui92, Theorem 22, p. 25].

Remark 3.2.11 For an algebraic variety Y over $\mathbf{C}$, we will denote by $Y(\mathbf{C})$ the set of $\mathbf{C}$-rational points of Y (since $\mathbf{C}$ is algebraically closed field, these are precisely the closed points). Recall that, if Y is an abelian variety over $\mathbf{C}$, then $Y(\mathbf{C})$ is a compact connected complex Lie group. If $(\sigma, \tilde{\sigma})$ is a real structure on Y, then it gives an antiholomorphic involution on $Y(\mathbf{C})$.

Recall that an *abelian variety over* K is a complete algebraic group over K, for any field K.

Let $\mathbf{Ab}/\mathbf{R}$ denotes the category of abelian varieties over $\mathbf{R}$. Let X be an abelian variety over $\mathbf{R}$. Then, $(X_{(\mathbf{C})}, 1 \times_{\mathrm{Spec}(\mathbf{R})} \mathrm{Spec}(\sigma))$ will be a real ringed space with $X_{(\mathbf{C})}$ an abelian variety over $\mathbf{C}$. By Remark 3.2.11, $X_{(\mathbf{C})}(\mathbf{C}) = X(\mathbf{C})$ [Liu2002, 3.2.3, Proposition 2.18 (a), p. 92] is a complex abelian variety, with involution σ_X induced from $(\sigma, \tilde{\sigma})$. Hence, $(X(\mathbf{C}), \sigma_X)$ is a real abelian variety. Thus, we get a functor from the category $\mathbf{Ab}/\mathbf{R}$ to the category $\mathbf{Ab}^{\mathbf{real}}$. On other hand, if (X, σ_X) is a real abelian variety, then there exist Y an abelian variety over $\mathbf{C}$, and a real structure $(\sigma, \tilde{\sigma})$ on Y induced from σ_X. By Remark 3.2.10, there exist an abelian variety X' over $\mathbf{R}$, such that $X'_{(\mathbf{C})}$ real isomorphic to Y. Define a functor $\mathbf{Ab}^{\mathbf{real}} \to \mathbf{Ab}/\mathbf{R}$ by $X \mapsto X'$. This will give an equivalence of categories between $\mathbf{Ab}^{\mathbf{real}}$ and $\mathbf{Ab}/\mathbf{R}$.

For more detail see [Hui92, Chapter 2, §2.3 and §2.4].

3.3 Real Connections

In this section we discuss the concept of real connections over real holomorphic manifolds. Subsection 3.3.2 and 3.3.3 discusses the relation between real $\mathcal{C}^\infty$ and real holomorphic connections. Subsection 3.3.4 is about real holomorphic connections in a real vector bundle over a real abelian variety. We will show that, if a real holomorphic vector bundle over a real abelian variety admits a real holomorphic connection, then it is real homogeneous.

3.3.1 Preliminaries

Let (X, σ) be a real holomorphic manifold. We will denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X. The sheaf of $\mathcal{C}^\infty$ complex valued functions on the underlying $\mathcal{C}^\infty$-manifold X is denoted by $\mathcal{C}^\infty_X$.

For any holomorphic (respectively $\mathcal{C}^\infty_X$) vector bundle E over X, we will denote the sheaf of holomorphic (respectively $\mathcal{C}^\infty$) sections of E over X by the same symbol E. In particular, for any open subset U of X, $E(U)$ will stand for the $\mathcal{O}_X(U)$-module (respectively $\mathcal{C}^\infty_X(U)$-module) of holomorphic (respectively $\mathcal{C}^\infty$) sections of E on U. If $x \in X$, then E_x will denote the stalk of a sheaf E at x, and $E(x)$ will denote the fibre of E at x, that is, $E(x) = E_x/\mathfrak{m}_x E_x$, where $\mathfrak{m}_x$ denotes the maximal ideal in the local ring $\mathcal{O}_{X,x}$ (respectively $\mathcal{C}^\infty_{X,x}$) (so $\mathfrak{m}_x$ consists of germs of functions vanishing at x). For any open neighborhood U of x, and for any section $s \in E(U)$, $s_x \in E_x$ denotes the germ of s at x, while $s(x) \in E(x)$ denotes the value of s at x, that is, the image of s_x in $E(x)$ under the canonical projection $E_x \to E(x)$.

The sheaf of $\mathcal{C}^\infty$ p-forms with values in E is denoted by $A^p(E)$. We denote by Ω^p_X the sheaf of holomorphic p-forms. We will denote the sheaf of holomorphic p-forms with values in E by $\Omega^p_X(E)$.

Let E be a holomorphic vector bundle over X. A real structure α^E on E we means a real structure on the sheaf of holomorphic sections of E. A real holomorphic vector bundle (E, α^E) we means E is a sheaf of holomorphic sections of E, and α^E is a real structure on E. We will denote by the same symbol α^E the induced antiholomorphic involution on the total space induced by α^E (Remark 3.2.8). In fact for $x \in X$, $\alpha_x^E : E(x) \to E(\sigma(x))$ is given by

$$\alpha_x^E(e) = \alpha_x^E\Big(\sum_{i=1}^{r} a_i s_i(x)\Big) = \sum_{i=1}^{r} \overline{a_i}\big(\alpha_U^E(s_i)\big)\big(\sigma(x)\big), \tag{3.6}$$

where $\{s_i\}_{i=1}^r$ is any local holomorphic frame in an open neighborhood U of x, and $e = \sum_{i=1}^r a_i s_i(x)$, for some $a_i \in \mathbf{C}$, $i = 1 \ldots r$.

Remark 3.3.1 Let (E, α^E) be a real holomorphic manifold over a real manifold (X, σ). If $s = \{s_1, \ldots, s_r\}$ is a holomorphic frame of E on U, then $\alpha^E(s) = \{\alpha_U^E(s_1), \ldots, \alpha_U^E(s_r)\}$ is a holomorphic frame of E on $\sigma(U)$. Also $\alpha^E(s)$ is holomorphic frame of E^σ on U.

Remark 3.3.2 The involution σ induces a real structure on $\mathcal{C}_X^\infty$. For $U \subset X$ open,

$$\tilde{\tau}_U : \mathcal{C}_X^\infty(U) \to \mathcal{C}_X^\infty(\sigma(U))$$

is given by, $\tilde{\tau}_U(f) = \overline{f \circ \sigma}$. It is easy to see that $\big((X, \mathcal{C}_X^\infty), (\sigma, \tilde{\tau})\big)$ is a real ringed space. By abuse of notation, we also write the real structure $(\sigma, \tilde{\tau})$ on $(X, \mathcal{C}_X^\infty)$ by $(\sigma, \tilde{\sigma})$.

Remark 3.3.3 Let (E, α^E) be a real holomorphic vector bundle over (X, σ). Then, α^E induces a real structure on the underlying $\mathcal{C}^\infty$ vector bundle of E. Let $U \subset X$ be open. Define $\alpha_U : A_U^0(E) \to A_{\sigma(U)}^0(E)$ as follows. Let $s \in A_U^0(E)$, and $x \in U$. We must define $\alpha_U(s)(\sigma(x)) \in E(\sigma(x))$. Let $(s_i)_{i=1}^r$ be a holomorphic frame of E on an open neighborhood V of x in U. Write $s|_V = \sum_{j=1}^r f_j s_j$, where $f_j : V \to \mathbf{C}$ $(1 \le j \le r)$ are $\mathcal{C}^\infty$ functions. Define

$$\big(\alpha_U(s)\big)(\sigma(x)) = \sum_{j=1}^{r} \overline{f_j(x)}\alpha_V^E(s_j)(\sigma(x)).$$

One checks that the above definition is independent of the choice of the frame (s_i), and gives a welldefined map $\alpha_U : A_U^0(E) \to A_{\sigma(U)}^0(E)$. As U varies over all open subsets of E, the α_U define a $\mathcal{C}^\infty$ real structure α on E, so (E, α) is a real $\mathcal{C}^\infty_X$-module. We will denote α by α^E.

Let TX be the tangent $\mathbf{R}$-bundle of the underlying $\mathcal{C}^\infty$-manifold X of rank $2n$. We will denote by $TX_{\mathbf{C}}$ the $\mathbf{C}$-vector bundle $TX \otimes_{\mathbf{R}} \mathbf{C}$. We will denote by $\mathcal{T}X$ (respectively $\mathcal{T}^*X$) the holomorphic tangent bundle (respectively cotangent bundle) of X. The real structure $(\sigma, \tilde{\sigma})$ induces a canonical real structure of $\mathcal{T}X$, as follows. Define, $d\sigma(X)(f) = \overline{X(\overline{f \circ \sigma})} \circ \sigma$, for $X \in \mathcal{T}X(U)$, and $f \in \mathcal{O}_X(\sigma(U))$, U open in X. It is easy to verify that $d\sigma(X)$ is a holomorphic vector field on $\sigma(U)$, and $d\sigma$ is a real structure on $\mathcal{T}X$. Similarly $TX_{\mathbf{C}}$ also admits a real structure (we denote it also by $d\sigma$), hence $(TX_{\mathbf{C}}, d\sigma)$ is a real $\mathcal{C}^\infty_X$-module. By Subsection 3.1.4, $\mathcal{T}^*X$ is real. Also, A^p, $A^p(E)$ and Ω^p_X, $\Omega^p_X(E)$ admits real structures. We will denote by $d\sigma$ real structures on A^p, Ω^p_X, and α^E will denote real structures on $A^p(E)$, $\Omega^p_X(E)$. Hence, $(A^p, d\sigma)$, $(A^p(E), \alpha^E)$ are real $\mathcal{C}^\infty_X$-modules. Similarly, $(\Omega^p_X, d\sigma)$, $(\Omega^p_X(E), \alpha^E)$ are real $\mathcal{O}_X$-modules.

Remark 3.3.4 Let (X, σ) be a real holomorphic manifold. Let $(U, z_1, \ldots, z_n)$ be a holomorphic chart on X, where $z_i = x_i + \iota y_i$ $(i = 1, \ldots, n)$. Then, $(\sigma(U), {z_1}^\sigma, \ldots, {z_n}^\sigma)$, where ${z_i}^\sigma = x_i \circ \sigma - \iota y_i \circ \sigma$ will be a holomorphic chart. Recall that, $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$ (respectively, $\{\frac{\partial}{\partial {z_i}^\sigma}\}_{i=1}^n$) will be a frame field on U (respectively $\sigma(U)$) for $\mathcal{T}X$, where the operator $\frac{\partial}{\partial z_i}$ (respectively, $\frac{\partial}{\partial {z_i}^\sigma}$) is defined by $\frac{\partial}{\partial z_i} = \frac{1}{2}\big(\frac{\partial}{\partial x_i} - \iota \frac{\partial}{\partial y_i}\big)$ (respectively, $\frac{\partial}{\partial {z_i}^\sigma} = \frac{1}{2}\big(\frac{\partial}{\partial x_i \circ \sigma} + \iota \frac{\partial}{\partial y_i \circ \sigma}\big)$), for $i = 1, \ldots, n$. Similarly, $\{dz_i\}_{i=1}^n$ (respectively, $\{d{z_i}^\sigma\}_{i=1}^n$) will be a frame field on U (respectively $\sigma(U)$) for $\mathcal{T}^*X$, where $dz_i = dx_i + \iota dy_i$. Then it is easy to see that, $d\sigma(\frac{\partial}{\partial z_i}) = \frac{\partial}{\partial {z_i}^\sigma}$, and $d\sigma(dz_i) = d{z_i}^\sigma$. Moreover, if $f(z_1, \ldots, z_n) \in \mathcal{C}^\infty_X(U)$ then $d(\tilde{\sigma}(f)) = d\sigma(df)$. Indeed, since $df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$, and result follows from the relations $\frac{\partial \overline{f \circ \sigma}}{\partial z_i^\sigma} = \overline{\frac{\partial f}{\partial z_i}} \circ \sigma$ and $\frac{\partial \overline{f \circ \sigma}}{\partial \bar{z}_i^\sigma} = \overline{\frac{\partial f}{\partial \bar{z}_i}} \circ \sigma$. This also shows that $d\sigma(A^{p,q}) = A^{p,q}$, and $\alpha^E(A^{p,q}(E)) = A^{p,q}(E)$.

Remark 3.3.5 If (E, α^E) is a real holomorphic vector bundle, then

$$\alpha^E \circ \bar{\partial}_E = (\bar{\partial}_E)^\sigma \circ \alpha^E = \bar{\partial}_{E^\sigma} \circ \alpha^E. \tag{3.7}$$

Indeed, let $U \subset X$ be an open subset of X, and $s \in A^0(U)$. Since the problem is local, assume that, there exists a holomorphic frame $\{s_1, \ldots, s_r\}$ of E over U.

Let $s = \sum_{i=1}^{r} a_i s_i$, where $a_i : U \to \mathbf{C}$ are $\mathcal{C}^\infty$ functions, $i = 1, \ldots r$. Then,

$$\begin{aligned}\alpha_U^E \circ \big(\bar{\partial}_E\big)_U(s) &= \alpha_U^E\Big(\sum_{i=1}^{r} \bar{\partial}_U(a_i)s_i\Big) = \sum_{i=1}^{r} d\sigma(\bar{\partial}_U(a_i))\alpha_U^E(s_i)\\ &= \sum_{i=1}^{r} \bar{\partial}_{\sigma(U)}(\tilde{\sigma}(a_i))\alpha_U^E(s_i) = \big(\bar{\partial}_E\big)_{\sigma(U)}(\alpha_U^E(s)) = (\bar{\partial}_E)_U^\sigma \circ \alpha_U^E(s).\end{aligned}$$

And we have,

$$\begin{aligned}\big(\bar{\partial}_{E^\sigma}\big)_U \circ \alpha_U^E(s) &= \big(\bar{\partial}_{E^\sigma}\big)_U\Big(\sum_{i=1}^{r} a_i \cdot \alpha_U^E(s_i)\Big) = \sum_{i=1}^{r} \bar{\partial}_U(a_i) \cdot \alpha_U^E(s_i)\\ &= \sum_{i=1}^{r} d\sigma(\bar{\partial}_U(a_i))\alpha_U^E(s_i) = \alpha_U^E \circ \big(\bar{\partial}_E\big)_U(s).\end{aligned}$$

(Remark 3.3.1 and 3.3.4 give equalities in the above equations.)

Remark 3.3.6 For any $\mathcal{C}^\infty$ vector bundle E over X, we can identify $\big(A^p(E)\big)^\sigma$ with $A^p(E^\sigma)$, for all $p \geq 0$, by the following isomorphism. For an open subset U of X

$$\varphi : \big(A^p(E)\big)^\sigma(U) \longrightarrow \big(A^p(E^\sigma)\big)(U)$$

is such that, for $w \in A^p\big(\sigma(U)\big)$, and $s \in E(\sigma(U))$, $\varphi(w \otimes s) = \overline{(w \circ \sigma)} \otimes s$. (Note that $s \in E\big(\sigma(U)\big) = E^\sigma(U)$, therefore, $\overline{(w \circ \sigma)} \otimes s \in A^p(U) \otimes E^\sigma(U) \subset \big(A^p(E^\sigma)\big)(U)$.) Similarly, we identify $(\Omega_X^p(E))^\sigma$ with $\Omega_X^p(E^\sigma)$.

3.3.2 $\mathcal{C}^\infty$ and Holomorphic Connections

Let X be a complex manifold.

Let P be a $\mathcal{C}^\infty$ complex vector bundle of rank r over X. Recall that, a $\mathcal{C}^\infty$ *connection* ∇ in P is a $\mathbf{C}$-linear sheaf morphism,

$$\nabla : A^0(P) \longrightarrow A^1(P)$$

which satisfies the Leibnitz identity, $\nabla(fs) = f\nabla(s) + df \cdot s$, for $f \in A^0$, $s \in A^0(P)$. We extend a $\mathcal{C}^\infty$ connection ∇ to $\nabla : A^p(E) \to A^{p+1}(E)$ using the Leibnitz rule.

Let E be a holomorphic vector bundle over X. A *holomorphic connection* D

in E is a $\mathbf{C}$-linear sheaf morphism,

$$D : \Omega^0_X(E) \longrightarrow \Omega^1_X(E)$$

which satisfies the Leibnitz identity, $D(fs) = fD(s) + df \cdot s$, for $f \in \Omega^0_X$, $s \in \Omega^0_X(E)$. Similar to $\mathcal{C}^\infty$ connections, we extend a holomorphic connection $D : \Omega^0_X(E) \to \Omega^1_X(E)$ to $D : \Omega^p_X(E) \to \Omega^{p+1}_X(E)$ using the Leibnitz rule. (Note that, since f is holomorphic $df = \partial f$.)

Remark 3.3.7 Let E be a holomorphic vector bundle, and D a holomorphic connection in E. We can extend D to $\tilde{D} : A^{p,q}(E) \to A^{p+1,q}(E)$ by setting

$$\tilde{D}(s)(x) = \sum \big(\partial a_i \cdot s_i\big)\big(x) + (-1)^{p+r} \sum \big(a_i \cdot D(s_i)\big)(x) \quad (s \in A^r(E))$$

where $s|_U = \sum a_i \cdot s_i$ in some open neighborhood U of x, $\{s_1, \ldots, s_r\}$ a local holomorphic frame of E on U, and $a_i \in A^{p,q}(U)$. Note that, if s is holomorphic then a_i is holomorphic for all $i = 1, \ldots r$, hence $\partial a_i = da_i$. Therefore, for $s \in \Omega^p_X(E)$, $\tilde{D}(s) = D(s)$.

If D is a flat, then $\tilde{D}^2 = 0 : A^0(E) \to A^{2,0}(E)$. Let $s \in A^0(E)$. Since problem is local we can assume $s = \sum a_i s_i$, where $\{s_1, \ldots, s_r\}$ is a holomorphic frame of E, and a_i $(i = 1, \ldots, r)$ are $\mathcal{C}^\infty$ functions. Then, $\tilde{D}^2(s) = D\big(\sum \partial(a_i)s_i + \sum a_i D(s_i)\big) = \sum \partial^2(a_i)s_i - \sum \partial(a_i)D(s_i) + \sum \partial(a_i)D(s_i) + \sum a_i D^2(s_i) = 0$.

We denote $\tilde{D}$ also by D.

Remark 3.3.8 Let E be a holomorphic manifold over X, with a holomorphic connection D in E. Let P be the underlying $\mathcal{C}^\infty$ complex manifold of E. Define

$$\nabla : A^0(P) \longrightarrow A^1(P)$$

as follows. Let U be an open subset of X, and $s \in A^0_U(P)$. For $x \in U$ there exist an open neighborhood $V \subset U$ of x, and a holomorphic frame $\{s_1, \ldots, s_r\}$ of E over V such that $s|_V = \sum_{i=1}^r f_i s_i$ for $\mathcal{C}^\infty$ functions $f_i : V \to \mathbf{C}$, $i = 1, \ldots, r$. Define

$$\nabla(s)(x) = (df \otimes s)(x) + \Big(f_i D(s_i)\Big)(x).$$

To check ∇ is independent of a holomorphic frame chosen, let $\{t_1, \ldots, t_r\}$ be another holomorphic frame over V, and $s|_V = \sum_{i=1}^r g_i t_i$, for some $\mathcal{C}^\infty$ functions

$g_i : V \to \mathbf{C}$, $i = 1, \ldots, r$. We must show that

$$(dg \otimes t)(x) + \Big(g_i D(t_i)\Big)(x) = (df \otimes s)(x) + \Big(f_i D(s_i)\Big)(x).$$

Since $\{s_1, \ldots, s_r\}$ and $\{t_1, \ldots, t_r\}$ are local holomorphic frames there exist a holomorphic transition functions $a_{ij} : V \to \mathrm{GL}_r(\mathbf{C})$ such that $s_j = \sum_{i=1}^r a_{ij} t_i$, $i, j = 1, \ldots, r$. Now the require equality follows from $g_i = \sum_{k=1}^r f_k a_{ki}$ and

$$D\left(\sum_{k=1}^r a_{ki} t_k\right) = \sum_{k=1}^r da_{ki} t_k + \sum_{k=1}^r a_{ki} D(t_k)$$

(second equality is due to a_{ki} are holomorphic for all i, k and by Leibnitz identity). Hence, ∇ is welldefined. It is clear that ∇ is a $\mathcal{C}^\infty$ connection in P.

Let $s = \sum a_i s_i \in A^0(E)$, where $\{s_1, \ldots, s_r\}$ be a local holomorphic frame of E, and a_i be $\mathcal{C}^\infty$ functions for $i = 1, \ldots, r$. Then, $\nabla(s) = \sum da_i s_i + \sum a_i D(s_i) = \left(\sum \partial(a_i) s_i + \sum a_i D(s_i)\right) + \sum \bar{\partial}(a_i) s_i = D(s) + \bar{\partial}_E(s)$, by Remark 3.3.7. Hence, $\nabla = D + \bar{\partial}_E$. Comparing bidegrees we get $\nabla^{1,0} = D$.

Remark 3.3.9 Giving a pair (E, D) where E is a holomorphic vector bundle, and D is a flat holomorphic connection is equivalent to giving a pair (P, ∇), where P is a $\mathcal{C}^\infty$ complex vector bundle, and $\nabla : A^0(P) \to A^1(P)$ is a flat $\mathcal{C}^\infty$ connection.

Let (E, D) be as given. Let (P, ∇) be as given in Remark 3.3.8. We have to just check that if D is flat so is ∇. Since ∇^2 is $\mathcal{C}^\infty$-linear to show it vanishes it is enough to show that for any holomorphic section s, $\nabla^2(s) = 0$. Consider

$$\nabla^2(s) = (D + \bar{\partial}_E)^2(s) = (D^2 + \bar{\partial}_E D + D\bar{\partial}_E + \bar{\partial}_E^2)(s) = (\bar{\partial}_E D + D\bar{\partial}_E)(s)$$

(by Remark 3.3.7 $D^2 = 0$). Since s is holomorphic $\bar{\partial}_E(s) = 0$, also $D(s) \in \Omega^1_X(E)$, hence $\bar{\partial}_E(D(s)) = 0$. Thus, $\nabla^2 = 0$.

Conversely, let (P, ∇) as given. Since ∇ is flat,

$$0 = \nabla^2 = (\nabla^{1,0} + \nabla^{0,1})^2 = (\nabla^{1,0})^2 + (\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}) + (\nabla^{0,1})^2.$$

By comparing bidegrees we get, $(\nabla^{1,0})^2 = 0$, $\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0} = 0$, and $(\nabla^{0,1})^2 = 0$. By integrability, there exists a unique structure of a holomorphic vector bundle E on P such that $\bar{\partial}_E = \nabla^{0,1}$ [Kob87, Chapter I, Proposition 3.7,

p. 9]. Define, $D : \Omega^0_X(E) \to \Omega^1_x(E)$ by $D(s) = \nabla^{1,0}(s)$, for holomorphic section s of E. Since s is holomorphic we get

$$\bar{\partial}_E\big(D(s)\big) = \nabla^{0,1}\nabla^{1,0}(s) = -\nabla^{1,0}\nabla^{0,1}(s) = -\nabla^{1,0}\bar{\partial}_E(s) = 0.$$

Therefore, $D(s)$ is holomorphic, that is, $D(s) \in \Omega^1_X(E)$. Moreover, D is clearly $\mathbf{C}$-linear, and satisfies Leibnitz identity. Hence D is a holomorphic connection, and D is flat since $D^2 = (\nabla^{1,0})^2 = 0$.

The following proposition is immediate consequence of the above remark.

Proposition 3.3.10 *Let E be a holomorphic vector bundle over a complex manifold X. Giving a flat holomorphic connection D is equivalent to giving a flat $\mathcal{C}^\infty$ connection ∇ in the underlying $\mathcal{C}^\infty$ vector bundle of E such that $\nabla^{0,1} = \bar{\partial}_E$.*

3.3.3 Real Connections

Let (X, σ) be a real holomorphic manifold.

Let P be a real $\mathcal{C}^\infty$ vector bundle over X, and let ∇ be a $\mathcal{C}^\infty$ connection in P. Define

$$\nabla^\sigma : P^\sigma \longrightarrow A^1(P^\sigma) = \big(A^1(P)\big)^\sigma$$

(Remark 3.3.6) by $\nabla^\sigma_U(s) = \nabla_{\sigma(U)}(s)$ for an open subset U of X, and $s \in P^\sigma(U)$. Let $a \in \mathbf{C}$ and $s \in P^\sigma(U)$ then $\nabla^\sigma_U(a \cdot s) = \nabla_{\sigma(U)}(\bar{a}s) = \bar{a}\nabla_{\sigma(U)}(s) = a \cdot \nabla_{\sigma(U)}(s)$. Therefore, ∇^σ is $\mathbf{C}$-linear. For $f \in A^0_U$ and $s \in P^\sigma(U)$,

$$\nabla^\sigma_U(f \cdot s) = \nabla_{\sigma(U)}\big(\tilde{\sigma}_U(f)s\big) = d\big(\tilde{\sigma}_U(f)\big) \otimes s + \tilde{\sigma}_U(f)\nabla_{\sigma(U)}(s).$$

By Remark 3.3.4, $d\big(\tilde{\sigma}_U(f)\big) = d(\overline{f \circ \sigma}) = d\sigma(df)$, that is, $d\big(\tilde{\sigma}_U(f)\big) \otimes s = df \cdot s$ (under the identification made in Remark 3.3.6). Thus, we get $\nabla^\sigma_U(f \cdot s) = df \cdot s + f \cdot \nabla^\sigma s$. Hence, ∇^σ is a $\mathcal{C}^\infty$ connection in P^σ.

Definition 3.3.11 Let ∇ be a $\mathcal{C}^\infty$ connection in a real $\mathcal{C}^\infty$ vector bundle (P, α^P). Then, ∇ is called *real* if the diagram

$$\begin{tikzcd} A^0(P) \arrow[r, "\nabla"] \arrow[d, "\alpha^P"'] & A^1(P) \arrow[d, "\alpha^P"] \\ A^0(P^\sigma) \arrow[r, "\nabla^\sigma"'] & A^1(P^\sigma). \end{tikzcd}$$

commutes.

Similarly, let E be a real holomorphic vector bundle over X, and let D be a holomorphic connection in E. Define, $D^\sigma : E^\sigma \to \Omega^1_X(E)^\sigma$ by $D^\sigma(s) = D_{\sigma(U)}(s)$ for U open subset of X, and $s \in E^\sigma(U)$. Similar to ∇^σ, D^σ is also a holomorphic connection in E^σ. A holomorphic connection D is called a *real holomorphic connection in* (E, α^E) if $\alpha^E \circ D = D^\sigma \circ \alpha^E$.

Remark 3.3.12 Let $s = \{s_1, \ldots, s_r\}$ be a local frame for E over U. Let $\alpha^E_U(s) = \{\alpha^E_U(s_1), \ldots, \alpha^E_U(s_r)\}$. Then, recall that $\alpha^E_U(s)$ is a local frame for E^σ over U, and it is a local frame for E over $\sigma(U)$. Let $\omega = (w_{ij})$ be the connection matrix of D with respect to s, and let $\theta = (\theta_{ij})$ be the connection matrix of D with respect to $\alpha^E_U(s)$. Then, $\theta^\sigma = (\tilde{\sigma}_U(\theta_{ij}))$ is the connection matrix of D^σ with respect to s. Now if D is real it is easy to see that $d\sigma_U(\theta_{ij}) = \overline{\theta_{ij} \circ \sigma} = w_{ij}$ for all $i, j = 1, \ldots, r$, and conversely.

Proposition 3.3.13 *Let* (E, α^E) *be a real holomorphic vector bundle over a real holomorphic manifold* (X, σ). *Giving a real flat holomorphic connection in* (E, α^E) *is equivalent to giving a real flat* $\mathcal{C}^\infty$ *connection* ∇ *in the real* $\mathcal{C}^\infty_X$*-module* (E, α^E) *such that* $\nabla^{0,1} = \bar{\partial}_E$.

Proof. By Proposition 3.3.10 it remains to prove that if D is a real holomorphic connection then $\nabla = D + \bar{\partial}_E$ is also real and conversely. Consider,

$$\alpha^E \circ \nabla = \alpha^E(D + \bar{\partial}_E) = \alpha^E(D) + \alpha^E(\bar{\partial}_E).$$

Since, D is real $D^\sigma \circ \alpha^E = \alpha^E \circ D$, and by Remark 3.3.5, $\left(\bar{\partial}_E\right)^\sigma \circ \alpha^E = \alpha^E \circ \bar{\partial}_E$. Hence, $\nabla^\sigma \circ \alpha^E = \alpha^E \circ \nabla$. On the other hand if ∇ is real then

$$(D^\sigma + (\bar{\partial}_E)^\sigma) \circ \alpha^E = \alpha^E \circ (D + \bar{\partial}_E) = \alpha^E \circ D + \alpha^E \circ \bar{\partial}_E$$

By Remark 3.3.4, $d\sigma(A^{p,q}) = A^{p,q}$, hence comparing bidegrees, we get $D^\sigma \circ \alpha^E = \alpha^E \circ D$. □

3.3.4 Real Connections and Real Homogeneous Bundles

By abuse of notation we will denote by π the corresponding projections for all vector bundles.

Remark 3.3.14 Let X be a complex manifold. Let E be a holomorphic vector bundle over X, and D a holomorphic connection in E. Let $(x, e) \in X \times E$ be such that $e \in E(x)$. Let $w : \mathbf{C} \to X$ be a holomorphic map such that $w(0) = x$. Then, we get the commutative diagram

$$\begin{tikzcd} w^*(E) \arrow[r, "w^*"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbf{C} \arrow[r, "w"'] & X, \end{tikzcd}$$

where the holomorphic vector bundle $w^*(E)$ is the pull back of E by w over $\mathbf{C}$. Since D is a holomorphic connection, w^*D is also a holomorphic connection in $w^*(E)$. Moreover, since the complex manifold $\mathbf{C}$ is 1-dimensional, w^*D is a flat connection. Therefore, there exists a unique holomorphic section

$$s : \mathbf{C} \longrightarrow w^*(E) \tag{3.8}$$

such that $(w^*D)(s) = 0$ and $s(0) = (0, e)$. Let $\tilde{w}_e = w^* \circ s : \mathbf{C} \to E$. Then, $\tilde{w}_e$ is called the *horizontal lift of w with respect to D such that $\tilde{w}(0) = e$.*

Remark 3.3.15 Let (X, σ) be a real holomorphic manifold, and let (E, α^E) be a real holomorphic vector bundle over X. Furthermore assume that E admits a holomorphic connection D. For every pair, $(x, e) \in X \times E$ we will denote by $\mathcal{C}_x^X$ the set

$$\{w : \mathbf{C} \to X \,|\, w \text{ is a holomorphic map, and } w(0) = x\}$$

and $\mathcal{C}_e^E$ the set

$$\{w : \mathbf{C} \to E \,|\, w \text{ is a holomorphic map, and } w(0) = e\}.$$

By Remark 3.3.14, for every pair (x, e) such that $\pi(e) = x$, we get a function $\tilde{\bullet} : \mathcal{C}_x^X \to \mathcal{C}_e^E$, $w \mapsto \tilde{w}_e$. Since X has a real structure, we get a canonical map $\bullet^\sigma : \mathcal{C}_x^X \to \mathcal{C}_{\sigma(x)}^X$ given by $w \mapsto \sigma \circ w \circ c$, where $c : \mathbf{C} \to \mathbf{C}$ denotes the conjugation morphism, $c(z) = \bar{z}$. Similarly there exist a map $\bullet^\sigma : \mathcal{C}_e^E \to \mathcal{C}_{\alpha^E(e)}^E$, $w \mapsto \alpha^E \circ w \circ c$, where α^E denotes the antiholomorphic involution on the total space of E, induced by the real structure α^E (3.6). Thus, we get the diagram

$$\begin{tikzcd} \mathcal{C}^X_x \arrow[r, "\bullet^\sigma"] \arrow[d, "\tilde{\bullet}"'] & \mathcal{C}^X_{\sigma(x)} \arrow[d, "\tilde{\bullet}"] \\ \mathcal{C}^E_e \arrow[r, "\bullet^\sigma"'] & \mathcal{C}^E_{\alpha^E(e)}. \end{tikzcd}$$

If moreover D is a real connection, then the above diagram commutes, that is, for every $w \in \mathcal{C}^X_x$,

$$\widetilde{(w^\sigma)}_{\alpha^E(e)} = \alpha^E \circ \tilde{w}_e \circ c. \tag{3.9}$$

Let s_w (respectively s_{w^σ}) be the unique section of a vector bundle $w^*(E)$ (respectively $(w^\sigma)^*(E)$ such that $w^*D(s_w) = 0$ (respectively $(w^\sigma)^*D(s_{w^\sigma}) = 0$, and $s_w(0) = (0, e)$ (respectively $s_{w^\sigma}(0) = (0, \alpha^E(e))$), see (3.8). Note that, $(w^*(E))^\sigma = (w^\sigma)^*(E)$. Define, $(s_w)^\sigma = \alpha^{w^*(E)} \circ s_w \circ c : \mathbf{C} \to (w^*(E))^\sigma = (w^\sigma)^*(E)$, the section of the vector bundle $(w^\sigma)^*(E)$. If D is real, then $(s_w)^\sigma$ is s_{w^σ}. Now (3.9) follows from the facts that $\alpha^E \circ w^* = (w^\sigma)^* \circ \alpha^{w^*(E)}$, and the horizontal lift of w^σ is unique.

Lemma 3.3.16 *Let (X, σ) be a real abelian variety. Let V denotes the Lie algebra of X. Then for all $x \in \mathcal{R}(X)$, there exists $A \in V$ such that $d\sigma(A) = A$, and $\exp A = x$.*

Proof. Since V is the Lie algebra of X, $\exp : V \to X$ is a surjective holomorphic homomorphism of groups [Mum2008, Chapter I, p. 2, (2)]. Now X is divisible [Mum2008, Chapter I, p. 2, (3)], therefore there exists $x' \in X$ such that $2x' = x$. Then, $x' \in \mathcal{R}(X)$ since $x \in \mathcal{R}(X)$. By the surjectivity of exp map, there exists $v \in V$ such that $\exp(v) = x'$. Let $A = v + d\sigma(v)$. Then $\exp(A) = \exp(v + d\sigma(v)) = \exp(v) + \exp \circ d\sigma(v)$. Note that σ is a homomorphism of X, hence $\exp \circ d\sigma = \sigma \circ \exp$ [Bou75, Chapter III, p. 312, Proposition 10]. Therefore, $\exp(A) = x' + \sigma(x') = 2x' = x$. Also, by definition of A, $d\sigma(A) = A$. □

Definition 3.3.17 Let (X, σ) be a real abelian variety. Let $x \in \mathcal{R}(X)$. Then, by Lemma 3.3.16, there exists $A \in V$ such that $\exp A = x$, and $d\sigma(A) = A$. For every $y \in X$, define

$$w_y : \mathbf{C} \to X, \quad t \mapsto y + \exp tA. \tag{3.10}$$

Then w_y is a holomorphic map such that $w_y(0) = y$ (since exp is a $\mathbf{Z}$-homomorphism, $\exp(0) = 0$), and $w_y(1) = y + x = \tau_x(y)$.

Remark 3.3.18 (Notations as in the above definition.) By the definition of w_y, $(w_y)^\sigma = \sigma \circ w_y \circ c = w_{\sigma(y)}$. Indeed, $(w_y)^\sigma(t) = \sigma(w_y(\bar{t})) = \sigma(y + \exp(\bar{t}A))$. Since, $\sigma \circ \exp = \exp \circ d\sigma$, and $d\sigma$ is conjugate linear, we get $(w_y)^\sigma(t) = \sigma(y) + \exp(td\sigma(A))$. But $d\sigma(A) = A$, hence

$$w_y{}^\sigma = w_{\sigma(y)}. \tag{3.11}$$

Also, for $t \in \mathbf{C}$, and $y \in X$, $w_y(t+1) = y+\exp((t+1)A) = y+\exp(tA)+\exp(A) = x + y + \exp(tA) = w_{x+y}(t)$, hence

$$w_{x+y}(t) = w_y(t+1). \tag{3.12}$$

Definition 3.3.19 Let (X, σ) be a real abelian variety. Let E be a vector bundle over X, and let D be a holomorphic connection in E. Let $(y, e) \in X \times E$ be such that $e \in E(y)$. We denote by $\tilde{w}_{(y,e)}$ the horizontal lift of w_y by D such that $\tilde{w}_{(y,e)}(0) = e$ (w_y as defined in (3.10)).

Proposition 3.3.20 *Let (X, σ) be a real abelian variety, and (E, α^E) a real vector bundle over X. If (E, α^E) admits a real holomorphic connection, then (E, α^E) is a real homogeneous vector bundle.*

Proof. Let $x \in \mathcal{R}(X)$, real point of X. Then we have to show that $\tau_x^*(E) \cong E$ in the category $\mathcal{O}_X$-$\mathbf{mod}^{\mathbf{real}}$.

Define, $\tilde{\varphi} : E \to \tau^*(E)$, $\tilde{\varphi}(e) = (y, \tilde{w}_{(y,e)}(1))$, where $\pi(e) = y$, and w_y, $\tilde{w}_{(y,e)}$ are as defined in Definition 3.3.17 and 3.3.19. Then $\tilde{\varphi}$ is holomorphic, since the holomorphic lift of $\tilde{w}_{(y,e)}$ is given by the solution of a system of a holomorphic ordinary differential equations with initial conditions given by y and e. Define, $\tilde{\psi} : \tau_x^*(E) \to E$ by $\tilde{\psi}(y, e) = \tilde{w}_{(x+y,e)}(-1)$ (note that, since $(y, e) \in \tau^*(E)$, $\pi(e) = x + y$). Then, $\tilde{\psi}$ is also holomorphic by the same reasoning as above.

We claim that $\tilde{\varphi}$ and $\tilde{\psi}$ are inverses of each other. Let $e \in E(y)$, then

$$\tilde{\psi} \circ \tilde{\varphi}(e) = \tilde{\psi}(y, \tilde{w}_{(y,e)}(1)) = \tilde{w}_{(x+y,e')}(-1),$$

where $e' = \tilde{w}_{(y,e)}(1)$. Note that $\tilde{w}_{(y,e)} \circ f$ is the horizontal lift of w_{x+y} by D such that at 0 it is equal to e', where $f : \mathbf{C} \to \mathbf{C}$ is $f(z) = z + 1$. This follows from $e' = \tilde{w}_{(y,e)}(1)$, and by (3.12). Hence, by the uniqueness of the horizontal lift we get, $\tilde{w}_{(x+y,e')}(-1) = \tilde{w}_{(y,e)}(-1 + 1) = \tilde{w}_{(y,e)}(0) = e$. Hence, $\tilde{\psi} \circ \tilde{\varphi} = \mathbf{1}_E$. Similarly,

$\tilde{\varphi} \circ \tilde{\psi} = \mathbf{1}_{\tau_x^*(E)}$. Hence, $\tau_x^*(E)$ is isomorphic to E as an $\mathcal{O}_X$-module. So it remains to prove that $\tilde{\varphi}$ and $\tilde{\psi}$ are real morphisms.

Since D is real, by Remark (3.9) we have $\widetilde{(w_y)^\sigma}_{\alpha^E(e)} = \left(\tilde{w}_{(y,e)}\right)^\sigma$. But $(w_y)^\sigma = w_{\sigma(y)}$. Therefore, we get

$$\tilde{w}_{(\sigma(y),\alpha^E(e))} = \alpha^E \circ \tilde{w}_{(y,e)} \circ c. \tag{3.13}$$

In particular, for $z = 1$ we get,

$$\tilde{w}_{(\sigma(y),\alpha^E(e))}(1) = \alpha^E \circ \tilde{w}_{(y,e)}(1). \tag{3.14}$$

To prove that $\tilde{\varphi}$ is real we have to check that, $\alpha^{\tau_x^*(E)} \circ \tilde{\varphi} = \tilde{\varphi}^\sigma \circ \alpha^E$. It is enough to check the equality on each fibre, that is, for $y \in X$, and $e \in E$ we have to check $\alpha^{\tau_x^*(E)} \circ \tilde{\varphi}(e) = \tilde{\varphi} \circ \alpha^E(e)$. Consider, $\alpha^{\tau_x^*(E)} \circ \tilde{\varphi}(e) = \alpha^{\tau_x^*(E)}\big(y, \tilde{w}_{(y,e)}(1)\big) = (\sigma(y), \alpha^E \circ \tilde{w}_{(y,e)}(1))$. By (3.14), we get $\alpha^{\tau_x^*(E)} \circ \tilde{\varphi}(e) = \big(\sigma(y), \tilde{w}_{(\sigma(y),\alpha^E(e))}(1)\big) = \tilde{\varphi}(\alpha^E(e)) = \tilde{\varphi} \circ \alpha^E(e)$. (Since $\tilde{\varphi}$ is defined on the total space of E, $\tilde{\varphi}^\sigma$ means $\tilde{\varphi}$ at the fiber $\sigma(y)$.) Hence, $\tilde{\varphi}$ is real. Similarly by substituting $z = -1$ in (3.13) we can show that $\tilde{\psi}$ is real.

Therefore, $\tau_x^*(E) \cong E$ in the category $\mathcal{O}_X$-$\mathbf{mod}^{\mathbf{real}}$ for all $x \in \mathcal{R}(X)$. □

3.4 Real Homogeneous Vector Bundles over a Real Abelian Variety

In this section we will prove the main theorem of this chapter. In subsections 3.4.1, 3.4.2 we develops real analogues of Hermitian structures, Kähler metrics etc. Subsection 3.4.3 is about real stability. In the last subsection we prove the main theorem.

3.4.1 Real Hermitian Structures

Let (X, σ) be a real holomorphic manifold. Recall that σ induces a real structure on the underlying $\mathcal{C}^\infty$ manifold of E, which we denote by the same symbol $\tilde{\sigma}$.

Let (E, α^E) be a real $\mathcal{C}_X^\infty$ vector bundle over X. By Subsection 3.1.4, $(E^* \otimes \bar{E}^*, \alpha)$ is a real $\mathcal{C}_X^\infty$ real vector bundle, where α is a real structure induced by α^E on $E^* \otimes \bar{E}^*$, $\alpha = \alpha^{E^*} \otimes \alpha^{\bar{E}^*}$.

Definition 3.4.1 A *real Hermitian structure* or a *real Hermitian metric* h in a real $\mathcal{C}^\infty_X$ vector bundle E is a $\mathcal{C}^\infty$ Hermitian metric h in E which is a real section of $E^* \otimes \bar{E}^*$.

Note that for a real $\mathcal{C}^\infty_X$ or real holomorphic vector bundle (E, α^E) over (X, σ), a section $s \in \Gamma(X, E)$ is real if $\alpha^E(s) = s \circ \sigma$.

Let h be a real Hermitian metric in (E, α^E). Let $s = \{s_1, \ldots, s_r\}$ be a local frame field of E on U. Then

$$h_U(s_i, s_j) = \overline{h_{\sigma(U)}\big(\alpha^E(s_i), \alpha^E(s_j)\big)}.$$

If α^E is the antiholomorphic involution on the total space of E corresponding to α^E, then for every $x \in X$ and $\xi, \eta \in E(x)$, $h_x(\xi, \eta) = \overline{h_{\sigma(x)}\big(\alpha^E(\xi), \alpha^E(\eta)\big)}$.

Remark 3.4.2 Let h be a Hermitian metric in a real $\mathcal{C}^\infty_X$ vector bundle (E, α^E). Define h_{real} by $h_{\text{real}}(\xi, \zeta) = h_U(\xi, \zeta) + \overline{h_{\sigma(U)}(\alpha^E(\xi), \alpha^E(\zeta))}$ for $U \subset X$ open, $\xi, \zeta \in E(U)$. Then h_{real} is a real Hermitian metric in (E, α^E). Hence every real $\mathcal{C}^\infty_X$ vector bundle admits a real Hermitian metric.

Remark 3.4.3 Recall that if E is a $\mathcal{C}^\infty$ complex vector bundle over a complex manifold X, and h a Hermitian matric in E, then a connection ∇ is called a *h-connection* if it preserves h or makes h parallel in the following sense,

$$dh(\xi, \eta) = h(\nabla\xi, \eta) + h(\xi, \nabla\eta)$$

[Kob87, p. 11]. If E is a holomorphic vector bundle over X, then a Hermitian structure h determine a natural h-connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ such that $\nabla^{0,1} = \bar{\partial}_E$.

Proposition 3.4.4 *If h is a real Hermitian metric in a real holomorphic vector bundle (E, α^E) over (X, σ), then there exists a unique real h-connection ∇ such that $\nabla^{0,1} = \bar{\partial}_E$.*

Proof. By [Kob87, Chapter I, Proposition 4.9, p. 11] there exists a unique h-connection ∇ such that $\nabla^{0,1} = \bar{\partial}_E$. To prove the proposition it is enough to prove that ∇ is real. Let $s = \{s_1, \ldots, s_r\}$ be a local holomorphic frame of E over U. Let $\omega_U = (w_{ij})$ be a connection form of ∇ with respect to s over U. Let $h_{ij} = h(s_i, s_j)$. Then by [Kob87, 4.10, P. 11] we have $d'h_{ij} = \sum_{k=1}^r w_{ki} h_{kj}$. Since s is a holomorphic frame of E on U, $\alpha^E(s) = \{\alpha^E_U(s_1), \ldots, \alpha^E_U(s_r)\}$ is a holomorphic frame of E on $\sigma(U)$. Let $h\big(\alpha_U^E(s_i), \alpha_U^E(s_j)\big) = g_{ij}$, and let (θ_{ij}) be a connection form of ∇ on $\sigma(U)$. Since h is real Hermitian metric $h_{ij} = \tilde{\sigma}(g_{ij})$. Therefore,

$$\partial h_{ij} = \partial(\tilde{\sigma}(g_{ij})) = d\sigma(\partial g_{ij}) = d\sigma\Big(\sum_{k=1}^{r} \theta_{ki} g_{ki}\Big)$$

(as ∇ is a h-connection $\partial g_{ij} = \sum_{k=1}^{r} \theta_{ki} g_{ki}$). Thus, $\partial h_{ij} = \sum_{k=1}^{r} d\sigma(\theta_{ki})\tilde{\sigma} g_{ij} = \sum_{k=1}^{r} d\sigma(\theta_{ki}) h_{ik}$. Hence, $d\sigma(\theta_{ki}) = w_{ki}$ for all $k, i = 1, \ldots, r$, which implies that ∇ is real. □

Remark 3.4.5 Let (X, σ) be a real holomorphic manifold. Recall that $(\mathcal{T}X, d\sigma)$ is a real holomorphic vector bundle over X. Let g be a real Hermitian structure in the holomorphic tangent bundle $(\mathcal{T}X, d\sigma)$ of X. We call it a *real Hermitian metric on* (X, σ). Let $(U, z_1, \ldots, z_n)$ be a local co-ordinate system for X. Then $g = \sum_{i,j} g_{ij} dz_i \otimes d\bar{z}_j$, where $g_{ij} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$. The *fundamental* 2*-form* or the *Kähler* 2*-form* Φ *associate with* g is given by

$$\Phi_U = \iota \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j.$$

Note that $(\sigma(U), z_1^\sigma = x_1 \circ \sigma - \iota y_1 \circ \sigma, \ldots, z_n^\sigma = x_n \circ \sigma - \iota y_n \circ \sigma)$ is a holomorphic co-ordinate chart on X, , where $z_i = x_i + \iota y_i$ for $i = 1, \ldots, r$. Let $h_{ij} = h(\frac{\partial}{\partial z_i^\sigma}, \frac{\partial}{\partial \bar{z}_j^\sigma})$. Since g is real $\bar{h}_{ij} = g_{ij} \circ \sigma$, that is, $h_{ji} = g_{ij} \circ \sigma$. Thus,

$$\begin{aligned}
\sigma^* \Phi_U = \sigma^*\Big(\iota \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j\Big) &= \iota \sum_{i,j} (g_{ij} \circ \sigma) dz_i \circ \sigma \wedge d\bar{z}_j \circ \sigma \\
&= -\iota \sum_{i,j} (g_{ij} \circ \sigma) d\bar{z}_j \circ \sigma \wedge dz_i \circ \sigma = -\iota \sum_{i,j} h_{ij} dz_i^\sigma \wedge d\bar{z}_j^\sigma = -\Phi_{\sigma(U)}.
\end{aligned}$$

Hence,

$$\sigma^* \Phi = -\Phi. \tag{3.15}$$

3.4.2 Real Kähler Manifolds

A *real Kähler manifold* is a triple (X, σ, g) such that (X, σ) is a real manifold, and g is a real Hermitian metric on X which is Kähler.

Let (X, g) be a Kähler manifold, and X admits a real structure σ. Then $g + \sigma^* g$ is a real Hermitian metric on (X, σ). Moreover, since d commutes with σ^*

its fundamental 2-form is closed. Hence $(X, \sigma, g + \sigma^* g)$ is a real Käher manifold.

If no confusion is likely to occur, we denote a real Kähler manifold (X, σ, g) by just X.

Remark 3.4.6 Recall that the *degree* of a torsion-free coherent sheaf $\mathcal{F}$ on a compact Kähler manifold (X, g) is

$$\text{degree}(\mathcal{F}) = \int_X c_1(\mathcal{F}) \wedge \Phi^{n-1},$$

where n is the dimension of X, $c_1(\mathcal{F})$ is the first Chern class of $\mathcal{F}$ [Kob87, p. 167, (6.15)], and Φ is the Kähler form of (X, g).

Proposition 3.4.7 *Let (X, σ, g) be a real compact Kähler manifold. Let E be a vector bundle over X. Then,* $\text{degree}(E) = \text{degree}(E^\sigma)$*. In particular, if $\mathcal{F}$ is a torsion free coherent sheaf on X, then*

$$\text{degree}(\mathcal{F}) = \text{degree}(\mathcal{F}^\sigma).$$

Proof. Recall that $E^\sigma = \sigma^*(\bar{E})$. By (3.15) we have $\sigma^*\Phi = -\Phi$. Also, $c_1(\bar{E}) = -c_1(E)$. Consider,

$$\begin{aligned} \text{degree}(E^\sigma) &= \int_X c_1(\sigma^*(\bar{E})) \wedge \Phi^{n-1} = \int_X \sigma^*(c_1(\bar{E})) \wedge (-\sigma^*(\Phi^{n-1}) \\ &= (-1)^n \int_X \sigma^*(c_1(E) \wedge \Phi^{n-1}) \end{aligned}$$

(since $\sigma^* : A^0(X) \to A^1(X)$ is a $\mathbf{R}$-algebra homomorphism). Antiholomorphic involutions are orientation reversing, hence

$$\text{degree}(E^\sigma) = (-1)^n(-1)^n \int_X c_1(E) \wedge \Phi^{n-1} = \text{degree}(E).$$

The second statement follows from the definition of the first Chern class of a torsion free coherent sheaf. ☐

3.4.3 Real Stable Vector Bundles

Recall that the *slope* of a torsion free coherent sheaf over a compact Kähler manifold (X, g) is defined as

$$\mu(\mathcal{F}) = \frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})}.$$

Definition 3.4.8 A real holomorphic vector bundle (E, α^E) over a compact real Kähler manifold is said to be *real stable* (respectively *real semistable*) if for every proper real holomorphic coherent subsheaf $\mathcal{F}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(E)$, we have

$$\mu(\mathcal{F}) < \mu(E) \quad (\text{respectively } \mu(\mathcal{F}) \leq \mu(E)).$$

Proposition 3.4.9 *Let (E, α^E) be a real holomorphic vector bundle over a compact real Kähler manifold (X, σ, g). Then, (E, α^E) is real semistable if and only if E is semistable.*

Proof. If E is semistable, it is clear that (E, α^E) is real semistable. Conversely, suppose that (E, α^E) is real semistable. Let $\mathcal{F}$ be the maximal semistable subsheaf of E. Thus, $\mathcal{F}$ is semistable, and for every subsheaf $\mathcal{F}'$ of E, $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. We will prove that $\alpha^E|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\sigma$ is a real structure on $\mathcal{F}$. For that, it is enough to prove that $\text{Im}(\alpha^E)(\mathcal{F}) = \mathcal{F}^\sigma$. Since α^E is an isomorphism of sheaves, $\text{Im}(\alpha^E)(\mathcal{F})$ is a maximal semistable subsheaf of E^σ. Also, $\mathcal{F}^\sigma$ is a maximal semistable subsheaf of E^σ. Hence, by uniqueness of maximal semistable sheaf, $\text{Im}(\alpha^E)(\mathcal{F}) = \mathcal{F}^\sigma$. Therefore, $(\mathcal{F}, \alpha^E|_{\mathcal{F}})$ is real. But then since (E, α^E) is real semistable, $\mu(\mathcal{F}) \leq \mu(E)$. Since $\mathcal{F}$ is maximal semistable, this implies that $\mu(E) = \mu(\mathcal{F})$. Therefore, E is semistable. □

Remark 3.4.10 Let (E, α^E) be a real holomorphic vector bundle over a compact real Kähler manifold (X, σ, g). Then, (E, α^E) is real stable if and only if it is stable or it is of the form $E = E_1 \oplus \left(\alpha^E\right)^\sigma ({E_1}^\sigma)$, where E_1 is a stable vector bundle of the slope $\mu(E)$. (This follows by imitating the proof of [Gar93, Theorem 6].)

Remark 3.4.11 If E is a semistable vector bundle over a compact Kähler manifold, E contains a unique non-trivial maximal polystable subsheaf S such that $\mu(S) = \mu(E)$. This sheaf is called the *socle of* E [HL97, Lemma 1.5.5]. In fact, S is the sum of all stable subsheaves F such that $\mu(F) = \mu(E)$ [AB2001, Definition 2.4, Lemma 2.5].

Let (E, α^E) be a real holomorphic vector bundle over a compact real Kähler manifold (X, σ, g). Then (E, α^E) is said to be *real polystable* if $E = \oplus_{i=1}^k E_i$, where $(E_i, \alpha^E|_{E_i})$ is a real stable sub-bundles of E satisfying $\mu(E_i) = \mu(E)$ for $i = 1, \ldots, k$.

Proposition 3.4.12 *Let (X, σ, g) be a real compact Kähler manifold, and let E be a semistable vector bundle over X. If S is the socle of E, then S^σ is the socle of E^σ. Moreover, if (E, α^E) is a real semistable vector bundle, then the socle of E is a real subsheaf of E.*

Proof. For any torsion free coherent sheaf $\mathcal{F}$ we have $\deg(F^\sigma) = \deg(\sigma^* \overline{F}) = \deg F$ (Proposition 3.4.9). Therefore, if F is a stable vector bundle then F^σ is also stable. Since $\bullet^\sigma$ is exact (recall property 3 of the functor $\bullet^\sigma$), S^σ is polystable and $\mu(S^\sigma) = \mu(S) = \mu(E) = \mu(E^\sigma)$. That the polystable sub-bundle S^σ is maximal follows from the uniqueness of S. Since the socle is unique, $(S, \alpha^E|_S)$ is a real subsheaf of (E, α^E). Let $S = \oplus_{i=1}^k E_i$. Suppose E_1 is not real stable, then there exists some $j \neq 1$ such that $E_j = (\alpha^E)^\sigma \big({E_1}^\sigma\big)$. This follows from the fact that $\mu((\alpha^E)^\sigma \big({E_1}^\sigma\big)) = \mu({E_1}^\sigma) = \mu(E)$, and [AB2001, Definition 2.4, Lemma 2.5]. Now $E_1 \oplus (\alpha^E)^\sigma \big({E_1}^\sigma\big)$ is real stable (Remark 3.4.10). Similarly for every k such that E_k is not real stable there exists $l \neq k$ such that $E_l = (\alpha^E)^\sigma \big({E_k}^\sigma\big)$. Hence S is real polystable. $\square$

3.4.4 Extension Classes

Proposition 3.4.13 *Let X be a complex manifold, and let*

$$\alpha : \quad 0 \to S \xrightarrow{i} E \xrightarrow{p} Q \to 0, \tag{3.16}$$

be an exact sequence of holomorphic vector bundles. Then, for every $\mathcal{C}^\infty$ splitting $l : Q \to E$, there exist a unique $\beta_l \in A^{0,1}(H)$ such that

$$i(\beta_l) = \bar{\partial}_E l - l \bar{\partial}_Q, \tag{3.17}$$

where $H = \mathcal{H}om(Q, S)$. Moreover $\beta_l \in Z^{0,1}(H)$, the sheaf of $\bar{\partial}_H$-closed $(0,1)$-forms, and the cohomology class of β_l in $H^{0,1}(X, H)$ is the Dolbeault extension class. Conversely, if $\beta \in Z^{0,1}(H)$ is a cocycle representing the Dolbeault extension class in $H^{0,1}(X, H)$, then there exist a $\mathcal{C}^\infty$ splitting $l : Q \to E$ such that $\beta = \beta_l$.

Proof. First statement is in [DK90, pp. 389-390]. For converse, let $l' : Q \to E$ be a $\mathcal{C}^\infty$ splitting, and let $\beta_{l'}$ be the corresponding representative of the extension class α. Then there exist $\varphi \in A^0(H)$, such that $\beta - \beta_{l'} = \bar\partial_H \varphi$. Consider,

$$l = i \circ \varphi + l' : Q \longrightarrow E,$$

then l is a $\mathcal{C}^\infty$ splitting of α. To prove that $\beta = \beta_l$, it is enough to prove that $i(\beta) = i(\beta_l)$. We have, $i(\beta_l) = \bar\partial_E l - l\bar\partial_Q = \bar\partial_E(i \circ \varphi) - (i \circ \varphi)\bar\partial_Q + i(\beta_{l'})$. Since i is holomorphic $\bar\partial_E i = i\bar\partial_S$. Also we have $\bar\partial_H \varphi = \bar\partial_S \varphi - \varphi\bar\partial_Q$, hence, $i(\beta_l) = i(\bar\partial_H \varphi + \beta_{l'}) = i(\beta)$. □

Remark 3.4.14 Let the notations be as in Proposition 3.4.13. If $l : Q \to E$ is a $\mathcal{C}^\infty$-splitting of the given extension, then

$$\bar\partial_E(i(s) + l(q)) = i\bar\partial_S(s) + i\beta_l(q) + l\bar\partial_Q(q)$$

for all $s \in A^0(S)$ and $q \in A^0(Q)$. This follows from the definition of β_l, and since i is holomorphic $\bar\partial_E i = i\bar\partial_S$.

Remark 3.4.15 Let notations as in Proposition 3.4.13. Let

$$\mathrm{Split}(\alpha) = \{l \in A^0(\mathcal{H}om(Q, E)) \,|\, p \circ l = \mathbf{1}_Q\}.$$

If $l, l' \in \mathrm{Split}(\alpha)$, then $p(l - l') = 0$. So, there exists $u \in A^0(H)$, such that $l - l' = iu$. Conversely, if $u \in A^0(H)$ and $l \in \mathrm{Split}(\alpha)$, then $l + iu \in \mathrm{Split}(\alpha)$. Therefore, $\mathrm{Split}(\alpha)$ has a canonical structure of an affine space model after $A^0(H)$.

Let l and l' be two $\mathcal{C}^\infty$-splittings of the given extension α, (3.16). Then $\beta_l = \beta_{l'}$ if and only if there exists a unique $u \in \Omega^0_X(H)$ such that $l' = l + iu$. Indeed, if $\beta_l = \beta_{l'}$ then $\bar\partial_E(l - l') = (l - l')\bar\partial_Q$. Hence, $l - l' \in \Omega^0_X(\mathcal{H}om(Q, E))$. On other the hand since $p(l - l') = 0$, there exists $u \in A^0(H)$ such that $l - l' = iu$. Since i is holomorphic, and $l - l' \in \Omega^0_X(\mathcal{H}om(Q, E))$ we get $u \in \Omega^0_X(H)$. Conversely suppose that, $l - l' = iu$, for $u \in \Omega^0_X(H)$. Then,

$$i(\beta_l) = \bar\partial_E l - l\bar\partial_Q = \bar\partial_E iu - iu\bar\partial_Q + i(\beta_{l'}).$$

Since $u \in \Omega^0_X(H)$, and i is holomorphic, $\bar\partial_E i = i\bar\partial_S$ and $\bar\partial_S u = u\bar\partial_Q$. Thus $i(\beta_l) = i(\beta_{l'})$, hence $\beta_l = \beta_{l'}$.

Proposition 3.4.13, and the above statement gives us a canonical map

$$\begin{aligned} \mathrm{Split}(\alpha) &\longrightarrow Z^{0,1}(H) \\ l &\longmapsto \beta_l \end{aligned}$$

such that the cohomology class of β_l in $H^{0,1}(H)$ is the Dolbeault extension class. This induces a bijection

$$\mathrm{Split}(\alpha)/\Omega^0(H) \longrightarrow \text{Dolbeault extension class.}$$

Remark 3.4.16 Let (X, σ) be a real holomorphic manifold, and let

$$\alpha : 0 \longrightarrow S \xrightarrow{i} E \xrightarrow{p} Q \longrightarrow 0$$

be an exact sequence of real holomorphic vector bundles, that is, (S, α^S), (E, α^E) and (Q, α^Q) are real holomorphic vector bundles, and i, p are real morphism. Let α^H be the canonical real structure on $H = \mathcal{H}om(Q, S)$ induced by real structures α^Q and α^S. Let $\alpha^\sigma : 0 \to S^\sigma \xrightarrow{i^\sigma} E^\sigma \xrightarrow{p^\sigma} Q^\sigma \to 0$. (Recall that $\bullet^\sigma$ is an exact functor, hence α^σ is exact). Also if l is a splitting of α, then l^σ is a splitting of α^σ. If l is any $\mathcal{C}^\infty$ splitting of α, then

$$\beta_{l^\sigma} = \big(\beta_l\big)^\sigma.$$

This follows from (3.7) and (3.17). Moreover, if l is real then β_l is real, that is,

$$\alpha^H(\beta_l) = \big(\beta_l\big)^\sigma = \beta_{l^\sigma}. \tag{3.18}$$

To see this consider, $i^\sigma\big(\alpha^H(\beta_l)\big) = i^\sigma\big(\alpha^S \circ \beta_l \circ (\alpha^Q)^\sigma\big)$. Since i is real we get $i^\sigma \circ \alpha^S = \alpha^E \circ i$, and since l is real we get $l^\sigma \circ \alpha^E = \alpha^Q \circ l$. Thus, we get $i^\sigma\big(\alpha^H(\beta_l)\big) = (i\beta_l)^\sigma$. This will give us the required equality.

Remark 3.4.17 Let (E, α^E) be a real holomorphic vector bundle over a real Kähler manifold (X, σ, Φ). Let D be a real flat $\mathcal{C}^\infty$ connection in E, such that $D^{0,1} = \bar{\partial}_E$. If β is a D-harmonic form in $A^\bullet(E)$, then $\alpha^E(\beta)$ is a D^σ-harmonic form in $A^\bullet(E^\sigma)$. This follows from the facts that, D is real, and $d\sigma$ commutes with operators $*$ ([Kob87, p. 60, (2.4)]) and $\bar{\partial}$.

3.4.5 Real Flat Connections

In this subsection we give various equivalent criteria for real holomorphic vector bundles to admits real flat connection over a real abelian variety.

Proposition 3.4.18 *Let X be a compact real Kähler manifold. If (E, α^E) is a real polystable vector bundle over X such that $c_1(E) = c_2(E) = 0$, then (E, α^E) admits a real flat holomorphic connection.*

Proof. Let $G = \mathbf{Z}/2\mathbf{Z} = \{e, \sigma\}$, where e denotes the identity element and σ be such that $\sigma^2 = e$. Then, to give a real structure on X is the same as giving an antiholomorphic action of G on the complex manifold X, and G-invariant vector bundles are real vector bundles. Now by imitating the proof of [Gar93, Theorem 5] we see that if E is a real polystable vector bundle, then E admits a real Einstein-Hermitian metric h. Moreover, if $c_1(E) = c_2(E) = 0$, then by Lübke's inequality [Kob87, Chapter IV, Theorem 4.11, p. 115], every Einstein-Hermitian metric in E is flat. Let ∇ be the h-connection. Then by Proposition 3.4.4, ∇ is real, and $\nabla^{0,1} = \bar{\partial}_E$. By Proposition 3.3.13, $\nabla^{1,0}$ is a real flat holomorphic connection in (E, α^E). $\square$

Proposition 3.4.19 *Let X be a compact real Kähler manifold and let*

$$0 \to S \xrightarrow{i} E \xrightarrow{p} Q \to 0,$$

be an exact sequence of real holomorphic vector bundles over X. Suppose that S and Q admit real flat holomorphic connections. Then, E also has a real flat holomorphic connection.

Proof. Let $\tilde{\alpha} \in H^1(X, H)$ be the extension class of the given extension. Let $\alpha \in H^{0,1}(X, H)$ be the Dolbeault cohomology class corresponding to $\tilde{\alpha}$ under the Dolbeault isomorphism $H^1(X, H) \cong H^{0,1}(X, H)$. Let $\beta \in Z^{0,1}(H)$ be the unique $\bar{\partial}_H$-harmonic representative of α. Let $l \in A^0(\mathcal{H}om(Q, E))$ be a $\mathcal{C}^\infty$ splitting of the given extension such that $\beta = \beta_l$ (Proposition 3.4.13).

Let ∇_S and ∇_Q be flat real holomorphic connections in S and Q, respectively. Then, $D_S = \nabla_S + \bar{\partial}_S$ and $D_Q = \nabla_Q + \bar{\partial}_Q$ are real flat $\mathcal{C}^\infty$ connections in S and Q, respectively. Define a $\mathcal{C}^\infty$ connection D_E in E by

$$D_E(i(s) + l(q)) = iD_S(s) + i\beta(q) + lD_Q(q)$$

for all $s \in A^0(E)$ and $q \in A^0(Q)$. Then, the $(0,1)$-part of D_E is given by

$$\begin{aligned} D_E^{0,1}(i(s)+l(q)) &= iD_S^{0,1}(s) + i\beta(q) + lD_Q^{0,1}(q) \\ &= i\bar{\partial}_S(s) + i\beta(q) + l\bar{\partial}_Q(s) = \bar{\partial}_E(i(s)+l(q)) \end{aligned}$$

using Remark 3.4.14. Thus, $D_E^{0,1} = \bar{\partial}_E : A^0(E) \to A^{0,1}(E)$. We claim that the $\mathcal{C}^\infty$ connection D_E is flat. The connections D_S and D_Q induce a $\mathcal{C}^\infty$ connection D_H in $H = \mathcal{H}om(Q,S)$, which is also real and flat. By the Hodge Decomposition Theorem for flat vector bundles (see [Kob87, Chapter III, § 2, pp. 64-66] or [Sim92, p. 22]) every $\bar{\partial}_H$-harmonic form in $A^\bullet(H)$ is D_H-harmonic also. Therefore, $D_H(\beta) = 0$, that is,

$$D_S \circ \beta + \beta \circ D_Q = 0. \tag{3.19}$$

Now, the curvature of D_E is given by

$$\begin{aligned} D_E^2(i(s)+l(q)) &= D_E\big(iD_S(s) + i\beta(q) + lD_Q(q)\big) \\ &= D_E\Big(i\big(D_S(s)+\beta(q)\big) + lD_Q(q)\Big) \\ &= iD_S\big(D_S(s)+\beta(q)\big) + i\beta\big(D_Q(q)\big) + lD_Q\big(D_Q(q)\big) \\ &= iD_S^2(s) + i\big(D_S\beta(q) + \beta D_Q(q)\big) + lD_Q^2(q) = 0 \end{aligned}$$

using (3.19) and the fact that D_S and D_Q are flat. This, proves the claim that D_E is flat.

Now it remains to prove that D_E is real. We will show that β is real, which implies that D_E is real. Let l' be a real splitting of the exact sequence $0 \to S \xrightarrow{i} E \xrightarrow{p} Q \to 0$. Hence, $\beta_{l'}$ is real (3.18). Since $[\beta] = [\beta_{l'}]$, there exists $\varphi \in A^0(H)$ such that $\beta - \beta_{l'} = \bar{\partial}_H(\varphi)$. Since the canonical real structure α^H on H induced by the real structures on Q and S commutes with $\bar{\partial}_H$, we get $[\alpha^H(\beta)] = [\alpha^H(\beta_{l'})]$. Thus, $[\alpha^H(\beta)] = [(\beta_{l'})^\sigma] = [\beta_{l'^\sigma}] = [\beta_{l^\sigma}] = [\beta^\sigma]$. Since β is D_H-harmonic so is β^σ. Also by Remark 3.4.17 $\alpha^H(\beta)$ is D_H-harmonic. Hence, by uniqueness of D_H-harmonic form, $\alpha^H(\beta) = \beta^\sigma$, that is, β is real.

Thus, D_E is a flat real $\mathcal{C}^\infty$ connection in E such that $D_E^{0,1} = \bar{\partial}_E$. Therefore, by Proposition 3.3.13, $D_E^{1,0}$ induces a flat real holomorphic connection in E. □

Remark 3.4.20 If (X,σ) is a real abelian variety, then there exists a real translation invariant Kähler metric on X. Indeed, since X is an abelian variety there exists a translation invariant Kähler metric g [GH94, pp. 301-302]. Consider a real Kähler metric $g+\sigma^*g$. It remains to show that $g+\sigma^*g$ is translation invariant. Consider,

$$\tau_x^*(g+\sigma^*g) = \tau_x^*g + (\sigma\circ\tau_x)^*g = \tau_x^*g + (\tau_{\sigma(x)}\circ\sigma)^*g,$$

as $\tau_x \circ \sigma = \sigma \circ \tau_{\sigma(x)}$. Since g is translation invariant $\tau_x^*g = g$ and $\tau_{\sigma(x)}^*g = g$. Thus, $\tau_x^*(g+\sigma^*g) = g+\sigma^*g$, and the Kähler form Φ of g is also translation invariant. Thus, (X,σ,g) is a real Kähler manifold such that the Kähler form Φ of g is translation invariant. In particular, for any coherent torsion free sheaf over (X,σ,g), $\text{degree}(\tau_x^*(\mathcal{F})) = \text{degree}(\mathcal{F})$. As $\text{degree}(\tau_x^*(\mathcal{F})) = \int_X c_1(\tau_x^*(\mathcal{F})\wedge\Phi^{n-1} = \int_X \tau_x^*(c_1(\mathcal{F})\wedge\Phi^{n-1} = \text{degree}(\mathcal{F})$, since Φ is translation invariant.

Theorem 3.4.21 *Let (X,σ) be a real abelian variety, and let (E,α^E) be a real holomorphic vector bundle over X. Then the following are equivalent:*

1. *The real holomorphic vector bundle (E,α^E) admits a real holomorphic connection.*

2. *The real holomorphic vector bundle (E,α^E) is real homogeneous.*

3. *The real holomorphic vector bundle (E,α^E) is real semistable with $c_1(E) = c_2(E) = 0$.*

4. *The real holomorphic vector bundle (E,α^E) admits a filtration*

$$E^\bullet : \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that E_i is a real sub-bundle of (E,α^E), $c_j(E_i) = 0$, for $j = 1,2$ and $i = 1,\ldots,n$, and E_i/E_{i-1} is real polystable.

5. *The real holomorphic vector bundle (E,α^E) admits a real flat holomorphic connection.*

Proof. Th proof of (1) implies (2) follows from Proposition 3.3.20.

By Proposition 3.4.9, condition (3) is equivalent to the statement that E is semistable with $c_1(E) = c_2(E) = 0$. For any vector bundle V on X, let $\mathcal{K}(\det V) = \{x \in X \mid \tau_x^* \det V \cong \det V\}$. By [Mum2008, Proposition, p. 57], $\mathcal{K}(\det V)$ is a Zariski-closed subgroup of X for any holomorphic vector bundle V. By the Artin-Lang homomorphism Theorem [Bec82, Lemma 1.5, p. 8] $\mathcal{R}(X)$ is Zariski-dense in X. (Take U to be X in [Bec82, Lemma 1.5], and recall that 0 is a simple point in U.) Now, $\mathcal{R}(X) \subset \mathcal{K}(\det E)$. Hence, $\mathcal{K}(\det E)$ is equal to X. Therefore, $\tau_x^*(\det E) \cong \det E$ for all $x \in X$. By definition [Mum2008, (i), p. 70], $\det E \in \mathrm{Pic}^0(X)$. Hence, degree$(\det E) =$ degree(E) is 0 [Mum2008, p. 82, Theorem of Appel-Humbert, p. 19]. To prove E is semistable, let $\mathcal{F}$ be the maximal semistable subsheaf of E. Let $\varphi_x : E \to \tau_x^*(E)$ be an isomorphism for $x \in \mathcal{R}(X)$. Then $\varphi_x(\mathcal{F})$ is the maximal semistable subsheaf of $\tau_x^*(E)$. But $\mathrm{rk}(\tau_x^*(\mathcal{F})) = \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\varphi_x(\mathcal{F}))$, and degree$(\tau_x^*(\mathcal{F})) =$ degree$(\mathcal{F}) =$ degree$(\varphi_x(\mathcal{F}))$. Hence, $\tau_x^*(\mathcal{F})$ is also a maximal semistable subsheaf of $\tau_x^*(E)$. Thus, by the uniqueness of maximal semistable subsheaf $\tau_x^*(\mathcal{F}) = \varphi_x(\mathcal{F}) \cong \mathcal{F}$ for all $x \in \mathcal{R}(X)$. Again, since $\mathcal{K}(\det \mathcal{F})$ is Zariski-closed and $\mathcal{R}(X)$ is Zariski-dense, we get $\det \mathcal{F} \in \mathrm{Pic}^0(X)$. Hence degree$(\mathcal{F}) = 0$. But $\deg E = 0$. Since $\mathcal{F}$ is maximal semistable, E is semistable. Since $\tau_x^*(E) \cong E$ for all $\mathcal{R}(X)$, which is Zariski-dense subset of X, by the proof of [Bis2004, Corollary 3.2, p. 383] $c_2(E) = 0$. Thus, (2) implies (3).

We will prove (3) implies (4) now. By Proposition 3.4.9 E is semistable. Since E is semistable, by [HL97, Lemma 1.5.5, p. 23] there exists a unique filtration of coherent analytic subsheaves

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \tag{3.20}$$

such that E_i/E_{i-1} is the socle of E/E_{i-1}. Applying $\bullet^\sigma$ to the above filtration we get,

$$0 = E_0 \subset {E_1}^\sigma \subset {E_2}^\sigma \subset \cdots \subset {E_n}^\sigma = E^\sigma.$$

By Proposition 3.4.12, and since $\bullet^\sigma$ is exact, ${E_i}^\sigma/{E_{i-1}}^\sigma = (E_i/E_{i-1})^\sigma$ is the socle of $E^\sigma/{E_{i-1}}^\sigma$. Since α^E is an isomorphism of E to E^σ, we get another filtration in E^σ, from the filtration (3.20)

$$0 = \alpha^E(E_0) \subset \alpha^E(E_1) \subset \alpha^E(E_2) \subset \cdots \subset \alpha^E(E_n) = \alpha^E(E) = E^\sigma.$$

Since α^E is an isomorphism of $\mathcal{O}_X$-modules, $\alpha^E(E_i/E_{i-1}) = \alpha^E(E_i)/\alpha^E(E_{i-1})$ is the socle of $E^\sigma/\alpha^E(E_{i-1})$. By uniqueness of filtration, we get $\alpha^E(E_i) = {E_i}^\sigma$. Hence, E_i is real subsheaf of (E, α^E), for all $i = 1, \ldots, n$. By [BG2008, last paragraph p. 42, fist paragraph p.43], $\tau_x^*(E_i) \cong E_i$ for all $x \in X$ in particularly for $x \in \mathcal{R}(X)$, and E_i is vector bundle for all $i = 0, \ldots, n$. Hence by (2) implies (3), E_i is semistable, with $c_1(E_i) = c_2(E_i) = 0$ for all $i = 1, \ldots, n$. Thus, we get a filtration of E,

$$E^{\bullet}: \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that each E_i is a real holomorphic sub-bundle of E, each E_i is semistable with $c_1(E_i) = c_2(E_i) = 0$, and E_i/E_{i-1} is polystable, since being the socle of a vector bundle.

Now we will prove (4) implies (5). As $c_2(E_i) = c_2(E_{i-1} \oplus E_i/E_{i-1}) = c_2(E_{i-1}) + c_1(E_{i-1})c_1(E_i/E_{i-1}) + c_2(E_i/E_{i-1})$, we have $c_2(E_i/E_{i-1}) = 0$. Since, c_1 is additive $c_1(E_i/E_{i-1}) = 0$. Therefore by Proposition 3.4.18, E_i/E_{i-1} admits real flat holomorphic connection. Now, finally Proposition 3.4.19 and induction on i, we get that E_i admits a real flat holomorphic connection for all $i \in \{0, \ldots, n\}$, In particular $E_n = E$ admits a real flat connection.

Finally (5) implies (1) is clearly true. □

The following is the real analogue of a special case of a result of Simpson [Sim92, Theorem, p. 39].

Corollary 3.4.22 *Let X be a real abelian variety, and E a real semistable holomorphic vector bundle over X, such that $c_1(E) = c_2(E) = 0$. Then, E is obtained by successive extensions of real stable holomorphic vector bundles with vanishing Chern classes.*

Proof. This follows from the implication (3) implies (4) of Theorem 3.4.21, since every real polystable vector bundle is obtained by successive extensions of real stable vector bundles. □

Appendix A

Category Theory

In this section we will recall some facts in category theory, which we are using in the first chapter.

A:1 Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and $\varphi : F \to G$ be a morphism of functors. Then, φ is an isomorphism of functors if and only if for all $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\varphi(X) : F(X) \to G(X)$ is an isomorphism of objects in $\mathcal{D}$. [Lan98, p. 16]

A:2 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then we get a new functor $\mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet)) : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ defined by the following assignments:

(a) For all $(Y, X) \in \mathrm{Ob}(\mathcal{D}^{\mathrm{op}} \times \mathcal{C})$

$$\mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet))(X, Y) = \mathrm{Hom}_{\mathcal{D}}(Y, F(X)).$$

(b) If (Y, X) and (Y', X') are objects in $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$, the function

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}} \times \mathcal{C}}((Y, X), (Y', X')) &\to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{D}}(Y, F(X)), \mathrm{Hom}_{\mathcal{D}}(Y', F(X'))) \\ (g^{\mathrm{op}}, f) &\mapsto \mathrm{Hom}(g, F(f)) \end{aligned}$$

is given by

$$\mathrm{Hom}(g, F(f))(u) = F(f) \circ u \circ g$$

for all morphisms $g : Y' \to Y$ in $\mathcal{D}$, $f : X \to X'$ in $\mathcal{C}$, and $u : Y \to F(X)$ in $\mathcal{D}$.

If $G : \mathcal{D} \to \mathcal{C}$ is a functor then, $\mathrm{Hom}_{\mathcal{C}}(G(\bullet), \bullet) : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ can be similarly defined.

A:3 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. We say that G *is a left adjoint of* F, or that F *is a right adjoint* *of* G, if there exists an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(G(\bullet), \bullet) \tag{3.21}$$

of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$; in that case, φ is called an *adjunction between* F *and* G.

A:4 By **A:1** the adjunction φ attaches to each object (Y, X) in $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$, a bijection of sets

$$\varphi(Y, X) : \mathrm{Hom}_{\mathcal{D}}(Y, F(X)) \to \mathrm{Hom}_{\mathcal{C}}(G(Y), X)$$

which is functorial in Y and X, i.e., if $g : Y' \to Y$ is a morphism in $\mathcal{D}$, and if $f : X \to X'$ is a morphism in $\mathcal{C}$, then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(Y, F(X)) \arrow[r, "\varphi(Y,X)"] \arrow[d, "\mathrm{Hom}(g,F(f))"'] & \mathrm{Hom}_{\mathcal{C}}(G(Y), X) \arrow[d, "\mathrm{Hom}(G(g),f)"] \\
\mathrm{Hom}_{\mathcal{D}}(Y', F(X')) \arrow[r, "\varphi(Y',X')"'] & \mathrm{Hom}_{\mathcal{C}}(G(Y'), X')
\end{tikzcd}$$

commutes.

A:5 Let $Y = F(X)$, substituting in Equation (3.21) we get,

$$\varphi(F(X), X) : \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \to \mathrm{Hom}_{\mathcal{C}}(G(F(X)), X).$$

Recall that $\varphi(X, F(X))(\mathbf{1}_{F(X)})$ is the *counit morphism of* X with respect to the adjunction φ. We will denote it by σ_X.

A:6 A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* (respectively, *full*, *fully faithful*) if for all $X, X' \in \mathrm{Ob}(\mathcal{C})$, the function

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(X'))$$

is injective (respectively, surjective, bijective). A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object Y in $\mathcal{D}$, there exists an object X in $\mathcal{C}$ such that $F(X)$ is isomorphic to Y. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathbf{1}_{\mathcal{C}}$ and $F \circ G \cong \mathbf{1}_{\mathcal{D}}$.

A:7 A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

A:8 Let the notation be as in (**A:4**). For every object Y in $\mathcal{D}$, we define the *unit morphism of* Y with respect to the adjunction φ, to be the morphism $\rho_Y : Y \to F(G(Y))$ in $\mathcal{D}$ such that

$$\varphi_{Y,G(Y)}(\rho_Y) = \mathbf{1}_{G(Y)}.$$

A:9 Let $\mathcal{C}$ be a category. Then for every object $X \in \mathcal{C}$, we get the *functor* Hom,

$$\mathrm{Hom}(X, \bullet) : \mathcal{C} \to \mathbf{Set},$$

which is defined by the following assignments:

(a) $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathbf{Set})$, $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(b) If $Y, Y' \in \mathrm{Ob}(\mathcal{C})$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Y, Y') &\to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(X, Y), \mathrm{Hom}_{\mathcal{C}}(X, Y')) \\ u &\mapsto \mathrm{Hom}(X, u), \end{aligned}$$

where, for $u \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ and $w \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have

$$\mathrm{Hom}(X, u)(w) = u \circ w.$$

It is easy to verify that $\mathrm{Hom}(X, \bullet) : \mathcal{C} \to \mathbf{Set}$ is indeed a functor.

A:10 Let $\mathcal{C}$ be a category. We say that a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if there exists an object $X \in \mathrm{Ob}(\mathcal{C})$ such that $F \cong \mathrm{Hom}(X, \bullet)$. An object $X \in \mathrm{Ob}(\mathcal{C})$ is called a *representing object for* F; we also say that F *is representable by* X. By Yoneda embedding, a representing object is unique up to a canonical isomorphism.

A:11 Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. Let $X \in \mathrm{Ob}(\mathcal{C})$. An element $\xi \in F(X)$ is called a *universal element* for F if it satisfies the following condition: for every object $Y \in \mathrm{Ob}(\mathcal{C})$ and for every element $\mu \in F(Y)$, there exists a unique morphism $w : X \to Y$ in $\mathcal{C}$ such that $\mu = F(w)(\xi)$.

A:12 If a functor $F : \mathcal{C} \to \mathbf{Set}$ is representable by X, then there exist a universal element $\xi \in F(X)$ for F. Then, a *representation of* F is a pair (X, ξ), where $X \in \mathrm{Ob}(\mathcal{C})$, and $\xi \in F(X)$ is a universal element for F.

Appendix B

After reading the work presented in Chapter 2 of this thesis, Professor J. Oesterlé kindly pointed out that hypothesis in Theorem 2.2.13, of the existence of an admissible category can be replaced by the weaker hypothesis, that locally free sheaves of bounded rank are acyclic. We given below a proof of this remark.

Theorem B.1 *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $A = \Gamma(X, \mathcal{O}_X)$ its ring of global sections. Assume that each locally free $\mathcal{O}_X$-module of bounded rank is acyclic, and generated by finitely many global sections. Then the functor $\mathcal{S}$ defines an equivalence of categories from the category $\mathbf{Fgp}(A)$ to the category $\mathbf{Lfb}(X)$. A quasi-inverse is the canonical functor $\Gamma(X, \bullet)$.*

Remark B.2 If P is a finitely generated projective A-module, then $\mathcal{S}(P) = \mathcal{P}(P)$, that is, $\mathcal{P}(P)$ is a sheaf. (Recall from Subsection 2.1.2 that, $\mathcal{P}(P)(U)$ is defined by $P \otimes_A \mathcal{O}_X(U)$ for every open subset U of X.) This follows from the fact that, the tensor product by P over A commutes with projective limits, since P is finitely generated projective module. Hence, the canonical A-linear map $h \mapsto h \otimes \mathbf{1}_A$ from P to $\Gamma(X, \mathcal{S}(P))$ is an isomorphism.

Proposition B.3 *The functor $\mathcal{S} : \mathbf{Fgp}(A) \to \mathbf{Lfb}(X)$ is fully faithful.*

Proof. By Lemma 2.2.12, if P is in $\mathbf{Fgp}(A)$ then $\mathcal{S}(P)$ is a locally free $\mathcal{O}_X$-module of bounded rank. Let P and Q be two finitely generated projective A-modules. The map $S : \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(P), \mathcal{S}(Q))$, $u \mapsto \mathcal{S}(u)$ is obtained by composing the two canonical isomorphisms

$$\mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(P, \Gamma(X, \mathcal{S}(Q))) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S}(P), \mathcal{S}(Q))$$

(second isomorphism is given by the adjunction (2.4)). □

Lemma B.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be two locally free $\mathcal{O}_X$-modules of bounded rank. If $u : \mathcal{F} \to \mathcal{G}$ is a surjective homomorphism,* $\ker(u)$ *is a locally free $\mathcal{O}_X$-module of bounded rank.*

Proof. We have the exact sequence $0 \to \ker(u) \to \mathcal{F} \xrightarrow{u} \mathcal{G} \to 0$. By choosing local frames for $\mathcal{F}$ and $\mathcal{G}$, we can see that the above exact sequence locally splits. Hence, $\ker(u)$ is of finite type. By Corollary 2.1.19, the rank of the sheaf $\ker(u)$ is locally constant, hence is locally free of bounded rank (Proposition 2.1.39). $\square$

Proof of Theorem B.1. We already know by Proposition B.3 and Remark B.2 that, the functor $\mathcal{S}$ is fully faithful, and that $\Gamma(X, \mathcal{S}(P))$ is canonically isomorphic to P when P is a finitely generated projective A-module. Hence, it is suffices to show that the functor $\mathcal{S}$ is essentially surjective, that is, each locally free $\mathcal{O}_X$-module $\mathcal{F}$ of bounded rank is isomorphic to $\mathcal{S}(P)$ for some P in $\mathbf{Fgp}(A)$.

Since $\mathcal{F}$ is finitely generated by global sections, there exists a surjective morphism $u : \mathcal{O}_X^n \to \mathcal{F}$ for some $n \in \mathbf{N}$. Hence, by Lemma B.4, $\ker(u)$ is in $\mathbf{Lfb}(X)$, so is acyclic. Thus, the map $\Gamma(X, u) : A^n \to \Gamma(X, \mathcal{F})$ is surjective.

For all $x \in X$, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}^n) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{F}_x) \to 0$ is surjective, this follows from the fact that $\mathcal{F}_x$ is free. Since $\mathcal{F}$ is of finite presentation by Proposition 2.1.23 $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n))_x \to (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}))_x \to 0$ is exact, that is, $\mathcal{H}om(\mathcal{F}, u)$ is a surjective homomorphism between $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. Also we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ are locally free sheaves of bounded rank. It follows as before that the homomorphism

$$\mathrm{Hom}(\mathcal{F}, u) : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is surjective (deduced from the previous one by passing to the global section is surjective). Hence there exists $v : \mathcal{F} \to \mathcal{O}_X^n$ such that $u \circ v = \mathbf{1}_{\mathcal{F}}$. We then have $\Gamma(X, u) \circ \Gamma(X, v) = \mathbf{1}_{\Gamma(X,\mathcal{F})}$. This prove that $\Gamma(X, u)$ has a section, and therefore $P = \Gamma(X, \mathcal{F})$ is a projective A-module of finite rank.

Let $w : \mathcal{S}(P) \to \mathcal{F}$ be such that $\lambda_{\mathcal{F},\Gamma(X,\mathcal{F})}(\mathbf{1}_{\Gamma(X,\mathcal{F})}) = w$ (that is, w is the counit morphism of $\mathcal{F}$ with respect to the adjunction λ (2.4)). Then w is surjective, since $\mathcal{F}$ is generated by global section. (See the proof of Proposition 2.2.4.) By Lemma B.4, $\mathcal{G} = \ker(w)$ is a locally free $\mathcal{O}_X$-module of bounded rank. Since $\Gamma(X, \bullet)$ is left exact we get that $\ker(\Gamma(X, w)) = \Gamma(X, \mathcal{G})$. But $\Gamma(X, w) : \Gamma(X, \mathcal{F}) \otimes A \to \Gamma(X, \mathcal{F})$ is by definition $\mathbf{1}_{\Gamma(X,\mathcal{F})} \otimes \mathbf{1}_A$, hence an isomorphism. Thus, $\ker(\Gamma(X, w)) = 0 = \Gamma(X, \mathcal{G})$. But by hypothesis $\mathcal{G}$ is generated by global section, hence $\mathcal{G} = 0$. This proves that $\mathcal{F}$ is isomorphic to $\mathcal{S}(P)$. $\square$

Remark B.5 In Theorem B.1, we can replace the assumption that every object in $\mathbf{Lfb}(X)$ is acyclic, with the weaker hypothesis that for every surjective homomorphism $u : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ belongs to $\mathbf{Lfb}(X)$, the homomorphism $\Gamma(X, u) : \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is surjective. That this hypothesis is indeed weaker follows from Lemma B.4

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List of Errata in the Thesis

The following is a list of corrections to the thesis: most of them are typographical. The symbol p/-l indicates the l-th line from the bottom of the page p of the thesis.

Page/line	**Read**	**Instead of**
22/-8	U	V
28/3,8	$\mathcal{F}$	$\mathcal{G}$
38/13	u_f	u
41/-5	$\mathcal{O}_X\|_{\mathrm{D}(f_i)} = (\tilde{A}_{f_i})$	$X\|_{\mathrm{D}(f_i)} = (\tilde{A}_{f_i})$
53/6	(refer Subsection 3.1.1	
	for the definition of real $\mathcal{O}_X$-module)	
58/1	the inclusion morphism $i : \mathcal{G} \to \mathcal{F}$ is	$i : \mathcal{G} \to \mathcal{F}$
61/9	$\tilde{\sigma}'_U(s) = \overline{\tilde{\sigma}_U(s)}$	$\tilde{\sigma}'_U(s) = \overline{\tilde{\sigma}(s)}$
69/9	$(-1)^{p+q}$	$(-1)^{p+r}$
69/1	$A^{p+q}(E)$	$A^r(E)$
75/-12,-8	$\tau_x^*(E)$	$\tau^*(E)$
79/-7	$\int_X \sigma^*(c_1(\bar{E})) \wedge (-\sigma^*(\Phi))^{n-1}$	$\int_X \sigma^*(c_1(\bar{E})) \wedge (-\sigma^*(\Phi^{n-1})$
85/3	$i\bar{\partial}_S(s) + i\beta(q) + l\bar{\partial}_Q(q)$	$i\bar{\partial}_S(s) + i\beta(q) + l\bar{\partial}_Q(s)$