Cohomology of Orbit Spaces, Fixed Point Sets of Group Actions, and Parametrized Borsuk-Ulam Problem

By

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Certificate

This is to certify that the Ph.D. thesis titled "**Cohomology of orbit spaces, fixed point sets of group actions, and parametrized Borsuk-Ulam problem**" by **Mahender Singh** is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent and original work by the candidate.

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Abstract

The major part of this thesis is devoted to determining cohomology of orbit spaces and fixed point sets of certain compact transformation groups on finitistic spaces. Equivariant maps are also studied and some parametrized Borsuk-Ulam theorems are proved.

Chapter 1 contains some basic definitions and results in the theory of topological transformation groups and spectral sequences required for our work.

In Chapter 2, free involutions on finitistic mod 2 cohomology lens spaces are studied. The possible mod 2 cohomology algebra of orbit space of any free involution on a finitistic mod 2 cohomology lens space is completely determined. As an application, it is shown that if X is a finitistic mod 2 cohomology lens space of dimension $2m-1$ with $m \geq 3$, then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2m$, where $\mathbb{S}^n$ is equipped with the antipodal involution.

In Chapter 3, some parametrized Borsuk-Ulam theorems are proved for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with free involutions.

In Chapter 4, involutions on finitistic spaces having mod 2 cohomology algebra of the wedge sum $S^n \vee S^{2n} \vee S^{3n}$ or $P^2(n) \vee S^{3n}$ are studied and the possible fixed point sets are determined up to mod 2 cohomology. Also examples realizing the possible cases are given.

In Chapter 5, a similar problem is considered for $\mathbb{S}^1$ actions on finitistic spaces having rational cohomology algebra of the wedge sum $S^n \vee S^{2n} \vee S^{3n}$ or $P^2(n) \vee S^{3n}$. The possible fixed point sets are determined up to rational cohomology and examples realizing the possible cases are given.

Chapter 6 contains some miscellaneous results obtained during our study.

Synopsis

1. Introduction

The theory of topological transformation groups deals with the symmetries of topological spaces. More precisely, it is the study of group actions on topological spaces. An action of a group on a space gives rise to two associated spaces, namely, the fixed point set and the orbit space. One of the basic problems of transformation groups is to determine these two associated spaces, either precisely or up to (co)homology. The pioneering result of Smith [65] on fixed point sets up to homology of prime periodic maps on homology spheres was the first in this direction. More explicit relations between the space, the fixed point set and the orbit space were obtained by Floyd [23, 24]. Bredon [6] contains an excellent account of results in this direction for the large class of *finitistic* spaces. This class of spaces was introduced by Swan [75] in his study of fixed point theory. It is a large class of spaces including all compact Hausdorff spaces and paracompact spaces of finite covering dimension. Due to Deo-Singh-Tripathi [14, 16], it is known that if X is a topological space and G is a compact Lie group acting continuously on X, then X is finitistic if and only if the orbit space X/G is finitistic. The major part of our work [61, 62, 64] is concerned with determining the cohomology of orbit spaces and fixed point sets of certain compact transformation groups on finitistic spaces.

Another direction of work in the theory of topological transformation groups is the study of equivariant maps. The well known Borsuk-Ulam theorem states that, if $n > k$ then there does not exist any $\mathbb{Z}_2$-equivariant map $\mathbb{S}^n \to \mathbb{S}^k$, where $\mathbb{S}^n$ and $\mathbb{S}^k$ are equipped with antipodal involutions. Over the years there have been several generalizations of the theorem in many directions (see for example [72, 42]). Jaworowski [30], Dold [19], Nakaoka

[51] and others extended this theorem to the setting of fiber bundles, by considering fiber preserving maps $f : SE \to E'$, where SE denotes the total space of the sphere bundle $SE \to B$ associated to a vector bundle $E \to B$, and $E' \to B$ is other vector bundle. Thus they parametrized the Borsuk-Ulam theorem. A part of our work [63] proves some parametrized Borsuk-Ulam theorems for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with free involutions.

The basic setting for our approach in the thesis is the *equivariant cohomology theory* introduced by Borel [10]. Let G be a compact Lie group and X be a G-space. This gives the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$, where $X_G = (X \times E_G)/G$ is the orbit space of the diagonal action on $X \times E_G$ and B_G is the base space of the universal principal G-bundle $G \hookrightarrow E_G \longrightarrow B_G$. Then the equivariant cohomology of X is defined to be any fixed cohomology (Čech cohomology in our case) of the total space X_G of the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$. By the work of Leray [40], there is a first quadrant spectral sequence of algebras $\{{E_r}^{*,*}, d_r\}$, converging to $H^*(X_G)$ as an algebra, with ${E_2}^{k,l} = H^k(B_G; \mathcal{H}^l(X))$, the cohomology of the base B_G with locally constant coefficients $\mathcal{H}^l(X)$ twisted by a canonical action of $\pi_1(B_G)$. This spectral sequence can be used to compute the equivariant cohomology of X and we exploit it heavily in our work. Throughout we use Čech cohomology, since it is found to be most compatible with the cohomology theory of topological transformation groups and the cohomological dimension theory.

Let X, Y be topological spaces and let p be a prime. By $X \simeq_p Y$ we mean that there is an isomorphism of graded algebras $H^*(X; \mathbb{Z}_p) \cong H^*(Y; \mathbb{Z}_p)$. Similarly, by $X \simeq_{\mathbb{Q}} Y$ we mean that there is an isomorphism of graded algebras $H^*(X; \mathbb{Q}) \cong H^*(Y; \mathbb{Q})$. With these notations, the chapter-wise details of the thesis are given as follows.

2. Cohomology algebra of orbit spaces of free involutions on lens spaces

An involution on a topological space X is an action of the group $G = \mathbb{Z}_2$ on X. In this chapter, we study cohomology of orbit spaces of free involutions on cohomology lens spaces. Lens spaces are odd dimensional spherical space forms described as follows. Let $p \geq 2$ be an integer and $q_1, q_2, ..., q_m$ be integers coprime to p, where $m \geq 1$. Let $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$ be the unit sphere and let $\iota^2 = -1$. Then

$$(z_1, ..., z_m) \mapsto (e^{\frac{2\pi\iota q_1}{p}} z_1, ..., e^{\frac{2\pi\iota q_m}{p}} z_m)$$

defines a free action of the cyclic group $\mathbb{Z}_p$ on $\mathbb{S}^{2m-1}$. The orbit space is called *Lens* space and is denoted by $L_p^{2m-1}(q_1, ..., q_m)$. It is a compact Hausdorff orientable manifold of dimension $(2m-1)$ and is finitistic.

Involutions on lens spaces have been studied in detail, particularly on 3-dimensional lens spaces [26, 33, 35, 36, 37, 49]. Kim [35] showed that if $p = 4k$ for some k, then the orbit spaces of any sense-preserving free involution on $L_p^3(1, q)$ is the lens space $L_{2p}^3(1, q^{'})$, where $q^{'}q \equiv \pm 1$ or $q^{'} \equiv \pm q$ mod p. Myers in [49] showed that every free involution on a 3-dimensional lens space is conjugate to an orthogonal free involution, in which case the orbit space is again a lens space.

Let $X \simeq_2 L_p^{2m-1}(q_1, ..., q_m)$ be a mod 2 cohomology lens space. Motivated by the work of Kim and Myers, we consider free involutions on finitistic mod 2 cohomology lens spaces and determine the possible mod 2 cohomology algebra of orbit space, using the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_{\mathbb{Z}_2} \longrightarrow B_{\mathbb{Z}_2}$. More precisely, if X/G denotes the orbit space, then we prove the following theorem.

Theorem. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q_1, ..., q_m)$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded commutative algebras:*

1. $\mathbb{Z}_2[x]/\langle x^{2m}\rangle$,

   where $deg(x) = 1$.

2. $\mathbb{Z}_2[x, y]/\langle x^2, y^m\rangle$,

   where $deg(x) = 1$ *and* $deg(y) = 2$.

3. $\mathbb{Z}_2[x, y, z]/\langle x^3, y^2, z^{\frac{m}{2}}\rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ *and* m *is even.*

4. $\mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^{\frac{m}{2}}, x^2y\rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ *and* m *is even.*

5. $\mathbb{Z}_2[x, y, w, z]/\langle x^5, y^2, w^2, z^{\frac{m}{4}}, x^2y, wy\rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(w) = 3$, $deg(z) = 8$ *and* $4 \mid m$.

Our theorem generalizes the results [35, 49] known for orbit spaces of free involutions on 3-dimensional lens spaces, to that of the large class of finitistic spaces $X \simeq_2 L_p^{2m-1}(q_1, ..., q_m)$.

Application to $\mathbb{Z}_2$-equivariant maps

Let $\mathbb{S}^n$ be the unit n-sphere equipped with the antipodal involution and let X be a paracompact Hausdorff space with a fixed free involution. The index of the involution on X is defined as

$$\text{ind}(X) = max\ \{\ n \mid \text{there is a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^n \to X\}.$$

There is a rich literature on index theory and a survey can be found in Conner-Floyd [13]. Using the notion of index and our theorem, we prove the following.

Theorem. *Let $m \geq 3$ and $X \simeq_2 L_p^{2m-1}(q_1, ..., q_m)$ be a finitistic space with a free involution. Then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2m$.*

3. Parametrized Borsuk-Ulam problem for projective space bundles

The unit n-sphere $\mathbb{S}^n$ is equipped with the antipodal involution given by $x \mapsto -x$. One formulation of the classical Borsuk-Ulam theorem states that, if $n \geq k$ then for every continuous map $f : \mathbb{S}^n \to \mathbb{R}^k$ there exist a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. Over the years there have been several generalizations of this theorem in many directions (see for example [72, 42]). Jaworowski [30, 31, 32], Dold [19], Nakaoka [51] and others extended this theorem to the setting of fiber bundles, by considering fiber preserving maps $f : SE \to E^{'}$, where SE denotes the total space of the sphere bundle $SE \to B$ associated to a vector bundle $E \to B$, and $E^{'} \to B$ is other vector bundle. Thus, they parametrized the Borsuk-Ulam theorem, whose general formulation is as follows.

Let G be a compact Lie group. Consider a fiber bundle $\pi : E \to B$ and a vector bundle $\pi^{'} : E^{'} \to B$ such that G acts fiber preserving and freely on E and $E^{'} - 0$, where 0 stands for the zero section of the bundle $\pi^{'} : E^{'} \to B$. For a fiber preserving G-equivariant map $f : E \to E^{'}$, the parametrized version of the Borsuk-Ulam theorem deals in estimating the cohomological dimension of the set $Z_f = \{x \in E \mid f(x) = 0\}$.

We refer to cohomological dimension in the sense of Nagami [50]. Dold [19] and Nakaoka [51] defined certain polynomials, which they called the characteristic polynomials, for vector bundles with free G-actions ($G = \mathbb{Z}_p$ or $\mathbb{S}^1$) and used them for obtaining such results. In this chapter, we use this technique to prove some parametrized Borsuk-Ulam theorems for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with any free involution. As an application, we also estimate the size of the $\mathbb{Z}_2$-coincidence set of a fiber preserving map.

Characteristic polynomials for bundles

Let (X, E, π, B) be a fiber bundle with a fiber preserving free $\mathbb{Z}_2$ action such that the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$ has a cohomology extension of the fiber and let $\pi^{'} : E^{'} \to B$ be a k-dimensional vector bundle with a fiber preserving $\mathbb{Z}_2$-action on $E^{'}$ which is free on $E^{'} - 0$, where 0 stands for the zero section. We first obtain the characteristic polynomials associated to the bundles.

Characteristic polynomials for $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$

$$W_1(x, y) = w_{n+1} + w_n x + w_{n-1} y + \cdots + w_2 y^{\frac{n-1}{2}} + w_1 x y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \quad W_2(x, y) = \nu_2 + \nu_1 x + \alpha y + x^2,$$

where x and y are indeterminates of degrees 1 and 2 respectively. We obtain a natural isomorphism of $H^*(B)$-algebras

$$H^*(B)[x, y]/\langle W_1(x, y), W_2(x, y)\rangle \cong H^*(\overline{E}).$$

Characteristic polynomials for $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$

$$W_1(x, y) = w_{2n+2} + w_{2n+1} x + w_{2n} x^2 + \cdots + w_2 x^2 y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \quad W_2(x) = \nu_3 + \nu_2 x + \nu_1 x^2 + x^3,$$

where x and y are indeterminates of degrees 1 and 4 respectively. In this case also we obtain a natural isomorphism of $H^*(B)$-algebras

$$H^*(B)[x,y]/\langle W_1(x,y), W_2(x)\rangle \cong H^*(\overline{E}).$$

Characteristic polynomial for $\pi' : E' \to B$

$$W'(x) = w'_k + w'_{k-1}x + \cdots + w'_1 x^{k-1} + x^k,$$

where x is an indeterminate of degree 1. Just as above, we obtain an isomorphism of $H^*(B)$-algebras

$$H^*(B)[x]/\langle W'(x)\rangle \cong H^*(\overline{SE'}).$$

Parametrized Borsuk-Ulam theorems

Let $f : E \to E'$ be a fiber preserving $\mathbb{Z}_2$-equivariant map. Define $Z_f = f^{-1}(0)$ and $\overline{Z_f} = Z_f/\mathbb{Z}_2$, the quotient by the free $\mathbb{Z}_2$-action induced on Z_f.

Each polynomial $q(x,y)$ in $H^*(B)[x,y]$ defines an element of $H^*(\overline{E})$, which we denote by $q(x,y)|_{\overline{E}}$. We denote by $q(x,y)|_{\overline{Z_f}}$ the image of $q(x,y)|_{\overline{E}}$ under the $H^*(B)$- homomorphism $i^* : H^*(\overline{E}) \to H^*(\overline{Z_f})$, where i^* is the map induced by the inclusion $i : \overline{Z_f} \hookrightarrow \overline{E}$. For the real case we prove the following.

Theorem. *Let $X \simeq_2 \mathbb{R}P^n$ be a finitistic space. If $q(x,y)$ in $H^*(B)[x,y]$ is a polynomial such that $q(x,y)|_{\overline{Z_f}} = 0$, then there are polynomials $r_1(x,y)$ and $r_2(x,y)$ in $H^*(B)[x,y]$ such that $q(x,y)W'(x) = r_1(x,y)W_1(x,y) + r_2(x,y)W_2(x,y)$ in the ring $H^*(B)[x,y]$, where $W'(x)$, $W_1(x,y)$ and $W_2(x,y)$ are the characteristic polynomials.*

As a corollary we have the following parametrized version of the Borsuk-Ulam theorem.

Corollary. *Let $X \simeq_2 \mathbb{R}P^n$ be a finitistic space. If the fiber dimension of $E^{'} \to B$ is k, then $q(x,y)|_{\overline{Z_f}} \neq 0$ for all non-zero polynomials $q(x,y)$ in $H^*(B)[x,y]$, whose degree in x and y is less than $(n-k+1)$. Equivalently, the $H^*(B)$-homomorphism*

$$\bigoplus_{i+j=0}^{n-k} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n-k).$$

Similarly, for the complex case we prove the following.

Theorem. *Let $X \simeq_2 \mathbb{C}P^n$ be a finitistic space. If $q(x,y)$ in $H^*(B)[x,y]$ is a polynomial such that $q(x,y)|_{\overline{Z_f}} = 0$, then there are polynomials $r_1(x,y)$ and $r_2(x,y)$ in $H^*(B)[x,y]$ such that $q(x,y)W^{'}(x) = r_1(x,y)W_1(x,y) + r_2(x,y)W_2(x)$ in the ring $H^*(B)[x,y]$, where $W^{'}(x)$, $W_1(x,y)$ and $W_2(x)$ are the characteristic polynomials.*

Corollary. *Let $X \simeq_2 \mathbb{C}P^n$ be a finitistic space. If the fiber dimension of $E^{'} \to B$ is k, then $q(x,y)|_{\overline{Z_f}} \neq 0$ for all non-zero polynomials $q(x,y)$ in $H^*(B)[x,y]$, whose degree in x and y is less than $(2n-k+2)$. Equivalently, the $H^*(B)$-homomorphism*

$$\bigoplus_{i+j=0}^{2n-k+1} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ *and* $y^j \to y^j|_{\overline{Z_f}}$ *is a monomorphism. As a result, if* $2n \geq k$, *then*

$$cohom.dim(Z_f) \geq cohom.dim(B) + (2n - k + 1).$$

Application to $\mathbb{Z}_2$-coincidence sets

Let (X, E, π, B) be a fiber bundle with the hypothesis as above. Let $E'' \to B$ be a k-dimensional vector bundle and let $f : E \to E''$ be a fiber preserving map. Here we do not assume that E'' has an involution. Even if E'' has an involution, f is not assumed to be $\mathbb{Z}_2$-equivariant. If $T : E \to E$ is a generator of the $\mathbb{Z}_2$ action, then the $\mathbb{Z}_2$-coincidence set of f is defined as

$$A(f) = \{x \in E \mid f(x) = f(T(x))\}.$$

As an application of our results, we have the following theorems.

Theorem. *If* $X \simeq_2 \mathbb{R}P^n$ *is a finitistic space, then*

$$cohom.dimA(f) \geq cohom.dim(B) + (n - k).$$

Theorem. *If* $X \simeq_2 \mathbb{C}P^n$ *is a finitistic space, then*

$$cohom.dimA(f) \geq cohom.dim(B) + (2n - k + 1).$$

4. Fixed point sets of involutions on spaces of type $(a, 0)$

In this chapter, we investigate the fixed point sets of involutions on certain types of spaces first studied by Toda [77]. Toda studied the cohomology algebra of a space X having only non-trivial cohomology groups $H^{in}(X;\mathbb{Z}) = \mathbb{Z}$ for $i = 0$, 1, 2 and 3, where n is a fixed positive integer. Let $u_i \in H^{in}(X;\mathbb{Z})$ be a generator for $i = 1$, 2 and 3. Then the ring structure of $H^*(X;\mathbb{Z})$ is completely determined by the integers a and b such that

$$u_1^2 = au_2 \quad \text{and} \quad u_1u_2 = bu_3.$$

Such a space is said to be of type (a, b). For a prime p, let $X \simeq_p P^h(n)$ mean that there is an isomorphism of graded algebras $H^*(X;\mathbb{Z}_p) \cong \mathbb{Z}_p[z]/z^{h+1}$, where z is a homogeneous element of degree n.

For spaces X and Y, let $X \vee Y$ denote their wedge sum and let $X \sqcup Y$ denote their disjoint union. One can see that a space X of type (a, b) is determined by the integers a and b in terms of the familiar spaces as follows. If $b \not\equiv 0 \bmod p$, then

$$X \simeq_p S^n \times S^{2n} \quad \text{for} \quad a \equiv 0 \bmod p$$

or

$$X \simeq_p P^3(n) \quad \text{for} \quad a \not\equiv 0 \bmod p.$$

And, if $b \equiv 0 \bmod p$, then

$$X \simeq_p S^n \vee S^{2n} \vee S^{3n} \quad \text{for} \quad a \equiv 0 \bmod p$$

or

$$X \simeq_p P^2(n) \vee S^{3n} \quad \text{for} \quad a \not\equiv 0 \bmod p.$$

Let X be a G-space and let $X \hookrightarrow X_G \longrightarrow B_G$ be the associated Borel fibration as defined earlier. We say that X is totally non-homologous to zero in X_G with respect to $\mathbb{Z}_p$ if the inclusion of a typical fiber $X \hookrightarrow X_G$ induces a surjection in the cohomology $H^*(X_G; \mathbb{Z}_p) \longrightarrow H^*(X; \mathbb{Z}_p)$.

The cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \not\equiv 0$ mod p has been investigated in detail by Bredon [5, 6] and Su [73, 74] for all primes p. And the cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \equiv 0$ mod p has been completely determined by Dotzel and Singh [20, 21] for odd primes p. In [64] we settle the remaining case of $b \equiv 0$ mod 2 and investigate the fixed point sets of involutions on finitistic spaces of type $(a, 0)$ mod 2. More precisely, we prove the following results.

Theorem. *Let $G = \mathbb{Z}_2$ act on a finitistic space X of type $(a, 0)$ mod 2 with trivial action on $H^*(X; \mathbb{Q})$ and with fixed point set F. Suppose X is totally non-homologous to zero in X_G, then F has at most four components satisfying the following:*

1. *If F has four components, then each is acyclic, n is even and $a \equiv 0$ mod 2.*
2. *If F has three components, then n is even and*

$$F \simeq_2 S^r \sqcup \{point_1\} \sqcup \{point_2\} \text{ for some even integer } 2 \leq r \leq 3n.$$

3. *If F has two components, then either*

$$F \simeq_2 S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for some integers } 1 \leq r, s \leq 3n$$

or

$$F \simeq_2 P^2(r) \sqcup \{point\} \text{ for some even integer } 2 \leq r \leq n.$$

4. *If F has one component, then either*

$$F \simeq_2 S^r \vee S^s \vee S^t \text{ for some integers } 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_2 S^s \vee P^2(r) \text{ for some integers } 1 \leq r \leq n \text{ and } 1 \leq s \leq 3n.$$

Moreover, if n is even, then X is always totally non-homologous to zero in X_G. Further, all the cases are realizable.

Theorem. *Let $G = \mathbb{Z}_2$ act on a finitistic space X of type $(a, 0)$ mod 2 with trivial action on $H^*(X; \mathbb{Q})$ and with fixed point set F. Suppose X is not totally non-homologous to zero in X_G, then either $F = \phi$ or $F \simeq_2 S^r$, where $1 \leq r \leq 3n$ is an odd integer. Moreover, the second possibility is realizable.*

5. Fixed point sets of circle actions on spaces of type $(a, 0)$

In this chapter, we study fixed point sets of circle actions on rational cohomology finitistic spaces of type $(a, 0)$. By $X \simeq_{\mathbb{Q}} P^h(n)$ we mean that $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[z]/z^{h+1}$, where z is a homogeneous element of degree n.

It is clear that if $b \neq 0$, then

$$X \simeq_{\mathbb{Q}} S^n \times S^{2n} \text{ for } a = 0$$

or

$$X \simeq_{\mathbb{Q}} P^3(n) \text{ for } a \neq 0.$$

And, if $b = 0$, then

$$X \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n} \text{ for } a = 0$$

or

$$X \simeq_{\mathbb{Q}} P^2(n) \vee S^{3n} \text{ for } a \neq 0.$$

Let X be a G-space and let $X \hookrightarrow X_G \longrightarrow B_G$ be the associated Borel fibration as defined earlier. We say that X is totally non-homologous to zero in X_G with respect to $\mathbb{Q}$ if the inclusion of a typical fiber $X \hookrightarrow X_G$ induces a surjection in the cohomology $H^*(X_G; \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$.

The cohomological nature of the fixed point sets of actions of the cyclic group $\mathbb{Z}_p$ of prime order p on spaces of type (a, b) has been studied in detail [5, 6, 20, 21, 64, 73, 74].

For $b \neq 0$, the cohomological nature of the fixed point sets of $\mathbb{S}^1$ actions has been studied in detail by Bredon [5, 6]. In [62] we study $\mathbb{S}^1$ actions on rational cohomology finitistic spaces of type $(a, 0)$ and determine the possible fixed point sets up to rational cohomology. More presisely, we prove the following results.

Theorem. *Let $G = \mathbb{S}^1$ act on a rational cohomology finitistic space X of type $(a, 0)$ with fixed point set F. Suppose X is totally non-homologous to zero in X_G, then F has at most four components satisfying the following:*

1. *If F has four components, then each is acyclic and n is even.*
2. *If F has three components, then n is even and*

$$F \simeq_{\mathbb{Q}} S^r \sqcup \{point_1\} \sqcup \{point_2\} \text{ for some even integer } 2 \leq r \leq 3n.$$

3. *If F has two components, then either*

$$F \simeq_{\mathbb{Q}} S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for some integers } 1 \leq r, s \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} P^2(r) \sqcup \{point\} \text{ for some even integer } 2 \leq r \leq n.$$

4. *If F has one component, then either*

$$F \simeq_{\mathbb{Q}} S^r \vee S^s \vee S^t \text{ for some integers } 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} S^s \vee P^2(r) \text{ for some integers } 1 \leq r \leq n \text{ and } 1 \leq s \leq 3n.$$

Moreover, if n is even, then X is always totally non-homologous to zero in X_G. Further, all the cases are realizable.

Theorem. *Let $G = \mathbb{S}^1$ act on a rational cohomology finitistic space X of type $(a, 0)$ with fixed point set F. Suppose X is not totally non-homologous to zero in X_G, then either $F = \phi$ or $F \simeq_{\mathbb{Q}} S^r$, where $1 \leq r \leq 3n$ is an odd integer. Moreover, the second possibility is realizable.*

6. Some miscellaneous results

In this chapter, we prove some miscellaneous results that we obtained during the course of our study.

Nice $\mathbb{Z}_p$ actions

Let G be a group acting on a space X. Then there is an induced action of G on the cohomology of X. This induced action is important in the cohomology theory of transformation groups. One can see that for any action of $\mathbb{S}^1$ on a space X whose rational cohomology is of finite type, the induced action on the rational cohomology is always trivial. This is, however, not true for actions of the cyclic group $\mathbb{Z}_p$ of prime order p, when the cohomology is taken with coefficients in the finite field $\mathbb{F}_p$. Sikora in [58] defined certain actions of $\mathbb{Z}_p$ which behave well on passing to mod p cohomology.

An action of $\mathbb{Z}_p$ on a $\mathbb{F}_p$-vector space N is said to be nice if $N = T \oplus F$ as $\mathbb{F}_p[\mathbb{Z}_p]$-module, where T is a trivial and F is a free $\mathbb{F}_p[\mathbb{Z}_p]$-module. In other words, $T = \bigoplus \mathbb{F}_p$ and $F = \bigoplus \mathbb{F}_p[\mathbb{Z}_p]$. We say that a $\mathbb{Z}_p$ action on a space X

is **nice** if the induced $\mathbb{Z}_p$ action on $H^n(X;\mathbb{F}_p)$ is nice for each $n \geq 0$. Note that, trivial actions are nice.

There is a $\mathbb{Z}_3$ action on $S^n \times S^n$, for $n = 1$, 3 or 7, which is not nice. However, every action of $\mathbb{Z}_3$ on S^n is nice. Thus, an arbitrary action of $\mathbb{Z}_p$ on $X \times Y$ need not be nice even if every action of $\mathbb{Z}_p$ on both X and Y is nice. If X and Y are G-spaces, then there is a G action on $X \times Y$ given by $\big(g,(x,y)\big) \mapsto (g.x, g.y)$, called the diagonal action. In this note, we show that the diagonal action is nice. More precisely, we prove the following.

Theorem. *If $\mathbb{Z}_p$ acts nicely on spaces X and Y of finite type, then the diagonal action on $X \times Y$ is also nice.*

Commutativity of inverse limit and orbit map

This note is motivated by the following example of Bredon [6, p.145]. Let S^2 be the 2-sphere identified with the unreduced suspension of the circle $S^1 = \{z \in \mathbb{C} \ ; \ |z| = 1\}$, and $f : S^2 \to S^2$ be the suspension of the map $S^1 \to S^1$, $z \mapsto z^3$. Then f commutes with the antipodal involution on S^2. If Σ is the inverse limit of the inverse system

$$\cdots \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2$$

then $\Sigma/\mathbb{Z}_2$ is homeomorphic to $\varprojlim \mathbb{R}P^2$.

In [60] we show that this can be generalized, that is, the inverse limit and the orbit map commute for actions of compact groups on compact Hausdorff spaces. The proof of the result is simple, but does not seem to be available in the literature. The result is as follows.

Theorem. *Let $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ be an inverse system of non-empty compact Hausdorff topological spaces and let $\{G_\alpha, \nu_\alpha^\beta, \Lambda\}$ be an inverse system of compact* *topological groups, where each X_α is a G_α-space and each bonding map π_α^β is ν_α^β-equivariant. Further, assume that Λ has the least element λ, G_λ action on X_λ is free and the bonding map ν_λ^α is injective for each $\alpha \in \Lambda$. Then, there is a natural homeomorphism*

$$\psi : (\varprojlim X_\alpha)/(\varprojlim G_\alpha) \to \varprojlim (X_\alpha / G_\alpha).$$

List of publications and preprints:

1. Mahender Singh, $\mathbb{Z}_2$ *actions on complexes with three non-trivial cells*,

   Topology and its Applications, 155 (2008), 965-971.

2. Mahender Singh, *Fixed points of circle actions on spaces with rational cohomology of* $S^n \vee S^{2n} \vee S^{3n}$ *or* $P^2(n) \vee S^{3n}$,

   Archiv der Mathematik, 92 (2009), 174-183.

3. Mahender Singh, *A note on the commutativity of inverse limit and orbit map*,

   Mathematica Slovaca, 61 (2011), 653- 656.

4. Mahender Singh, *Parametrized Borsuk-Ulam problem for projective space bundles*,

   Fundamenta Mathematicae, 211 (2011), 135-147.

5. Mahender Singh, *Cohomology algebra of orbit spaces of free involutions on lens spaces*,

   Journal of the Mathematical Society of Japan, 65 (2013), 1038-1061.

Dedicated

to

my family

Table of Contents

0 Introduction

0.1 History and overview

The theory of topological transformation groups deals with the symmetries of topological spaces. This theory was formalized in the year 1955 by D. Montgomery in his classic text *Topological Transformation Groups* [47] written jointly with L. Zippin. It is now a very active area of research expanding in many directions and interacting with many areas of mathematics including algebraic and differential topology, algebraic geometry and differential geometry. Historically, algebraic topology was first used in the study of topological transformation groups with the work of P. A. Smith [67, 68, 69] in the 1930s and 1940s and is often called as *Smith Theory*. Since the publication of [47], a number of new ideas and tools have been applied in dealing the problems of transformation groups. The Borel seminar [11] gives an account of the work done in this field before 1960 by leading mathematicians such as G. E. Bredon, E. E. Floyd, D. Montgomery and Borel himself. Later the subject received substantial clarification by the work of Atiyah and Segal [2, 56] in the equivariant K-theory and by the ideas implicit in the work of Borel, which were reformulated in the Localization Theorem proven independently by W. -Y. Hsiang [27] and D. G. Quillen [53] for actions of compact Lie groups on spaces of finite cohomological dimension. The theorem was later proved for actions of compact Lie groups on finitistic spaces by Deo-Singh-Shukla [15] and Allday-Puppe [1]. Important contributions, which can now be treated as the classical part of the subject initiated by P. A. Smith, have been made by A. Borel, G. E. Bredon, P. E. Conner, E. E. Floyd,

W. -Y. Hsiang, R. B. Oliver, D. G. Quillen, J. C. Su, T. Chang, T. Skjelbred and many others. The recent developments in this subject are due to A. Adem, C. Allday, W. Browder, G. Carlsson, D. H. Gottlieb, S. Halperin, V. Puppe and others. The text [6] by G. E. Bredon contains a comprehensive treatment of almost all aspects of the theory of compact transformation groups before 1972. Thereafter in 1975, the important text [27] by W. -Y. Hsiang appeared, in which the transformation groups were studied in terms of the geometric weight systems. Later, texts by T. tom Dieck [17], K. Kawakubo [34] and finally by C. Allday and V. Puppe [1] appeared. All these texts present the latest developments in the field.

0.2 Work done in thesis

Recall that, to an action of a group G on a space X, there are two associated spaces, namely, the fixed point set and the orbit space. It has always been one of the basic problems of transformation groups to determine these two associated spaces, either precisely or up to (co)homology. The pioneering result of Smith [67] on fixed point sets up to homology of prime periodic maps on homology spheres was the first in this direction. More explicit relations between the space, the fixed point set and the orbit space were obtained by Floyd [23, 24]. Bredon [6] contains an excellent account of results in this direction for the large class of *finitistic* spaces. This class of spaces was introduced by Swan [75] in his study of fixed point theory. It is a large class of spaces including all compact Hausdorff spaces and paracompact spaces of finite covering dimension. Due to Deo-Singh-Tripathi [14, 16], it is known that if X is a topological space and G is a compact Lie group acting continuously on X, then X is finitistic if and only if the orbit space X/G is finitistic. The major part of our work is concerned with determining the cohomology of orbit spaces and fixed point sets of certain compact transformation groups on finitistic spaces.

Our first problem is regarding the cohomology of orbit spaces of free involutions on cohomology lens spaces. Kim [35] showed that if $p = 4k$ for some k, then the orbit space of any sense-preserving free involution on $L^3_p(1, q)$ is the lens space $L^3_{2p}(1, q^{'})$, where $q^{'}q \equiv \pm 1$ or $q^{'} \equiv \pm q \pmod{p}$ and an involution is sense-preserving if the induced map on $H_1(L^3_p(1, q); \mathbb{Z})$ is the identity map. Myers in [49] showed that every free involution on a 3-dimensional lens space $L^3_p(1, q)$ is conjugate to an orthogonal free involution, in which case the orbit space is again a lens space. Motivated by these results, we consider free involutions on more general spaces, namely, finitistic mod 2 cohomology lens spaces. If $G = \mathbb{Z}_2$ acts freely on a finitistic mod 2 cohomology lens space X of dimension $2m-1$, then we determine completely the possible mod 2 cohomology algebra of the orbit space X/G.

Our second problem deals with the cohomology of fixed point sets of involutions on certain types of spaces first studied by Toda [77]. Toda studied the cohomology algebra of a space X having only non trivial cohomology groups $H^{in}(X; \mathbb{Z}) = \mathbb{Z}$ for $i = 0$, 1, 2 and 3, where n is a fixed positive integer. If $u_i \in H^{in}(X; \mathbb{Z})$ is a generator for $i = 1$, 2 and 3, then the ring structure of $H^*(X; \mathbb{Z})$ is completely determined by the integers a and b such that

$$u_1^2 = au_2 \text{ and } u_1u_2 = bu_3.$$

Such a space is said to be of type (a, b). Let p be a prime and $\mathbb{Z}_p$ be the cyclic group of order p. The cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \not\equiv 0$ mod p has been investigated in detail by Bredon [5, 6] and Su [73, 74] for all primes p. And the cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \equiv 0$ mod p has been completely determined by Dotzel and Singh [20, 21] for odd primes p. We study involutions on finitistic spaces of type $(a, 0)$ mod 2 and determine the possible fixed point sets up to mod 2 cohomology. We also give examples realizing the possible cases.

Our third problem deals with the cohomology of fixed point sets of $\mathbb{S}^1$ actions on spaces of type $(a, 0)$. For $b \neq 0$, the cohomological nature of the fixed point sets of $\mathbb{S}^1$

actions has been studied in detail by Bredon [5, 6]. In our work we study $\mathbb{S}^1$ actions on finitistic spaces of type $(a, 0)$ and determine the possible fixed point sets up to rational cohomology. Examples realizing the possible cases are also given.

Another direction of work in the theory of topological transformation groups is the study of equivariant maps. The well known Borsuk-Ulam theorem states that, if $n > k$ then there does not exist any $\mathbb{Z}_2$-equivariant map $S^n \to S^k$, where S^n and S^k are equipped with the antipodal involutions. Over the years there have been several generalizations of the theorem in many directions. Jaworowski [30], Dold [19], Nakaoka [51] and others extended this theorem to the setting of fiber bundles, by considering fiber preserving maps $f : SE \to E'$, where SE denotes the total space of the sphere bundle $SE \to B$ associated to a vector bundle $E \to B$ and $E' \to B$ is other vector bundle. Thus they parametrized the Borsuk-Ulam theorem.

Our fourth problem deals with the parametrized Borsuk-Ulam theorem. We prove parametrized Borsuk-Ulam theorems for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with free involutions.

The basic setting for our approach in this thesis is the *equivariant cohomology theory* introduced by Borel [9]. Let G be a compact Lie group and X be a G-space. Then the equivariant cohomology of the G-space X is defined to be any fixed cohomology (say Čech cohomology) of the total space X_G of the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$. By the work of Leray [40] (also see [44] for more details), there is a first quadrant spectral sequence of algebras $\{E_r{}^{*,*}, d_r\}$, converging to $H^*(X_G)$ as an algebra, with

$$E_2{}^{k,l} = H^k(B_G; \mathcal{H}^l(X)),$$

the cohomology of the base B_G with locally constant coefficients $\mathcal{H}^l(X)$ twisted by a canonical action of $\pi_1(B_G)$. This spectral sequence can be used to compute the equivariant cohomology of X and we exploit it heavily in our work. Throughout we use Čech cohomology, since it is found to be most compatible with the cohomology theory of topological transformation groups and the cohomological dimension theory.

0.3 Organization of thesis

The thesis is organized as follows.

In Chapter 1, we recall the necessary material for our work in subsequent chapters. Basic definitions and results in the theory of topological transformation groups including Smith theory and group actions on projective spaces are recorded. The main tool employed in our work is the Leray spectral sequence associated to a Borel fibration. We also develop the necessary background in the theory of spectral sequences.

In Chapter 2, we discuss our first research problem. We study free involutions on mod 2 cohomology lens spaces. We determine the possible mod 2 cohomology algebra of orbit space of any free involution on a finitistic mod 2 cohomology lens space X of dimension $2m-1$, using the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_{\mathbb{Z}_2} \longrightarrow B_{\mathbb{Z}_2}$. As an application, we show that if X is a finitistic mod 2 cohomology lens space of dimension $2m-1$ with $m \geq 3$, then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2m$, where $\mathbb{S}^n$ is equipped with the antipodal involution. This work was presented by the author in the HRI International Conference in Mathematics, Allahabad, March 16-20, 2009.

In Chapter 3, we prove some parametrized Borsuk-Ulam theorems for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with free involutions. The size of the $\mathbb{Z}_2$-coincidence sets is also estimated. This work has appeared in [63] and was presented by the author in the Second East Asia Conference on Algebraic Topology held at the Institute for Mathematical Sciences, National University of Singapore, Singapore, December 15-19, 2008.

In Chapter 4, we study involutions on finitistic spaces X having mod 2 cohomology algebra of the wedge sum $P^2(n) \vee S^{3n}$ or $S^n \vee S^{2n} \vee S^{3n}$ and determine the possible fixed point sets up to mod 2 cohomology depending on whether or not X is totally non-homologous to zero in $X_{\mathbb{Z}_2}$ in the Borel fibration $X \hookrightarrow X_{\mathbb{Z}_2} \longrightarrow B_{\mathbb{Z}_2}$. We also give examples realizing the possible cases. This work has appeared in [64].

Chapter 5 deals with $\mathbb{S}^1$ actions on finitistic spaces X with rational cohomology algebra of the wedge sum $P^2(n) \vee S^{3n}$ or $S^n \vee S^{2n} \vee S^{3n}$. We determine the possible fixed point sets up to rational cohomology depending on whether or not X is totally non-homologous to zero in $X_{\mathbb{S}^1}$ in the Borel fibration $X \hookrightarrow X_{\mathbb{S}^1} \longrightarrow B_{\mathbb{S}^1}$. Examples realizing the possible cases are also given. This work has appeared in [62].

Chapter 6 contains some miscellaneous results obtained during the course of our study. We study the notion of *nice* actions introduced by Sikora [58] and show that if the cyclic group $\mathbb{Z}_p$ (p prime) acts nicely on spaces X and Y of finite type, then the diagonal action on $X \times Y$ is also nice. In the end, we prove the commutativity of inverse limit and orbit map for free actions of compact groups. This work has appeared in [60].

The precise chapter-wise details are given in the following pages.

Chapter 1

Brief Review of Transformation Groups and Spectral Sequences

In this chapter we give some basic definitions and results that will be used in the thesis. Most of the material is taken from Allday-Puppe [1], Bredon [6] and McCleary [44].

1.1 Group actions and their properties

Let G be a topological group and X be a Hausdorff topological space.

Definition 1.1.1. An **action** of G on X is a continuous map $\theta : G \times X \to X$ such that

1. $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G.

2. $\theta\big(g, \theta(h, x)\big) = \theta(gh, x)$ for all $g,\ h \in G$ and $x \in X$.

The triple (G, X, θ) is called a **topological transformation group** and X is called a G**-space**. We will write $g.x$ to denote $\theta(g, x)$ when the action is clear from the context. We note that an action of the group $\mathbb{Z}_2$ is also called an **involution**.

The subject of transformation groups is motivated by examples. Let G be a topological group. Then G acts on itself by conjugation, $G \times G \to G$ given by $(g, h) \mapsto ghg^{-1}$. By a **representation** of a topological group G, we mean a continuous homomorphism from G to an orthogonal group $O(n)$. Since $O(n)$ acts on a wide variety of spaces, such as $\mathbb{R}^n$, $\mathbb{D}^n$, $\mathbb{S}^{n-1}$, $\mathbb{R}P^{n-1}$ and $G_k(\mathbb{R}^n)$, one obtains a multitude of G-actions from a representation. Likewise a complex representation $G \to U(n)$ gives actions on $\mathbb{C}P^{n-1}$, $G_k(\mathbb{C}^n)$, etc. A group action arising from a continuous homomorphism $G \to GL(n, \mathbb{R})$ is called a **linear action**.

Let X be a G-space. For $g \in G$, let $\theta_g : X \to X$ be the map defined by $\theta_g(x) = \theta(g, x)$. Then, $\theta_g\theta_h = \theta_{gh}$ and $\theta_e = 1_X$. Thus each θ_g is a homeomorphism of X. If $Homeo(X)$ denote the group of all homeomorphisms of X, then $g \mapsto \theta_g$ defines a homomorphism $\theta : G \to Homeo(X)$. Conversely, any such homomorphism gives an action of G on X.

For each $x \in X$, $\overline{x} = \{g.x |\ g \in G\}$ is called the **orbit** of x. Let X/G denote the set of all orbits and let $\pi : X \to X/G$ be the canonical map given by $\pi(x) = \overline{x}$, called the **orbit map**. Then X/G equipped with the quotient topology induced by π is called the **orbit space**. The following is a quite useful result.

Theorem 1.1.1. *[6, Chapter I, Theorem 3.1] If X is a G-space with G compact, then*

1. *X/G is Hausdorff.*
2. *$\pi : X \to X/G$ is closed.*
3. *X is compact if and only if X/G is compact.*
4. *X is locally compact if and only if X/G is locally compact.*

For each $x \in X$, one can associate a subgroup $G_x = \{g \in G \mid g.x = x\}$ of G called the **isotropy subgroup** at x. These subgroups play an important role in the theory of topological transformation groups.

Proposition 1.1.2. *[6, Chapter I] Let X be a G-space and assume that X is T_1. Then G_x is a closed subgroup of G.*

We say that G acts **freely** on X if $G_x = \{e\}$ for all $x \in X$. The subspace $X^G = \{x \in X \mid g.x = x \text{ for all } g \in G\}$ of X is called the **fixed point set** of the action. For convenience we write F to denote the fixed point set X^G. The orbit space and the fixed point set are two important spaces associated to a group action.

Definition 1.1.2. Let X be a G-space with action θ and Y be a H-space with action θ'. If $\nu : G \to H$ is a topological group homomorphism, then a map $f : X \to Y$ is called **ν-equivariant** if $f\big(\theta(g,x)\big) = \theta'\big(\nu(g), f(x)\big)$ for all $g \in G$ and $x \in X$, in other words, the following diagram commute

$$\begin{array}{ccc} G \times X & \xrightarrow{\theta} & X \\ \Big\downarrow{\scriptstyle \nu\times f} & & \Big\downarrow{\scriptstyle f} \\ H \times Y & \xrightarrow{\theta'} & Y. \end{array}$$

Equivariant maps are the right ones to be considered in studying group actions. If both X and Y are G-spaces, then f is called G-equivariant (here $\nu =$ identity map of G). Note that the ν-equivariant map f induces a map $\overline{f} : X/G \to Y/H$ given by $\overline{f}(\overline{x}) = \overline{f(x)}$ and hence gives the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ X/G & \xrightarrow{\overline{f}} & Y/H. \end{array}$$

1.2 Direct and inverse systems

In this section, we give the necessary background required for defining Čech cohomology. Our main reference is the classic text [22] by Eilenberg and Steenrod. Let us first recall the following definition.

Definition 1.2.1. A **directed set** Λ is a set with a relation $<$ such that:

1. $\alpha < \alpha$ for all $\alpha \in \Lambda$.

2. $\alpha < \beta$ and $\beta < \gamma$ implies $\alpha < \gamma$.

3. Given α and β, there exists γ such that $\alpha < \gamma$ and $\beta < \gamma$.

Definition 1.2.2. A **direct system** of sets, denoted by $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$, consists of a directed set Λ and a family of sets $\{X_\alpha\}_{\alpha\in\Lambda}$ such that for each α , $\beta \in \Lambda$ with $\alpha < \beta$, there is a map $\pi_\alpha^\beta : X_\alpha \to X_\beta$ satisfying:

1. $\pi_\alpha^\alpha : X_\alpha \to X_\alpha$ is the identity map for all $\alpha \in \Lambda$.

2. $\pi_\beta^\gamma \circ \pi_\alpha^\beta = \pi_\alpha^\gamma$ for all $\alpha < \beta < \gamma$.

The maps $\{\pi_\alpha^\beta\}$ are called **projections** of the direct system (also called **bonding maps**). A direct system is usually represented by the following diagram

$$\cdots \to X_\alpha \overset{\pi_\alpha^\beta}{\to} X_\beta \overset{\pi_\beta^\gamma}{\to} X_\gamma \to \cdots, \quad \text{where } \alpha < \beta < \gamma.$$

If each X_α is a topological space, or a R-module, or a topological group, and each projection is continuous, or a R-module homomorphism, or a continuous homomorphism respectively, then the system is called a direct system of topological spaces, R-modules, or topological groups respectively.

Let $\{G_\alpha, \pi_\alpha^\beta, \Lambda\}$ be a direct system of R-modules and R-module homomorphisms. Let Σ denote the disjoint union of the R-modules. Introduce an equivalence relation on Σ by declaring $g_\alpha \sim g_\beta$ if there exists a γ such that $\alpha,\ \beta < \gamma$ and $\pi_\alpha^\gamma(g_\alpha) = \pi_\beta^\gamma(g_\beta)$. The **direct limit** of the direct system, denoted by $\varinjlim G_\alpha$, is the set $\Sigma/\sim$ of equivalence classes. If $[g_\alpha]$ denote an element of $\varinjlim G_\alpha$, then it can be made into an abelian group by defining

$$[g_\alpha] + [g_\beta] = [\pi_\alpha^\gamma(g_\alpha) + \pi_\beta^\gamma(g_\beta)],$$

where γ is such that $\alpha,\ \beta < \gamma$. The scalar multiplication is defined as $r[g_\alpha] = [rg_\alpha]$. Thus the direct limit $\varinjlim G_\alpha$ is a R-module.

Definition 1.2.3. An **inverse system** of sets, denoted by $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$, consists of a directed set Λ and a family of sets $\{X_\alpha\}_{\alpha\in\Lambda}$ such that for each α , $\beta \in \Lambda$ with $\alpha < \beta$ there is a map $\pi_\alpha^\beta : X_\beta \to X_\alpha$ satisfying:

1. $\pi_\alpha^\alpha : X_\alpha \to X_\alpha$ is the identity map for all $\alpha \in \Lambda$.

2. $\pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma$ for all $\alpha < \beta < \gamma$.

The maps $\{\pi_\alpha^\beta\}$ are called **projections** of the inverse system (also called **bonding maps**). An inverse system is usually represented by the following diagram

$$\cdots \to X_\gamma \overset{\pi_\beta^\gamma}{\to} X_\beta \overset{\pi_\alpha^\beta}{\to} X_\alpha \to \cdots, \quad \text{where } \alpha < \beta < \gamma.$$

If each X_α is a topological space, or a R-module, or a topological group, and each projection is continuous, or a R-module homomorphism, or a continuous homomorphism respectively, then the system is called an inverse system of topological spaces, R-modules, or topological groups respectively.

Given an inverse system $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ of sets, let $\varprojlim X_\alpha$ (possibly empty) be the subset of $\Pi_{\alpha\in\Lambda} X_\alpha$ consisting of elements (x_α) such that $x_\alpha = \pi_\alpha^\beta(x_\beta)$ for each $\alpha < \beta$ in Λ. The set $\varprojlim X_\alpha$ is called the **inverse limit** of the inverse system. We denote by $\pi_\beta : \varprojlim X_\alpha \to X_\beta$, the restriction of the canonical projection $\Pi_{\alpha\in\Lambda} X_\alpha \to X_\beta$.

If $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an inverse system of spaces, then $\varprojlim X_\alpha$ is assigned the topology is has as a subspace of $\Pi_{\alpha\in\Lambda} X_\alpha$. If $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an inverse system of R-modules, it is easily seen that $\varprojlim X_\alpha$ is a subgroup of $\Pi_{\alpha\in\Lambda} X_\alpha$ and hence is a R-module. Similarly, an inverse limit of topological groups is a topological group. The following basic results are well known.

Proposition 1.2.1. *[22, p.215] If $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an inverse system of topological spaces, then each π_α is continuous.*

Theorem 1.2.2. *[22, p.217] The inverse limit of an inverse system of non-empty compact Hausdorff topological spaces is a non-empty compact Hausdorff topological space.*

Theorem 1.2.3. *[22, p.219] Let X be a topological space and $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ be an inverse system of topological spaces. If for each $\alpha \in \Lambda$ there is a map $\psi_\alpha : X \to X_\alpha$ such that $\pi_\alpha^\beta \circ \psi_\beta = \psi_\alpha$ for each $\alpha < \beta$ in Λ, then there is a unique map $\psi : X \to \varprojlim X_\alpha$.*

1.3 Čech cohomology

The cohomology used in this thesis throughout is the Čech cohomology, unless otherwise stated. It is a well known fact that this is the most suitable cohomology for studying the cohomology theory of topological transformation groups. In this section, we give the necessary background concerning Čech cohomology theory. For details on other aspects of algebraic topology, we refer to [25, 48, 70].

Let X be a space and let $\mathcal{A}$ be an open covering of X. We define an abstract simplicial complex called the **Nerve** of $\mathcal{A}$, denoted by $N(\mathcal{A})$, as follows. Its vertices are the elements of $\mathcal{A}$ and its simplices are the finite sub-collections $\{A_1, ..., A_n\}$ of $\mathcal{A}$ such that $A_1 \cap A_2 \cap ... \cap A_n \neq \phi$. Now if $\mathcal{B}$ is a refinement of $\mathcal{A}$, we define a map $g : \mathcal{B} \to \mathcal{A}$ by choosing $g(B)$ to be an element of $\mathcal{A}$ that contains B. This map g induces a simplicial map $g : N(\mathcal{B}) \to N(\mathcal{A})$. Any other choice $g^{'}$ for g is contiguous to g. Thus, if $\mathcal{B}$ is a refinement of $\mathcal{A}$, for any coefficient group G, we have a uniquely defined homomorphism in simplicial cohomology

$$g^* : H^*(N(\mathcal{A}); G) \to H^*(N(\mathcal{B}); G)$$

induced by the simplicial map g. We call it the homomorphism induced by the refinement. Let Λ be the directed set of all open coverings of X, directed by letting $\mathcal{A} < \mathcal{B}$ if $\mathcal{B}$ is a refinement of $\mathcal{A}$. Construct a direct system by assigning to the element $\mathcal{A}$ of Λ, the group $H^k(N(\mathcal{A}); G)$ and by assigning to the pair $\mathcal{A} < \mathcal{B}$ in Λ, the homomorphism

$$f^{\mathcal{B}}_{\mathcal{A}} : H^k(N(\mathcal{A}); G) \to H^k(N(\mathcal{B}); G),$$

induced by the refinement. We define the Čech cohomology group of X in dimension k, with coefficients in G, by the equation

$$H^k(X; G) = \varinjlim_{\mathcal{A} \in \Lambda} H^k(N(\mathcal{A}); G).$$

Most of the fundamental properties of Čech cohomology can be stated in the following theorem.

Theorem 1.3.1. *[22, Chapter IX] The Čech cohomology theory satisfies the Eilenberg-Steenrod axioms.*

Using inverse system of groups, one can define the Čech homology of a space X, just as we defined the Čech cohomology. We will exploit the continuity property of the Čech cohomology theory which we explain as follows. Let $\mathcal{T}$ denote a category of spaces and maps. Let Inv$\mathcal{T}$ denote the category of inverse systems having values in $\mathcal{T}$ and let $\mathcal{G}_R$ be the category of R-modules, where R is a ring. Suppose that a cohomology theory H is defined on $\mathcal{T}$ with values in $\mathcal{G}_R$. Then H can be applied to $\{X_\alpha, \pi_\alpha^\beta, \Lambda\} \in \text{Inv}\mathcal{T}$ to yield a direct system

$$\{H^q(X_\alpha), \pi_\alpha^{\beta^*}, \Lambda\}.$$

Let $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ be an inverse system having values in $\mathcal{T}$. Let $\pi_\beta : \varprojlim X_\alpha \to X_\beta$ be the projection, then $\pi_\alpha^\beta \pi_\beta = \pi_\alpha$ for $\alpha < \beta$ and $\pi_\beta^* \pi_\alpha^{\beta^*} = \pi_\alpha^*$. Thus the maps $\{\pi_\beta^*\}$ constitute a homomorphism

$$H^q(X_\beta) \to H^q(\varprojlim X_\alpha),$$

thereby defining a limit homomorphism

$$l(q) : \varinjlim H^q(X_\beta) \to H^q(\varprojlim X_\alpha).$$

Definition 1.3.1. A cohomology theory H with values in the category $\mathcal{G}_R$ is said to be **continuous** on the category $\mathcal{T}$ if the transformation l is a natural equivalence, that is, for each inverse system $\{X_\alpha, \pi_\alpha^\beta, \Lambda\} \in \text{Inv}\mathcal{T}$,

$$l(q) : \varinjlim H^q(X_\beta) \xrightarrow{\cong} H^q(\varprojlim X_\alpha).$$

Theorem 1.3.2. *[22, Chapter X, Theorem 3.1] The Čech cohomology theory based on a coefficient group which is in $\mathcal{G}_R$ is continuous on the category of compact pairs.*

See Chapter X of [22] for more on continuity property. From now onwards the cohomology used in all our results will be the Čech cohomology.

1.4 Group actions on finitistic spaces

We now define a class of spaces that plays an important role in the cohomology theory of transformation groups.

Definition 1.4.1. A paracompact Hausdorff space X is said to be **finitistic** if its every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially.

The notion of a finitistic space was given by R. G. Swan [75] for studying fixed point theory. It is a large class of spaces including all compact Hausdorff spaces and all paracompact spaces of finite covering dimension [50, Chapter II]. The Čech cohomology is found to be most suitable for the cohomology theory of transformation groups on finitistic spaces. If G is a compact Lie group acting continuously on a space X, then due to Deo-Singh-Tripathi [14, 16], the space X is finitistic if and only if the orbit space X/G is finitistic. We now state some important results regarding group actions on finitistic spaces.

Theorem 1.4.1. *[6, Chapter III, Theorem 7.9] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space X with fixed point set F. Then for each fixed $n \geq 0$,*

$$\sum_{i \geq n} rkH^i(F;\mathbb{Z}_p) \leq \sum_{i \geq n} rkH^i(X;\mathbb{Z}_p).$$

We now have the following result which is originally due to Floyd.

Theorem 1.4.2. *[6, Chapter III, Theorem 7.10] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space X with fixed point set F. If $rkH^*(X;\mathbb{Z}_p) < \infty$ and the Euler characteristics are defined in terms of mod p Čech cohomology, then*

$$\chi(X) + (p-1)\chi(F) = p\chi(X/G).$$

An action of a group G on a space X induces an action on the cohomology given by $g^{-1} : X \to X$.

Theorem 1.4.3. *[6, Chapter III, Theorem 7.2] Let X be a paracompact G-space with G finite and let $\pi : X \to X/G$ be the orbit map. If Λ is a field of characteristic zero or prime to $|G|$, then*

$$\pi^* : H^*(X/G; \Lambda) \to H^*(X; \Lambda)^G$$

is an isomorphism.

It is a well known fact that when n is even, $\mathbb{Z}_2$ is the only non-trivial group acting freely on the sphere $\mathbb{S}^n$. When n is odd, the problem is quite difficult and in fact a complete solution is not known. We now state a well known result which is originally due to Smith [66].

Theorem 1.4.4. *[6, Chapter III, Theorem 8.1] If p is a prime, then the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ cannot act freely on a finitistic mod p cohomology n-sphere.*

We now present some results regarding actions of compact Lie groups. In studying group actions it is necessary to count the number of conjugacy classes of isotropy subgroups of the group. If G is a compact Lie group and H is a closed subgroup of G, then let $[H]$ denote the conjugacy class of H in G. The group G is said to act on a space X with **finitely many orbit types (FMOT)** if the set $\{[G_x] \mid x \in X\}$ is finite, where G_x is the isotropy subgroup at x. For a subgroup H of G, let H^0 denote the component of identity in H. Then G is said to act on X with **finitely many connective orbit types (FMCOT)** if the set $\{[G_x^0] \mid x \in X\}$ is finite. The group actions of such type are the correct ones to consider in generalizing results about the actions of finite groups to those of compact groups. See for example [1] and [6] for a detailed account of results concerning such actions. Clearly FMOT implies FMCOT. When working with a field of characteristic zero only FMCOT is needed [1, p.131]. For torus actions we have the following useful result.

Lemma 1.4.5. *[1, Lemma 4.2.1(1)] Let $G = (\mathbb{S}^1)^r$ acts on the finitistic space X with FMCOT, then there is a sub-circle $\mathbb{S}^1 \subset G$ such that their fixed point sets are same, that is, $X^{\mathbb{S}^1} = X^G$.*

It is clear that the group $G = \mathbb{S}^1$ act on any space with FMCOT and hence we drop this hypothesis from the results about $\mathbb{S}^1$ actions that we present in this section.

Theorem 1.4.6. *[1, Corollary 3.1.13, Remark 3.10.5(2)] Let $G = \mathbb{S}^1$ act on the finitistic space X with fixed point set F. Suppose that $\sum_{i\geq 0} rkH^i(X;\mathbb{Q}) < \infty$, then*

$$\chi(X) = \chi(F).$$

Theorem 1.4.7. *[1, Lemma 3.10.16, Corollary 3.10.12] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space X. Suppose that $H^i(X;\mathbb{Z}_p) = 0$ for $i > n$, then*

1. *$H^i(F;\mathbb{Z}_p) = 0$ for $i > n$.*

2. *$H^i(X/G;\mathbb{Z}_p) = 0$ for $i > n$.*

This also holds for $G = \mathbb{S}^1$ and rational coefficients.

We now recall an important construction due to Borel [11, Chapter IV]. For a compact Lie group G, let $G \hookrightarrow E_G \longrightarrow B_G$ be the universal principal G-bundle. Let X be a G-space. Consider the diagonal action of G on $X \times E_G$. Then the projection $pr_2 : X \times E_G \to E_G$ is G-equivariant and gives a fibration $X \hookrightarrow X_G \longrightarrow B_G$ called the **Borel fibration**, where $X_G = (X \times E_G)/G$ is the orbit space of the diagonal action on $X \times E_G$. The fibration $X \hookrightarrow X_G \longrightarrow B_G$ is in fact a fiber bundle. Similarly, the projection $pr_1 : X \times E_G \to X$ is G-equivariant and we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[d, "q_1"] & X \times E_G \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, "q"] & E_G \arrow[d, "q_2"] \\
X/G & X_G \arrow[l, "\overline{pr_1}"'] \arrow[r, "\overline{pr_2}"] & B_G.
\end{tikzcd}$$

Any fixed cohomology (say Čech cohomology) of X_G is called the **equivariant cohomology** of X. Note that if X is a paracompact G-space, then X_G is also paracompact (see [1, p.141]). If $\overline{pr_1} : (X_G, F_G) \to (X/G, F)$ is the map induced by the G-equivariant projection $(X \times E_G, F \times E_G) \to (X, F)$, then we have the following results.

Theorem 1.4.8. *[6, Chapter VII, Proposition 1.1] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space X with fixed point set F. Then*

$$\overline{pr_1}^* : H^i(X/G, F) \to H^i(X_G, F_G)$$

is an isomorphism for all $i \geq 0$ and for arbitrary coefficients. This also holds for $G = \mathbb{S}^1$ and rational coefficients.

In fact for free actions X/G and X_G have the same homotopy type.

Theorem 1.4.9. *[6, Chapter VII, Theorem 1.5] Let $G = \mathbb{Z}_2$ act freely on the finitistic space X. Suppose that $H^i(X; \mathbb{Z}_2) = 0$ for $i > n$, then $H^i(X_G; \mathbb{Z}_2) = 0$ for $i > n$.*

We now digress to discuss the Leray-Hirsch theorem, which will be used crucially in Chapter 3. Before that we have the following definition [6, p.372].

Definition 1.4.2. Let (X, E, π, B) be a fiber bundle and let Λ be a principal ideal domain. By a **cohomology extension of the fiber** we mean a Λ-module homomorphism of degree zero

$$\theta : H^*(X; \Lambda) \to H^*(E; \Lambda)$$

such that for any $b \in B$, the composition

$$H^*(X; \Lambda) \xrightarrow{\theta} H^*(E; \Lambda) \xrightarrow{i_b^*} H^*(X_b; \Lambda)$$

is an isomorphism, where $i_b : X_b \hookrightarrow E$ is the inclusion of the fiber X_b over b.

Here θ is not required to preserve products. With this we prove the following [6, p.372].

Theorem 1.4.10. *(Leray-Hirsch) Let (X, E, π, B) be a fiber bundle and let Λ be a principal ideal domain. Assume that X is paracompact and that B is a CW-complex. Let θ be a cohomology extension of the fiber, with respect to the base ring Λ, and assume that $H^*(X; \Lambda)$ is a torsion free Λ-module. Then the map*

$$H^*(B; \Lambda) \otimes_\Lambda H^*(X; \Lambda) \to H^*(E; \Lambda),$$

taking $\beta \otimes \alpha \mapsto \pi^*(\beta) \cup \theta(\alpha)$, is an isomorphism of $H^*(B; \Lambda)$-modules.

In other words, $H^*(E;\Lambda)$ is a free $H^*(B;\Lambda)$-module, where we view $H^*(E;\Lambda)$ as a module over the ring $H^*(B;\Lambda)$ by defining scalar multiplication as $\beta.c = \pi^*(\beta) \cup c$ for $\beta \in H^*(B;\Lambda)$ and $c \in H^*(E;\Lambda)$. We now give another important definition [6, p.373].

Definition 1.4.3. Let (X, E, π, B) be a fiber bundle and let Λ be a principal ideal domain. If $i : X \to E$ is the inclusion of a typical fiber, then we say that X is **totally non-homologous to zero** in E with respect to ring Λ if the homomorphism

$$i^* : H^*(E;\Lambda) \to H^*(X;\Lambda)$$

is surjective.

We write **TNHZ** to mean totally non-homologous to zero. In case of coefficients in a field Λ, the cohomology extension of the fiber clearly exists if and only if X is totally non-homologous to zero in E. With this definition, we have the following result.

Theorem 1.4.11. *[1, Theorem 3.10.4] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space X with fixed point set F. Suppose that $\sum_{i\geq 0} rkH^i(X;\mathbb{Z}_p) < \infty$, then*

$$\sum_{i\geq 0} rkH^i(F;\mathbb{Z}_p) \leq \sum_{i\geq 0} rkH^i(X;\mathbb{Z}_p).$$

Furthermore, the following statements are equivalent:

1. *$\sum_{i\geq 0} rkH^i(F;\mathbb{Z}_p) = \sum_{i\geq 0} rkH^i(X;\mathbb{Z}_p)$.*

2. *X is totally non-homologous to zero in X_G.*

3. *G acts trivially on $H^*(X;\mathbb{Z}_p)$ and the Leray spectral sequence $E_2^{k,l} = H^k(B_G, H^l(X;\mathbb{Z}_p)) \implies H^{k+l}(X_G;\mathbb{Z}_p)$ of the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$ degenerates.*
   This also holds for $G = \mathbb{S}^1$ and rational coefficients.

For the details on spectral sequences we refer to Section 1.7. We now present a useful result regarding involutions.

Proposition 1.4.12. *[6, Corollary 7.3] Suppose that $G = \mathbb{Z}_2$ acts on the finitistic space X with fixed point set F and that X is totally non-homologous to zero in X_G. Then any class $a \in H^n(X;\mathbb{Z}_2)$ with $a^2 \neq 0$ restricts non-trivially to F.*

1.5 Group actions on projective spaces

In this section, we record some fundamental results regarding transformation groups on cohomology projective spaces. For a prime p, we write $X \simeq_p Y$ if X and Y have isomorphic mod p cohomology algebras. Similarly, we write $X \simeq_p P^h(n)$ to mean that

$$H^*(X;\mathbb{Z}_p) \cong \mathbb{Z}_p[a]/a^{h+1},$$

where a is a homogeneous element of degree n. For $n = 1$, 2 or 4, X has the mod p cohomology algebra of the real projective space $\mathbb{R}P^h$, the complex projective space $\mathbb{C}P^h$ or the quaternionic projective space $\mathbb{H}P^h$, respectively. For $n = 8$ and $h = 2$, X has the mod p cohomology algebra of the Cayley projective plane $\mathbb{O}P^h$. The following result is well known.

Proposition 1.5.1. *[71, Chapter I, 4.5] Let X be a space such that $X \simeq_p P^h(n)$, then for $p = 2$*

1. $n = 1, 2, 4$ *for* $h \geq 2$.

2. $n = 8$ *for* $h = 2$.

and for p odd, n must be even for $h \geq 2$.

For $\mathbb{Z}_2$ actions on $\mathbb{R}P^h$ the following theorem was proved by Smith [65]. Smith's arguments were extended to the case of cohomology $\mathbb{R}P^h$ by Su [74]. The following general case was proved by Bredon [5, 8].

Theorem 1.5.2. *[6, Chapter VII, Theorem 3.1] Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the finitistic space $X \simeq_p P^h(n)$. Then either the fixed point set F is* *empty or F is the disjoint union of components $F_i \simeq_p P^{h_i}(n_i)$ with $h+1 = \sum(h_i+1)$ and $n_i \leq n$. The number of components is at most p. For p odd and $h \geq 2$, n and n_i are all even. Moreover, if $n_i = n$ for some i, then the restriction $H^n(X;\mathbb{Z}_p) \to H^n(F_i;\mathbb{Z}_p)$ is an isomorphism.*

An analogue of above theorem holds for $\mathbb{S}^1$ actions on rational cohomology projective spaces. For $\mathbb{Z}_2$ actions the following result is well known.

Theorem 1.5.3. *[6, Chapter VII, Theorem 3.2] Suppose that $G = \mathbb{Z}_2$ acts on the finitistic space $X \simeq_2 P^h(n)$, $h \geq 2$. Then one of the following possibilities must hold:*

1. *F is empty and h is odd.*
2. *F is connected and $F \simeq_2 P^h(m)$, where $n = m$ or $n = 2m$.*
3. *F has two components F_1 and F_2, where $F_i \simeq_2 P^{h_i}(n)$ and $h = h_1 + h_2 + 1$.*

Moreover, in case (2) the restriction $H^n(X;\mathbb{Z}_2) \to H^n(F;\mathbb{Z}_2)$ is an isomorphism.

1.6 Cohomological dimension

In this section, we recall some definitions and results in dimension theory which will be required in our work. For more details, we refer to Nagami [50]. The cohomology used is the Čech cohomology.

Definition 1.6.1. The **large cohomological dimension** $Dim(X,\Lambda)$ of a space X with respect to an abelian group Λ is the largest positive integer n such that $H^n(X,A;\Lambda) \neq 0$ for some closed subspace A of X.

Definition 1.6.2. The **small cohomological dimension** $dim(X,\Lambda)$ of a space X with respect to an abelian group Λ is the smallest positive integer n such that for each $m \geq n$ the map $i^* : H^m(X;\Lambda) \to H^m(A;\Lambda)$ induced by the inclusion $i : A \hookrightarrow X$ is surjective for each closed subspace A of X.

The following is an important result.

Theorem 1.6.1. *[50, Theorem 37-7] If X is paracompact, then $Dim(X, \Lambda) = dim(X, \Lambda)$ for any abelian group Λ.*

Since we will be dealing with paracompact spaces, we denote the cohomological dimension of a space X with respect to an abelian group Λ by $cohom.dim(X, \Lambda)$. For actions of compact Lie groups on paracompact spaces, we have the following.

Proposition 1.6.2. *[52, Proposition A.11] Let G be a compact Lie group acting on a paracompact space X with orbit space X/G, then for any abelian group Λ, we have*

$$cohom.dim(X/G, \Lambda) \leq cohom.dim(X, \Lambda).$$

1.7 Leray spectral sequence

It has been a challenging problem to relate the cohomology algebra of the total space, the base space and the fiber space of a fiber bundle. Leray [39] solved this problem and gave the first explicit example of a spectral sequence in the cadre of sheaves and a general cohomology theory which specializes to the Alexander-Spanier cohomology, Čech cohomology, de Rham cohomology and singular cohomology. Before giving the definition of a spectral sequence, we recall the following definition.

Definition 1.7.1. A **differential bigraded module** over a ring R, is a collection of R-modules $E^{k,l}$, where k and l are integers, together with R-linear mapping $d : E^{*,*} \to E^{*,*}$, the **differential** of bidegree $(s, 1-s)$ or $(-s, s-1)$ for some integer s and satisfying $d^2 = 0$.

With the differential, we can take the homology of the differential bigraded module:

$$H^{k,l}(E^{*,*}, d) = ker\{d : E^{k,l} \to E^{k+s,l-s+1}\}/im\{d : E^{k-s,l+s-1} \to E^{k,l}\}.$$

Now we can give the definition of a spectral sequence.

Definition 1.7.2. A **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where r =1, 2,...; the differentials are either all of bidegree $(-r, r-1)$ (for a spectral sequence of **homology type**)or all of bidegree $(r, 1-r)$ (for a spectral sequence of **cohomology type**) and for all r, k, l, $E_{r+1}^{k,l} \cong H^{k,l}(E_r^{*,*}, d_r)$.

We will be using only the cohomology type spectral sequences.

Definition 1.7.3. A **filtration** F^* of a R-module A is a family of submodules $\{F^kA\}$ for $k \in \mathbb{Z}$ such that

$$\cdots \subset F^{k+1}A \subset F^kA \subset F^{k-1}A \subset \cdots \subset A \ (decreasing\ filtration)$$

$$or\ \ \cdots \subset F^{k-1}A \subset F^kA \subset F^{k+1}A \subset \cdots \subset A \ (increasing\ filtration).$$

If H^* is a graded R-module and H^* is filtered, then we can examine the filtration on each degree by letting $F^kH^n = F^kH^* \cap H^n$. Thus the associated graded module is bigraded when we consider

$$F^kH^{k+l}/F^{k+1}H^{k+l}, \text{ if } F^* \text{ is decreasing}$$

$$or \quad F^kH^{k+l}/F^{k-1}H^{k+l}, \text{ if } F^* \text{ is increasing.}$$

We want to find where the spectral sequence converge to. To do so, we present a spectral sequence as a tower of submodules of a given module. From this tower, it is clear where the algebraic information is converging. For the sake of clarity we suppress the bigrading. Denote

$$Z_2 = \ker d_2 \ \text{ and } \ B_2 = \operatorname{im} d_2.$$

The condition, $d_2^2 = 0$, implies $B_2 \subset Z_2 \subset E_2$, and by definition, $E_3 \cong Z_2/B_2$. Write $\overline{Z_3}$ for $\ker d_3 : E_3 \to E_3$. Since, $\overline{Z_3}$ is a submodule of E_3, it can be written as Z_3/B_2, where Z_3 is a submodule of Z_2. Similarly, $\overline{B_3} = \operatorname{im} d_3$ is isomorphic to B_3/B_2 and so

$$E_4 \cong \overline{Z_3}/\overline{B_3} \cong (Z_3/B_2)/(B_3/B_2) \cong Z_3/B_3.$$

This data can be presented as a tower of inclusions:

$$B_2 \subset B_3 \subset Z_3 \subset Z_2.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of E_2:

$$B_2 \subset B_3 \subset \cdots B_n \subset \cdots \quad \cdots \subset Z_n \subset \cdots Z_3 \subset Z_2 \subset E_2$$

with the property that $E_{n+1} \cong Z_n/B_n$, and the differential d_{n+1} can be taken as a mapping $Z_n/B_n \to Z_n/B_n$, which has kernel Z_{n+1}/B_n and image B_{n+1}/B_n. The short exact sequence induced by d_{n+1},

$$0 \to Z_{n+1}/B_n \to Z_n/B_n \to B_{n+1}/B_n \to 0,$$

gives rise to isomorphisms $B_{n+1}/B_n \cong Z_n/Z_{n+1}$ for all n. Conversely, a tower of submodules of E_2, together with such a set of isomorphisms, determines a spectral sequence. We say that an element in E_2 that lies in Z_r **survives to the** r^{th} **stage**, having being in the kernel of the previous $r-2$ differentials. The submodule B_r of E_2 is the set of elements that are **boundaries by the** r^{th} **stage**. Let $Z_\infty = \bigcap_n Z_n$ be the submodule of E_2 of elements that **survive forever**, that is, elements that are cycles at every stage. The submodule $B_\infty = \bigcup_n B_n$ consists of those elements that **eventually bound**. From the tower of inclusions it is clear that $B_\infty \subset Z_\infty$. We define

$$E_\infty = Z_\infty/B_\infty,$$

which is a bigraded module that remains after the computation of the infinite sequence of successive homologies. It is the E_∞ term of a spectral sequence that is the general goal of a computation. We next combine the associated graded module with the definition of a spectral sequence.

Definition 1.7.4. A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to **converge** to H^*, a graded R-module, if there is a filtration F^* of H^* such that

$$E_\infty^{k,l} \cong F^k H^{k+l}/F^{k+1}H^{k+l},$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Determination of a graded module H^* is generally the goal of a computation. Although, one can associate a spectral sequence to a filtered differential graded module [44, Theorem 2.6] and to a bigraded exact couple [44, Theorem 2.8], we will be maily concerned with the spectral sequence associated to a fibration. Let R be a commutative ring with unit.

Theorem 1.7.1. *[44, Theorem 5.2] Suppose that $X \overset{i}{\hookrightarrow} E \xrightarrow{\rho} B$ is a fibration, where B is path connected and X is connected. Then there is a first quadrant spectral sequence of algebras $\{E_r{}^{*,*}, d_r\}$, converging to $H^*(E;R)$ as an algebra, with*

$$E_2^{k,l} = H^k(B; \mathcal{H}^l(X;R)),$$

the cohomology of the base B with locally constant coefficients $\mathcal{H}^l(X;R)$ twisted by a canonical action of $\pi_1(B)$. This spectral sequence is natural with respect to fiber preserving maps of fibrations.

For the Čech or the Alexander-Spanier cohomology theories the multiplicative structure in the spectral sequence is carried along transparently in the construction of the spectral sequence and we get a spectral sequence of algebras converging to $H^*(E;R)$ as an algebra [40, 10]. The result for singular theory, however, is more difficult and is one of the celebrated works in Serre's thesis [57].

Theorem 1.7.2. *[44, Theorem 5.9] Suppose that $X \overset{i}{\hookrightarrow} E \xrightarrow{\rho} B$ is a fibration, where B is path connected and X is connected, and that the system of local coefficients on B is simple, then the edge homomorphisms*

$$\begin{aligned} H^k(B;R) = E_2^{k,0} \longrightarrow E_3^{k,0} \longrightarrow \cdots \\ \longrightarrow E_k^{k,0} \longrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(E;R) \end{aligned}$$

and

$$H^l(E;R) \longrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \cdots \subset E_2^{0,l} = H^l(X;R)$$

are the homomorphisms

$$\rho^* : H^k(B;R) \to H^k(E;R) \quad \textit{and} \quad i^* : H^l(E;R) \to H^l(X;R).$$

By introducing some simplifying hypothesis, the spectral sequence takes the following form.

Proposition 1.7.3. *[44, Proposition 5.5] Suppose that the system of local coefficients on B determined by the fiber is simple, that F is connected, and that F and B are of finite type, then for a field R, we have*

$$E_2^{k,l} \cong H^k(B;R) \otimes_R H^l(X;R).$$

Remark 1.7.1. The graded commutative algebra $H^*(E;R)$ is isomorphic to $\mathrm{Tot}E_\infty^{*,*}$, the total complex of $E_\infty^{*,*}$. As mentioned earlier, $H^*(E)$ is a $H^*(B)$-module with the scalar multiplication given by $\beta.c = \rho^*(\beta) \cup c$ for $\beta \in H^*(B)$ and $c \in H^*(E)$.

Remark 1.7.2. Our main concern will be the Leray spectral sequence associated to the Borel fibration $X \overset{i}{\hookrightarrow} X_G \xrightarrow{\rho} B_G$.

Chapter 2

Cohomology Algebra of Orbit Spaces of Free Involutions on Lens Spaces

2.1 Introduction

This chapter is concerned with the orbit spaces of free involutions on cohomology lens spaces. The study of 3-dimensional lens spaces dates back to the work of Tietze in 1908 [76]. They are the first known examples of 3-manifolds which are not determined by their homology and fundamental group alone. The term **lens space** was not introduced until 1930 by Seifert and Threlfall. Since then the spaces have appeared frequently in works concerning 3-manifolds, surgery and knot theory and have important distinction of being the first non-trivial class of 3-manifolds to be entirely classified up to homeomorphism [54]. They played an important role in the Milnor's counterexample to the Hauptvermutung [46]. Because of the diverse nature of their applicability, there are at least five more or less distinct definitions of the 3-dimensional lens space, depending on the context in which they appear. For our purpose, we define lens spaces as odd dimensional spherical space forms.

Let $p \geq 2$ be an integer and $q_1, q_2, ..., q_m$ be integers coprime to p, where $m \geq 1$. Let $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$ be the unit sphere and let $\iota^2 = -1$. Then

$$(z_1, ..., z_m) \mapsto (e^{\frac{2\pi\iota q_1}{p}} z_1, ..., e^{\frac{2\pi\iota q_m}{p}} z_m)$$

defines a free action of the cyclic group $\mathbb{Z}_p$ on $\mathbb{S}^{2m-1}$. The orbit space is called **Lens** space and is denoted by $L_p^{2m-1}(q_1, ..., q_m)$. It is a compact Hausdorff orientable manifold of dimension $(2m-1)$. Being compact Hausdorff it is finitistic.

We now describe a geometric model of the 3-dimensional lens space, whose equivalence with the above definition is simple and can be found in [25]. For convenience take $q_1 = 1$. Consider a lens shaped closed 3-cell, whose surface consists of two identical, radially symmetric caps which meet at a circular rim. Label the north and the south poles N and S respectively and partition the circular rim into p equal arcs separated by points $x_0, x_1, \cdots, x_{p-1}$. Joint each x_i with N and S with curvilinear segments to divide each cap into p identical triangular sectors. The $\frac{2\pi q_2}{p}$-radian positive rotation and an orthogonal projection results in each sector Nx_ix_{i+1} being identified with $Sx_{q+i}x_{q+i+1}$, where the all subscripts are taken mod p. The resulting space is $L_p^3(1, q_2)$.

Involutions on lens spaces have been studied in detail, particularly on 3-dimensional lens spaces [26, 33, 35, 36, 37, 49]. Hodgson and Rubinstein [26] obtained a classification of smooth involutions on 3-dimensional lens spaces having one dimensional fixed point sets. Kim [36] obtained a classification of orientation preserving and sense preserving PL involutions on 3-dimensional lens spaces. Also Kim [37] obtained a classification of free involutions on 3-dimensional lens spaces whose orbit spaces contain Klein bottles. Kim [35] showed that if $p = 4k$ for some k, then the orbit space of any sense-preserving free involution on $L_p^3(1, q)$ is the lens space $L_{2p}^3(1, q^{'})$, where $q^{'}q \equiv \pm 1$ or $q^{'} \equiv \pm q$ mod p and an involution is sense-preserving if the induced map on $H_1(L_p^3(1, q); \mathbb{Z})$ is the identity map. Myers in [49] showed that every free involution on a 3-dimensional lens space is conjugate to an orthogonal free involution, in which case the orbit space is again a lens space.

From now onwards, for convenience we write $L_p^{2m-1}(q)$ to denote $L_p^{2m-1}(q_1, ..., q_m)$. Let $X \simeq_2 L_p^{2m-1}(q)$ mean that X is a space with an isomorphism of graded algebras

$$H^*(X; \mathbb{Z}_2) \cong H^*(L_p^{2m-1}(q); \mathbb{Z}_2).$$

We call such a space a mod 2 cohomology lens space and refer to dimension of $L_p^{2m-1}(q)$ as its dimension. Motivated by the work of Kim and Myers, we investigate the cohomology of orbit spaces of free involutions on cohomology lens spaces. We work on the general class of finitistic spaces. More precisely, we consider free involutions on finitistic mod 2 cohomology lens spaces and determine completely the possible mod 2 cohomology algebra of orbit space. If X/G denotes the orbit space, then we prove the following theorem.

Theorem 2.1.1. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q)$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded commutative algebras:*

1. $\mathbb{Z}_2[x]/\langle x^{2m} \rangle$,

   where $deg(x) = 1$.

2. $\mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle$,

   where $deg(x) = 1$ and $deg(y) = 2$.

3. $\mathbb{Z}_2[x, y, z]/\langle x^3, y^2, z^{\frac{m}{2}} \rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ *and* m *is even.*

4. $\mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^{\frac{m}{2}}, x^2y \rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ *and* m *is even.*

5. $\mathbb{Z}_2[x, y, w, z]/\langle x^5, y^2, w^2, z^{\frac{m}{4}}, x^2y, wy \rangle$,

   where $deg(x) = 1$, $deg(y) = 1$, $deg(w) = 3$, $deg(z) = 8$ *and* $4 \mid m$.

Our theorem generalizes the results known for orbit spaces of free involutions on 3-dimensional lens spaces, to that of the large class of finitistic spaces $X \simeq_2 L_p^{2m-1}(q)$ (see remarks 2.5.1 and 2.5.2). We also give an application to non-existence of $\mathbb{Z}_2$-equivariant maps $\mathbb{S}^n \to X$, where $\mathbb{S}^n$ is equipped with the antipodal involution.

2.2 Cohomology of lens spaces

The homology groups of a lens space can be easily computed using its cell decomposition (see for example [25, p.144]) and are given by

$$H_i(L_p^{2m-1}(q); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0,\ 2m-1 \\ \mathbb{Z}_p & \text{if } i \text{ is odd and } 0 < i < 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

If p is odd, then the mod 2 cohomology groups are

$$H^i(L_p^{2m-1}(q); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0,\ 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

And if p is even, then

$$H^i(L_p^{2m-1}(q); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq i \leq 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Free involutions on lens spaces

We now construct a free involution on the lens space $L_p^{2m-1}(q)$. Let $q_1, ..., q_m$ be odd integers coprime to p. Consider the map $\mathbb{C}^m \to \mathbb{C}^m$ given by

$$(z_1, ..., z_m) \mapsto (e^{\frac{2\pi\iota q_1}{2p}} z_1, ..., e^{\frac{2\pi\iota q_m}{2p}} z_m).$$

This map commutes with the $\mathbb{Z}_p$ action on $\mathbb{S}^{2m-1}$ defining the lens space and hence descends to a map $\alpha : L_p^{2m-1}(q) \to L_p^{2m-1}(q)$ such that α^2= identity. Thus α is an involution. Denote elements of $L_p^{2m-1}(q)$ by $[z]$ for $z = (z_1, ..., z_m) \in \mathbb{S}^{2m-1}$. If $\alpha([z]) = [z]$, then

$$(e^{\frac{2\pi\iota q_1}{2p}} z_1, ..., e^{\frac{2\pi\iota q_m}{2p}} z_m) = (e^{\frac{2\pi\iota k q_1}{p}} z_1, ..., e^{\frac{2\pi\iota k q_m}{p}} z_m)$$

for some integer k. Let $1 \leq i \leq m$ be an integer such that $z_i \neq 0$, then $e^{\frac{2\pi\iota q_i}{2p}} z_i = e^{\frac{2\pi\iota k q_i}{p}} z_i$ and hence $e^{\frac{2\pi\iota q_i}{2p}} = e^{\frac{2\pi\iota k q_i}{p}}$. This implies

$$\frac{q_i}{2p} - \frac{kq_i}{p} = \frac{q_i(1-2k)}{2p}$$

is an integer, a contradiction. Hence the involution α is free. Observe that the orbit space of the above involution is $L_p^{2m-1}(q)/\langle\alpha\rangle = L_{2p}^{2m-1}(q)$.

2.4 Orbit spaces of free involutions on lens spaces

Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q)$ and let X/G denote the orbit space. Theorem 2.1.1 determines the possible mod 2 cohomology algebra of the orbit space X/G. Recall that for $G = \mathbb{Z}_2$, $H^*(B_G; \mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, where t is a homogeneous element of degree 1. Čech cohomology with $\mathbb{Z}_2$ coefficients will be used and we will suppress the coefficient group from the cohomology notation. We will exploit the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$. This section is divided into three parts according to the various possibilities for p and Theorem 2.1.1 follows from a sequence of propositions proved in this section.

2.4.1 When p is odd.

Recall that, for p odd, we have $L_p^{2m-1}(q) \simeq_2 \mathbb{S}^{2m-1}$. It is well known that the orbit space of any free involution on a mod 2 cohomology sphere is a mod 2 cohomology real projective space of same dimension (see for example Bredon [7, p.144]). For the sake of completeness, we give a quick proof using the Leray spectral sequence.

Proposition 2.4.1. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 \mathbb{S}^n$, where $n \geq 1$. Then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/\langle x^{n+1}\rangle,$$

where $deg(x) = 1$.

Proof. Note that $E_2^{k,l}$ is non-zero only for $l = 0,\ n$. Therefore the differentials $d_r = 0$ for $2 \leq r \leq n$ and for $r \geq n + 2$. As there are no fixed points, by Theorem 1.4.11, the spectral sequence do not degenerate and hence

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0}$$

is non-zero and it is the only non-zero differential. Thus $E_\infty^{*,*} = E_{n+2}^{*,*}$ and

$$H^k(X_G) = E_\infty^{k,0} = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0} \subset H^1(X_G)$ be determined by $t \otimes 1 \in E_2^{1,0}$. Since the cup product

$$x \cup (-) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k \leq n-1$, we have $x^k \neq 0$ for $1 \leq k \leq n$ and therefore $H^*(X_G) \cong \mathbb{Z}_2[x]/\langle x^{n+1}\rangle$. As the action of G is free, by Theorem 1.4.8, we have $H^*(X/G) \cong H^*(X_G)$. This gives the case (1) of Theorem 2.1.1. □

2.4.2 When p is even and $4 \nmid p$.

Let p be even, say $p = 2p'$ for some integer $p' \geq 1$. Since $q_1, ..., q_m$ are coprime to p, all of them are odd. Also all of them are coprime to p'. Note that $L_p^{2m-1}(q) = L_{2p'}^{2m-1}(q) =$

$L_{p'}^{2m-1}(q)/\langle\alpha\rangle$, where α is the involution on $L_{p'}^{2m-1}(q)$ as defined in Section 2.3. When $4 \nmid p$, that is p' is odd, we have $L_{p'}^{2m-1}(q) \simeq_2 \mathbb{S}^{2m-1}$ and hence $L_p^{2m-1}(q) \simeq_2 \mathbb{R}P^{2m-1}$. Therefore it amounts to determining the cohomology algebra of orbit spaces of free involutions on odd dimensional mod 2 cohomology real projective spaces.

Proposition 2.4.2. *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 \mathbb{R}P^{2m-1}$, where $m \geq 1$, then*

$$H^*(X/G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y]/\langle x^2, y^m\rangle,$$

where deg(x)=1 and deg(y)=2.

Proof. Note that for $m = 1$, the proposition is obvious. Assume $m \geq 2$. Let $a \in H^1(X)$ be the generator of the cohomology algebra $H^*(X)$. As there are no fixed points and $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X)$, by Theorem 1.4.11, the spectral sequence do not degenerate at the E_2 term. Therefore $d_2(1 \otimes a) = t^2 \otimes 1$. One can see that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is the trivial homomorphism for l even and an isomorphism for l odd. Note that $d_r = 0$ for all $r \geq 3$ and for all k, l. Hence $E_\infty^{*,*} = E_3^{*,*}$. This gives

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0,\, 1 \text{ and } l = 0,\, 2,\ldots,\, 2m-2 \\ 0 & \text{otherwise.} \end{cases}$$

But

$$H^j(X_G) = \begin{cases} E_\infty^{0,j} & \text{if } j \text{ even} \\ E_\infty^{1,j-1} & \text{if } j \text{ odd.} \end{cases}$$

Therefore

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq j \leq 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0}$ be determined by $t \otimes 1 \in E_2^{1,0}$ and $x^2 \in E_\infty^{2,0} = 0$. The element $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $y \in E_\infty^{0,2} = H^2(X_G)$. Also $i^*(y) = a^2$ and $y^m = 0$. Since the multiplication

$$x \cup (-) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k \leq 2m-2$, we have $xy^r \neq 0$ for $0 \leq r \leq m-1$. Therefore we get

$$H^*(X_G) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,$$

where $deg(x) = 1$ and $deg(y) = 2$. As the action of G is free, by Theorem 1.4.8, $H^*(X/G) \cong H^*(X_G)$. This gives the case (2) of Theorem 2.1.1. $\square$

Remark 2.4.1. It is well known that there is no free involution on a finitistic space $X \simeq_2 \mathbb{R}P^{2m}$. For, the Floyd's Euler characteristic formula (Theorem 1.4.2)

$$\chi(X) + \chi(X^G) = 2\chi(X/G)$$

gives a contradiction as $\chi(X) = 1$ and $\chi(X^G) = 0$.

Remark 2.4.2. The above result follows easily for free involutions on $\mathbb{R}P^3$. Let there be a free involution on $\mathbb{R}P^3$. This lifts to a free action on $\mathbb{S}^3$ by a group H of order 4 and $\mathbb{R}P^3/\mathbb{Z}_2 = \mathbb{S}^3/H$. There are only two groups of order 4, namely, the cyclic group $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Theorem 1.4.4, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot act freely on $\mathbb{S}^3$. Hence H must be the cyclic group $\mathbb{Z}_4$. Now by Rice [55], this action is equivalent to an orthogonal free action and hence $\mathbb{R}P^3/\mathbb{Z}_2 = L_4^3(q)$.

2.4.3 When $4 \mid p$**.**

As above $L_p^{2m-1}(q) = L_{2p'}^{2m-1}(q) = L_{p'}^{2m-1}(q)/\langle \alpha \rangle$. Since $4 \mid p$, that is p' is even, the cohomology groups $H^i(L_{p'}^{2m-1}(q)) = \mathbb{Z}_2$ for $0 \leq i \leq 2m-1$ and 0 otherwise. The Smith-Gysin sequence of the orbit map $\eta : L_{p'}^{2m-1}(q) \to L_{2p'}^{2m-1}(q)$, which is a 0-sphere bundle, is given by

$$\begin{gathered} 0 \to H^0(L_{2p'}^{2m-1}(q)) \xrightarrow{\eta^*} H^0(L_{p'}^{2m-1}(q)) \xrightarrow{\tau} H^0(L_{2p'}^{2m-1}(q)) \xrightarrow{\cup v} H^1(L_{2p'}^{2m-1}(q)) \xrightarrow{\eta^*} \\ \cdots \xrightarrow{\cup v} H^{2m-1}(L_{2p'}^{2m-1}(q)) \xrightarrow{\eta^*} H^{2m-1}(L_{p'}^{2m-1}(q)) \xrightarrow{\tau} H^{2m-1}(L_{2p'}^{2m-1}(q)) \to 0, \end{gathered}$$

where τ is the transfer map. By exactness the cup-square v^2 of the characteristic class $v \in H^1(L_{2p'}^{2m-1}(q))$ is zero. This gives the cohomology algebra

$$H^*(X) \cong H^*(L_{2p'}^{2m-1}(q)) \cong \wedge[v] \otimes \mathbb{Z}_2[w]/\langle w^m \rangle \cong \mathbb{Z}_2[v, w]/\langle v^2, w^m \rangle,$$

where $v \in H^1(L^{2m-1}_{2p'}(q))$ and $w \in H^2(L^{2m-1}_{2p'}(q))$.

Two non-trivial examples of this case are $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$ and the Dold manifold $P(1, m-1)$. We will elaborate these examples in forthcoming sections.

Let u, v be generators of $H^*(X) = H^*(L^{2m-1}_{2p'}(q))$ as above. As the group $G = \mathbb{Z}_2$ acts freely on X with trivial action on $H^*(X)$, the spectral sequence does not degenerate at the E_2 term. If $d_2 = 0$, then $d_3 \neq 0$, otherwise, the spectral sequence degenerate at the E_2 term. Thus we have the following proposition.

Proposition 2.4.3. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L^{2m-1}_p(q)$, where $4 \mid p$. Let $\{E^{*,*}_r, d_r\}$ be the Leray spectral sequence associated to the fibration $X \overset{i}{\hookrightarrow} X_G \xrightarrow{\rho} B_G$. If u, v are generators of $H^*(X)$ such that $d_2 = 0$, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z]/\langle x^3, y^2, z^{\frac{m}{2}} \rangle,$$

where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ and m is even.

Proof. As $d_2 = 0$, we have that $d_3 \neq 0$, otherwise, the spectral sequence degenerate at the E_2 term. Since $d_3(1 \otimes u) = 0$, we must have $d_3(1 \otimes v) = t^3 \otimes 1$. By the multiplicative property of d_3, we have

$$d_3(1 \otimes v^q) = \begin{cases} t^3 \otimes v^{q-1} & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even} \end{cases}$$

Similarly

$$d_3(1 \otimes uv^q) = \begin{cases} t^3 \otimes uv^{q-1} & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even} \end{cases}$$

This shows that

$$d_3 : E_3^{k,l} \to E_3^{k+3,l-2}$$

is an isomorphism for $l = 4q+2, 4q+3$ and zero for $l = 4q, 4q+1$. Also note that $v^m = 0$. If m is odd, then

$$0 = d_3(1 \otimes v^m) = d_3\big((1 \otimes v^{m-1})(1 \otimes v)\big) = t^3 \otimes v^{m-1},$$

a contradiction. Hence m must be even, say $m = 2n$ for some $n \geq 1$. Therefore

$$E_4^{k,l} = \begin{cases} E_3^{k,l} & \text{if } k = 0, 1, 2 \text{ and } l = 4q, 4q+1, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $d_r = 0$ for all $r \geq 4$ and for all k, l as $E_r^{k+r, l-r+1} = 0$. Therefore $E_\infty^{*,*} = E_4^{*,*}$ and the additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } j = 4q, 4q+3, \text{ where } 0 \leq q \leq n-1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } j = 4q+1, 4q+2, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0}$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{3,0} = 0$, we have $x^3 = 0$. Note that $1 \otimes u \in E_2^{0,1}$ is a permanent cocycle and hence determines an element say $y \in E_\infty^{0,1}$. Also $i^*(y) = u$ and $E_\infty^{0,2} = 0$ implies $y^2 = 0$. Similarly $1 \otimes v^2$ is a permanent cocycle and therefore it determines an element say $z \in E_\infty^{0,4} = H^4(X_G)$. Also $i^*(z) = v^2$ and $E_\infty^{0,4n} = 0$ implies $z^{\frac{m}{2}} = 0$. Since the cup product by x

$$x \smile (-) : H^j(X_G) \to H^{j+1}(X_G)$$

is an isomorphism for $0 \leq j \leq 2m-2$, we have $xz^r \neq 0$ for $0 \leq r \leq \frac{m-1}{2}$. Therefore

$$H^*(X_G) \cong \mathbb{Z}_2[x, y, z]/\langle x^3, y^2, z^{\frac{m}{2}} \rangle,$$

where $deg(x) = 1$, $deg(y) = 1$ and $deg(z) = 4$. As the action of G is free, we have $H^*(X/G) \cong H^*(X_G)$. This is the case (3) of the main theorem. $\square$

Next, we consider $d_2 \neq 0$, for which we have the following possibilities:

(A) $d_2(1 \otimes u) = t^2 \otimes 1$ and $d_2(1 \otimes v) = t^2 \otimes u$,

(B) $d_2(1 \otimes u) = t^2 \otimes 1$ and $d_2(1 \otimes v) = 0$ and

(C) $d_2(1 \otimes u) = 0$ and $d_2(1 \otimes v) = t^2 \otimes u$.

We consider the above possibilities one by one. We first observe that the possibility (A) does not arise. Suppose that $d_2(1 \otimes u) = t^2 \otimes 1$ and $d_2(1 \otimes v) = t^2 \otimes u$. By the multiplicative property of d_2, we have

$$d_2(1 \otimes v^q) = \begin{cases} t^2 \otimes uv^{q-1} & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even} \end{cases}$$

and $d_2(1 \otimes uv^q) = t^2 \otimes v^q$ for $0 < q < m$. This shows that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism if l even and $4 \nmid l$ or l odd. And d_2 is zero if $4 \mid l$. Just as in the previous proposition, m must be even, say $m = 2n$ for some $n \geq 1$. This gives

$$E_3^{k,l} = \begin{cases} E_2^{k,l} & \text{if } k = 0, 1 \text{ and } l = 4q, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $d_r = 0$ for all $r \geq 3$ and for all k, l as $E_r^{k+r,l-r+1} = 0$. Therefore $E_\infty^{*,*} = E_3^{*,*}$ and

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } l = 4q, 4q+1, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this shows that $H^{2m-1}(X/G) \cong H^{2m-1}(X_G) = 0$. But the Smith-Gysin sequence

$$\cdots \to H^{2m-1}(X/G) \xrightarrow{\eta^*} H^{2m-1}(X) \xrightarrow{\tau} H^{2m-1}(X/G) \to 0$$

implies that $H^{2m-1}(X) = 0$, which is a contradiction. Hence this possibility does not arise.

For the possibility (B), we have the following proposition.

Proposition 2.4.4. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q)$, where $4 \mid p$. Let $\{E_r^{*,*}, d_r\}$ be the Leray spectral sequence associated to the fibration $X \overset{i}{\hookrightarrow} X_G \xrightarrow{\rho} B_G$. If u, v are generators of $H^*(X)$ such that $d_2(1 \otimes u) \neq 0$ and $d_2(1 \otimes v) = 0$, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,$$

where $deg(x) = 1$ and $deg(y) = 2$.

Proof. Let $d_2(1 \otimes u) = t^2 \otimes 1$ and $d_2(1 \otimes v) = 0$. Consider

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}.$$

If $l = 2q$, then $d_2(t^k \otimes v^q) = 0$ and if $l = 2q+1$, then $d_2(t^k \otimes uv^q) = t^{k+2} \otimes v^q$ for $0 \leq q \leq m-1$. This gives

$$E_3^{k,l} = \begin{cases} E_2^{k,l} & \text{if } k = 0, 1 \text{ and } l = 0, 2, \ldots, 2m-2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$$

is zero for all $r \geq 3$ and for all k, l as $E_r^{k+r,l-r+1} = 0$. This gives $E_\infty^{*,*} = E_3^{*,*}$. But

$$H^j(X_G) = \begin{cases} E_\infty^{0,j} & \text{if } j \text{ even} \\ E_\infty^{1,j-1} & \text{if } j \text{ odd.} \end{cases}$$

Therefore

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq j \leq 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0}$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{2,0} = 0$, we have $x^2 = 0$. Note that $1 \otimes v$ is a permanent cocycle and therefore it determines an element say $y \in E_\infty^{0,2} = H^2(X_G)$. Also $i^*(y) = v$ and $E_\infty^{0,2m} = 0$ implies $y^m = 0$. Since the cup product by x

$$x \smile (-) : H^j(X_G) \to H^{j+1}(X_G)$$

is an isomorphism for $0 \leq j \leq 2m-2$, we have $xy^r \neq 0$ for $0 \leq r \leq m-1$. Therefore

$$H^*(X_G) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,$$

where $deg(x) = 1$ and $deg(y) = 2$. As the action of G is free, $H^*(X/G) \cong H^*(X_G)$. Again we get the case (2) of the main theorem. $\square$

Finally, for the possibility (C), we have the following proposition.

Proposition 2.4.5. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q)$, where $4 \mid p$. Let $\{E_r^{*,*}, d_r\}$ be the Leray spectral sequence associated to the fibration $X \overset{i}{\hookrightarrow} X_G \xrightarrow{\rho} B_G$. If u, v are generators of $H^*(X)$ such that $d_2(1 \otimes u) = 0$ and $d_2(1 \otimes v) \neq 0$, then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded commutative algbras:*

(i) $\mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^{\frac{m}{2}}, x^2y\rangle$,

where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$ and m is even.

(ii) $\mathbb{Z}_2[x, y, w, z]/\langle x^5, y^2, w^2, z^{\frac{m}{4}}, x^2y, wy\rangle$,

where $deg(x) = 1$, $deg(y) = 1$, $deg(w) = 3$, $deg(z) = 8$ and $4 \mid m$.

Proof. Let $d_2(1 \otimes u) = 0$ and $d_2(1 \otimes v) = t^2 \otimes u$. The derivation property of the differential gives

$$d_2(1 \otimes v^q) = \begin{cases} t^2 \otimes uv^{q-1} & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even.} \end{cases}$$

Also $d_2(1 \otimes uv^q) = 0$ for $0 < q < m$. Note that $v^m = 0$. If m is odd, then

$$0 = d_2(1 \otimes v^m) = d_2\big((1 \otimes v^{m-1})(1 \otimes v)\big) = t^2 \otimes uv^{m-1},$$

a contradiction. Hence m must be even, say $m = 2n$ for some $n \geq 1$. From this we get

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism if l even and $4 \nmid l$, and is zero if l odd or $4 \mid l$. This gives

$$(\star) \qquad E_3^{k,l} = \begin{cases} E_2^{k,l} & \text{if } k \geq 0 \text{ arbitrary and } l = 4q, 4q+3, \text{ where } 0 \leq q \leq n-1 \\ E_2^{k,l} & \text{if } k = 0, 1 \text{ and } l = 4q+1, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the differentials one by one. First, we consider

$$d_3 : E_3^{k,l} \to E_3^{k+3,l-2}.$$

Clearly $d_3 = 0$ for all k and for $l = 4q, 4q+3$ as $E_3^{k+3,l-2} = 0$ in this case. For $k = 0, 1$ and for $l = 4(q+1) + 1 = 4q + 5$,

$$d_3 : E_3^{k,4q+5} \to E_3^{k+3,4q+3}$$

is also zero, because if $a \in E_3^{k,4q+5}$ and $d_3(a) = [t^{k+3} \otimes uv^{2q+1}]$, then for $b = [t^2 \otimes 1] \in E_3^{2,0}$, we have $ab \in E_3^{k+2,4q+5} = 0$ and hence

$$0 = d_3(ab) = d_3(a)b + ad_3(b) = d_3(a)b + 0 = [t^{k+5} \otimes uv^{2q+1}],$$

which is a contradiction. Hence $d_3 = 0$ for all k, l.

Next we break the remaining proof in the following two cases:

(a) $d_4 : E_4^{0,3} \to E_4^{4,0}$ is non-zero.

(b) $d_4 : E_4^{0,3} \to E_4^{4,0}$ is zero.

(a) When $d_4 : E_4^{0,3} \to E_4^{4,0}$ is non-zero

Let $d_4([1 \otimes uv]) = [t^4 \otimes 1]$. This gives

$$d_4 : E_4^{k,l} \to E_4^{k+4,l-3}$$

is an isomorphism for all k and for $l = 4q+3$, where $0 \leq q \leq n-1$ and zero otherwise. This gives

$$E_5^{k,l} = \begin{cases} E_4^{k,l} & \text{if } k = 0,1,2,3 \text{ and } l = 4q, \text{ where } 0 \leq q \leq n-1 \\ E_4^{k,l} & \text{if } k = 0,1 \text{ and } l = 4q+1, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $d_r = 0$ for all $r \geq 5$ and for all k, l as $E_r^{k+r,l-r+1} = 0$. Hence, $E_\infty^{*,*} = E_5^{*,*}$ and the additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } j = 4q, 4q+3, \text{ where } 0 \leq q \leq n-1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } j = 4q+1, 4q+2, \text{ where } 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

We see that $1 \otimes v^2 \in E_2^{0,4}$ and $1 \otimes u \in E_2^{0,1}$ are permanent cocycles. Hence, they determine elements $z \in E_\infty^{0,4} \subseteq H^4(X_G)$ and $y \in E_\infty^{0,1} \subseteq H^1(X_G)$, respectively. As

$H^4(X_G) = E_\infty^{0,4} = E_2^{0,4}$, we have $i^*(z) = v^2$. Since $E_\infty^{0,4n} = 0$, we get $z^n = 0$. Similarly, $i^*(y) = u$ and $E_\infty^{0,2} = 0$ implies $y^2 = 0$.

Let $x = \rho^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{4,0} = 0$, we have $x^4 = 0$. Also, the cup product $x^2y \in E_\infty^{2,1} = 0$. Hence,

$$H^*(X_G) \cong \mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^{\frac{m}{2}}, x^2y\rangle,$$

where $deg(x) = 1$, $deg(y) = 1$ and $deg(z) = 4$. As the action of G is free, $H^*(X/G) \cong H^*(X_G)$. This gives the case (4) of the main theorem.

(b) When $d_4 : E_4^{0,3} \to E_4^{4,0}$ is zero

We show that

$$d_4 : E_4^{k,l} \to E_4^{k+4,l-3}$$

is zero for all k, l. Note that $E_4^{k+4,l-3} = 0$ for all k and for $l = 4q$. Similarly $E_4^{k+4,l-3} = 0$ for $k = 0, 1$ and for $l = 4q + 1$. Now for any k and $l = 4q + 3$, we have that

$$d_4 : E_4^{k,4q+3} \to E_4^{k+4,4q}$$

is given by

$$d_4([t^k \otimes uv^{2q+1}]) = (d_4[t^k \otimes uv])[1 \otimes v^{2q}] + [t^k \otimes uv](d_4[1 \otimes v^{2q}]) = 0.$$

This shows that $d_4 = 0$ for all k, l.

Now we have the following two subcases:

(b1) $d_5 : E_5^{0,4} \to E_5^{5,0}$ is non-zero.

(b2) $d_5 : E_5^{0,4} \to E_5^{5,0}$ is zero.

(b1) When $d_5 : E_5^{0,4} \to E_5^{5,0}$ is non-zero

Let $d_5([1 \otimes v^2]) = [t^5 \otimes 1]$. Then

$$d_5([1 \otimes v^{2q}]) = q[t^5 \otimes v^{2(q-1)}] = \begin{cases} [t^5 \otimes v^{2(q-1)}] & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even} \end{cases}$$

and

$$d_5([1 \otimes uv^{2q+1}]) = q[t^5 \otimes uv^{2q-1}] = \begin{cases} [t^5 \otimes uv^{2q-1}] & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even.} \end{cases}$$

Note that $1 \otimes uv^{m+1} = 0$ and hence $0 = d_5([1 \otimes uv^{m+1}]) = n[t^5 \otimes uv^{m-1}]$. But this is possible only when $2 \mid n$ and hence $4 \mid m$.

From above we obtain

$$E_6^{k,l} = \begin{cases} E_5^{k,l} & \text{if } k = 0, 1, 2, 3, 4 \text{ and } l = 8q, 8q+3, \text{ where } 0 \leq q \leq \frac{n-2}{2} \\ E_5^{k,l} & \text{if } k = 0, 1 \text{ and } l = 8q+1, 8q+5, \text{ where } 0 \leq q \leq \frac{n-2}{2} \\ 0 & \text{otherwise.} \end{cases}$$

One can see that $d_r = 0$ for all $r \geq 6$ and for all k, l as $E_r^{k+r,l-r+1} = 0$. Hence, $E_\infty^{*,*} = E_6^{*,*}$ and the additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } j = 8q, 8q+7, \text{ where } 0 \leq q \leq \frac{n-2}{2} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } 8q < j < 84q+7, \text{ where } 0 \leq q \leq \frac{n-2}{2} \\ 0 & \text{otherwise.} \end{cases}$$

We see that $1 \otimes v^4 \in E_2^{0,8}$, $1 \otimes uv \in E_2^{0,3}$ and $1 \otimes u \in E_2^{0,1}$ are permanent cocycles. Hence, they determine elements $z \in E_\infty^{0,8} \subseteq H^8(X_G)$, $w \in E_\infty^{0,3} \subseteq H^3(X_G)$ and $y \in E_\infty^{0,1} \subseteq H^1(X_G)$, respectively. As $H^8(X_G) = E_\infty^{0,8} = E_2^{0,8}$, we have $i^*(z) = v^4$. Also $E_\infty^{0,2m} = 0$ implies $z^{\frac{m}{4}} = 0$. Similarly, $i^*(w) = uv$ and $E_\infty^{0,6} = 0$ implies $w^2 = 0$. Finally, $i^*(y) = u$ and $E_\infty^{0,2} = 0$ implies $y^2 = 0$.

Let $x = \rho^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{5,0} = 0$, we have $x^5 = 0$. Also, the only trivial cup products are $x^2y \in E_\infty^{2,1} = 0$ and $wy \in E_\infty^{0,4} = 0$. Hence,

$$H^*(X_G) \cong \mathbb{Z}_2[x, y, w, z]/\langle x^5, y^2, w^2, z^{\frac{m}{4}}, x^2y, wy \rangle,$$

where $deg(x) = 1$, $deg(y) = 1$, $deg(w) = 3$ and $deg(z) = 8$. As the action of G is free, $H^*(X/G) \cong H^*(X_G)$. This gives the case (5) of the main theorem.

(b2) When $d_5 : E_5^{0,4} \to E_5^{5,0}$ is zero

We show that

$$d_5 : E_5^{k,l} \to E_5^{k+5,l-4}$$

is zero for all k, l. Note that $E_5^{k+5,l-4} = 0$ for $k = 0, 1$ and for $l = 4q + 1$. For any k and for $l = 4q, 4q + 3$, d_5 is zero by the derivation property of d_5 and the above condition (b2). Hence $d_5 = 0$ for all k, l.

We have that d_2, d_3, d_4 and d_5 are all zero for the values of k and l given by equation $(\star)$. Note that only $1 \otimes u$, $1 \otimes v^2$ and $1 \otimes uv$ survives to the E_6 term. For $r \geq 6$, a typical non-zero element in $E_r^{k,l}$ is of the form $[t^k \otimes v^{2q}]$, $[t^k \otimes uv^{2q+1}]$ or $[t^k \otimes uv^{2q}]$ according as $l = 4q, 4q + 3, 4q + 1$ for $0 \leq q \leq n - 1$, respectively. But all these elements can be written as a product of previous three elements for which $d_r = 0$ for $r \geq 6$. Hence $E_\infty^{*,*} = E_3^{*,*}$. This gives $H^j(X_G) \neq 0$ for $j > 2m - 1$ (in particular $H^{2m}(X_G) \neq 0$), which is a contradiction. Hence (b2) does not arise. $\square$

With this we have completed the proof of the main theorem.

2.5 Examples realizing the cohomology algebras

In this section we provide some examples realizing the possible cohomology algebras of the main theorem.

- The case (1) of the main theorem can be realized by taking any free involution on a sphere.

- The Smith-Gysin sequence shows that the example discussed in section 2.3 realizes the case (2). For another example, let $X = \mathbb{S}^1 \times \mathbb{C}P^{m-1}$, where $m \geq 2$. The mod 2 cohomology algebra of X is given by

$$H^*(X) \cong \mathbb{Z}_2[u, v]/\langle u^2, v^m \rangle,$$

where $deg(u) = 1$ and $deg(v) = 2$. Note that X always admits a free involution as $\mathbb{S}^1$ does so. Taking any free involution on $\mathbb{S}^1$ and the trivial action on $\mathbb{C}P^{m-1}$ gives $X/G = \mathbb{S}^1 \times \mathbb{C}P^{m-1}$. Hence

$$H^*(X/G) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m\rangle,$$

where $deg(x) = 1$ and $deg(y) = 1$. This also realizes the case (2) of the main theorem.

- We now construct an example for the case (3). Let X be as above. If m is even, then $\mathbb{C}P^{m-1}$ always admits a free involution. In fact, if we denote an element of $\mathbb{C}P^{m-1}$ by $[z_1, z_2, \ldots, z_{m-1}, z_m]$, then the map

$$[z_1, z_2, \ldots, z_{m-1}, z_m] \mapsto [-\overline{z}_2, \overline{z}_1, \ldots, -\overline{z}_m, \overline{z}_{m-1}]$$

defines an involution on $\mathbb{C}P^{m-1}$. Now if

$$[z_1, z_2, \ldots, z_{m-1}, z_m] = [-\overline{z}_2, \overline{z}_1, \ldots, -\overline{z}_m, \overline{z}_{m-1}],$$

then there exits a $\lambda \in \mathbb{S}^1$ such that

$$(\lambda z_1, \lambda z_2, \ldots, \lambda z_{m-1}, \lambda z_m) = (-\overline{z}_2, \overline{z}_1, \ldots, -\overline{z}_m, \overline{z}_{m-1}).$$

This gives $z_1 = z_2 = \cdots = z_{m-1} = z_m = 0$, a contradiction. Hence the involution is free. The mod 2 cohomology algebra of orbit spaces of free involutions on odd dimensional complex projective spaces was determined in [59]. More precisely, it was proved that: For any free involution on $\mathbb{C}P^{m-1}$, where $m \geq 2$ is even, the mod 2 cohomology algebra of the orbit space is given by

$$H^*(\mathbb{C}P^{m-1}/G) \cong \mathbb{Z}_2[x, z]/\langle x^3, z^{\frac{m}{2}}\rangle,$$

where $deg(x) = 1$ and $deg(z) = 4$. Taking the trivial involution on $\mathbb{S}^1$ and a free involution on $\mathbb{C}P^{m-1}$, we have that $X/G = \mathbb{S}^1 \times \big(\mathbb{C}P^{m-1}/G\big)$. Using the above result, we have

$$H^*(X/G) \cong \mathbb{Z}_2[x, y, z]/\langle x^3, y^2, z^{\frac{m}{2}}\rangle,$$

where $deg(x) = 1$, $deg(y) = 1$ and $deg(z) = 4$. This realizes the case (3) of the main theorem.

- We do not have examples realizing the cases (4) and (5) of the main theorem. However, we feel that Dold manifolds may give some examples realizing these cases. For integers $r, s \geq 0$, a Dold manifold $P(r, s)$ is defined as

$$P(r,s) = \mathbb{S}^r \times \mathbb{C}P^s / \sim,$$

where $\big((x_1, \dots, x_{r+1}), [z_1, \dots, z_{s+1}]\big) \sim \big((-x_1, \dots, -x_{r+1}), [\overline{z}_1, \dots, \overline{z}_{s+1}]\big)$. Consider the equivariant projection $\mathbb{S}^r \times \mathbb{C}P^s \to \mathbb{S}^r$. On passing to orbit spaces, the Dold manifold can also be seen as the total space of the fiber bundle

$$\mathbb{C}P^s \hookrightarrow P(r,s) \to \mathbb{R}P^r.$$

The mod 2 cohomology algebra of a Dold manifold is well known [18] and is given by

$$H^*(P(r,s); \mathbb{Z}_2) \cong \mathbb{Z}_2[x,y]/\langle x^{r+1}, y^{s+1}\rangle,$$

where $deg(x) = 1$ and $deg(y) = 2$. Note that the Dold manifold $P(1, m-1) \simeq_2 L_p^{2m-1}(q)$ for $4 \mid p$ and can be considered as the twisted analogue of $X = \mathbb{S}^1 \times \mathbb{C}P^{m-1}$. For m even, the free involution on $\mathbb{C}P^{m-1}$ induces a free involution on $P(1, m-1)$. We feel that some exotic free involutions on $P(1, m-1)$ would possibly realize the cases (4) and (5) of the main theorem.

We conclude with the following remarks.

Remark 2.5.1. For the three dimensional lens space $L_p^3(q)$, where $p = 4k$ for some k, Kim [35, Theorem 3.6] has shown that the orbit space of any sense-preserving free involution on $L_p^3(q)$ is the lens space $L_{2p}^3(q')$, where $q'q \equiv \pm 1$ or $q' \equiv \pm q$ mod p. Here an involution is sense-preserving if the induced map on $H_1(L_p^3(q); \mathbb{Z})$ is the identity map. This is the case (2) of the main theorem.

Remark 2.5.2. If T is a free involution on $L_p^3(q)$ where p is an odd prime, then $\mathbb{Z}_p$ and the lift of T to $\mathbb{S}^3$ generate a group H of order $2p$ acting freely on $\mathbb{S}^3$. The involution T is said to be orthogonal if the action of H on $\mathbb{S}^3$ is orthogonal. Myers [49] showed that every free involution on $L_p^3(q)$ is conjugate to an orthogonal free involution. It is well known that there are only two groups of order $2p$, namely the cyclic group $\mathbb{Z}_{2p}$ and the dihedral group D_{2p}. But by Milnor [45], the dihedral group cannot act freely and orthogonally on $\mathbb{S}^3$. Hence H must be the cyclic group $\mathbb{Z}_{2p}$ acting freely and orthogonally on $\mathbb{S}^3$. Therefore the orbit space $L_p^3(q)/\langle T\rangle = \mathbb{S}^3/H = L_{2p}^3(q)$. Since p is odd, $L_p^3(q) \simeq_2 \mathbb{S}^3$ and $L_p^3(q)/\langle T\rangle \simeq_2 \mathbb{R}P^3$, which is the case (1) of the main theorem.

2.6 Application to $\mathbb{Z}_2$-equivariant maps

Let X be a paracompact Hausdorff space with a fixed free involution and let $\mathbb{S}^n$ be the unit n-sphere equipped with the antipodal involution. Conner and Floyd [13] asked the following question.

Question: For which integer n, is there a $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n$ to X, but no such map from $\mathbb{S}^{n+1}$ to X?

In view of the Borsuk-Ulam theorem, the answer to the question for $X = \mathbb{S}^n$ is n. Motivated by the classical results of Lyusternik- Shnirel'man [41], Borsuk-Ulam [3], Yang [79, 80, 81] and Bourgin [4], Conner and Floyd defined the index of the involution on X as

$$\mathrm{ind}(X) = max\{n \mid \text{there exist a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^n \to X\}.$$

It is natural to consider the purely cohomological criteria to study the above question. The best known and most easily managed cohomology class are the characteristic classes with coefficients in $\mathbb{Z}_2$. Let $w \in H^1(X/G;\mathbb{Z}_2)$ be the Stiefel-Whitney class of the principal G-bundle $X \to X/G$. Generalizing the Yang's index [80], Conner and Floyd defined

$$\text{co-ind}_{\mathbb{Z}_2}(X) \;=\; \text{largest integer } n \text{ such that } w^n \neq 0.$$

Since $\text{co-ind}_{\mathbb{Z}_2}(\mathbb{S}^n) = n$, by [13, (4.5)], we have

$$\text{ind}(X) \leq \text{co-ind}_{\mathbb{Z}_2}(X).$$

Also, since X is paracompact Hausdorff, we can take a classifying map

$$c : X/G \to B_G$$

for the principal G-bundle $X \to X/G$. If $k : X/G \to X_G$ is a homotopy equivalence, then $\rho k : X/G \to B_G$ also classifies the principal G-bundle $X \to X/G$ and hence it is homotopic to c. Therefore it suffices to consider the map

$$\rho^* : H^1(B_G; \mathbb{Z}_2) \to H^1(X_G; \mathbb{Z}_2).$$

The image of the Stiefel-Whitney class of the universal principal G-bundle $G \hookrightarrow E_G \longrightarrow B_G$ is the Stiefel-Whitney class of the principal G-bundle $X \to X/G$.

Let $X \simeq_2 L_p^{2m-1}(q)$ be a finitistic space with a free involution. The Smith-Gysin sequence associated to the principal G-bundle $X \to X/G$ shows that the Stiefel-Whitney class is non-zero.

In case (1), $x \in H^1(X/G; \mathbb{Z}_2)$ is the Stiefel-Whitney class with $x^{2m} = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 2m - 1$ and hence $\text{ind}(X) \leq 2m - 1$. Therefore, in this case, there is no $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2m$.

Taking $X = \mathbb{S}^k$ with the antipodal involution, by proposition 2.4.1, we obtain the classical Borsuk-Ulam theorem, which states that: There is no map from $\mathbb{S}^n \to \mathbb{S}^k$ equivariant with respect to the antipodal involutions when $n \geq k + 1$.

In case (2), $x \in H^1(X/G; \mathbb{Z}_2)$ is the Stiefel-Whitney class with $x^2 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 1$ and $\text{ind}(X) \leq 1$. Hence, there is no $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2$.

In case (3), $x \in H^1(X/G; \mathbb{Z}_2)$ is the Stiefel-Whitney class with $x^3 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 2$ and $\text{ind}(X) \leq 2$. Hence, there is no $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 3$.

In case (4), $x \in H^1(X/G;\mathbb{Z}_2)$ is the Stiefel-Whitney class with $x^4 = 0$. This gives co-ind$_{\mathbb{Z}_2}(X) = 3$ and hence ind$(X) \leq 3$. Hence, there is no $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 4$.

Finally, in case (5) of the main theorem, $x \in H^1(X/G;\mathbb{Z}_2)$ is the Stiefel-Whitney class with $x^5 = 0$. This gives co-ind$_{\mathbb{Z}_2}(X) = 4$ and hence ind$(X) \leq 4$. In this case also, there is no $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 5$.

Combining the above discussion, we have proved the following Borsuk-Ulam type result.

Theorem 2.6.1. *Let $m \geq 3$ and $X \simeq_2 L_p^{2m-1}(q)$ be a finitistic space with a free involution. Then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n \to X$ for $n \geq 2m$.*

Chapter 3

Parametrized Borsuk-Ulam Problem for Projective Space Bundles

3.1 Introduction

The unit n-sphere $\mathbb{S}^n$ is equipped with the antipodal involution given by $x \mapsto -x$. One formulation of the classical Borsuk-Ulam theorem (proved by Borsuk in 1933 [3]) states that, if $n \geq k$ then for every continuous map $f : \mathbb{S}^n \to \mathbb{R}^k$ there exist a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. Over the years there have been several generalizations of this theorem in many directions. We refer the reader to the article [72] by Steinlein which lists 457 publications concerned with various generalizations of the Borsuk-Ulam theorem. Also, the recent book by Matoušek [42] contains a detailed account of various applications of the Borsuk-Ulam theorem.

Jaworowski [30], Dold [19], Nakaoka [51] and others extended this theorem to the setting of fiber bundles, by considering fiber preserving maps $f : SE \to E^{'}$, where SE denotes the total space of the sphere bundle $SE \to B$ associated to a vector bundle $E \to B$, and $E^{'} \to B$ is other vector bundle. Thus, they parametrized the Borsuk-Ulam theorem, whose general formulation is as follows:

Let G be a compact Lie group. Consider a fiber bundle $\pi : E \to B$ and a vector bundle

$\pi^{'} : E^{'} \to B$ such that G acts fiber preserving and freely on E and $E^{'} - 0$, where 0 stands for the zero section of the bundle $\pi^{'} : E^{'} \to B$. For a fiber preserving G-equivariant map $f : E \to E^{'}$, the parametrized version of the Borsuk-Ulam theorem deals in estimating the cohomological dimension of the set

$$Z_f = \{x \in E \mid f(x) = 0\}.$$

Such results appeared first in the papers of Jaworowski [30], Dold [19] and Nakaoka [51]. Dold [19] and Nakaoka [51] defined certain polynomials, which they called the **characteristic polynomials**, for vector bundles with free G-actions ($G = \mathbb{Z}_p$ or $\mathbb{S}^1$) and used them for obtaining such results. The characteristic polynomials were used by Koikara and Mukerjee [38] to prove a parametrized version of the Borsuk-Ulam theorem for bundles whose fibers are a product of two spheres, with the free involution given by the product of the antipodal involutions. Recently, Mattos and Santos [43] also used the same technique to obtain parametrized Borsuk-Ulam theorems for bundles whose fiber has the mod p cohomology algebra (with $p > 2$) of a product of two spheres with any free $\mathbb{Z}_p$-action and for bundles whose fiber has the rational cohomology algebra of a product of two spheres with any free $\mathbb{S}^1$-action. Jaworowski obtained parametrized Borsuk-Ulam theorems for lens space bundles in [33] and parametrized Borsuk-Ulam theorems for sphere bundles in [30, 31, 32].

Our work proves some parametrized Borsuk-Ulam theorems for bundles whose fibers are finitistic mod 2 cohomology real or complex projective spaces with any free involution. The theorems are proved in Section 3.5. As an application, in Section 3.6, the size of the $\mathbb{Z}_2$-coincidence set of a fiber preserving map is also estimated.

The cohomology used will be the Čech cohomology with $\mathbb{Z}_2$ coefficients. For a space X, *cohom.dim*(X) will mean the cohomological dimension of X with respect to $\mathbb{Z}_2$. We write $X \simeq_2 \mathbb{R}P^n$ to mean that X is a space having the mod 2 cohomology algebra of $\mathbb{R}P^n$. Similarly, we write $X \simeq_2 \mathbb{C}P^n$ to mean that X is a space having the mod 2 cohomology algebra of $\mathbb{C}P^n$. If G is a compact Lie group acting freely on a paracompact Hausdorff space X, then $X \to X/G$ is a principal G-bundle and one can take a classifying map $X/G \to B_G$ for the principal G-bundle $X \to X/G$, where B_G is the classifying space of the group G. Recall that for $G = \mathbb{Z}_2$, $H^*(B_G; \mathbb{Z}_2) \cong \mathbb{Z}_2[s]$, where s is a homogeneous element of degree 1. We will also use some elementary notions about vector bundles for the details of which we refer to Husemoller [29]. Note that using a partition of unity, any vector bundle over a paracompact base space can be given an Euclidean metric. Throughout we will assume that the action on a vector bundle is metric preserving.

3.2 Free involutions on projective spaces

We construct free involutions on odd dimensional real projective spaces. Recall that $\mathbb{R}P^{2m+1}$, where $m \geq 0$, is the orbit space of the antipodal involution on $\mathbb{S}^{2m+1}$ given by

$$(x_1, x_2, ..., x_{2m+1}, x_{2m+2}) \mapsto (-x_1, -x_2, ..., -x_{2m+1}, -x_{2m+2}).$$

If we denote an element of $\mathbb{R}P^{2m+1}$ by $[x_1, x_2, ..., x_{2m+1}, x_{2m+2}]$, then the map $\mathbb{R}P^{2m+1} \to \mathbb{R}P^{2m+1}$ given by

$$[x_1, x_2, ..., x_{2m+1}, x_{2m+2}] \mapsto [-x_2, x_1, ..., -x_{2m+2}, x_{2m+1}]$$

defines an involution. If

$$[x_1, x_2, ..., x_{2m+1}, x_{2m+2}] = [-x_2, x_1, ..., -x_{2m+2}, x_{2m+1}],$$

then

$$(-x_1, -x_2, ..., -x_{2m+1}, -x_{2m+2}) = (-x_2, x_1, ..., -x_{2m+2}, x_{2m+1}),$$

which gives $x_1 = x_2 = ... = x_{2m+1} = x_{2m+2} = 0$, a contradiction. Hence the involution is free.

Similarly, the complex projective space $\mathbb{C}P^m$ admit free involutions when $m \geq 1$ is odd. Recall that $\mathbb{C}P^m$ is the orbit space of free $\mathbb{S}^1$ action on $\mathbb{S}^{2m+1}$ given by

$$(z_1, z_2, ..., z_m, z_{m+1}) \mapsto (\zeta z_1, \zeta z_2, ..., \zeta z_m, \zeta z_{m+1}) \text{ for } \zeta \in \mathbb{S}^1.$$

If we denote an element of $\mathbb{C}P^m$ by $[z_1, z_2, ..., z_m, z_{m+1}]$, then the map

$$[z_1, z_2, ..., z_m, z_{m+1}] \mapsto [-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{m+1}, \overline{z}_m]$$

defines an involution. If

$$[z_1, z_2, ..., z_m, z_{m+1}] = [-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{m+1}, \overline{z}_m],$$

then

$$(\lambda z_1, \lambda z_2, ..., \lambda z_m, \lambda z_{m+1}) = (-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{m+1}, \overline{z}_m)$$

for some $\lambda \in \mathbb{S}^1$, which gives $z_1 = z_2 = ... = z_m = z_{m+1} = 0$, a contradiction. Hence the involution is free.

3.3 Orbit spaces of free involutions on projective spaces

For the purpose of our work, we need to know the cohomology algebra of orbit spaces of free involutions on cohomology projective spaces. For that, we exploit the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_{\mathbb{Z}_2} \longrightarrow B_{\mathbb{Z}_2}$. For the real case we prove the following.

Proposition 3.3.1. *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 \mathbb{R}P^n$, where $n \geq 1$ is odd, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[u, v]/\langle u^2, v^{\frac{n+1}{2}} \rangle,$$

where deg(u)=1 and deg(v)=2.

Proof. This was proved in Chapter 2 as Proposition 2.4.2. $\square$

For the complex case we have the following.

Proposition 3.3.2. *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 \mathbb{C}P^n$, where $n \geq 1$ is odd, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[u, v]/\langle u^3, v^{\frac{n+1}{2}} \rangle,$$

where deg(u)=1 and deg(v)=4.

Proof. If $n = 1$, then $X \simeq_2 S^2$ and hence X/G is a mod 2 cohomology $\mathbb{R}P^2$. Therefore the result holds for $n = 1$. Now assume that $n \geq 2$. As $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X)$, by Theorem 1.4.11, the spectral sequence associated to the Borel fibration do not degenerate at the E_2 term. Since the system of local coefficients is simple, we have

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).$$

Note that $E_2^{k,l} = 0$ for l odd. This gives $d_2 = 0$ and hence $E_2^{*,*} = E_3^{*,*}$. Let $a \in H^2(X)$ be the generator of the cohomology algebra $H^*(X)$. Since, the spectral sequence do not degenerate, we have $d_3(1 \otimes a) = t^3 \otimes 1$. Therefore,

$$d_3 : E_3^{k,2l} \to E_3^{k+3,2l-2}$$

is zero for l even and an isomorphism for l odd. Note that $d_r = 0$ for all $r \geq 4$ and for all k, l. Hence $E_\infty^{*,*} = E_4^{*,*}$. This gives

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0,\ 1\ ,\ 2 \text{ and } l = 0,\ 4, ...,\ 2(n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Since $TotE_\infty^{*,*} \cong H^*(X_G)$, we have $H^i(X_G) \cong E_\infty^{k,i-k}$ where $k = 0,\ 1,\ 2$ and $4 \mid (i - k)$. Hence

$$H^i(X_G) = \begin{cases} 0 & \text{if } i = 3,\ 7,\ 11,\ ... \text{ or } j > 2n \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Let $u = \rho^*(t) \in H^1(X_G) = E_\infty^{1,0}$. Thus, $u^3 = 0$ and u is determined by $t \otimes 1 \in E_2^{1,0}$. The multiplication

$$u \cup (-) : E_\infty^{k,l} \to E_\infty^{k+1,l}$$

is an isomorphism for $k = 0, 1$. The element $1 \otimes a^2 \in E_2^{0,4}$ is a permanent cocycle and hence determines an element say $v \in E_\infty^{0,4}$. We have $i^*(v) = a^2$ and $v^{\frac{n+1}{2}} = 0$ as $H^{2n+2}(X_G) = 0$. Hence

$$TotE_\infty^{*,*} \cong \mathbb{Z}_2[u, v]/\langle u^3, v^{\frac{n+1}{2}} \rangle$$

and therefore

$$H^*(X_G) \cong \mathbb{Z}_2[u, v]/\langle u^3, v^{\frac{n+1}{2}} \rangle.$$

As the action of G is free, by Theorem 1.4.8, $H^*(X/G) \cong H^*(X_G)$. This proves the proposition. $\square$

We note that H. K. Singh and T. B. Singh also obtained the above results in [59].

Remark 3.3.1. Just as in Remark 2.4.1, the Floyd's Euler characteristic formula shows that, $\mathbb{Z}_2$ cannot act freely on a finitistic space $X \simeq_2 \mathbb{R}P^n$ or $\mathbb{C}P^n$ for n even.

Remark 3.3.2. Let $X \simeq_2 \mathbb{H}P^n$, where $\mathbb{H}P^n$ is the quaternionic projective space. For $n = 1$, $X \simeq_2 S^4$, which is dealt in [19]. For $n \geq 2$, there is no free involution on X, which follows from the stronger fact that such spaces have the fixed point property.

Remark 3.3.3. Let $X \simeq_2 \mathbb{O}P^2$, where $\mathbb{O}P^2$ is the Cayley projective plane. Note that $H^*(\mathbb{O}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[u]/\langle u^3 \rangle$, where u is a homogeneous element of degree 8. Just as in Remark 3.3.1, it follows from the Floyd's formula that there is no free involution on X.

3.4 Characteristic polynomials for bundles

Let (X, E, π, B) be a fiber bundle with a fiber preserving free $\mathbb{Z}_2$ action such that the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$ has a cohomology extension of the fiber (Definition 1.4.2). This condition on the bundle is assumed so that the Leray-Hirsch theorem (Theorem 1.4.10) can be applied. With this hypothesis, we now proceed to define the characteristic polynomials for the bundles. We deal the real and the complex case separately.

3.4.1 Characteristic polynomials for $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$

The bundle $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$ has a fiber preserving free $\mathbb{Z}_2$ action. This gives a free G action on a typical fiber, which is a finitistic space $X \simeq_2 \mathbb{R}P^n$. Thus, n is odd and by Proposition 3.3.1, $H^*(X/G; \mathbb{Z}_2)$ is a free graded algebra generated by the elements

$$1,\ u,\ v,\ uv, ..., v^{\frac{n-1}{2}},\ uv^{\frac{n-1}{2}},$$

subject to the relations $u^2 = 0$ and $v^{\frac{n+1}{2}} = 0$, where $u \in H^1(X/G; \mathbb{Z}_2)$ and $v \in H^2(X/G; \mathbb{Z}_2)$. By the Leray-Hirsch theorem (Theorem 1.4.10), there exist elements $a \in H^1(\overline{E})$ and $b \in H^2(\overline{E})$ such that the restriction to a typical fiber $j^* : H^*(\overline{E}) \to H^*(X/G)$ maps $a \mapsto u$ and $b \mapsto v$. Note that $H^*(\overline{E})$ is a $H^*(B)$-module, via the induced homomorphism $\overline{\pi}^*$ and is generated by the basis

$$1,\ a,\ b,\ ab, ..., b^{\frac{n-1}{2}},\ ab^{\frac{n-1}{2}}.$$

We can express the element $b^{\frac{n+1}{2}} \in H^{n+1}(\overline{E})$ in terms of the above basis. Therefore, there exist unique elements $w_i \in H^i(B)$ such that

$$b^{\frac{n+1}{2}} = w_{n+1} + w_n a + w_{n-1} b + \cdots + w_2 b^{\frac{n-1}{2}} + w_1 a b^{\frac{n-1}{2}}.$$

Similarly, we express the element $a^2 \in H^2(\overline{E})$ as

$$a^2 = \nu_2 + \nu_1 a + \alpha b,$$

where $\nu_i \in H^i(B)$ and $\alpha \in \mathbb{Z}_2$ are unique elements. The characteristic polynomials in the indeterminates x and y, of degrees 1 and 2 respectively, associated to the fiber bundle $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$ are defined by

$$W_1(x, y) = w_{n+1} + w_n x + w_{n-1} y + \cdots + w_2 y^{\frac{n-1}{2}} + w_1 x y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \quad W_2(x, y) = \nu_2 + \nu_1 x + \alpha y + x^2.$$

On substituting the values for the indeterminates x and y, we obtain the following homomorphism of $H^*(B)$-algebras

$$\sigma : H^*(B)[x, y] \to H^*(\overline{E})$$

given by $(x, y) \mapsto (a, b)$. The kernel of σ is the ideal generated by the polynomials $W_1(x, y)$ and $W_2(x, y)$ and hence

$$H^*(B)[x, y]/\langle W_1(x, y), W_2(x, y)\rangle \cong H^*(\overline{E}). \tag{3.4.1}$$

3.4.2 Characteristic polynomials for $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$

There is a free G action on a typical fiber, which is a finitistic space $X \simeq_2 \mathbb{C}P^n$. Thus, n is odd and by Proposition 3.3.2, $H^*(X/G; \mathbb{Z}_2)$ is a free graded algebra generated by the elements

$$1,\ u,\ u^2,\ v,\ uv, ..., v^{\frac{n-1}{2}},\ uv^{\frac{n-1}{2}},\ u^2v^{\frac{n-1}{2}},$$

subject to the relations $u^3 = 0$ and $v^{\frac{n+1}{2}} = 0$, where $u \in H^1(X/G; \mathbb{Z}_2)$ and $v \in H^4(X/G; \mathbb{Z}_2)$.

By the Leray-Hirsch theorem (Theorem 1.4.10), there exist elements $a \in H^1(\overline{E})$ and $b \in H^4(\overline{E})$ such that the restriction to a typical fiber $j^* : H^*(\overline{E}) \to H^*(X/G)$ maps $a \mapsto u$ and $b \mapsto v$. Note that $H^*(\overline{E})$ is a $H^*(B)$-module and is generated by the basis

$$1,\ a,\ a^2,\ b,\ ab, ..., b^{\frac{n-1}{2}},\ ab^{\frac{n-1}{2}},\ a^2b^{\frac{n-1}{2}}.$$

We write $b^{\frac{n+1}{2}} \in H^{2n+2}(\overline{E})$ in terms of the above basis. Thus, there exist unique elements $w_i \in H^i(B)$ such that

$$b^{\frac{n+1}{2}} = w_{2n+2} + w_{2n+1}a + w_{2n}a^2 + \cdots + w_2a^2b^{\frac{n-1}{2}}.$$

Similarly, we write the element $a^3 \in H^3(\overline{E})$ as

$$a^3 = \nu_3 + \nu_2 a + \nu_1 a^2,$$

where $\nu_i \in H^i(B)$ are unique elements. The characteristic polynomials in the indeterminates x and y, of degrees 1 and 4 respectively, associated to the fiber bundle $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$ are defined by

$$W_1(x, y) = w_{2n+2} + w_{2n+1}x + w_{2n}x^2 + \cdots + w_2x^2y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \;\; W_2(x) = \nu_3 + \nu_2 x + \nu_1 x^2 + x^3.$$

This gives a homomorphism of $H^*(B)$-algebras

$$\sigma : H^*(B)[x,y] \to H^*(\overline{E})$$

given by $(x,y) \mapsto (a,b)$ and having kernel σ as the ideal generated by the polynomials $W_1(x,y)$ and $W_2(x)$. Hence

$$H^*(B)[x,y]/\langle W_1(x,y), W_2(x)\rangle \cong H^*(\overline{E}). \tag{3.4.2}$$

3.4.3 Characteristic polynomial for $\pi^{'} : E^{'} \to B$

We now define the characteristic polynomial associated to the k-dimensional vector bundle $\pi' : E^{'} \to B$ with fiber preserving $\mathbb{Z}_2$-action on $E^{'}$ which is free on $E^{'} - 0$. Let $SE^{'}$ denote the total space of the sphere bundle of $\pi' : E^{'} \to B$. Since the action is free on $SE^{'}$, we obtain the projective space bundle $(\mathbb{R}P^{k-1}, \overline{SE^{'}}, \overline{\pi^{'}}, B)$ and the principal $\mathbb{Z}_2$-bundle $SE^{'} \to \overline{SE^{'}}$. We know that $H^*(\mathbb{R}P^{k-1};\mathbb{Z}_2) \cong \mathbb{Z}_2[u^{'}]/\langle {u^{'}}^k\rangle$, where $u^{'} = g^*(s)$ with $s \in H^1(B_G)$ and $g : \mathbb{R}P^{k-1} \to B_G$ is a classifying map for the principal $\mathbb{Z}_2$-bundle $S^{k-1} \to \mathbb{R}P^{k-1}$. Let $h : \overline{SE^{'}} \to B_G$ be a classifying map for the principal $\mathbb{Z}_2$-bundle $SE^{'} \to \overline{SE^{'}}$ and let $a^{'} = h^*(s) \in H^1(\overline{SE^{'}})$. Now the $\mathbb{Z}_2$-module homomorphism $\theta^{'} : H^*(\mathbb{R}P^{k-1}) \to H^*(\overline{SE^{'}})$ given by $u^{'} \mapsto a^{'}$ is a cohomology extension of the fiber. Again, by the Leray-Hirsch theorem, $H^*(\overline{SE^{'}})$ is a $H^*(B)$-module via the induced homomorphism ${\overline{\pi^{'}}}^*$ and is generated by the basis

$$1, a^{'}, {a^{'}}^2, ..., {a^{'}}^{k-1}.$$

We write ${a^{'}}^k \in H^k(\overline{SE^{'}})$ as

$${a^{'}}^k = w^{'}_k + w^{'}_{k-1}a^{'} + \cdots + w^{'}_1 {a^{'}}^{k-1},$$

where $w^{'}_i \in H^i(B)$ are unique elements.

Now the characteristic polynomial in the indeterminate x of degree 1, associated to the vector bundle $\pi' : E^{'} \to B$ is defined as

$$W^{'}(x) = w_k^{'} + w_{k-1}^{'} x + \cdots + w_1^{'} x^{k-1} + x^k.$$

By similar arguments as used earlier, we have the following isomorphism of $H^*(B)$-algebras

$$H^*(B)[x]/\langle W^{'}(x)\rangle \cong H^*(\overline{SE^{'}})$$

given by $x \mapsto a^{'}$.

3.5 Parametrized Borsuk-Ulam theorems

Let (X, E, π, B) be a fiber bundle with a fiber preserving free $\mathbb{Z}_2$-action such that the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$ has a cohomology extension of the fiber and let $\pi' : E^{'} \to B$ be a k-dimensional vector bundle with fiber preserving $\mathbb{Z}_2$-action on $E^{'}$ which is free on $E^{'} - 0$. Let $f : E \to E^{'}$ be a fiber preserving $\mathbb{Z}_2$-equivariant map. Define $Z_f = f^{-1}(0)$ and $\overline{Z_f} = Z_f/\mathbb{Z}_2$, the quotient by the free $\mathbb{Z}_2$-action induced on Z_f.

3.5.1 Borsuk-Ulam theorems for $X \simeq_2 \mathbb{R}P^n$

Let $H^*(B)[x, y]$ be the polynomial ring over $H^*(B)$ in the indeterminates x and y. Note that each polynomial $q(x, y)$ in $H^*(B)[x, y]$ defines an element of $H^*(\overline{E})$, which we denote by $q(x, y)|_{\overline{E}}$. We denote by $q(x, y)|_{\overline{Z_f}}$, the image of $q(x, y)|_{\overline{E}}$ by the $H^*(B)$-homomorphism $i^* : H^*(\overline{E}) \to H^*(\overline{Z_f})$, where i^* is the map induced by the inclusion $i : \overline{Z_f} \hookrightarrow \overline{E}$. With the above hypothesis and notations, we prove the following results.

Theorem 3.5.1. *Let $X \simeq_2 \mathbb{R}P^n$ be a finitistic space. If $q(x, y)$ in $H^*(B)[x, y]$ is a polynomial such that $q(x, y)|_{\overline{Z_f}} = 0$, then there are polynomials $r_1(x, y)$ and $r_2(x, y)$ in $H^*(B)[x, y]$ such that $q(x, y)W^{'}(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x, y)$ in the ring $H^*(B)[x, y]$, where $W^{'}(x)$, $W_1(x, y)$ and $W_2(x, y)$ are the characteristic polynomials.*

Proof. Let $q(x, y)$ in $H^*(B)[x, y]$ be a polynomial such that $q(x, y)|_{\overline{Z_f}} = 0$. It follows from the continuity property of the Čech cohomology theory (Theorem 1.3.2), that there is an open subset $V \subset \overline{E}$ such that $\overline{Z_f} \subset V$ and $q(x, y)|_V = 0$. Consider the long exact cohomology sequence for the pair $(\overline{E}, V)$, namely,

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_1^*} H^*(\overline{E}) \to H^*(V) \to H^*(\overline{E}, V) \to \cdots .$$

By exactness, there exist $\mu \in H^*(\overline{E}, V)$ such that $j_1^*(\mu) = q(x, y)|_{\overline{E}}$, where $j_1 : \overline{E} \to (\overline{E}, V)$ is the natural inclusion. The $\mathbb{Z}_2$-equivariant map $f : E \to E'$ gives the map $\overline{f} : \overline{E} - \overline{Z_f} \to \overline{E'} - 0$. The induced map $\overline{f}^* : H^*(\overline{E'} - 0) \to H^*(\overline{E} - \overline{Z_f})$ is a $H^*(B)$-homomorphism. We also have $W'(a') = 0$. Therefore,

$$W'(x)|_{\overline{E}-\overline{Z_f}} = W'(a) = W'\left(\overline{f}^*(a')\right) = \overline{f}^*\left(W'(a')\right) = 0.$$

Now consider the long exact cohomology sequence for the pair $(\overline{E}, \overline{E} - \overline{Z_f})$, that is,

$$\cdots \to H^*(\overline{E}, \overline{E} - \overline{Z_f}) \xrightarrow{j_2^*} H^*(\overline{E}) \to H^*(\overline{E} - \overline{Z_f}) \to H^*(\overline{E}, \overline{E} - \overline{Z_f}) \to \cdots .$$

By exactness, there exist $\lambda \in H^*(\overline{E}, \overline{E} - \overline{Z_f})$ such that $j_2^*(\lambda) = W'(x)|_{\overline{E}}$, where $j_2 : \overline{E} \to (\overline{E}, \overline{E} - \overline{Z_f})$ is the natural inclusion. Thus,

$$q(x, y)W'(x)|_{\overline{E}} = j_1^*(\mu)j_2^*(\lambda) = j^*(\mu \cup \lambda)$$

by the naturality of the cup product. But, $\mu \cup \lambda \in H^*(\overline{E}, V \bigcup(\overline{E} - \overline{Z_f})) = H^*(\overline{E}, \overline{E}) = 0$ and hence $q(x, y)W'(x)|_{\overline{E}} = 0$. Therefore, by equation (3.4.1), there exist polynomials $r_1(x, y)$ and $r_2(x, y)$ in $H^*(B)[x, y]$ such that $q(x, y)W'(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x, y)$ in the ring $H^*(B)[x, y]$. This proves the theorem. □

As a corollary, we have the following parametrized version of the Borsuk-Ulam theorem.

Corollary 3.5.2. *Let $X \simeq_2 \mathbb{R}P^n$ be a finitistic space. If the fiber dimension of $E' \to B$ is k, then $q(x, y)|_{\overline{Z_f}} \neq 0$ for all non-zero polynomials $q(x, y)$ in $H^*(B)[x, y]$, whose degree* *in x and y is less than $(n-k+1)$. Equivalently, the $H^*(B)$-homomorphism*

$$\bigoplus_{i+j=0}^{n-k} H^*(B)x^iy^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n-k).$$

Proof. Let $q(x,y)$ in $H^*(B)[x,y]$ be a non-zero polynomial such that $\deg(q(x,y)) < (n-k+1)$. If $q(x,y)|_{\overline{Z_f}} = 0$, then by Theorem 3.5.1, we have

$$q(x,y)W^{'}(x) = r_1(x,y)W_1(x,y) + r_2(x,y)W_2(x,y)$$

in the ring $H^*(B)[x,y]$ for some polynomials $r_1(x,y)$ and $r_2(x,y)$ in $H^*(B)[x,y]$. Note that $\deg(W^{'}(x)) = k$, $\deg(W_1(x,y)) = n+1$ and $\deg(W_2(x,y)) = 2$. Since

$$\deg(q(x,y)) + k = \max\{\deg(r_1(x,y)) + n + 1,\ \deg(r_2(x,y)) + 2\},$$

we have

$$\deg(q(x,y)) + k \geq \deg(r_1(x,y)) + n + 1.$$

Taking $\deg(r_1(x,y)) = 0$, this gives $\deg(q(x,y)) + k \geq n+1$ and hence $\deg(q(x,y)) \geq (n-k+1)$, which is a contradiction. Hence $q(x,y)|_{\overline{Z_f}} \neq 0$. Equivalently, the $H^*(B)$-homomorphism

$$\bigoplus_{i+j=0}^{n-k} H^*(B)x^iy^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n-k),$$

since $cohom.dim(Z_f) \geq cohom.dim(\overline{Z_f})$ by Proposition 1.6.2. □

Remark 3.5.1. If B is a point in the above corollary, then for any $\mathbb{Z}_2$-equivariant map $f : X \simeq_2 \mathbb{R}P^n \to \mathbb{R}^k$, where $n \geq k$, we have $cohom.dim(Z_f) \geq (n-k)$.

3.5.2 Borsuk-Ulam theorems for $X \simeq_2 \mathbb{C}P^n$

With similar notations as in the real case, we prove the following results.

Theorem 3.5.3. *Let $X \simeq_2 \mathbb{C}P^n$ be a finitistic space. If $q(x, y)$ in $H^*(B)[x, y]$ is a polynomial such that $q(x, y)|_{\overline{Z_f}} = 0$, then there are polynomials $r_1(x, y)$ and $r_2(x, y)$ in $H^*(B)[x, y]$ such that $q(x, y)W'(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x)$ in the ring $H^*(B)[x, y]$, where $W'(x)$, $W_1(x, y)$ and $W_2(x)$ are the characteristic polynomials.*

Proof. Let $q(x, y)$ in $H^*(B)[x, y]$ be a polynomial such that $q(x, y)|_{\overline{Z_f}} = 0$. By similar arguments as used in the proof of Theorem 3.5.1, we conclude that $q(x, y)W'(x)|_{\overline{E}} = 0$. Therefore, by equation (3.4.2), there exist polynomials $r_1(x, y)$ and $r_2(x, y)$ in $H^*(B)[x, y]$ such that $q(x, y)W'(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x)$ in the ring $H^*(B)[x, y]$. This proves the theorem. □

Corollary 3.5.4. *Let $X \simeq_2 \mathbb{C}P^n$ be a finitistic space. If the fiber dimension of $E' \to B$ is k, then $q(x, y)|_{\overline{Z_f}} \neq 0$ for all non-zero polynomials $q(x, y)$ in $H^*(B)[x, y]$, whose degree in x and y is less than $(2n - k + 2)$. Equivalently, the $H^*(B)$-homomorphism*

$$\bigoplus_{i+j=0}^{2n-k+1} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $2n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (2n - k + 1).$$

Proof. Let $q(x, y)$ in $H^*(B)[x, y]$ be a non-zero polynomial such that $\deg(q(x, y)) < (2n - k + 2)$. If $q(x, y)|_{\overline{Z_f}} = 0$, then by Theorem 3.5.3, we have

$$q(x, y)W'(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x)$$

in the ring $H^*(B)[x, y]$ for some polynomials $r_1(x, y)$ and $r_2(x, y)$ in $H^*(B)[x, y]$. Note that $\deg(W'(x)) = k$, $\deg(W_1(x, y)) = 2n + 2$ and $\deg(W_2(x)) = 3$. Since

$$\deg(q(x, y)) + k = \max\{\deg(r_1(x, y)) + 2n + 2,\ \deg(r_2(x, y)) + 3\},$$

we have

$$\deg(q(x,y)) + k \geq \deg(r_1(x,y)) + 2n + 2.$$

Taking $\deg(r_1(x,y)) = 0$, this gives $\deg(q(x,y)) + k \geq 2n+2$ and hence $\deg(q(x,y)) \geq (2n-k+2)$, which is a contradiction. Hence $q(x,y)|_{\overline{Z_f}} \neq 0$. Equivalently, the $H^*(B)$-homomorphism

$$\bigoplus_{i+j=0}^{2n-k+1} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $2n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (2n-k+1).$$

$\square$

Remark 3.5.2. If B is a point in the above corollary, then for any $\mathbb{Z}_2$-equivariant map $f : X \simeq_2 \mathbb{C}P^n \to \mathbb{R}^k$, where $2n \geq k$, we have $cohom.dim(Z_f) \geq (2n-k+1)$.

3.6 Application to $\mathbb{Z}_2$-coincidence sets

Let (X, E, π, B) be a fiber bundle with the hypothesis of Section 3.4. Let $E^{''} \to B$ be a k-dimensional vector bundle and let $f : E \to E^{''}$ be a fiber preserving map. Here we do not assume that $E^{''}$ has an involution. Even if $E^{''}$ has an involution, f is not assumed to be $\mathbb{Z}_2$-equivariant. If $T : E \to E$ is the generator of the $\mathbb{Z}_2$ action, then the $\mathbb{Z}_2$-coincidence set of f is defined as

$$A(f) = \{x \in E \mid f(x) = f(T(x))\}.$$

Let $V = E^{''} \oplus E^{''}$ be the Whitney sum of two copies of $E^{''} \to B$. Then $\mathbb{Z}_2$ acts on V by permuting the coordinates. This action has the diagonal D in V as the fixed point set. Note that D is a k-dimensional sub-bundle of V and the orthogonal complement $D^{\perp}$ of D is also a k-dimensional sub-bundle of V. Also note that $D^{\perp}$ is $\mathbb{Z}_2$ invariant and has a $\mathbb{Z}_2$ action which is free outside the zero section. Consider the map $f^{'} : E \to V$ given by

$$f^{'}(x) = \big(f(x), f(T(x))\big).$$

It is clearly $\mathbb{Z}_2$-equivariant. The linear projection along the diagonal defines a $\mathbb{Z}_2$-equivariant fiber preserving map $g : V \to D^{\perp}$ such that $g(V - D) \subset D^{\perp} - 0$, where 0 is the zero section of $D^{\perp}$. Let $h = g \circ f'$ be the composition

$$(E, E - A(f)) \xrightarrow{f'} (V, V - D) \xrightarrow{g} (D^{\perp}, D^{\perp} - 0).$$

Note that $Z_h = h^{-1}(0) = {f'}^{-1}(D) = A(f)$ and $h : E \to D^{\perp}$ is fiber preserving $\mathbb{Z}_2$-equivariant map.

Applying Corollary 3.5.2 to h, we have

Theorem 3.6.1. *If $X \simeq_2 \mathbb{R}P^n$ is a finitistic space, then*

$$cohom.dim A(f) \geq cohom.dim(B) + (n - k).$$

Similarly, applying Corollary 3.5.4 to h, we have

Theorem 3.6.2. *If $X \simeq_2 \mathbb{C}P^n$ is a finitistic space, then*

$$cohom.dim A(f) \geq cohom.dim(B) + (2n - k + 1).$$

Chapter 4

Fixed Point Sets of Involutions on Spaces of Type (a, 0)

4.1 Introduction

In this chapter, we investigate the fixed point sets of involutions on certain types of spaces first studied by Toda [77]. Toda studied the cohomology algebra of a space X having only non-trivial cohomology groups $H^{in}(X;\mathbb{Z}) = \mathbb{Z}$ for $i = 0$, 1, 2 and 3, where n is a fixed positive integer. Let $u_i \in H^{in}(X;\mathbb{Z})$ be a generator for $i = 1$, 2 and 3. Then the ring structure of $H^*(X;\mathbb{Z})$ is completely determined by the integers a and b such that

$$u_1^2 = au_2 \text{ and } u_1u_2 = bu_3.$$

Such a space is said to be of **type** (a, b). Let p be a prime. We write $X \simeq_p Y$ if there is an isomorphism of graded algebras $H^*(X;\mathbb{Z}_p) \cong H^*(Y;\mathbb{Z}_p)$. If Y is a space of type (a, b), we say that X is a space of type (a, b) mod p. Similarly, we use the notation $X \simeq_p P^h(n)$ to mean that $H^*(X;\mathbb{Z}_p) \cong \mathbb{Z}_p[z]/z^{h+1}$, where z is a homogeneous element of degree n.

Given spaces X_i with chosen base points $x_i \in X_i$ for $i = 1$, 2, ..., k, their wedge sum $\vee_{i=1}^k X_i$ is the quotient of the disjoint union $\sqcup_{i=1}^k X_i$ obtained by identifying the points

$x_1, x_2, ..., x_k$ to a single point called the wedge point.

One can see that a space X of type (a, b) is determined by the integers a and b in terms of some familiar spaces as follows.
If $b \not\equiv 0$ mod p, then

$$X \simeq_p S^n \times S^{2n} \text{ for } a \equiv 0 \bmod p$$

or

$$X \simeq_p P^3(n) \text{ for } a \not\equiv 0 \bmod p.$$

And, if $b \equiv 0$ mod p, then

$$X \simeq_p S^n \vee S^{2n} \vee S^{3n} \text{ for } a \equiv 0 \bmod p$$

or

$$X \simeq_p P^2(n) \vee S^{3n} \text{ for } a \not\equiv 0 \bmod p.$$

Let the cyclic group $G = \mathbb{Z}_p$ act on a space X of type (a, b). This gives the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$ (Chapter 1, p.16). Recall that, X is said to be totally non-homologous to zero (TNHZ) in X_G with respect to $\mathbb{Z}_p$ if the inclusion of a typical fiber $X \hookrightarrow X_G$ induces a surjection in the cohomology $H^*(X_G; \mathbb{Z}_p) \longrightarrow H^*(X; \mathbb{Z}_p)$ (Definition 1.4.3).

The cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \not\equiv 0$ mod p has been investigated in detail by Bredon [5, 6] and Su [73, 74] for all primes p. And the cohomological nature of the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \equiv 0$ mod p has been completely determined by Dotzel and Singh [20, 21] for odd primes p. We settle the remaining case of $b \equiv 0$ mod 2 and investigate the fixed point sets of involutions on finitistic spaces of type $(a, 0)$ mod 2. More precisely, we prove the following results.

Theorem 4.1.1. *Let $G = \mathbb{Z}_2$ act on a finitistic space X of type $(a, 0)$ mod 2 with trivial action on $H^*(X; \mathbb{Q})$ and with fixed point set F. Suppose X is totally non-homologous to zero in X_G, then F has at most four components satisfying the following:*

1. *If F has four components, then each is acyclic, n is even and $a \equiv 0$ mod 2.*

2. If F has three components, then n is even and

$$F \simeq_2 S^r \sqcup \{point_1\} \sqcup \{point_2\} \text{ for some even integer } 2 \leq r \leq 3n.$$

3. If F has two components, then either

$$F \simeq_2 S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for some integers } 1 \leq r, s \leq 3n$$

or

$$F \simeq_2 P^2(r) \sqcup \{point\} \text{ for some even integer } 2 \leq r \leq n.$$

4. If F has one component, then either

$$F \simeq_2 S^r \vee S^s \vee S^t \text{ for some integers } 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_2 S^s \vee P^2(r) \text{ for some integers } 1 \leq r \leq n \text{ and } 1 \leq s \leq 3n.$$

Moreover, if n is even, then X is always totally non-homologous to zero in X_G. Further, all the cases are realizable.

Theorem 4.1.2. *Let $G = \mathbb{Z}_2$ act on a finitistic space X of type $(a, 0)$ mod 2 with trivial action on $H^*(X;\mathbb{Q})$ and with fixed point set F. Suppose X is not totally non-homologous to zero in X_G, then either $F = \phi$ or $F \simeq_2 S^r$, where $1 \leq r \leq 3n$ is an odd integer. Moreover, the second possibility is realizable.*

Before proceeding to prove our theorems, we consider a $\mathbb{Z}_2$ action on the unit sphere $S^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$ that we will use in constructing examples in the following sections. For $0 \leq r \leq n$, $S^r \subseteq S^n$, where $S^r = \{(x_1, x_2, ..., x_{n+1}) \in S^n | x_{r+2} = x_{r+3} = ... = x_{n+1} = 0\}$. The $\mathbb{Z}_2$ action on S^n given by

$$(x_1, x_2, ..., x_{n+1}) \mapsto (x_1, x_2, ..., x_{r+1}, -x_{r+2}, -x_{r+3}, ..., -x_{n+1})$$

has S^r as its fixed point set. Given any point $x \in S^n$, we consider $\{x, -x\}$ as $S^0 \subset S^n$. Then the above action on S^n, for $r = 0$, has $\{x, -x\}$ as its fixed point set.

We will also use the join $X \star Y$ of two spaces X and Y, which is defined as the quotient of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$, where I is the unit interval. In other words, we are collapsing the subspace $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. Note that, if a group G acts on both X and Y with fixed point sets F_1 and F_2 respectively, then the induced action of G on $X \star Y$ has $F_1 \star F_2$ as its fixed point set. Throughout we will use the Čech cohomology with $\mathbb{Z}_2$ coefficients unless otherwise stated. We first prove the following lemma.

Lemma 4.1.3. *Let $G = \mathbb{Z}_2$ act on a finitistic space X with trivial action on the rational cohomology $H^*(X; \mathbb{Q})$, then*

$$\chi(X) = \chi(F).$$

Proof. By Theorem 1.4.3, we have

$$\pi^* : H^i(X/G; \mathbb{Q}) \xrightarrow{\cong} H^i(X; \mathbb{Q})^G \text{ for all } i \geq 0,$$

where $\pi : X \to X/G$ is the orbit map. Since G acts trivially on the cohomology, the fixed point set $H^i(X; \mathbb{Q})^G = H^i(X; \mathbb{Q})$ for all $i \geq 0$. This gives $H^i(X/G; \mathbb{Q}) \cong H^i(X; \mathbb{Q})$ for all $i \geq 0$ and hence $\chi(X) = \chi(X/G)$. By Theorem 1.4.2, we have

$$\chi(X) + \chi(F) = 2\chi(X/G)$$

and hence $\chi(X) = \chi(F)$. □

Theorem 4.1.1 is proved in Section 4.2 and Theorem 4.1.2 is proved in Section 4.3.

4.2 Fixed point sets when X is TNHZ in $X_{\mathbb{Z}_2}$

Let X be totally non-homologous to zero in X_G. Then by Theorem 1.4.11

$$\sum_{i \geq 0} rkH^i(F) = \sum_{i \geq 0} rkH^i(X) = 4.$$

It follows that F has at most four components.

Case (1) Suppose F has four components, then it is clear that each is acyclic. Let $\overline{u}_i$ denote the reductions of u_i mod 2. If $a \not\equiv 0$ mod 2, then ${\overline{u}_1}^2 = \overline{u}_2 \neq 0$. Hence by Proposition 1.4.12, $H^n(F) \neq 0$, showing that F has a non-acyclic component. Therefore, in this case $a \equiv 0$ mod 2. By Lemma 4.1.3, we have $\chi(X) = \chi(F) = 4$ and hence n must be even.

For $a \equiv 0$ mod 2, we can take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on the spheres S^n, S^{2n} and S^{3n} with two fixed points each and take their wedge sum at some fixed points. This gives a $\mathbb{Z}_2$ action on X with the disjoint union of four points as its fixed point set.

Case (2) Suppose that F has three components, then

$$F \simeq_2 S^r \sqcup \{point_1\} \sqcup \{point_2\} \text{ for some integer } 1 \leq r \leq 3n.$$

Note that $\chi(F) = 2$ or 4 according as r is odd or even. As $\chi(X) = \chi(F)$, both n and r are even.

For $a \equiv 0$ mod 2 and even integers r, n such that $2 \leq r \leq 3n$, we take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on the spheres S^n and S^{2n} with exactly two fixed points each and the action on S^{3n} with S^r as its fixed point set. Taking their wedge sum at some fixed points gives a $\mathbb{Z}_2$ action on X with $F = S^r \sqcup \{point_1\} \sqcup \{point_2\}$.

For $a \not\equiv 0$ mod 2, we know that $X \simeq_2 P^2(n) \vee S^{3n}$. If Y is a space such that $H^*(Y;\mathbb{Z}_2) = \mathbb{Z}_2[z]/z^{h+1}$, where z is of degree n, then by Proposition 1.5.1, we have $n =$ 2, 4 or 8 for $h = 2$. Therefore, we can take Y to be the complex projective plane $\mathbb{C}P^2$, the quaternionic projective plane $\mathbb{H}P^2$ or the Cayley projective plane $\mathbb{O}P^2$, according as $n = 2$, 4 or 8 respectively.

For $n = 2$, let $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \sum_{i=1}^{3} |z_i|^2 = 1\}$. Consider the $\mathbb{Z}_2$ action on S^5 given by $(z_1, z_2, z_3) \mapsto (z_1, z_2, -z_3)$. This action commutes with the usual S^1 action on S^5 and hence descends to an action on $\mathbb{C}P^2$. As $S^3 \subset S^5$ is fixed under the $\mathbb{Z}_2$ action on S^5, it is easy to see that $S^2 \sqcup \{point\}$ is the fixed point set of the $\mathbb{Z}_2$ action on $\mathbb{C}P^2$.

Similarly, for $n = 4$, let $\mathbb{H}$ be the normed division algebra of quaternions and $S^{11} = \{(w_1, w_2, w_3) \in \mathbb{H}^3 | \sum_{i=1}^{3} |w_i|^2 = 1\}$ and consider the $\mathbb{Z}_2$ action on S^{11} given by $(w_1, w_2, w_3) \mapsto (w_1, w_2, -w_3)$. This action commutes with the usual S^3 action on S^{11}. As above, one can see that $S^4 \sqcup \{point\}$ is the fixed point set of the induced action of $\mathbb{Z}_2$ on $\mathbb{H}P^2$.

For $n = 8$, we now construct a $\mathbb{Z}_2$ action on $\mathbb{O}P^2$ with $S^8 \sqcup \{point\}$ as its fixed point set. It is originally due to Bredon [6, p.389]. Let ω be a Cayley number of order 2 and let ω act on S^7 by $c \mapsto \omega c \omega$. Also let ω act on $S^7 \times S^7$ by $(c_1, c_2) \mapsto (\omega c_1, c_2 \omega)$. Then the multiplication $S^7 \times S^7 \to S^7$ is equivariant by the Moufang identity. Thus there is an induced action of $\mathbb{Z}_2$ on $\mathbb{O}P^2$. We now identify the fixed point set F. Note that if $c \in S^7$ is perpendicular to 1 and ω, then $\overline{c} = -c$ and $2 = |c+\omega|^2 = (\overline{c}+\overline{\omega})(c+\omega) = 2 + \overline{c}\omega + \overline{\omega}c$ so that $c\omega = \overline{\omega}c$. That is, $\omega c \omega = c$ for c in this 5-sphere. If $c \in S^7$ is in the plane spanned by 1 and ω, then $\omega c \omega = \omega^2 c$ which equals c. Since there are no fixed points on $S^7 \times S^7$, we see that F is the vertex of the mapping cone of $S^7 \star S^7 \to S(S^7)$ together with the suspension of the fixed set of $c \mapsto \omega c \omega$ on S^7. Thus $F = S^8 \sqcup \{point\}$.

Now, consider the $\mathbb{Z}_2$ action on S^{3n} with exactly two fixed points. Taking $X = Y \vee S^{3n}$, where the wedge sum is taken at the isolated fixed point of Y and a fixed point of S^{3n}, we get a $\mathbb{Z}_2$ action on X with the fixed point set $F = S^r \sqcup \{point_1\} \sqcup \{point_2\}$ for some even integer $2 \leq r \leq 3n$.

Case(3) Suppose F has two components, then

$$F \simeq_2 S^r \sqcup S^s,\ (S^r \vee S^s) \sqcup \{point\} \text{ or } P^2(r) \sqcup \{point\} \text{ for some } r \text{ and } s.$$

By Lemma 4, $\chi(X) = \chi(F)$. If n is odd, $\chi(F) = 0$ and hence

$$F \simeq_2 S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for odd integers } 1 \leq r, s \leq 3n.$$

And if n is even, $\chi(F) = 4$ and hence

$$F \simeq_2 S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for even integers } 2 \leq r, s \leq 3n$$

or

$$F \simeq_2 P^2(r) \sqcup \{point\} \text{ for some even integer } 2 \leq r \leq n.$$

For $a \equiv 0 \mod 2$, let $Y = S^{n-1} \star P^2(n)$. Consider a free $\mathbb{Z}_2$ action on S^{n-1} and that action on $P^2(n)$ which has the fixed point set $S^r \sqcup \{point\}$ for some r (which we constructed in case (2)). Let $\mathbb{Z}_2$ act on S^n with its fixed point set S^s for some s. Take $X = S^n \vee Y$, where the wedge sum is taken at the isolated fixed point of Y and some point of S^s. Then $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$ and has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 S^r \sqcup S^s$.

If we take the wedge sum at some point of S^r and some point of S^s, then X has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 (S^r \vee S^s) \sqcup \{point\}$.

Further, if we consider a free $\mathbb{Z}_2$ action on S^{n-1}, the trivial action on $P^2(n)$ and the action on S^n with exactly two fixed points, then $X = S^n \vee Y$, where the wedge is taken at some point of $P^2(n)$ and some fixed point of S^n, has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 P^2(n) \sqcup \{point\}$.

For $a \not\equiv 0 \mod 2$, take $X = P^2(n) \vee S^{3n}$. Consider the $\mathbb{Z}_2$ action on $P^2(n)$ with $S^r \sqcup \{point\}$ as its fixed point set and the action on S^{3n} with S^s as its fixed point set. By taking the wedge sum at suitable points, we get a $\mathbb{Z}_2$ action on X with $F \simeq_2 S^r \sqcup S^s$ or $(S^r \vee S^s) \sqcup \{point\}$. Similarly, suitable actions on $P^2(n)$ and S^{3n} gives an action on X with $F \simeq_2 P^2(r) \sqcup \{point\}$.

Case(4) Suppose F has one component, then either

$$F \simeq_2 S^r \vee S^s \vee S^t \text{ for some integers } 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_2 S^s \vee P^2(r) \text{ for some integers } 1 \leq r \leq n \text{ and } 1 \leq s \leq 3n.$$

As $\chi(F) = \chi(X)$, for $F \simeq_2 S^r \vee S^s \vee S^t$ we must have either r, s and t all are even or exactly one of them is even. Similarly, for $F \simeq_2 S^s \vee P^2(r)$ we must have either s and r

both even or both odd.

For $a \equiv 0$ mod 2, take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on S^n, S^{2n} and S^{3n} with S^r, S^s and S^t respectively as their fixed point sets. This gives an action on X with $S^r \vee S^s \vee S^t$ as its fixed point set, where the wedge is taken at some fixed points on the sub-spheres.

If we take $X = S^n \vee Y$, where $Y = S^{n-1} \star P^2(n)$ and consider the $\mathbb{Z}_2$ action on S^n with S^s as its fixed point set for some s and the action on Y with $P^2(r)$ as its fixed point set for some r, then we get a $\mathbb{Z}_2$ action on X with its fixed point set $F \simeq_2 S^s \vee P^2(r)$.

For $a \not\equiv 0$ mod 2, taking a suitable $\mathbb{Z}_2$ action on $X = P^2(n) \vee S^{3n}$ gives $F \simeq_2 S^s \vee P^2(r)$ for some integers r and s. Note that in this case the fixed point set cannot be a wedge of three spheres.

Finally, suppose that n is even and X is not totally non-homologous to zero in X_G. Then by Theorem 1.4.11,

$$\sum_{i\geq 0} rkH^i(F) \neq \sum_{i\geq 0} rkH^i(X) = 4.$$

and hence

$$\sum_{i\geq 0} rkH^i(F) \leq 3.$$

This gives $\chi(F)$ = -1, 0, 1, 2 or 3. But, $\chi(F) = \chi(X) = 4$, a contradiction. This completes the proof of the Theorem 4.1.1. □

4.3 Fixed point sets when X is not TNHZ in $X_{\mathbb{Z}_2}$

Let X be not totally non-homologous to zero in X_G. Then n is odd and hence $\chi(X) = 0$. As above $\sum_{i\geq 0} rkH^i(F) \leq 3$ and hence $\chi(F)$ = -1, 0, 1, 2 or 3. But by Lemma 4.1.3, we have $\chi(F) = 0$. Therefore either $F = \phi$ or $F \simeq_{\mathbb{Q}} S^r$ for some odd integer $1 \leq r \leq 3n$.

Note that, when n is odd, we have $a \equiv 0$ mod 2 [77]. We now construct a $\mathbb{Z}_2$ action on a space $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$ with fixed point set $F \simeq_2 S^r$ using a construction of Su

[73]. Let $h : S^3 \longrightarrow S^2$ be the Hopf map and Y be the union of mapping cylinders of the sphere bundle maps

$$S^2 \times S^n \overset{h\times 1}{\longleftarrow} S^3 \times S^n \overset{projection}{\longrightarrow} S^3.$$

Then $H^*(Y;\mathbb{Z}) = H^*(S^2 \times S^{n+2};\mathbb{Z})$ and Y is a manifold . Let $\mathbb{Z}_2$ act freely on S^n and trivially on both S^2 and S^3, then it act on Y with the fixed point set homeomorphic to S^3. Remove a fixed point from Y to obtain a space $Z \simeq_2 S^2 \vee S^{n+2}$ with a $\mathbb{Z}_2$ action and contractible fixed point set. With $\mathbb{Z}_2$ acting trivially on S^{n-3}, consider the induced action on the join $W = S^{n-3} \star Z$ which is homotopically equivalent to $S^n \vee S^{2n}$. This action on W has a contractible fixed point set. For a given odd integer $1 \leq r \leq 3n$, consider the $\mathbb{Z}_2$ action on S^{3n} with S^r as the fixed point set. Then the wedge sum of W and S^{3n} at some fixed points is a space $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$ and has a $\mathbb{Z}_2$ action with its fixed point set $F \simeq_2 S^r$. It is clear that every $\mathbb{Z}_2$ action on $X = S^n \vee S^{2n} \vee S^{3n}$ has a non-empty fixed point set. This completes the proof of Theorem 4.1.2. $\square$

Remark 4.3.1. It is clear that there is no free $\mathbb{Z}_2$ action on $S^n \vee S^{2n} \vee S^{3n}$. The author does not know of any free $\mathbb{Z}_2$ action on a space $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$.

Chapter 5

Fixed Point Sets of Circle Actions on Spaces of Type (a, 0)

5.1 Introduction

This chapter is concerned with the fixed point sets of $\mathbb{S}^1$ actions on spaces of type (a, b) introduced in Chapter 4. Toda studied the cohomology algebra of a space X having only non trivial cohomology groups $H^{in}(X;\mathbb{Z}) = \mathbb{Z}$ for $i = 0$, 1, 2 and 3, where n is a fixed positive integer. If $u_i \in H^{in}(X;\mathbb{Z})$ is a generator for $i = 1$, 2 and 3, then the ring structure of $H^*(X;\mathbb{Z})$ is completely determined by the integers a and b such that

$$u_1^2 = au_2 \text{ and } u_1u_2 = bu_3.$$

Such a space is said to be of **type** (a, b). Note that if n is odd, then $u_1^2 = 0$ and hence $a = 0$. We write $X \simeq_{\mathbb{Q}} Y$ if there is an isomorphism of graded algebras $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$. If Y is a space of type (a, b), we say that X is a rational cohomology space of type (a, b). Similarly, by $X \simeq_{\mathbb{Q}} P^h(n)$ we mean that $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[z]/z^{h+1}$, where z is a homogeneous element of degree n.

It is clear that if $b \neq 0$, then

$$X \simeq_{\mathbb{Q}} S^n \times S^{2n} \text{ for } a = 0$$

or

$$X \simeq_{\mathbb{Q}} P^3(n) \text{ for } a \neq 0.$$

And, if $b = 0$, then

$$X \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n} \text{ for } a = 0$$

or

$$X \simeq_{\mathbb{Q}} P^2(n) \vee S^{3n} \text{ for } a \neq 0.$$

Let the group $G = \mathbb{S}^1$ act on a space X of type (a, b). This gives the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$ (Chapter 1, p.16). Recall that, X is said to be totally non-homologous to zero (TNHZ) in X_G with respect to $\mathbb{Q}$ if the inclusion of a typical fiber $X \hookrightarrow X_G$ induces a surjection in the cohomology $H^*(X_G; \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$ (Definition 1.4.3).

The cohomological nature of the fixed point sets of actions of the cyclic group $\mathbb{Z}_p$ of prime order p on spaces of type (a, b) has been studied in detail [5, 6, 20, 21, 64, 73, 74].

For $b \neq 0$, the cohomological nature of the fixed point sets of $\mathbb{S}^1$ actions has been studied in detail by Bredon [5, 6]. We study $\mathbb{S}^1$ actions on rational cohomology finitistic spaces of type $(a, 0)$ and determine the possible fixed point sets up to rational cohomology. More precisely, we prove the following results.

Theorem 5.1.1. *Let $G = \mathbb{S}^1$ act on a rational cohomology finitistic space X of type $(a, 0)$ with fixed point set F. Suppose X is totally non-homologous to zero in X_G, then F has at most four components satisfying the following:*

1. *If F has four components, then each is acyclic and n is even.*

2. *If F has three components, then n is even and*

$$F \simeq_{\mathbb{Q}} S^r \sqcup \{point_1\} \sqcup \{point_2\} \ \ \textit{for some even integer} \ \ 2 \leq r \leq 3n.$$

3. *If F has two components, then either*

$$F \simeq_{\mathbb{Q}} S^r \sqcup S^s \ \ \textit{or} \ \ (S^r \vee S^s) \sqcup \{point\} \ \ \textit{for some integers} \ \ 1 \leq r, s \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} P^2(r) \sqcup \{point\} \quad \textit{for some even integer} \;\; 2 \leq r \leq n.$$

4. If F has one component, then either

$$F \simeq_{\mathbb{Q}} S^r \vee S^s \vee S^t \quad \textit{for some integers} \;\; 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} S^s \vee P^2(r) \quad \textit{for some integers} \;\; 1 \leq r \leq n \;\; \textit{and} \;\; 1 \leq s \leq 3n.$$

Moreover, if n is even, then X is always totally non-homologous to zero in X_G. Further, all the cases are realizable.

Theorem 5.1.2. *Let $G = \mathbb{S}^1$ act on a rational cohomology finitistic space X of type $(a, 0)$ with fixed point set F. Suppose X is not totally non-homologous to zero in X_G, then either $F = \phi$ or $F \simeq_{\mathbb{Q}} S^r$, where $1 \leq r \leq 3n$ is an odd integer. Moreover, the second possibility is realizable.*

We will use the join $X \star Y$ of two spaces X and Y, which is defined as the quotient of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$, where I is the unit interval. If Y is a two point space, then $X \star Y$ is called the suspension of X and is denoted by $S(X)$. Observe that,

- if a group G acts on both X and Y with fixed point sets F_1 and F_2 respectively, then the induced action on $X \star Y$ has the fixed point set $F_1 \star F_2$.
- a given map $f : X \times Y \to Z$ induces a map $\tilde{f} : X \star Y \to S(Z)$ given by

$$\tilde{f}\big(\,\overline{(x, y, t)}\,\big) = \overline{(f(x, y), t)}.$$

We say that a map $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ has **bidegree** (α, β), if the restriction $f|_{S^{n-1} \times \{p_2\}}$ has degree α and the restriction $f|_{\{p_1\} \times S^{n-1}}$ has degree β, where $(p_1, p_2) \in$

$S^{n-1} \times S^{n-1}$. The bidegree of f is independent of the choice of (p_1, p_2). We will need the following well known result

Proposition 5.1.3. *[71, p.14] For every even integer $n \geq 2$ there is a map $\varphi : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree $(2, -1)$.*

Proof. Define $\varphi : S^{n-1} \times S^{n-1} \to S^{n-1}$ by

$$\varphi(x, y) = y - 2(\sum_{i=1}^{n} x_i y_i)x.$$

If we fix $x = (1, 0, ..., 0)$, then $\varphi(x, y) = (-y_1, y_2, ..., y_n)$ which has degree -1. If we fix $y = (1, 0, ..., 0)$, then $\varphi(x, y) = (1 - 2{x_1}^2, -2x_1x_2, ..., -2x_1x_n) = g(x)$ say. Let $H_{\pm}^{n-1}$ be the closed subspace of S^{n-1} consisting of x such that $\pm x_1 \geq 0$. Then $\varphi : H_{+}^{n-1} - S^{n-2} \to S^{n-1} - \{(1, 0, ..., 0)\}$ is a homeomorphism and hence defines a map $f_1 : H_{+}^{n-1}/S^{n-2} \to S^{n-1}$ of degree $+1$. In a similar way, φ defines an map $f_2 : H_{-}^{n-1}/S^{n-2} \to S^{n-1}$. Note that $g(x) = g(-x)$ and hence the degree of f_2 is equal to that of the map $x \mapsto -x$, which is $(-1)^n = +1$. Since, the map g can be factored into $S^{n-1} \to S^{n-1}/S^{n-2} = S^{n-1} \vee S^{n-1} \xrightarrow{f_1 \vee f_2} S^{n-1}$, we see that g has degree 2 and hence φ has bidegree $(2, -1)$. $\square$

Let $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map of bidegree (α, β) and

$$\tilde{f} : S^{2n-1} = S^{n-1} \star S^{n-1} \to S(S^{n-1}) = S^n$$

be the induced map. If Y is the complex obtained by attaching a $2n$-cell e^{2n} to S^n via $\tilde{f}$, then for generators $x \in H^n(Y; \mathbb{Z})$ and $y \in H^{2n}(Y; \mathbb{Z})$, we have $x^2 = h(\tilde{f})y$ for some integer $h(\tilde{f})$ called the **Hopf invariant** of $\tilde{f}$. A homotopy of $\tilde{f}$ leaves the homotopy type of Y unchanged and hence the Hopf invariant is an invariant of the homotopy class of $\tilde{f}$. The following well known result relates the Hopf invariant and the bidegree.

Proposition 5.1.4. *[71, p.13] $h(\tilde{f}) = \pm\alpha\beta$*

Proof. Let E_1 and E_2 denote the closed n-cells with boundary S^{n-1}. Let S^n be the suspension of S^{n-1}. Then $S^n = E_+ \cup E_-$, where E_+ and E_- are closed n-cells in S^n

with $E_+ \cap E_- = S^{n-1}$. Let $\tilde{f} : S^{2n-1} = S^{n-1} \star S^{n-1} \to S(S^{n-1}) = S^n$ be the induced map. Note that $S^{n-1} \star S^{n-1} = E_1 \times S^{n-1} \cup S^{n-1} \times E_2$ and $\tilde{f}(E_1 \times S^{n-1}) \subset E_+$ and $\tilde{f}(S^{n-1} \times E_2) \subset E_-$. Since, the boundary of $E_1 \times E_2$ is $E_1 \times S^{n-1} \cup S^{n-1} \times E_2$, we can construct the adjunction space $X = (E_1 \times E_2) \cup_{\tilde{f}} S^n$. The attaching map $\tilde{f}$ gives rise to a map $g : (E_1 \times E_2, E_1 \times S^{n-1}, S^{n-1} \times E_2) \to (X, E_+, E_-)$. Let $x \in H^n(X;\mathbb{Z})$ be a generator. We define x_+ and x_- to be in the inverse images of x under the isomorphisms $H^n(X, E_-) \to H^n(X)$ and $H^n(X, E_+) \to H^n(X)$ respectively. Now we have a map $(X, \phi, \phi) \to (X, E_+, E_-)$. This gives rise to a commutative diagram

$$\begin{tikzcd}
H^n(X,E_+) \otimes H^n(X,E_-) \arrow[r, "\cup"] \arrow[d, "\cong"] & H^{2n}(X,S^n) \arrow[d, "\cong"] \\
H^n(X) \otimes H^n(X) \arrow[r, "\cup"] & H^{2n}(X).
\end{tikzcd}$$

The vertical maps are isomorphisms. Therefore the cup product $x_+ \cup x_-$ has image x^2 under the map $H^{2n}(X, S^n) \to H^{2n}(X)$. We have the following commutative diagram

$$\begin{tikzcd}
 & H^n(X) \arrow[dl, "\cong"'] & H^n(X,E_-) \arrow[l, "\cong"'] \arrow[r, "g^*"] \arrow[d] & H^n(E_1 \times E_2, S^{n-1} \times E_2) \arrow[d, "\cong"] \\
H^n(S^n) & H^n(S^n,E_-) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^n(E_+,S^{n-1}) \arrow[r, "g^*"] & H^n(E_1 \times p_2, S^{n-1} \times p_2) \\
 & & H^{n-1}(S^{n-1}) \arrow[u, "\delta", "\cong"'] \arrow[r, "g^*"] & H^{n-1}(S^{n-1} \times p_2) \arrow[u, "\delta", "\cong"'] \\
 & & \mathbb{Z} \arrow[u, "\cong"'] \arrow[r, "\alpha"] & \mathbb{Z}. \arrow[u, "\cong"']
\end{tikzcd}$$

By the diagram $g^*x_+ = \alpha w_+$, where w_+ generates $H^n(E_1 \times E_2, S^{n-1} \times E_2)$. By a similar argument, we see that $g^*x_- = \beta w_-$, where w_- generates $H^n(E_1 \times E_2, E_1 \times S^{n-1})$. Let $p_i : E_1 \times E_2 \to E_i$ be the projections. We define the generators $x_i \in H^n(E_i, S^{n-1})$ by $p_1^*x_1 = w_+$ and $p_2^*x_2 = w_-$. Now $w_+ \cup w_- = p_1^*x_1 \cup p_2^*x_2 = (x_1 \times 1) \cup (1 \times x_2) = x_1 \times x_2$. Hence $g^*x_+ \cup g^*x_- = \alpha\beta(x_1 \times x_2)$ and $(x_1 \times x_2)$ generates $H^{2n}(E_1 \times E_2, E_1 \times S^{n-1} \cup S^{n-1} \times E_2)$.

Now $g : (E_1 \times E_2, E_1 \times S^{n-1} \cup S^{n-1} \times E_2) \to (X, S^n)$ is a relative homeomorphism and therefore induces an isomorphism of cohomology groups. So we have the isomorphisms

$$H^{2n}(X) \xleftarrow{\cong} H^{2n}(X, S^n) \xrightarrow{g^*} H^{2n}(E_1 \times E_2, E_1 \times S^{n-1} \cup S^{n-1} \times E_2).$$

Under these isomorphisms $x^2 \in H^{2n}(X)$ corresponds to $x_+ \cup x_- \in H^{2n}(X, S^n)$ and to $\alpha\beta(x_1 \times x_2)$. Let $y \in H^{2n}(X)$ be a generator which corresponds to $x_1 \times x_2$. Then $x^2 = \alpha\beta y$. This proves the proposition. □

We now consider a $\mathbb{S}^1$ action on the unit sphere S^n that will be used in the following sections. For an odd integer n and $r = (n+1)/2$, the unit sphere $S^n = \{(z_1, z_2, ..., z_r) \in \mathbb{C}^r | \sum_{i=1}^r |z_i|^2 = 1\}$ has a free $\mathbb{S}^1$ action given by $\big(z, (z_1, z_2, ..., z_r)\big) \mapsto (zz_1, zz_2, ..., zz_r)$. Taking suspension gives a $\mathbb{S}^1$ action on the even dimensional sphere S^{n+1} with exactly two fixed points.

An action of a group on a vector bundle will be by bundle maps. Throughout we will use the Čech cohomology with rational coefficients. Theorem 5.1.1 is proved in Section 5.2 and Theorem 5.1.2 is proved in Section 5.3.

5.2 Fixed point sets when X is TNHZ in $X_{\mathbb{S}^1}$

Let X be totally non-homologous to zero in X_G. Then by Theorem 1.4.11,

$$\sum_{i \geq 0} rkH^i(F) = \sum_{i \geq 0} rkH^i(X) = 4.$$

It is clear that F has at most four components.

Case (1) Suppose F has four components, then it is clear that each is acyclic. By Theorem 1.4.6, $\chi(X) = \chi(F) = 4$ and hence n is even.

For $a = 0$ and even integer n, we can take $X = S^n \vee S^{2n} \vee S^{3n}$. As discussed in the previous section, an even dimensional sphere has a $\mathbb{S}^1$ action with exactly two fixed points. Taking the wedge sum at some fixed points of S^n, S^{2n} and S^{3n} gives a four fixed point $\mathbb{S}^1$ action on X.

For $a \neq 0$, we know that $X \simeq_{\mathbb{Q}} P^2(n) \vee S^{3n}$. Let n be an even integer, then by Proposition 5.1.3, there is a map $\varphi : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree $(2, -1)$. It is clear that φ is equivariant with respect to the usual $O(n)$ action on S^{n-1} and the diagonal action on $S^{n-1} \times S^{n-1}$ and hence the induced map $\tilde{\varphi} : S^{2n-1} \to S^n$ is also equivariant with respect to the induced action. Let X_n denote the mapping cone of $\tilde{\varphi}$ which inherits an $O(n)$ action. By Proposition 5.1.4, the Hopf invariant of $\tilde{\varphi}$ is -2. Then $X_n \simeq_{\mathbb{Q}} P^2(n)$ and X_0 consists of three points. It is clear that if $G \subset O(n)$ acts on $\mathbb{R}^n$ with fixed point set $\mathbb{R}^k$, then the induced action on X_n has fixed point set X_k. Let $\mathbb{S}^1 \subset O(n)$ acts on $\mathbb{R}^n$ with exactly one fixed point $\mathbb{R}^0$, then it act on X_n with the fixed point set X_0. Let $\mathbb{S}^1$ act on S^{3n} with exactly two fixed points, then the wedge sum $X = X_n \vee S^{3n}$ at fixed points has a $\mathbb{S}^1$ action with exactly four fixed points.

Case (2) Suppose that F has three components, then

$$F \simeq_{\mathbb{Q}} S^r \sqcup \{point_1\} \sqcup \{point_2\} \text{ for some } 1 \leq r \leq 3n.$$

Note that $\chi(F) = 2$ or 4 according as r is odd or even. But $\chi(X) = \chi(F)$ implies that both n and r are even.

For $a = 0$ and even integer $2 \leq r \leq 3n-2$, take $X = S^n \vee S^{2n} \vee S^{3n}$. As $1 \leq (3n-r-1)$ is an odd integer, $\mathbb{S}^1$ has a free action on S^{3n-r-1} and the $(r+1)$-fold suspension gives a $\mathbb{S}^1$ action on S^{3n} with S^r as its fixed point set. Taking a two fixed point action on both S^n and S^{2n}, the wedge sum at fixed points gives a $\mathbb{S}^1$ action on X with the fixed point set $F \simeq_{\mathbb{Q}} S^r \sqcup \{point_1\} \sqcup \{point_2\}$.

For $a \neq 0$, take $X = X_n \vee S^{3n}$. Taking the above three fixed point action of $\mathbb{S}^1$ on $X_n \simeq_{\mathbb{Q}} P^2(n)$ and the action on S^{3n} with S^r as its fixed point set, the wedge sum gives a $\mathbb{S}^1$ action on X with the desired fixed point set.

Case(3) Suppose F has two components. Then

$$F \simeq_{\mathbb{Q}} S^r \sqcup S^s,\ (S^r \vee S^s) \sqcup \{point\} \text{ or } P^2(r) \sqcup \{point\}.$$

If n is odd, $\chi(F) = \chi(X) = 0$ and hence

$$F \simeq_{\mathbb{Q}} S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for odd integers } 1 \leq r, s \leq 3n.$$

And if n is even, $\chi(F) = \chi(X) = 4$ and hence

$$F \simeq_{\mathbb{Q}} S^r \sqcup S^s \text{ or } (S^r \vee S^s) \sqcup \{point\} \text{ for even integers } 2 \leq r, s \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} P^2(r) \sqcup \{point\} \text{ for some even integer } 2 \leq r \leq n.$$

Let $a = 0$ and $n = 2$, 4 or 8. Let S^{n-1} denote the set of complex numbers, quaternions, or octanions of norm 1 and $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ be the multiplication of complex numbers, quaternions, or octanions for $n = 2$, 4 or 8, respectively. For $n = 2$ and 4, let $\mathbb{S}^1$ act on S^{n-1} by

$$(z, w) \mapsto zw$$

and act on $S^{n-1} \times S^{n-1}$ by

$$\big(z, (w_1, w_2)\big) \mapsto (zw_1, w_2).$$

Then $\mathbb{S}^1$ acts freely on both S^{n-1} and $S^{n-1} \times S^{n-1}$ and f is a $\mathbb{S}^1$-equivariant map. Thus the induced map $\tilde{f} : S^{2n-1} \to S^n$ is also $\mathbb{S}^1$-equivariant and $\mathbb{S}^1$ act on the mapping cone M of $\tilde{f}$ with $S^n \sqcup \{point\}$ as its fixed point set.

For $n = 8$, denote an element of S^{n-1} by (w_1, w_2) where w_1 and w_2 are quaternions. Let $\mathbb{S}^1$ act on S^{n-1} by

$$\big(z, (w_1, w_2)\big) \mapsto (zw_1z, w_2)$$

and act on $S^{n-1} \times S^{n-1}$ by

$$\Big(z, \big((w_1, w_2), (w_3, w_4)\big)\Big) \mapsto \big((zw_1, w_2z), (w_3z, w_4\overline{z})\big).$$

Then $\mathbb{S}^1$ acts freely on each factor of $S^{n-1} \times S^{n-1}$ and acts on S^{n-1} with fixed point set S^5. Note that, f is $\mathbb{S}^1$-equivariant and hence the induced map $\tilde{f}$ is also $\mathbb{S}^1$-equivariant.

Thus $\mathbb{S}^1$ acts on the mapping cone M of $\tilde{f}$ and has the fixed point set $S^6 \sqcup \{point\}$.

Note that f has bidegree $(1,1)$ and hence $\tilde{f}$ has Hopf invariant 1. This shows $M \simeq_{\mathbb{Q}} P^2(n)$. Let $\mathbb{S}^1$ act freely on S^{n-1} and act on S^n with S^r as the fixed set for some even integer r. Then $\mathbb{S}^1$ acts on $Y = S^{n-1} \star M$ with the fixed point set $S^s \sqcup \{point\}$, where s is an even integer and hence act on $X = S^n \vee Y \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n}$ with the fixed point set $S^r \sqcup S^s$ or $(S^r \vee S^s) \sqcup \{point\}$ depending on the wedge sum.
For the remaining case, we consider the $\mathbb{S}^1$ action on $X_n \simeq_{\mathbb{Q}} P^2(n)$ with $X_r \simeq_{\mathbb{Q}} P^2(r)$ as its fixed point set, where both n and r are even. Let $\mathbb{S}^1$ act freely on S^{n-1} and act on S^n with exactly two fixed points. Then it acts on $Y = S^{n-1} \star X_n$ with X_r as the fixed point set and hence acts on $X = S^n \vee Y \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n}$ with the fixed point set $F \simeq_{\mathbb{Q}} P^2(r) \sqcup \{point\}$. Alternatively, one can use Tomter [78], where a $\mathbb{S}^1$ action has been constructed on the actual quaternionic projective plane $\mathbb{H}P^2$ with the complex projective plane $\mathbb{C}P^2$ as its fixed point set and then do the above construction.

For $a \neq 0$, take $X = P^2(n) \vee S^{3n}$. As above, we can construct $\mathbb{S}^1$ actions on X with the desired fixed point sets.

Case(4) Suppose F has one component. Then either

$$F \simeq_{\mathbb{Q}} S^r \vee S^s \vee S^t \text{ for some integers } 1 \leq r, s, t \leq 3n$$

or

$$F \simeq_{\mathbb{Q}} S^s \vee P^2(r) \text{ for some integers } 1 \leq r \leq n \text{ and } 1 \leq s \leq 3n.$$

As $\chi(F) = \chi(X)$, for $F \simeq_{\mathbb{Q}} S^r \vee S^s \vee S^t$ we must have either r, s and t all are even or exactly one of them is even. Similarly, for $F \simeq_{\mathbb{Q}} S^s \vee P^2(r)$ we must have either s and r both even or both odd.

For $a = 0$, take $X = S^n \vee S^{2n} \vee S^{3n}$. If n is even, we take $\mathbb{S}^1$ actions on S^n, S^{2n} and S^{3n} having S^r, S^s and S^t respectively as fixed point sets, where r, s and t are all even. And if n is odd, we take actions where exactly one of them is even namely s. This gives an action on X with $S^r \vee S^s \vee S^t$ as the fixed point set, where the wedge is taken at some fixed points on the sub-spheres.

For the other case take $X = S^n \vee Y$, where $Y = S^{n-1} \star P^2(n)$ and n is even. Consider the $\mathbb{S}^1$ action on S^n with S^s as its fixed point set for some even s and the action on Y fixing $P^2(r)$ for some even r. This can be obtained by taking a free action on S^{n-1} and an action on $P^2(n)$ with $P^2(r)$ as the fixed point set. For example the $\mathbb{S}^1$ action on $\mathbb{R}^n$ with $\mathbb{R}^r$ as its fixed point set (r is even) induces an action on $X_n \simeq_{\mathbb{Q}} P^2(n)$ with $X_r \simeq_{\mathbb{Q}} P^2(r)$ as its fixed point set or one can use Tomter [78] for $n = 4$ and $r = 2$. This gives a $\mathbb{S}^1$ action on X with $F \simeq_{\mathbb{Q}} S^s \vee P^2(r)$.

For $a \neq 0$, as above, we can construct an action on $X = P^2(n) \vee S^{3n}$ with $F \simeq_{\mathbb{Q}} S^s \vee P^2(r)$ for even integers r and s. In this case the fixed point set cannot be a wedge of three spheres.

Now suppose that n is even and X is not totally non-homologous to zero in X_G. Then by Theorem 1.4.11,

$$\sum_{i\geq 0} rkH^i(F) \neq \sum_{i\geq 0} rkH^i(X) = 4$$

and hence

$$\sum_{i\geq 0} rkH^i(F) \leq 3.$$

This gives $\chi(F)$ = -1, 0, 1, 2 or 3. But Theorem 1.4.6 gives $\chi(F) = \chi(X) = 4$, a contradiction. This completes the proof of Theorem 5.1.1. □

5.3 Fixed point sets when X is not TNHZ in $X_{\mathbb{S}^1}$

Let X be not totally non-homologous to zero in X_G. Then n is odd and $\chi(F) = \chi(X) = 0$. As above $\sum_{i\geq 0} rkH^i(F) \leq 3$ and hence $\chi(F)$ = -1, 0, 1, 2 or 3.

But $\chi(F) = 0$ and therefore either $F = \phi$ or $F \simeq_{\mathbb{Q}} S^r$, where $1 \leq r \leq 3n$ is an odd integer.

Recall that when n is odd, we have $a = 0$. Using a construction of Bredon [5, p.268], we first construct a $\mathbb{S}^1$ action on $S^2 \times S^{n+2}$ with S^3 as its fixed point set, where $n \geq 1$ is odd. Let η be the Hopf 2-plane bundle over S^2 and $-\eta$ be its inverse, that is $-\eta \oplus \eta$ = trivial 4-plane bundle. Let ϵ be the trivial $(n-1)$-plane bundle over S^2. Then $-\eta \oplus \epsilon$ is a $(n+1)$-plane bundle (where $n+1$ is even) admitting a fiber preserving orthogonal action of $\mathbb{S}^1 \subset O(n+1)$ by bundle maps leaving the zero section fixed (which is the base space S^2). Together with the trivial action of $\mathbb{S}^1$ on η, this gives an action on the trivial $(n+3)$-plane bundle

$$\eta \oplus (-\eta \oplus \epsilon) : \ \mathbb{R}^{n+3} \hookrightarrow S^2 \times \mathbb{R}^{n+3} \longrightarrow S^2$$

whose fixed set is η. Taking the unit sphere bundles we get an action of $\mathbb{S}^1$ on $S^2 \times S^{n+2}$ with fixed point set S^3 (which is the total space of the unit sphere bundle of η). Now, remove a fixed point from $S^2 \times S^{n+2}$ to obtain a space $Z \simeq_{\mathbb{Q}} S^2 \vee S^{n+2}$ with a $\mathbb{S}^1$ action and contractible fixed point set. Let $\mathbb{S}^1$ act trivially on S^{n-3} and consider the induced action on the join $W = S^{n-3} \star Z$ which is homotopically equivalent to $S^n \vee S^{2n}$. This action on W has a contractible fixed point set. For a given odd integer $1 \leq r \leq 3n$, consider the $\mathbb{S}^1$ action on S^{3n} with S^r as its fixed point set. Then the wedge of W and S^{3n} at a fixed point is a space $X \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n}$ and has a $\mathbb{S}^1$ action with fixed point set $F \simeq_{\mathbb{Q}} S^r$. This proves Theorem 5.1.2. □

Remark 5.3.1. It is clear that there is no free $\mathbb{S}^1$ action on $S^n \vee S^{2n} \vee S^{3n}$. The author does not know of any free $\mathbb{S}^1$ action on a space $X \simeq_{\mathbb{Q}} S^n \vee S^{2n} \vee S^{3n}$.

5.4 Conclusions for torus actions

While studying circle actions it is natural to ask what happens for torus actions. We note that:

- If the r-torus $G = (\mathbb{S}^1)^r$ acts on a finitistic space X of type $(a, 0)$ with FMCOT, then by Lemma 1.4.5, there is a sub-circle $\mathbb{S}^1 \subset G$ such that their fixed point sets are same, that is, $X^{\mathbb{S}^1} = X^G$.

- If $\mathbb{S}^1$ acts on a space X, then we can define $G = (\mathbb{S}^1)^r = \mathbb{S}^1 \times (\mathbb{S}^1)^{r-1}$ action on X by
$$\big((g, g_1, ..., g_{r-1}), x\big) \mapsto g.x$$
such that $X^{\mathbb{S}^1} = X^G$.

Thus, with an additional assumption of FMCOT, our results and examples hold for torus actions also.

Chapter 6

Some Miscellaneous Results

6.1 Nice $\mathbb{Z}_p$ actions

This note is concerned with certain actions of the cyclic group $\mathbb{Z}_p$ of prime order p, which behaves well on passing to cohomology. An action of a group G on a space X induces an action of G on the cohomology of X. This induced action is important in the cohomology theory of transformation groups. One can see that for any action of $\mathbb{S}^1$ on a space X whose rational cohomology is of finite type, the induced action on the rational cohomology is always trivial. This is, however, not true for actions of the cyclic group $\mathbb{Z}_p$, when the cohomology is taken with coefficients in the finite field $\mathbb{F}_p$. Sikora in [58] defined certain actions of $\mathbb{Z}_p$ which behave well on passing to mod p cohomology. For such actions on Poincaré duality spaces, he proved a mod 4 congruence between the total Betti numbers of the space and that of the fixed point set.

An action of $\mathbb{Z}_p$ on a $\mathbb{F}_p$-vector space N is said to be nice if $N = T \oplus F$ as $\mathbb{F}_p[\mathbb{Z}_p]$-module, where T is a trivial and F is a free $\mathbb{F}_p[\mathbb{Z}_p]$-module. In other words, $T = \bigoplus \mathbb{F}_p$ and $F = \bigoplus \mathbb{F}_p[\mathbb{Z}_p]$. We say that a $\mathbb{Z}_p$ action on a space X is **nice** if the induced $\mathbb{Z}_p$ action on $H^n(X; \mathbb{F}_p)$ is nice for each $n \geq 0$. Note that, trivial actions are nice.

There is a $\mathbb{Z}_3$ action on $S^n \times S^n$, for $n = 1, 3$ or 7, which is not nice. S^n can be regarded as the set of elements of norm 1 in the ring of complex numbers, quaternions or Cayley numbers for $n = 1$, 3 or 7 respectively. Let $M = \{(x, y, z) \in S^n \times S^n \times S^n \mid (xy)z = 1\}$. Then M is homeomorphic to $S^n \times S^n$. Since $(xy)z = 1 \Leftrightarrow (yz)x = 1 \Leftrightarrow (zx)y = 1$, there is an action of $\mathbb{Z}_3$ on M by cyclic permutations. Note that $\mathbb{Z}_3$ action on $H^0(M; \mathbb{F}_3) = \mathbb{F}_3 = H^{2n}(M; \mathbb{F}_3)$ is trivial. Clearly, $H^n(M; \mathbb{F}_3) \neq T \oplus F$. Hence, $\mathbb{Z}_3$ action on M is not nice. However, note that every action of $\mathbb{Z}_3$ on S^n is nice. Thus, an arbitrary action of $\mathbb{Z}_p$ on $X \times Y$ need not be nice even if every action of $\mathbb{Z}_p$ on both X and Y is nice. In this note, we show that the diagonal action is nice.

If a group G acts on spaces X and Y, then there is a G action on $X \times Y$ given by

$$\big(g, (x, y)\big) \mapsto (g.x, g.y)$$

called the diagonal action. We say that a space X is of finite type if $\dim_{\mathbb{F}_p} H^n(X; \mathbb{F}_p) < \infty$ for all $n \geq 0$. We prove the following:

Theorem 6.1.1. *If $\mathbb{Z}_p$ acts nicely on spaces X and Y of finite type, then the diagonal action on $X \times Y$ is also nice.*

Before proceeding to prove the theorem, we recall the following consequence of Künneth formula for singular cohomology.

Theorem 6.1.2. *If X and Y are spaces of finite type, then there is a natural isomorphism*

$$\mu : \bigoplus_{i+j=n} H^i(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y; \mathbb{F}_p) \xrightarrow{\cong} H^n(X \times Y; \mathbb{F}_p)$$

for each $n \geq 0$.

Proof. The proof follows from the Künneth formula [70, p.247], since $Tor\big(H^*(X; \mathbb{F}_p), H^*(Y; \mathbb{F}_p)\big) = 0$. □

Recall that by the naturality of the isomorphism μ, if $f : X \to X'$ and $g : Y \to Y'$ are continuous maps, then the diagram

$$\begin{array}{ccc}
H^i(X'; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y'; \mathbb{F}_p) & \xrightarrow{\mu} & H^{i+j}(X' \times Y'; \mathbb{F}_p) \\
\Big\downarrow {\scriptstyle f^* \otimes g^*} & & \Big\downarrow {\scriptstyle (f \times g)^*} \\
H^i(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y; \mathbb{F}_p) & \xrightarrow{\mu} & H^{i+j}(X \times Y; \mathbb{F}_p)
\end{array}$$

commutes. In other words, if $u^{'} \in H^i(X^{'};\mathbb{F}_p)$ and $v^{'} \in H^j(Y^{'};\mathbb{F}_p)$, then

$$(f \times g)^* \mu(u^{'} \otimes v^{'}) = \mu\big(f^*(u^{'}) \otimes g^*(v^{'})\big)$$

This shows that if a group G acts on spaces X and Y, then the action induced in the cohomology from the diagonal action on $X \times Y$ is component-wise. We will also use the following proposition.

Proposition 6.1.3. *Let S be a commutative ring with identity 1_S and R be a subring of S containing 1_S. If F is a free R-module with basis X, then $S \otimes_R F$ is also a free S-module with basis $\{1_S \otimes x \mid x \in X\}$ of cardinality $|X|$.*

Proof. See [28, p.216]. □

We now prove Theorem 6.1.1.

Proof. Let $n \geq 0$ be fixed. By Theorem 6.1.2, we have

$$H^n(X \times Y;\mathbb{F}_p) \cong \bigoplus_{i+j=n} H^i(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y;\mathbb{F}_p)$$

as $\mathbb{F}_p$-vector spaces. By the naturality, this isomorphism is $\mathbb{Z}_p$-equivariant. Therefore, they are isomorphic as $\mathbb{F}_p[\mathbb{Z}_p]$- modules. Let i and j be fixed such that $i + j = n$. As $\mathbb{Z}_p$ acts nicely on both X and Y, we have $H^i(X;\mathbb{F}_p) = T_1 \oplus F_1$ and $H^j(Y;\mathbb{F}_p) = T_2 \oplus F_2$, where T_1, T_2 are trivial and F_1, F_2 are free $\mathbb{F}_p[\mathbb{Z}_p]$-modules. Since, the tensor product distributes over the direct sum and vice versa, it suffices to take $T_1 = \mathbb{F}_p = T_2$ and $F_1 = \mathbb{F}_p[\mathbb{Z}_p] = F_2$. Now,

$$\begin{aligned} H^i(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y;\mathbb{F}_p) &= (\mathbb{F}_p \oplus \mathbb{F}_p[\mathbb{Z}_p]) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \oplus \mathbb{F}_p[\mathbb{Z}_p]) \\ &= \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p \oplus \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mathbb{Z}_p] \oplus \mathbb{F}_p[\mathbb{Z}_p] \otimes_{\mathbb{F}_p} \mathbb{F}_p \oplus \\ &\mathbb{F}_p[\mathbb{Z}_p] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mathbb{Z}_p] \\ &= \mathbb{F}_p \oplus \mathbb{F}_p[\mathbb{Z}_p] \oplus \mathbb{F}_p[\mathbb{Z}_p] \oplus E \end{aligned}$$

where E is a free $\mathbb{F}_p[\mathbb{Z}_p]$-module by Proposition 6.1.3. Hence,

$$H^i(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(Y;\mathbb{F}_p) = T \oplus F,$$

where T is a trivial and F is a free $\mathbb{F}_p[\mathbb{Z}_p]$-module. This shows that the $\mathbb{Z}_p$ action on $H^n(X \times Y; \mathbb{F}_p)$ is nice and hence the diagonal action on $X \times Y$ is nice. $\square$

6.2 Commutativity of inverse limit and orbit map

This note is motivated by the following example of Bredon [6, p.145]. Let S^2 be the 2-sphere identified with the unreduced suspension of the circle $S^1 = \{z \in \mathbb{C}\ ;\ |z| = 1\}$, and $f : S^2 \to S^2$ be the suspension of the map $S^1 \to S^1$, $z \mapsto z^3$. Then f commutes with the antipodal involution on S^2. If Σ is the inverse limit of the inverse system

$$\cdots \xrightarrow{f} S^2 \xrightarrow{f} S^2 \xrightarrow{f} S^2$$

then $\Sigma/\mathbb{Z}_2$ is homeomorphic to $\varprojlim \mathbb{R}P^2$.

We show that this can be generalized, that is, the inverse limit and the orbit map commute for actions of compact groups on compact Hausdorff spaces. The proof of the result is simple, but does not seem to be available in the literature.

If $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an inverse system of topological spaces and $\{G_\alpha, \nu_\alpha^\beta, \Lambda\}$ is an inverse system of topological groups, where each X_α is a G_α-space and each bonding map π_α^β is ν_α^β-equivariant, then we get another inverse system $\{X_\alpha/G_\alpha, \overline{\pi_\alpha^\beta}, \Lambda\}$ by passing to orbit spaces. Also, under above conditions $\varprojlim X_\alpha$ is a $\varprojlim G_\alpha$-space with the action given by

$$(g_\alpha).(x_\alpha) = (g_\alpha.x_\alpha) \ \text{ for } (g_\alpha) \in \varprojlim G_\alpha \text{ and } (x_\alpha) \in \varprojlim X_\alpha.$$

In view of the above discussion, it is natural to ask, when is $(\varprojlim X_\alpha)/(\varprojlim G_\alpha)$ homeomorphic to $\varprojlim (X_\alpha/G_\alpha)$. We present the following theorem in this direction.

Theorem 6.2.1. *Let $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ be an inverse system of non-empty compact Hausdorff topological spaces and let $\{G_\alpha, \nu_\alpha^\beta, \Lambda\}$ be an inverse system of compact topological groups, where each X_α is a G_α-space and each bonding map π_α^β is ν_α^β-equivariant. Further, assume that Λ has the least element λ, G_λ action on X_λ is free and the bonding map ν_λ^α is injective for each $\alpha \in \Lambda$. Then there is a natural homeomorphism*

$$\psi : (\varprojlim X_\alpha)/(\varprojlim G_\alpha) \to \varprojlim (X_\alpha/G_\alpha).$$

We first prove the following simple lemma.

Lemma 6.2.2. *Let $\{X_\alpha, \pi^\beta_\alpha, \Lambda\}$ be an inverse system of non-empty compact Hausdorff topological spaces and let $\{G_\alpha, \nu^\beta_\alpha, \Lambda\}$ be an inverse system of compact topological groups, where each X_α is a G_α-space and each bonding map π^β_α is ν^β_α-equivariant. Then there is a natural closed continuous surjection*

$$\psi : (\varprojlim X_\alpha)/(\varprojlim G_\alpha) \to \varprojlim (X_\alpha/G_\alpha).$$

Proof. Let $X = \varprojlim X_\alpha$ and $G = \varprojlim G_\alpha$. Let $\pi_\beta : X \to X_\beta$ and $\nu_\beta : G \to G_\beta$ be the canonical projections for each $\beta \in \Lambda$. Each π_β is continuous by Proposition 1.2.1 and is also ν_β-equivariant and therefore induces a map $\psi_\beta : X/G \to X_\beta/G_\beta$ given by $\overline{(x_\alpha)} \mapsto \overline{x_\beta}$ (note that $\psi_\beta = \overline{\pi_\beta}$). Also observe that for $\gamma < \beta$ the diagram

$$\begin{tikzcd} X/G \arrow[r, "\psi_\beta"] \arrow[d, "\psi_\gamma"] & X_\beta/G_\beta \arrow[dl, "\overline{\pi^\beta_\gamma}"] \\ X_\gamma/G_\gamma & \end{tikzcd}$$

commutes. Therefore by Theorem 1.2.3, we have a natural map

$$\psi : X/G \to \varprojlim (X_\alpha/G_\alpha)$$

given by $\overline{(x_\alpha)} \mapsto (\overline{x_\alpha})$. Clearly ψ is surjective. By Theorem 1.1.1 and Theorem 1.2.2, we see that X/G is compact and $\varprojlim (X_\alpha/G_\alpha)$ is Hausdorff. Therefore ψ is a closed map. This completes the proof of the lemma. $\square$

We now complete the proof of Theorem 6.2.1.

Proof. It just remains to show that the map ψ is injective. Suppose $(\overline{x_\alpha}) = (\overline{y_\alpha})$, that is, $\overline{x_\alpha} = \overline{y_\alpha}$ for each $\alpha \in \Lambda$. This means that for each $\alpha \in \Lambda$, we have $x_\alpha = g_\alpha . y_\alpha$ for some $g_\alpha \in G_\alpha$. In particular, $x_\lambda = g_\lambda . y_\lambda$ for some $g_\lambda \in G_\lambda$. Since $\lambda \in \Lambda$ is the least element, we have $\lambda < \alpha$ and hence $\pi^\alpha_\lambda(x_\alpha) = x_\lambda$ and $\pi^\alpha_\lambda(y_\alpha) = y_\lambda$. This gives $x_\lambda = \pi^\alpha_\lambda(x_\alpha) = \pi^\alpha_\lambda(g_\alpha . y_\alpha) = \nu^\alpha_\lambda(g_\alpha).\pi^\alpha_\lambda(y_\alpha) = \nu^\alpha_\lambda(g_\alpha).y_\lambda$ and hence $g_\lambda . y_\lambda = \nu^\alpha_\lambda(g_\alpha).y_\lambda$.

The freeness of G_λ action on X_λ implies $g_\lambda = \nu_\lambda^\alpha(g_\alpha)$. Now, for $\lambda < \alpha < \beta$, we have $\nu_\lambda^\alpha \nu_\alpha^\beta(g_\beta) = \nu_\lambda^\beta(g_\beta) = g_\lambda$. By injectivity of ν_λ^α we get $\nu_\alpha^\beta(g_\beta) = g_\alpha$. Thus, $(x_\alpha) = (g_\alpha.y_\alpha) = (g_\alpha).(y_\alpha)$ where $(g_\alpha) \in G$. This shows that ψ is injective and hence a homeomorphism by Lemma 6.2.2. $\square$

As a consequence we have the following corollary.

Corollary 6.2.3. *Let G be a compact topological group acting freely on a compact Hausdorff topological space X and let $f : X \to X$ be a G-equivariant map. Then for the inverse system*

$$\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X$$

$(\varprojlim X)/G$ is homeomorphic to $\varprojlim(X/G)$.

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