Different Aspects of Black Hole Physics in String Theory

By

Nabamita Banerjee

Harish-Chandra Research Institute, Allahabad

A Thesis submitted to the

Board of Studies in Physical Science Discipline

In partial fulfilment of requirements

For the degree of

DOCTOR OF PHILOSOPHY

of

Homi Bhabha National Institute

May, 2009

Certificate

This is to certify that the Ph. D. thesis titled "Different Aspects of Black Hole Physics in String Theory " submitted by Nabamita Banerjee is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date:

Prof. Dileep Jatkar
Thesis Advisor

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Dileep Jatkar, at Harish-Chandra Research Institute, Allahabad.

Date:

Nabamita Banerjee
Ph. D. Candidate

To my Family

Acknowledgments

First and foremost, I would like to thank my supervisor Prof. Dileep Jatkar for carrying out a liberal yet vigilant supervision of my Ph. D. project. He has always allowed me to enjoy the full academic freedom for pursuing my interests in physics. He has been extremely patient towards me and has been very generous in spending time for discussions. Besides being my advisor, he has also been a very good friend of mine and helped me a lot. I do not think I am articulate enough to express how much I am indebted to Prof. Ashoke Sen, for not only giving excellent courses and educating us about several advanced topics but also for giving superb audience to our queries and patiently explaining the issues and subtleties involved in the computations. I have had the fortune of having many an excellent discussions during various collaborations with him. It would be quite inappropriate if I fail to recognize the value of his advice and guidence in shaping my career as a string theorist.

I sincerely wish to express my gratitude to all my collaborators: Dumitru Astefanesei, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Suvankar Dutta, Rajeev Jain, Dileep Jatkar, Ipsita Mandal, R. Loganayagam, Ashoke Sen and P. Surowka for many useful discussions. I am also very much thankful to Prof. Justin David, Prof. Rajesh Gopakumar and Prof. Shiraz Minwalla for many useful discussions. I am thankful to all the String Theory Group members in HRI for wonderful discussions.

I would like to thank the entire H. R. I. community for making my stay extremely comfortable and memorable. Thanks are also due to all my fellow students in HRI, and in particular, Arijit. I would also like to thank my teachers, specially R. K. Bose, S. P. Das and N. Kar for their inspiration.

I am especially thankful to Suvankar for his tremendous support during the period of Ph. D and beautiful physics discussions. At the end I would like to express my deep regards to my parents for their enormous amount of love, care and affectionate support without which this thesis would not have been possible.

Different Aspects of Black Hole Physics in String Theory

Thesis Synopsis

Nabamita Banerjee
Harish-Chandra Research Institute

In this thesis I have mainly been working on different aspects of Black-Hole physics and AdS/CFT within the context of string theory . I have worked on degeneracy counting problem to obtain precision results on black hole entropy as well as on black holes as solution to the supergravity equations of motion. I have then utilized this further to understand the higher derivative corrections to different thermodynamic and hydrodynamic quantities (like entropy, shear viscosity etc.) of boundary field theory.

Black Hole Microstate Counting

I have studied aspects of microscopic origin of a special class of dyonic black holes in four dimensional N = 4 supersymmetric string theories. Within the two derivative approximation of the gravity, the black hole entropy is given by Bekenstein-Hawking-Wald area law. For extremal black holes, this is well understood from statistical view point, i.e., the black hole entropy is same as the logarithm of the degeneracy of the number of quantum states associated with the black hole. This initial success of equality of the macroscopic (Black Hole) entropy and the microscopic (statistical) entropy was obtained in the large charge limit, where two derivative gravity is a good approximation. Hence, the next obvious question to ask is, what happens when we consider large but finite charges or equivalently add higher derivative terms in the effective action? To answer this question, on the macroscopic side one needs to compute entropy in the presence of higher derivative terms. Different approaches like Euclidean entropy, Wald's formalism or Sen's entropy function formalism can be applied for this purpose. On the microscopic side, one needs to go beyond the Cardy formula and compute statistical entropy with greater accuracy. This was done long ago in case of quarter BPS states in N = 4 supersymmetric heterotic string theory. Recently these results were generalized to arbitrary N = 4 string theory in four dimensions. The computation of exact microscopic entropy for a class of quarter BPS dyonic black holes (carrying special charges) in N = 4 supersymmetric string theories is obtained in terms of a logarithm of a dyon degeneracy formula. It has been shown that the black hole entropy, with the addition of the Gauss-Bonnet(GB) term in the action, is same as the microscopic entropy expanded to correct order in the inverse power of charges.

In the work [1], we have extended the microscopic computation of entropy to generalized (carrying general charges) dyonic black holes in N = 4 supersymmetric string theories. We showed that the low energy supergravity already knows how to take care of the dyon degeneracy, in the sense that if we turn on more charges the supergravity automatically adjusts other charges to conform to the dyon degeneracy formula.

In [2], we develop a better understanding of the asymptotic expansion of the degeneracy formula in inverse power of charges. We have done two loop computation (i.e., going beyond the GB on gravity side) and found that the effective action at this order is invariant under continuous S-duality and T-duality transformations. We also found that the asymptotic expansion formula fails to give correct result for lower values of charges. Deviation from the leading asymptotics indicates that contribution from sub-leading saddle point is significant for small charges. We take contribution from the sub-leading saddle points into account to correct asymptotic expansion of the degeneracy function for small charges.

In [3], I have extended the asymptotic expansion analysis of black hole degeneracy to five dimensional BMPV black holes. The BMPV black holes share several features with the dyonic black holes in four dimensions. It is particularly evident when one looks at them from brane configuration point of view. Given the fact that we can compute dyon degen-eracy precisely in four dimension, we should be able to get exact statistical entropy for five dimensional BMPV black hole, bcause in the brane configuration picture BMPV black hole is a subset of four dimensional dyonic black hole. I utilize this to compute exact degeneracy for BMPV black holes. Carrying out the asymptotic expansion of the exact result, I obtain one loop corrected entropy for BMPV black holes. This one loop corrected entropy is valid beyond the Farey tail limit.

Higher Derivative Corrections to Shear Viscosity from Graviton's Effective Coupling

The shear viscosity coefficient of strongly coupled boundary gauge theory plasma depends on the horizon value of the effective coupling of transverse graviton moving in black hole background. The proof for the above statement is based on the canonical form of graviton's action. But in presence of generic higher derivative terms in the bulk Lagrangian the action is no longer canonical. In [4] we give a procedure to find an effective action for graviton (to first order in coefficient of higher derivative term) in canonical form in presence of any arbitrary higher derivative terms in the bulk. From that effective action we find the effective coupling constant for transverse graviton which in general depends on the radial coordinate r. We also argue that horizon value of this effective coupling is related to the shear viscosity coefficient of the boundary fluid in higher derivative gravity. We explicitly check this procedure for two specific examples: (1) four derivative action and (2) eight derivative action ($Weyl^4$ term). For both cases we show that our results for shear viscosity coefficient (upto first order in coefficient of higher derivative term) completely agree with the existing results in the literature.

Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity

In [5], we calculate shear viscosity to entropy density ratio in presence of four derivative (with coefficient α') and six derivative (with coefficient α'^2) terms in bulk action. In general, there can be three possible four derivative terms and ten possible six derivative terms in the Lagrangian. Among them two four derivative and eight six derivative terms are ambiguous, i.e., these terms can be removed from the action by suitable field redefinitions. Rest are unambiguous. According to the AdS/CFT correspondence all the unambiguous coefficients (coefficients of unambiguous terms) can be fixed in terms of field theory parameters. Therefore, any measurable quantities of boundary theory, for example shear viscosity to entropy density ratio, when calculated holographically can be expressed in terms of unambiguous coefficients in the bulk theory (or equivalently in terms of boundary parameters). We calculate η/s for generic six derivative gravity and find that apparently it depends on few ambiguous coefficients at order α'^2. We calculate six derivative corrections to central charges a and c and express η/s in terms of these central charges and unambiguous coefficients in the bulk theory.

1. *Adding Charges to N=4 Dyons,* With Dileep Jatkar, Ashoke Sen. JHEP 0707:024,2007. e-Print: arXiv:0705.1433 [hep-th]

2. *Assymptotic Expansion of the N=4 Dyon Degeneracy,* With Dileep Jatkar, Ashoke Sen. e-Print: arXiv:0810.4372 [hep-th]

3. *Subleading Correction to Statistical Entropy for BMPV Black Hole,* Phys. Rev. D 79, 081501(R) (2009) e-Print: arXiv:0807.1314 [hep-th]

4. *Higher Derivative Corrections to Shear Viscosity from Graviton's Effective Coupling,* With S. Dutta, JHEP 0903:116,2009. arXiv:0901.3848 [hep-th].

5. *Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity,* With S. Dutta arXiv:0903.3925 [hep-th].

Publications/Preprints Not included in this thesis:

1. *Non-gravitating scalar field in the FRW background,* With R. Jain, D. Jatkar Gen. Rel. Grav. **40**, 93 (2008) [arXiv:hep-th/0610109].

2. *Phase Transition of Electrically Charged Ricci-flat Black Holes,* With S. Dutta JHEP **0707**, 047 (2007) [arXiv:0705.2682 [hep-th]].

3. *(Un)attractor black holes in higher derivative AdS gravity,* With D. Astefanesei and S. Dutta JHEP **0811**, 070 (2008) [arXiv:0806.1334 [hep-th]].

4. *Hydrodynamics from charged black branes,* With J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, arXiv:0809.2596 [hep-th].

5. *Black Hole Hair Removal,* With I. Mandal and A. Sen, arXiv:0901.0359 [hep-th].

Contents

List of Figures v

List of Tables vi

I Introduction 1

1 Introduction 3

1.1 Black Hole Precision Counting 5

1.1.1 5-Dimensional Black Hole 6

1.1.2 4-Dimensional Black Hole 7

1.2 The AdS/CFT Correspondence 9

1.2.1 The Conjecture 10

1.2.2 The Thermal AdS/CFT Correspondence 10

1.2.3 Bulk-Boundary Thermodynamics 11

1.2.4 The AdS/CFT Correspondence and Hydrodynamics 13

II Black Hole Microstate Counting 17

2 Adding Charges to $\mathcal{N} = 4$ Dyons 19

2.1 Introduction 19

2.2 Background 20

2.3 Charge Carrying Deformations 23

2.4 Additional Shifts in the Charges 26

2.5 Dyon Spectrum 30

2.6 More General Charge Vector 32

3 Asymptotic Expansion of the $\mathcal{N} = 4$ Dyon Degeneracy 35

3.1 Introduction and Summary 35

3.2 An Overview of Statistical Entropy Function 37

3.2.1 Dyon degeneracy 38

3.2.2 Asymptotic expansion and statistical entropy function . . . 40

3.2.3 Exponentially suppressed corrections 43

3.2.4 Organising the Asymptotic Expansion 45

3.3 Power Suppressed Corrections 46

3.4 Exponentially Suppressed Corrections 54

3.5 Macroscopic Origin of the Exponentially Suppressed Corrections 57

4 Subleading Correction to Statistical Entropy for BMPV Black Hole 67

4.1 Introduction 67

4.2 Degeneracy Function For 5-dimensional BMPV Black Holes . . . 69

4.2.1 Degeneracy Function of 4-dimensional Dyonic Black holes . 69

4.2.2 Degeneracy for BMPV Black Holes 72

4.3 Correction to The Statistical Entropy Function 73

4.4 Correction to Statistical Entropy 76

4.5 Degeneracy for More General 5D Black Holes 77

4.6 Discussion 78

III Hydrodynamics from Black Holes 81

5 Hydrodynamics from AdS/CFT 83

5.1 Introduction 83

5.2 Viscosity from Kubo Formula 85

5.3 Hydrodynamic limit in AdS/CFT and Membrane paradigm . . . 87

6 Higher Derivative Corrections to Shear Viscosity coefficient From Graviton's Effective Coupling 91

6.1 Introduction 91

6.2 Shear Viscosity from Effective Coupling 94

6.3 The Effective Action 96
6.3.1 The General Action and Equation of Motion 96
6.3.2 Strategy to Find The Effective Action 99
6.4 Flow from Boundary to Horizon 102
6.5 Membrane Fluid in Higher Derivative Gravity 103
6.6 Four Derivative Lagrangian 105
6.6.1 The General Action 106
6.6.2 The Effective Action and Shear Viscosity 107
6.7 String Theory Correction to Shear Viscosity 108
6.7.1 The General Action 109
6.7.2 The Effective Action and Shear Viscosity 109
6.8 Discussion 110

7 Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity 113
7.1 Introduction and Summary 113
7.2 The Field Re-definition and $\frac{\eta}{s}$ 118
7.3 The Perturbed Background Metric 122
7.4 The Effective Action and Shear Viscosity 124
7.4.1 Shear Viscosity to Entropy Density Ratio 127
7.5 Conformal Anomaly in Six-derivative Gravity 128
7.6 η, s and $\frac{\eta}{s}$ 135

IV Appendices 137

A Normalization and Sign Conventions 139

B Fixing the Normalization Constant 143

C Expressions for A^{GB} 149

D Expressions for A^{W} 151

E Expressions for A_i's 153

F Expressions for $\mathcal{B}_0$ and $\mathcal{B}_1$ 155

G Shear Viscosity from Kubo's Formula 157

H Leading $r-$Dependence of Curvature Tensors 159

Bibliography 161

Index 171

List of Figures

3.1 2-loop graphs using the vertices from F_0. 47
3.2 1-loop graph using a 2-vertex from F_1. 48

List of Tables

3.1 Comparison of the exact statistical entropy to the tree level, one loop and two loop results obtained via the asymptotic expansion. In the last two columns D_1 is the difference of the exact result and the one loop result and D_2 is the difference of the exact result and the two loop result. We clearly see that for $Q \cdot P > 0$ where only single centered black holes contribute to S_{stat}, inclusion of the two loop results reduces the error, at least for large charges. 52
3.2 First exponentially suppressed contribution to $d(Q, P)$ and $S_{stat}(Q, P)$. Note that the correction vanishes accidentally for $Q \cdot P = Q^2/2 = P^2/2$ odd. 58

Part I

Introduction

Chapter 1

Introduction

Black holes first appeared as classical solutions to general theory of relativity. Purely from theoretical point of view, these solutions have a point-like curvature singularity. This singularity is separated from the outside world by a hypothetical surface known as the event horizon. Classically, any object that crosses the horizon cannot escape to infinity, and hence no information can come out of this surface. This is in contradiction with the second law of thermodynamics. To see this let us consider a hot body falling in the black hole. The mass of the combined system would increase slightly. Classically, once the hot body is absorbed by the black hole, all its information is lost. However, according to the second law of thermodynamics, total entropy of the combined system can not decrease. To get out of this impasse, Bekenstein suggested that, black holes must carry entropy. Another beautiful classical result regarding black hole is that the horizon area of a black hole cannot decrease with time and same is true whenever two or more black holes merge to form a single black hole[1]. Taking the cue from this, Bekenstein [2, 3] postulated that entropy of a black hole is proportional to the area of its event horizon. Later, by semi-classical computation, Hawking [4] showed that black holes are quantum mechanical black bodies which radiate energy according to Planck's law and behave as thermodynamic objects. Hence, they can be described in terms of thermodynamic quantities like temperature (hawking temperature) and entropy (Bekenstein-Hawking entropy). These properties are difficult to understand at the fundamental level using statistical mechanics. For example we can compute the entropy of a system by counting the number of degrees of freedom describing that system. If we wish to do similar computation for a black hole then we need to give microscopic description of it in terms of a fundamental theory of quantum gravity.

In this thesis, we would like to address the issue of understanding the black hole entropy by counting the degeneracy of quantum states associated with it. String theory, the most promising candidate of quantum gravity, has made a lot of progress in this regard. This theory contains a variety of fundamental objects like, open strings, closed strings as well as solitonic objects like D-branes. Any closed string theory contains gravity because gravitons appear in its elementary excitation spectrum. A low energy limit of string theory is nicely given in terms of supergravity theory. It is known that black holes are classical solutions to the supergravity equations of motion. We are, however, interested in counting black hole microstates, which can most suitably be done using string theory. It is therefore desirable to describe a black hole in terms of basic string theoretic ingredients, namely fundamental strings and solitons, like D-branes, NS5-branes and KK monopoles. The solitonic objects are very massive at weak coupling, since their masses are inversely proportional to the coupling constant. However, at strong coupling, they become light but interact strongly resulting into formation of a black hole. In the next section (1.1), we will briefly review how we can interpret the black hole entropy statistically using string theory.

There is yet another way to understand black hole thermodynamics from the point of view of string theory, *i.e.*, in the context of the AdS/CFT correspondence, another outstanding success of string theory. This conjecture was proposed by Maldacena in 1997, which states that type IIB string theory in $AdS_5 \times S^5$ spacetime is dual to $N = 4$ super Yang-Mills theory living on four dimensional manifold. Therefore, using the AdS/CFT correspondence, one can study different properties of string theory (or black hole) spacetime by doing some computation in its dual version, *i.e.*, on the gauge theory side and vice-versa. In this thesis, we have also taken this alternative approach to understand thermodynamic and hydrodynamic aspects of the boundary field theory. In section (1.2), we will discuss how the AdS/CFT conjecture gives a dual description of black hole thermodynamics and hydrodynamics.

1.1 Black Hole Precision Counting

Let us first consider a stationary Schwarzschild black hole and a string with high degree of excitation but zero momenta. The Schwarzschild black hole entropy S_{BH} which according to Bekenstein proposal is related to the area of event horizon, is known to be proportional to M^2 where, M is the mass of the black hole. The string entropy S_{st} for the same mass states, on the other hand, goes as M [5],

$$\frac{S_{BH}}{\kappa} = 4\pi G M^2 \qquad \text{and} \qquad \frac{S_{st}}{\kappa} = 4\pi\sqrt{\alpha'}M, \tag{1.1.1}$$

where, κ is Boltzmann constant, G is Newton's constant, α' is inverse string tension. There is a clear disagreement between these two entropies. While the black hole entropy $\sim M^2$, the string entropy $\sim M$. The reason behind the discrepancy is obvious: the black hole entropy was calculated in a regime where the interactions are necessary (in other words, in string theory, black holes can only exist when the interactions are turned on), while the string entropy was computed for free strings. We can only expect an agreement between the two if, for some reason, the interaction did not affect the string entropy calculations.

The condition mentioned above can be achieved easily if we have supersymmetry in the theory. In this case, BPS states, which preserves certain amount of supersymmetry, can be counted at zero string coupling and the counting of states remains valid when the coupling is turned on. The Schwarzschild black holes cannot be represented by such BPS states. Initially, some four dimensional half-BPS black hole solutions were obtained, but they had zero horizon area [7]. Hence, S_{BH} were zero for this black holes although S_{st} was finite. This situation is opposite of that encountered in case of the Schwarzschild black holes. To see the equivalence of S_{BH} and S_{st}, one needs a BPS black hole solution with non zero horizon area. This was first achieved for a particular kind of five-dimensional black holes in the superstring theory, that can be described in terms of BPS states in the theory [6]. For these black holes the computation of S_{BH} and S_{st} has been done and the results completely agree. We will briefly outline the computation below. Similar results were subsequently obtained in the case of four dimensional black holes which we will discuss later.

1.1.1 5-Dimensional Black Hole

In this section, we will summarize the results for 5-dimensional black hole entropy computation and discuss how the microscopic and macroscopic computation gives the same result. The particular configuration we consider here is the Strominger-Vafa black hole [6]. More general 5-dimensional black hole will [8] be considered in chapter 4.

Let us consider a 5D black hole carrying three different electric charges Q_1, Q_5 and N. A specific black hole is the one with three fixed integer values for these charges. We consider extremal limit (*i.e.*, minimal mass of the black hole compatible with the charges) such that half of the spacetime supersymmetries are preserved. In other words, we get a half-BPS black hole. The thermodynamic (macroscopic) entropy of this black hole is given as,

$$\begin{aligned} \frac{S_{BH}}{\kappa} &= \frac{A_H}{4G_5} = 2\pi\sqrt{NQ_1Q_5} \\ \text{where} \quad G_5 &= \text{5D Newton's Constant} \end{aligned} \tag{1.1.2}$$

Now, let us look into the microscopic computation of the entropy by treating this black holes as an object composed of specific states in the string theory. String theory should be able to explain the black hole entropy in terms of logarithm of microscopic degrees of freedom constituting the black hole. Since, this black hole is supersymmetric, it ensures that the counting of states at zero coupling continues to hold even when interactions are turned on.

The three charge 5D black hole is constructed in IIB superstring theory compactified on $T^4 \times S^1$. The microscopic configuration contains coincident Q_1 number of D1-branes wrapping S^1 and Q_5 number of D5-branes wrapping

$T^4 \times S^1$. The charge N is the momentum along the S^1 circle. This momentum is carried by the open stings attached to the D-branes. There are many string states stretched between different D-branes. Moreover the momentum quantum number N can be split between many open strings. Thus we see that string theory can account for different states associated with the black hole.

As no massive states are excited in the configuration, we are interested in, only string states stretched between D1-D5 branes need to be counted. Total number ground states of the strings stretched between D1-D5 branes is eight, four bosonic and four fermionic. Also, Q_1 number of wrapped D1-branes can be treated as a single D1-brane wrapped Q_1 times. Similarly for D5-branes we can think of the configuration to be a single D5-branes wrapped Q_5 times. With these information, we can go ahead and compute the degeneracy of states associated with this configuration. This is same as partition of the number NQ_1Q_5 into 4 bosonic and 4 fermionic numbers. The result is given by,

$$P(NQ_1Q_5 : 4 : 4) = e^{2\pi\sqrt{NQ_1Q_5}}. \tag{1.1.3}$$

Hence the string entropy associated with this configuration is,

$$\frac{S_{st}}{\kappa} = \ln P(NQ_1Q_5 : 4 : 4) = 2\pi\sqrt{NQ_1Q_5} \tag{1.1.4}$$

We see that the two entropies exactly matches even up to the numerical numbers. This matching of entropies is a major success for string theory. We would also like to understand the precise match of macroscopic and microscopic entropy for the four dimensional black holes.

1.1.2 4-Dimensional Black Hole

We will now consider four dimensional black holes. They can be obtained by compactifying any superstring theory on any six-dimensional manifold. The choice of manifold and choice of string theory determines the amount of supersymmetry preserved in four dimensional theory. In these theories, we can have electrically charged blak holes, magnetically charged black holes or dyonic black holes. The issue of matching S_{BH} and S_{st} is a bit subtle in both cases. The subtlety arises due to following reasons:

- For purely electrically (or magnetically) charged extremal (preserving certain amount of SUSY)[1] black holes, the horizon area is zero, if we consider only Einstein-Hilbert action. Hence, in this case, $S_{BH} = 0$, although one does get a non zero answer for S_{st}. The issue has been resolved by considering "the stretched horizon" [9, 10]. We actually need to add the higher derivative terms to E-H action and consider the corrected solution. With this corrected solution and modified way of computing black hole entropy for higher derivative terms (for example, Wald's approach), we can calculate S_{BH} and it gives exact agreement with S_{st}.

- Another way to approach the problem is to consider dyonic black holes [22]. These black holes carry both, electric and magnetic charges and have non zero horizon area even with out adding any higher derivative terms to the Einstein-Hilbert action. But, the computation of degeneracy of string states (microscopic) is bit subtle here. Only fundamental string states (they are electrically charged) can not solve the purpose.

In this thesis, we will concentrate on the degeneracy counting of dyonic black hole states in four dimensional $\mathcal{N} = 4$ theories. The details of our analysis and results are given in chapters 2 and 3. These chapters are self contained, as it contains sufficient required back ground material.

- The spectrum of dyons in a class of N=4 supersymmetric string theories has been for a specific set of electric and magnetic charge vectors. In chapter 2, we extend the analysis to more general charge vectors by considering various charge carrying collective excitations of the original system.

- In chapter 3, we study various aspects of power suppressed as well as exponentially suppressed corrections in the asymptotic expansion of the degeneracy of quarter BPS dyons in N=4 supersymmetric string theories. In particular we explicitly calculate the power suppressed corrections up to second order and the first exponentially suppressed corrections. We also propose a macroscopic origin of the exponentially suppressed cor-

[1] More general definition of extremal black hole, which is also applicable to the non-supersymmetric case, is the one which has near horizon geometry $AdS_2 \times S^2$.

rections using the quantum entropy function formalism. This suggests a universal pattern of exponentially suppressed corrections to all four dimensional extremal black hole entropies in string theory.

- In chapter 4 , we will return to more general five dimensional black holes and evaluate higher derivative corrections to the degeneracy formula.

1.2 The AdS/CFT Correspondence

So far we have discussed how to interpret black hole entropy as a logarithm of number of black hole microstates. As we have mentioned, one can also give an alternative description of black hole thermodynamics in terms of its dual theory (*i.e. gauge theory*) via AdS/CFT correspondence. Since type IIB string theory in AdS space and $N = 4$ super Yang-Mills theory are dual description of each other, we can also extract information of boundary gauge field theory by studying the string theory in AdS background. Before we start applying this correspondence to understand different thermodynamic and hydrodynamic properties of gauge theory, let us first briefly review few important aspects of this conjecture.

The AdS/CFT correspondence (or AdS/CFT conjecture) is one of the most exciting development in string theory of last decades. This conjecture was proposed by Maldacena in 1997 [11] and then extended by Witten [12, 13]. Anti de-Sitter space/Conformal Field Theory correspondence (AdS/CFT) is a conjectural equivalence between closed string theory on certain ten dimensional background involving AdS spacetime and four dimensional conformal field theory. The conjecture is a powerful tool in theoretical high energy physics because it relates a theory of gravity such as string theory, to a theory with no gravitational excitation at all. Not only that, the conjecture also relates highly nonperturbative problems in Yang-Mills theory to questions in classical superstring theory or supergravity. The most promising advantage of this correspondence is that the problem that appear to be almost intractable on one side may be solvable on the other side.

The correspondence is itself a vast subject and there are lots of good review articles on this subject. Therefore, in this chapter we will briefly discuss the important points of this conjecture.

1.2.1 The Conjecture

The conjecture states the *equivalence* between following two seemingly unrelated theories.

Type IIB string theory on $AdS_5 \times S^5$, where both AdS_5 and S^5 has radius b, with a five form field strength F_5, which has integer flux N over S^5, and complex string coupling $\tau_S = a + ie^{-\phi}$ where a is axion and ϕ is dilaton field
AND
$\mathcal{N} = 4$ SYM theory in 4 dimension, with gauge group $SU(N)$, Yang-Mills coupling g_{YM} and instanton angle θ_I (together define a complex coupling $\tau_{YM} = \frac{\theta_I}{2\pi} + \frac{4\pi i}{g^2_{YM}}$) in its superconformal phase,
WITH
$g_S = \frac{g^2_{YM}}{4\pi}$, $a = \frac{\theta_I}{2\pi}$ and $b^4 = 4\pi g_S N(\alpha')^2$.

More precisely, the AdS/CFT conjecture states that these two theories, including operator observables, states, correlation functions and full dynamics, are equivalent to one another [11, 12, 14]. For a general review on this subject, see [15].

1.2.2 The Thermal AdS/CFT Correspondence

Gauge theory at finite temperature possesses more interesting and richer phase diagram than that at zero temperature. Finite temperature gauge theory undergoes a phase transition (confinement-deconfinement phase transition) at large N. Also finite temperature effect breaks the supersymmetry and conformal invariance of the boundary gauge theory. In some sense finite temperature effects make life more complicated and at the same time more interesting. In this chapter we will briefly discuss the finite temperature version of the AdS/CFT correspondence. We will discuss how to construct the dual gravity solution of finite temperature gauge theory and will also discuss their thermodynamical properties like free energy, entropy *etc.*.

The gravity solution describing the gauge theory at finite temperature can be obtained by starting from the general black 3-brane solution. In the decoupling limit the metric is given by,

$$ds^2 = \frac{r^2}{b^2}\left[-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + \frac{dr^2}{\frac{r^2}{b^2}f(r)} + b^2 d\Omega_5^2 \tag{1.2.5}$$

with,

$$f(r) = \left(1 - \frac{r_0^4}{r^4}\right) \tag{1.2.6}$$

where, r_0 is the position of the horizon of the black $D3$ brane and $d\Omega_5^2$ is the metric element of a unit five sphere. The black brane has temperature

$$T = \frac{r_0}{\pi b^2}. \tag{1.2.7}$$

The string theory on this background geometry (black brane at temperature T) is dual to the boundary gauge theory at the same temperature T.

1.2.3 Bulk-Boundary Thermodynamics

First indication that large N finite T gauge theory might be a good microscopic description of N coincident D3 brane geometry, comes from **Free energy or Entropy** calculation on both sides [16]. On the supergravity side the entropy of a non-extremal black $D3$ brane is given by the usual Bekenstein-Hawking result,

$$S_{SUGRA} = \frac{Area}{4G_5} = \frac{1}{4G_5}\frac{r_0^3}{b^3}\int d^3x = \frac{1}{4G_5}\frac{r_0^3}{b^3}V_3. \tag{1.2.8}$$

This entropy is expected to be the entropy of thermal gauge theory at large N and large $g_{YM}^2 N$. But it is very difficult to calculate the entropy of a strongly coupled gauge theory at finite temperature. Nevertheless in [16] the authors considered the large N gauge theory at free field limit and computed the entropy and surprisingly two results for the entropy agreed up to a factor of $\frac{4}{3}$.

$$\begin{aligned} S_{SUGRA} &= \frac{\pi^2}{8} N^2 V_3 T^3 \\ S_{SYM} &= \frac{4}{3} S_{SUGRA}. \end{aligned} \tag{1.2.9}$$

Currently there is no concrete argument why these two results agreed up to a puzzling factor of $\frac{4}{3}$. The gauge theory computation was performed at zero 't Hooft coupling where as the supergravity approximation is valid at strong 't Hooft coupling limit. Indeed it was suggested in [17] that the leading term in $\frac{1}{N}$ expansion of entropy has the following form

$$S(g_{YM}^2 N) = f(g_{YM}^2 N) \frac{\pi^2}{6} N^2 V_3 T^3, \tag{1.2.10}$$

where, $f(g_{YM}^2 N)$ is a function which smoothly interpolates between a weak coupling limit of 1 and strong coupling limit of 3/4 The function $f(g_{YM}^2 N)$ is expected to be a smooth function of $g_{YM}^2 N$. Therefore it is very exciting to find out the leading correction to this function to the limiting values both at strong coupling and weak coupling. The results are given by [17, 18],

$$\begin{aligned} f(g_{YM}^2 N) &= 1 - \frac{3}{2\pi^2} g_{YM}^2 N + \cdots \qquad for\ small\ g_{YM}^2 N \\ &= \frac{3}{4} + \frac{45}{32} \frac{\zeta(3)}{(g_{YM}^2 N)^{3/2}} + \cdots \quad for\ large\ g_{YM}^2 N. \end{aligned} \tag{1.2.11}$$

The weak coupling computation is straightforward. One has to apply the diagrammatic techniques of perturbative field theory and find the corrections loop by loop. The constant term comes from one loop calculation and the leading correction comes from two loop calculation.

On the other hand in the string theory side, if we consider the 't Hooft coupling to be very large but finite, then we have to include the string theory corrections to thermodynamic quantities, *i.e.* we need to improve the supergravity results by including the higher derivative terms in the action for example the first leading correction to the type IIB supergravity action is proportional to α'^3. In general, for finite but large 't Hooft coupling the bulk effective action is given by classical supergravity action plus all possible higher derivative terms, which appear in type IIB string theory.

Thus we see that the conjectured duality between thermal gauge theory and gravity in one higher dimensional AdS spacetime is a useful tool to extract thermodynamical properties of string theory or gravity in terms of dual gauge theory and vice-versa.

1.2.4 The AdS/CFT Correspondence and Hydrodynamics

The power of AdS/CFT is not confined to characterizing only the thermodynamic properties of black brane geometries. If we consider a black object with translation invariant horizon, for example black D3 brane geometry, one can also discuss hydrodynamics - long wave length deviation (low frequency fluctuation) from thermal equilibrium. In addition to the thermodynamic quantities the black brane is also characterized by the hydrodynamic parameters like viscosity, diffusion constant, *etc.*. The black $D3$ brane geometry with low energy fluctuations (*i.e.* with hydrodynamic behavior) is dual to some finite temperature gauge theory plasma living on boundary with hydrodynamic fluctuations. Therefore studying the hydrodynamic properties of strongly coupled gauge theory plasma using the AdS/CFT duality is an interesting subject of current research. The energy momentum tensor of a relativistic viscous conformal fluid is given by,

$$T_{\mu\nu} = (e+p)u_\mu u_\nu + p\eta_{\mu\nu} - 2\eta\sigma_{\mu\nu} \tag{1.2.12}$$

where u_μ is fluid 4-velocity with $u_\mu u^\mu = -1$, e is energy, p is pressure and η is shear viscosity coefficient. $\sigma_{\mu\nu}$ is defined in (5.2.4). Conformal invariance implies that $e = 3p$. We will discuss different properties of viscus fluid in chapter 5.

The first attempt to study hydrodynamics via AdS/CFT was [76], where authors related the shear viscosity coefficient η of strongly coupled $\mathcal{N} = 4$ gauge theory plasma in large N limit with the absorption cross-section of low energy gravitons by black $D3$ brane. Other hydrodynamic quantities like speed of sound, diffusion coefficients, drag force on quarks *etc.* can also be computed in the context of AdS/CFT.

In this thesis we compute generic higher derivative correction to shear viscosity coefficient of boundary plasma using the AdS/CFT conjecture. We have given a brief review of the method of computations of hydrodynamic properties in chapter 5. More detailed discussions can be found in [76–78, 80, 83]. In the thesis, we will mainly focus on following two computations in chapters 6 and 7.

- The shear viscosity coefficient of strongly coupled boundary gauge theory plasma depends on the horizon value of the effective coupling of transverse graviton moving in black hole background. The proof for the above statement is based on the canonical form of graviton's action. But in presence of generic higher derivative terms in the bulk Lagrangian the action is no longer canonical. We give a procedure to find an effective action for graviton (to first order in coefficient of higher derivative term) in canonical form in presence of any arbitrary higher derivative terms in the bulk. From that effective action we find the effective coupling constant for transverse graviton which in general depends on the radial coordinate r. We also argue that horizon value of this effective coupling is related to the shear viscosity coefficient of the boundary fluid in higher derivative gravity. We explicitly check this procedure for two specific examples: (1) four derivative action and (2) eight derivative action ($Weyl^4$ term). For both cases we show that our results for shear viscosity coefficient (up to first order in coefficient of higher derivative term) completely agree with the existing results in the literature.

- We calculate shear viscosity to entropy density ratio in presence of four derivative (with coefficient α') and six derivative (with coefficient α'^2) terms in bulk action. In general, there can be three possible four derivative terms and ten possible six derivative terms in the Lagrangian. Among them two four derivative and eight six derivative terms are ambiguous, i.e., these terms can be removed from the action by suitable field redefinitions but the remaining terms are unambiguous. According to the AdS/CFT correspondence all the unambiguous coefficients (coefficients of unambiguous terms) can be fixed in terms of field theory parameters.

Therefore, any measurable quantities of boundary theory, for example shear viscosity to entropy density ratio, when calculated holographically can be expressed in terms of unambiguous coefficients in the bulk theory (or equivalently in terms of boundary parameters). We calculate η/s for generic six derivative gravity and find that apparently it depends on few ambiguous coefficients at order α'^2. We calculate six derivative corrections to central charges a and c and express η/s in terms of these central charges and unambiguous coefficients in the bulk theory.

Part II

Black Hole Microstate Counting

Chapter 2

Adding Charges to $\mathcal{N} = 4$ Dyons

2.1 Introduction

We now have a good understanding of the spectrum of quarter BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories[22, 25, 30, 34–42], obtained by taking a $\mathbb{Z}_N$ orbifold of type II string theory compactified on $K3 \times T^2$ or T^6. However, in the direct approach to the computation of the spectrum based on counting of states the spectrum has so far been computed only for states carrying a restricted set of charges[37–39]. Our goal in this paper will be to extend this analysis to states carrying a more general set of charges, obtained from collective excitations of the system that has been analyzed earlier. For simplicity we shall restrict our analysis to type II string theory compactified on $K3 \times T^2$. Generalizing this to the case of $\mathcal{N} = 4$ supersymmetric orbifolds of this theory is straightforward, requiring setting to zero some of the charges which are not invariant under the orbifold group. The analysis for $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds of type II string theory compactified on T^6 can also be done in an identical manner.

2.2 Background

We consider the case of type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ or equivalently heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$. The latter description – to be called the second description – is obtained from the first description by first making an S-duality transformation in ten dimensional type IIB string theory, followed by a T-duality along the circle $\widetilde{S}^1$ that converts it to type IIA string theory on $K3 \times S^1 \times \widehat{S}^1$ and then using the six dimensional string-string duality that converts it to heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$. The coordinates ψ, y and χ along $\widetilde{S}^1$, S^1 and $\widehat{S}^1$ are all normalized to have period $2\pi\sqrt{\alpha'}$. Other normalization and sign conventions have been described in appendix A.

The compactified theory has 28 U(1) gauge fields and hence a given state is characterized by 28 dimensional electric and magnetic charge vectors Q and P. We shall use the second description of the theory to classify charges as electric and magnetic. Since in this description there are no RR fields or D-branes, an electrically charged state will correspond to an elementary string state and a magnetically charged state will correspond to wrapped NS 5-branes and Kaluza-Klein monopoles. The relationship between Q and P and the fields which appear in the supergravity theory underlying the second description will follow the convention of [61] in the $\alpha' = 16$ unit. This has been reviewed in appendix A, eq.(A-5). In this description the theory has an $SO(6,22;\mathbb{Z})$ T-duality symmetry, and the T-duality invariant combination of charges is given by

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P\,, \tag{2.2.1}$$

where L is a symmetric matrix with 22 eigenvalues -1 and 6 eigenvalues $+1$. We shall choose a basis in which L has the form

$$L = \begin{pmatrix} \widehat{L} & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0_2 & I_2 \\ & & & I_2 & 0_2 \end{pmatrix}, \tag{2.2.2}$$

where $\widehat{L}$ is a matrix with 3 eigenvalues $+1$ and 19 eigenvalues -1. The charge vectors will be labelled as

$$Q = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{P} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{pmatrix}. \tag{2.2.3}$$

According to the convention of appendix A, k_3, k_4, $-k_5$ and $-k_6$ label respectively the momenta along $\widehat{S}^1$, S^1 and fundamental string winding along $\widehat{S}^1$ and S^1 in the second description of the theory. On the other hand $-l_3$, l_4, l_5 and l_6 label respectively the number of NS 5-branes wrapped along $S^1 \times T^4$ and $\widehat{S}^1 \times T^4$, and Kaluza-Klein monopole charges associated with $\widehat{S}^1$ and S^1. The rest of the charges label the momentum/winding or monopole charges associated with the other internal directions. By following the duality chain that relates the first and second description of the theory and using the sign convention of appendix A, the different components of P and Q can be given the following interpretation in the first description of the theory. k_3 represents the D-string winding charge along $\widetilde{S}^1$, k_4 is the momentum along S^1, k_5 is the D5-brane charge along $K3 \times \widetilde{S}^1$, k_6 is the number of Kaluza-Klein monopoles associated with the compact circle $\widetilde{S}^1$, l_3 is the D-string winding charge along S^1, $-l_4$ is the momentum along $\widetilde{S}^1$, l_5 is the D5-brane charge along $K3 \times S^1$ and l_6 is the number of Kaluza-Klein monopoles associated with the compact circle S^1. Other components of Q (P) represent various other branes of type IIB string theory wrapped on $\widetilde{S}^1$ (S^1) times various cycles of $K3$. We shall choose a convention in which the 22-dimensional charge vector $\widehat{Q}$ represents 3-branes wrapped along the 22 2-cycles of K3 times $\tilde{S}^1$, k_1 represents fundamental type IIB string winding charge along $\widetilde{S}^1$, k_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times \widetilde{S}^1$, the 22-dimensional charge vector $\widehat{P}$ represents 3-branes wrapped along the 22 2-cycles of K3 times S^1, l_1 represents fundamental type IIB string winding charge along S^1 and l_2 represents the number of NS 5-branes of type IIB wrapped along $K3 \times S^1$. In this convention

$\widehat{L}$ represents the intersection matrix of 2-cycles of $K3$. Using the various sign conventions described in appendix A, and the T-duality transformation laws for the RR fields given in [24] one can verify that the combinations $k_3 k_5 + \widehat{Q}^2/2$, $l_3 l_5 + \widehat{P}^2/2$ and $k_3 l_5 + l_3 k_5 + \widehat{Q} \cdot \widehat{P}$ are invariant under the mirror symmetry transformation on $K3$.

The original configuration studied in [37] has charge vectors of the form:

$$Q = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ 0 \\ -n \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{0} \\ 0 \\ 0 \\ Q_1 - Q_5 = Q_1 - 1 \\ -J \\ Q_5 = 1 \\ 0 \end{pmatrix}. \tag{2.2.4}$$

Thus in the first description we have $-n$ units of momentum along S^1, J units of momentum along $\widetilde{S}^1$, a single Kaluza-Klein monopole (with negative magnetic charge) associated with $\widetilde{S}^1$, a single D5-brane wrapped on $K3 \times S^1$ and Q_1 D1-branes wrapped on S^1. The D5-brane wrapped on $K3 \times S^1$ also carries -1 units of D1-brane charge along S^1; this is responsible for the shift by -1 of Q_1 as given in (2.2.4). The associated invariants are

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q \cdot P = J. \tag{2.2.5}$$

The degeneracy of this system was calculated in [37] as a function of n, Q_1 and J. If we call this function $f(n, Q_1, J)$, then we can express the degeneracy $d(Q, P)$ as a function of Q, P as:

$$d(Q, P) = f\left(\frac{1}{2}Q^2, \frac{1}{2}P^2 + 1, Q \cdot P\right). \tag{2.2.6}$$

Ref.[37] actually considered a more general charge vector where Q_5, representing the number of D5-branes wrapped along $K3 \times S^1$, was arbitrary and derived the same formula (2.2.6) for $d(Q, P)$. However, the analysis of dyon spectrum becomes simpler for $Q_5 = 1$. For this reason we have set $Q_5 = 1$. We shall comment on the more general case at the end.

2.3 Charge Carrying Deformations

Our goal will be to consider charge vectors more general than the ones given in (2.2.4) and check if the degeneracy is still given by (2.2.6). We shall do this by adding charges to the existing system by exciting appropriate collective modes of the system. These collective modes come from three sources:

1. The original configuration in the type IIB theory contains a Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$. This solution is given by

$$\begin{aligned} ds^2 &= \left(1+\frac{K\sqrt{\alpha'}}{2r}\right)\left(dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right) \\ &\quad +K^2\left(1+\frac{K\sqrt{\alpha'}}{2r}\right)^{-1}\left(d\psi+\frac{\sqrt{\alpha'}}{2}\cos\theta d\phi\right)^2 \end{aligned} \tag{2.3.7}$$

with the identifications:

$$\begin{aligned} (\theta,\phi,\psi) \equiv (2\pi-\theta,\phi+\pi,\psi+\frac{\pi}{2}\sqrt{\alpha'}) &\equiv (\theta,\phi+2\pi,\psi+\pi\sqrt{\alpha'}) \\ &\equiv (\theta,\phi,\psi+2\pi\sqrt{\alpha'})\,. \end{aligned} \tag{2.3.8}$$

The coordinate ψ can be regarded as the coordinate of $\widetilde{S}^1$, whereas (r,θ,ϕ) represent spherical polar coordinates of the non-compact space. K is a constant related to the physical radius of $\widetilde{S}^1$. This geometry, also known as the Taub-NUT space, admits a normalizable self-dual harmonic form ω, given by [26, 27]

$$\begin{aligned} \omega &\propto \frac{2}{\sqrt{\alpha'}}\frac{r}{r+\frac{1}{2}K\sqrt{\alpha'}}d\sigma_3+\frac{K}{(r+\frac{1}{2}K\sqrt{\alpha'})^2}dr\wedge\sigma_3\,, \\ \sigma_3 &\equiv \left(d\psi+\frac{\sqrt{\alpha'}}{2}\cos\theta d\phi\right)\,. \end{aligned} \tag{2.3.9}$$

(2.3.7) represents the geometry of the space-time transverse to the Kaluza-Klein monopole. Besides the $K3$ surface, the world-volume of the Kaluza-Klein monopole spans the circle S^1, which we shall label by y, and time

t.

Now type IIB string theory compactified on K3 has various 2-form fields, – the original NSNS and RR 2-form fields B and $C^{(2)}$ of the ten dimensional type IIB string theory as well as the components of the 4-form field $C^{(4)}$ along various 2-cycles of $K3$. Given any such 2-form field C_{MN}, we can introduce a scalar mode φ by considering deformations of the form [28]:

$$C = \varphi(y,t)\,\omega\,, \tag{2.3.10}$$

where y denotes the coordinate along S^1. If the field strength dC associated with C is self-dual or anti-self-dual in six dimensions then the corresponding scalar field φ is chiral in the $y - t$ space; otherwise it represents a non-chiral scalar field. We can now consider configurations which carry momentum conjugate to this scalar field φ or winding number along y of this scalar field φ, represented by a solution where φ is linear in t or y. In the six dimensional language this corresponds to $dC \propto dt \wedge \omega$ or $dy \wedge \omega$. From (2.3.9) we see that $dC \propto dt \wedge \omega$ will have a component proportional to $r^{-2}\,dt \wedge dr \wedge d\psi$ for large r, and hence the coefficient of this term represents the charge associated with a string, electrically charged under C, wrapped along $\widetilde{S}^1$. On the other hand $dC \propto dy \wedge \omega$ will have a component proportional to $\sin\theta\, dy \wedge d\theta \wedge d\phi$ and the coefficient of this term will represent the charge associated with a string, magnetically charged under C, wrapped along $\widetilde{S}^1$. If the 2-form field C represents the original RR or NSNS 2-form field of type IIB string theory in ten dimensions, then the electrically charged string would correspond to a D-string or a fundamental type IIB string and the magnetically charged string would correspond to a D5-brane or NS 5-brane wrapped on $K3$. On the other hand if the 2-form C represents the component of the 4-form field along a 2-cycle of K3, then the corresponding string represents a D3-brane wrapped on a 2-cycle times $\widetilde{S}^1$. Recalling the interpretation of the charges $\widehat{Q}$ and k_i appearing in (2.2.3) we now see that the momentum and winding modes of φ correspond to the charges $\widehat{Q}$, k_1, k_2, k_3 and k_5. More specifically, after taking into account the sign conventions described in appendix A, these charges correspond to switching on deformations of the form:

$$dB \propto -k_1 dt \wedge \omega, \quad dB \propto k_2 dy \wedge \omega, \quad dC^{(2)} \propto -k_3 dt \wedge \omega,$$
$$dC^{(2)} \propto k_5 dy \wedge \omega, \qquad dC^{(4)} \propto \sum_\alpha \widehat{Q}_\alpha (1+*)\, \Omega_\alpha \wedge dy \wedge \omega\,, \tag{2.3.11}$$

where $\{\Omega_\alpha\}$ denote a basis of harmonic 2-forms on $K3$ ($1 \leq \alpha \leq 22$) satisfying $\int_{K3} \Omega_\alpha \wedge \Omega_\beta = \widehat{L}_{\alpha\beta}$. Thus in the presence of these deformations we have a more general electric charge vector of the form

$$Q_0 = \begin{pmatrix} \widehat{Q} \\ k_1 \\ k_2 \\ k_3 \\ -n \\ k_5 \\ -1 \end{pmatrix} . \tag{2.3.12}$$

As can be easily seen from (2.3.11), k_2 represents NS 5-brane charge wrapped along $K3 \times \widetilde{S}^1$. However, for weakly coupled type IIB string theory, the presence of this charge could have large backreaction on the geometry. In order to avoid it we shall choose

$$k_2 = 0\,. \tag{2.3.13}$$

2. The original configuration considered in [37] also contains a D5-brane wrapped around $K3 \times S^1$. We can switch on flux of world-volume gauge field strengths $\mathcal{F}$ on the D5-brane along the various 2-cycles of K3 that it wraps. The net coupling of the RR gauge fields to the D5-brane in the presence of the world-volume gauge fields may be expressed as[24]

$$\int \left[C^{(6)} + C^{(4)} \wedge \mathcal{F} + \frac{1}{2} C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F} + \cdots \right] , \tag{2.3.14}$$

up to a constant of proportionality. The integral is over the D5-brane world-volume spanned by y, t and the coordinates of $K3$. In order to be compatible with the convention of appendix A that the D5-brane wrapped on $K3 \times S^1$ carries negative $(dC^{(6)})_{(K3)yrt}$ asymptotically, we need to take the integration measure in the yt plane in (2.3.14) as $dy \wedge dt$, i.e. $\epsilon^{yt} > 0$. Via the coupling

$$\int C^{(4)} \wedge \mathcal{F}\,, \tag{2.3.15}$$

the gauge field configuration will produce the charges of a D3-brane wrapped on a 2-cycle of $K3$ times S^1, – i.e. the 22 dimensional magnetic charge vector $\widehat{P}$ appearing in (2.2.3). More precisely, we find that the gauge field flux required to produce a specific magnetic charge vector $\widehat{P}$ is

$$\mathcal{F} \propto -\sum_\alpha \widehat{P}_\alpha\, \Omega_\alpha\,. \tag{2.3.16}$$

3. The D5-brane can also carry electric flux along S^1. This will induce the charge of a fundamental type IIB string wrapped along S^1. According to the physical interpretation of various charges given earlier, this gives the component l_1 of the magnetic charge vector P.

   The net result of switching on both the electric and magnetic flux along the D5-brane world-volume is to generate a magnetic charge vector of the form:

$$P_0 = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 \\ -J \\ 1 \\ 0 \end{pmatrix}. \tag{2.3.17}$$

2.4 Additional Shifts in the Charges

This, however, is not the end of the story. So far we have discussed the effect of the various collective mode excitations on the charge vector to first order in the charges. We have not taken into account the effect of the interaction of deformations produced by the collective modes with the background fields already present in the system, or the background fields produced by other collective modes. Taking into account these effects produces further shifts in the charge vector as described below.

1. As seen from (2.3.14), the D5-brane world-volume theory has a coupling proportional to $\int C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F}$. Thus in the presence of magnetic flux $\mathcal{F}$ the D5-brane wrapped on $K3 \times S^1$ acts as a source of the D1-brane charge wrapped on S^1. The effect is a shift in the magnetic charge quantum number l_3 that is quadratic in $\mathcal{F}$ and hence quadratic in $\widehat{P}$ due to (2.3.16). A careful calculation, taking into account various signs and normalization factors, shows that the net effect of this term is to give an additional contribution to l_3 of the form:

$$\Delta_1 l_3 = -\widehat{P}^2/2\,. \tag{2.4.18}$$

2. Let C be a 2-form in the six dimensional theory obtained by compactifying type IIB string theory on $K3$ and $F = dC$ be its field strength. As summarized in (2.3.11), switching on various components of the electric charge vector Q requires us to switch on F proportional to $dt \wedge \omega$ or $dy \wedge \omega$. The presence of such background induces a coupling proportional to

$$-\int \sqrt{-\det g}\, g^{yt} F_{ymn} F_t{}^{mn} \tag{2.4.19}$$

with the indices m, n running over the coordinates of the Taub-NUT space. This produces a source for g^{yt}, i.e. momentum along S^1. The effect of such terms is to shift the component k_4 of the charge vector Q. A careful calculation shows that the net change in k_4 induced due to this coupling is given by

$$\Delta_2 k_4 = k_3 k_5 + \widehat{Q}^2/2\,, \tag{2.4.20}$$

where we have used the fact that k_2 has been set to zero. The $k_3 k_5$ term comes from taking F if (2.4.19) to be the field strength of the RR 2-form field, and $\widehat{Q}^2/2$ term comes from taking F to be the field strength of the components of the RR 4-form field along various 2-cycles of $K3$.

3. The D5-brane wrapped on $K3 \times S^1$ or the magnetic flux on this brane

along any of the 2-cycles of $K3$ produces a magnetic type 2-form field configuration of the form:

$$F \equiv dC \propto \sin\theta \, d\psi \wedge d\theta \wedge d\phi \,, \tag{2.4.21}$$

where C is any of the RR 2-form fields in six dimensional theory obtained by compactifying type IIB string theory on $K3$. One can verify that the 3-form appearing on the right hand side of (2.4.21) is both closed and co-closed in the Taub-NUT background and hence F given in (2.4.21) satisfies both the Bianchi identity and the linearized equations of motion. The coefficients of the term given in (2.4.21) for various 2-form fields C are determined in terms of $\widehat{P}$ and the D5-brane charge along $K3 \times S^1$ which has been set equal to 1. This together with the term in F proportional to $dt \wedge \omega$ coming from the collective coordinate excitation of the Kaluza-Klein monopole generates a source of the component $g^{\psi t}$ of the metric via the coupling proportional to

$$-\int \sqrt{-\det g} g^{\psi t} F_{\psi mn} F_t{}^{mn} \tag{2.4.22}$$

This induces a net momentum along $\widetilde{S}^1$ and gives a contribution to the component l_4 of the magnetic charge vector P. A careful calculation shows that the net additional contribution to l_4 due to this coupling is given by

$$\Delta_3 l_4 = k_3 + \widehat{Q} \cdot \widehat{P} \,. \tag{2.4.23}$$

In this expression the contribution proportional to k_3 comes from taking F in (2.4.22) to be the field strength associated with the RR 2-form field of IIB, whereas the term proportional to $\widehat{Q} \cdot \widehat{P}$ arises from taking F to be the field strength associated with the components of the RR 4-form field along various 2-cycles of $K3$.

4. Eqs.(2.3.17) and (2.4.18) show that we have a net D1-brane charge along S^1 equal to

$$l_3 = Q_1 - 1 - \widehat{P}^2/2 \,. \tag{2.4.24}$$

If we denote by $C^{(2)}$ the 2-form field of the original ten dimensional type

IIB string theory, then the effect of this charge is to produce a background of the form:

$$dC^{(2)} \propto (Q_1 - 1 - \widehat{P}^2/2)\, r^{-2}\, dr \wedge dt \wedge dy\,. \tag{2.4.25}$$

Again one can verify explicitly that the right hand side of (2.4.25) is both closed and co-closed in the Taub-NUT background. We also have a component

$$dC^{(2)} \propto k_5\, dy \wedge \omega\,, \tag{2.4.26}$$

coming from the excitation of the collective coordinate of the Kaluza-Klein monopole. This gives a source term for $g^{\psi t}$ via the coupling proportional to

$$-\int \sqrt{-\det g}\, g^{\psi t} F_{\psi r y} F_t{}^{ry} \tag{2.4.27}$$

producing an additional contribution to the charge l_4 of the form

$$\Delta_4 l_4 = k_5(Q_1 - 1 - \widehat{P}^2/2)\,. \tag{2.4.28}$$

So far in our analysis we have taken into account possible additional sources produced by the terms quadratic in the fields. What about higher order terms? It is straightforward to show that the possible effect of the higher order terms on the shift in the charges will involve one or more powers of the type IIB string coupling. Since the shift in the charges must be quantized, they cannot depend on continuous moduli. Thus at least in the weakly coupled type IIB string theory there are no additional corrections to the charges. Incidentally, the same argument can be used to show that the shifts in the charges must also be independent of the other moduli; thus it is in principle sufficient to calculate these shifts at any particular point in the moduli space.

Combining all the results we see that we have a net electric charge vector Q and a magnetic charge vector P of the form:

$$Q^T = \left(\ \widehat{Q}^T,\ k_1,\ 0,\ k_3,\ -n + k_3 k_5 + \widehat{Q}^2/2,\ k_5,\ -1\ \right), \tag{2.4.29}$$

$$P = \begin{pmatrix} \widehat{P} \\ l_1 \\ 0 \\ Q_1 - 1 - \widehat{P}^2/2 \\ -J + k_3 + \widehat{Q}\cdot\widehat{P} + k_5(Q_1 - 1 - \widehat{P}^2/2) \\ 1 \\ 0 \end{pmatrix}. \tag{2.4.30}$$

This has

$$Q^2 = 2n, \qquad P^2 = 2(Q_1 - 1), \qquad Q \cdot P = J\,. \tag{2.4.31}$$

Thus the additional charges do not affect the relationship between the invariants Q^2, P^2, $Q \cdot P$ and the original quantum numbers n, Q_1 and J.

2.5 Dyon Spectrum

Let us now turn to the analysis of the dyon spectrum in the presence of these charges. For this we recall that in [37] the dyon spectrum was computed from three mutually non-interacting parts, – the dynamics of the Kaluza-Klein monopole, the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole and the motion of the D1-branes relative to the D5-brane. The precise dynamics of the D1-branes relative to the D5-brane is affected by the presence of the gauge field flux on the D5-brane since it changes the non-commutativity parameter describing the dynamics of the gauge field on the D5-brane world-volume[29]. As a result the moduli space of D1-branes, described as non-commutative instantons in this gauge theory[31], gets deformed. However, we do not expect this to change the elliptic genus of the corresponding conformal field theory[32] that enters the degeneracy formula. With the exception of the zero mode associated with the D1-D5 center of mass motion in the Kaluza-Klein monopole background, the rest of the contribution to the degeneracy came from the excitations involving non-zero mode oscillators of the collective coordinates of the Kaluza-Klein monopole and the collective coordinates associated with the overall motion of the D1-D5 system[37]. This is not affected either by switching on gauge field fluxes on the D5-brane world-volume or the momenta or winding number of the collective coordinates of the Kaluza-Klein monopole. On the other hand the dynamics of the D1-D5-brane center of mass motion in the background geometry is also not expected to be modified in the weakly coupled type IIB string theory since in this limit the additional background fields due to the additional charges are small compared to the one due to the Kaluza-Klein monopole. (For this it is important that the additional charges do not involve any other Kaluza-Klein monopole or NS 5-brane charge.) Thus we expect the degeneracy to be given by the same function $f(n, Q_1, J)$ that appeared in the absence of the additional charges. Using (2.4.31) we can now write

$$d(Q, P) = f\left(\frac{1}{2}Q^2, \frac{1}{2}P^2 + 1, Q \cdot P\right) . \tag{2.5.32}$$

This is a generalization of (2.2.6) and shows that for the charge vectors given in (2.4.29), the degeneracy $d(Q, P)$ depends on the charges only through the combination Q^2, P^2 and $Q \cdot P$.

As was discussed in [41], the formula for the degeneracy for a given charge vector can change across walls of marginal stability in the moduli space. Hence a given formula for the degeneracy makes sense only if we specify how the region of the moduli space in which we are carrying out our analysis is situated with respect to the walls of marginal stability. In the theory under consideration the moduli space is the coset $A \times B$, where,

$$\begin{aligned} A &= (SL(2,\, \mathbb{Z})\backslash SL(2,\, \mathbf{R})/U(1)) \\ B &= (SO(6,22;\, \mathbb{Z})\backslash SO(6,22;\, \mathbf{R})/SO(6) \times SO(22))\,. \end{aligned} \tag{2.5.33}$$

The coset is parametrized by a complex modulus τ and a 28×28 symmetric $SO(6,22)$ matrix M. For fixed M, the walls of marginal stability are either straight lines in the τ plane, intersecting the real axis at an integer, or circles intersecting the real axis at rational points a/c and b/d with $ad - bc = 1$, $a, b, c, d \in \mathbb{Z}$. The precise shape of the circles and the slopes of the straight lines depend on the modulus M and the charge vector of the state under consideration. It was shown in [41] that for the charge vector given in (2.2.4) the region where the type IIB string coupling and the angle between S^1 and $\widetilde{S}^1$

are small and the other moduli are of order 1 can fall into one of the two domains in the upper half τ plane. The first of these domains is bounded by a pair of straight lines in the τ plane, passing through the points 0 and 1 respectively, and a circle passing through the points 0 and 1. The second domain is bounded by a pair of straight lines passing through the points -1 and 0 respectively and a circle passing through the points -1 and 0. Carrying out a similar analysis for the modified charge vector (2.4.29) one finds that as long as all the charges are finite, the region of moduli space where type IIB coupling is small falls inside the same domains, i.e. domains bounded by a set of walls of marginal stability which intersect the real τ axis at the same points. This is just as well; had the new charge vector landed us into a different domain in the τ plane, our result (2.5.32) would be in contradiction with the result of [41] that in different domains bounded by different walls of marginal stability the degeneracies are given by different functions of P^2, Q^2 and $Q \cdot P$.

2.6 More General Charge Vector

The charge vector given in (2.4.29), while more general than the one considered in [37], is still not the most general charge vector. Is it possible to extend our analysis to include more general charge vectors? First of all note that l_5, representing the number of D5-branes wrapped on $K3 \times S^1$, was chosen to be an arbitrary integer instead of 1 in [37]. Thus we can certainly take as our starting point the more general charge vector where Q_5 in (2.2.4) is chosen to be an arbitrary integer instead of 1. Our analysis up to (2.4.31) proceeds in a straightforward manner (with $Q_1 - 1$ replaced by $Q_5(Q_1 - Q_5)$). The issue, however, is how the additional charges affect the dyon spectrum. In particular one needs to examine carefully the effect of the gauge field flux on the D5-brane on the dynamics of the D1-D5 system, generalizing the analysis given in [32]. However, as long as, we do not switch on gauge field flux on the D5-branes, i.e. consider configurations with $\widehat{P} = 0$, $l_1 = 0$, there is no additional complication and the final degeneracy will still be given by (2.5.32). On the other hand following the analysis of [41] one can show that the region of the moduli space where the type IIB string coupling and the angle between S^1

and $\widetilde{S}^1$ are small is still bounded by the same set $\mathcal{B}_R$, $\mathcal{B}_L$ of walls of marginal stability.

In (2.4.29) we have set the component k_2 of the electric charge vector to zero even though we could switch it on by switching on an NSNS sector 3-form field strength of the form $dy \wedge \omega$. The reason for this was that this charge represents the number of NS 5-branes wrapped along $K3 \times \widetilde{S}^1$ and the presence of NS 5-branes could have large backreaction on the geometry thereby invalidating our analysis. We can, however, keep its effect small compared to that of the original background produced by the Kaluza-Klein monopole by taking the radius R of S^1 to be large compared to $\sqrt{\alpha'}$. Since in the string metric the mass of the Kaluza-Klein monopole is proportional to R while the NS 5-brane wrapped along $\widetilde{S}^1$ does not have such a factor, we can expect that for large R the effect of the background produced by the NS 5-brane will be small compared to that of the Kaluza-Klein monopole. We can then analyze the system in the same manner as for the other charges and conclude that the formula for the degeneracy in the presence of this additional charge is still given by (2.4.31). One also finds that the region of the moduli space where the type IIB string coupling and the angle between S^1 and $\widetilde{S}^1$ are small is still bounded by the same set $\mathcal{B}_R$, $\mathcal{B}_L$ of walls of marginal stability.

Let us now turn to k_6 which has been set equal to -1 in (2.4.29). This is the number of Kaluza-Klein monopoles associated with the compactification circle $\widetilde{S}^1$. Changing this number would require us to study the dynamics of multiple Kaluza-Klein monopoles. While, in principle, this can be done, this will certainly require a major revision of the analysis done so far. Thus there does not seem to be a minor variation of our analysis that can change the charge k_6 to any other integer.

This leaves us with the components l_2 and l_6 both of which have been set to 0 in (2.4.29). l_2 represents the number of NS 5-branes wrapped on S^1. Switching this charge on would require us to introduce explicit NS 5-brane background and study the dynamics of D-branes in such a background. This would require techniques quite different from the one used so far. On the other hand, the component l_6 represents the Kaluza-Klein monopole charge associated with the compact circle S^1. This also causes significant change in the background geometry and calculation of the spectrum of such configurations would require fresh analysis.

Chapter 3

Asymptotic Expansion of the $\mathcal{N} = 4$ Dyon Degeneracy

3.1 Introduction and Summary

One of the major successes of string theory has been the matching of the Bekenstein-Hawking entropy of a class of extremal black holes and the statistical entropy of a system of branes carrying the same quantum numbers as the black hole[6]. The initial comparison between the two was done in the limit of large charges. In this limit the analysis simplifies on both sides. On the gravity side we can restrict our analysis to two derivative terms in the action, while on the statistical side the analysis simplifies because we can use certain asymptotic formula to estimate the degeneracy of states for large charges. However given the successful matching between the statistical entropy and Bekenstein-Hawking entropy in the large charge limit, it is natural to explore whether the agreement continues to hold beyond this approximation. On the gravity side this requires taking into account the effect of higher derivative corrections and quantum corrections in computing the entropy. The effect of higher derivative terms is captured by the Wald's generalization of the Bekenstein-Hawking formula[19]. For extremal black holes this leads to the entropy function formalism for computing the entropy[20]. Recently it has been suggested that the effect of quantum corrections to the entropy of extremal black holes is encoded in the quantum entropy function, defined as the partition function of string theory on the near horizon geometry of the black holes[21]. On the other hand computing higher derivative corrections to the statistical entropy requires us to compute microscopic degeneracies of the black hole to greater accuracy. Here significant progress has been made in a class of $\mathcal{N} = 4$ supersymmetric field theories, for which we now have exact formulæ for the microscopic degeneracies[22, 23, 25, 30, 33–53]. (For a similar proposal in $\mathcal{N} = 2$ supersymmetric theories, see [54].)

Our eventual goal is to compare the statistical entropy computed from the exact degeneracy formula to the predicted result on the black hole side from the computation of the quantum entropy function (or whatever formula gives the exact result for the entropy of extremal black holes). However in practice we can compute the black hole side of the result only as an expansion in inverse powers of charges, by matching these to an expansion in powers of derivatives / string coupling constant. Thus we must carry out a similar expansion of the statistical entropy if we want to compare the results on the two sides. A systematic procedure for developing such an expansion of the statistical entropy has been discussed in [22, 23, 34, 37]. Our main goal in this paper is to explore this expansion in more detail, and. to whatever extent possible, relate it to the results of macroscopic computation.

The rest of the paper is organized as follows. In §3.2 we give a brief overview of the exact dyon degeneracy formula in a class of $\mathcal{N} = 4$ supersymmetric string theories, and discuss the systematic procedure of extracting the degeneracy for large but finite charges. We also organise the computation of the statistical entropy by representing the result as a sum of contributions from single centered and multi-centered black holes, and then express the single centered black hole entropy as an asymptotic expansion in inverse powers of charges, together with exponentially suppressed corrections. In §3.3 we examine the leading exponential term in the expression for the statistical entropy and compute the statistical entropy to order $1/\text{charge}^2$. Previous computation of the statistical entropy was carried out to order charge^0. We compare these results with the exact result for the statistical entropy and find good agreement. We also find that the agreement is worse if we compare the result with the exact statistical entropy in a domain where besides single centered black holes, we also have contribution from two centered black holes. This confirms that the asymptotic expansion is best suited for computing the entropy of single centered black holes. From the gravity perspective these corrections should be captured by six derivative corrections to the effective action; however explicit analysis of such contributions has not been carried out so far.

In §3.4 we analyze the contribution from the exponentially subleading terms to the entropy of single centered black holes. While power suppressed corrections to the statistical entropy have been compared to the higher derivative corrections to the black hole entropy in various approximations, so far there has been no explanation of these exponentially suppressed terms from the black hole side.[1] In §3.5 we suggest a macroscopic origin of the exponentially suppressed contributions to the entropy from quantum entropy function formalism. In this formalism the leading contribution to the macroscopic degeneracy comes from path integral over the near horizon AdS_2 geometry of the black hole with appropriate boundary condition. We show that for the same boundary conditions there are other saddle points which have different values of the euclidean action. These values have precisely the form needed to reproduce the exponentially suppressed contributions to the leading microscopic degeneracy.

3.2 An Overview of Statistical Entropy Function

In this section, we briefly review the systematic procedure for computing the asymptotic expansion of the statistical entropy of a dyon in a class of $\mathcal{N} = 4$ supersymmetric string theories. The approach mainly follows [22, 23, 34, 37, 46]. Our notation will be that of [47].

[1] Note that this expansion is quite different from the Rademacher expansion studied in [55, 56] since we scale all the charges uniformly.

3.2.1 Dyon degeneracy

Let us consider an $\mathcal{N}=4$ supersymmetric string theory with a rank r gauge group. We shall work at a generic point in the moduli space where the unbroken gauge group is $U(1)^r$. The low energy supergravity describing this theory has a continuous $SO(6,r-6)\times SL(2,\mathbf{R})$ symmetry which is broken to a discrete subgroup in the full string theory. We denote by Q and P the r dimensional electric and magnetic charges of the theory, by L the $SO(6,r-6)$ invariant metric and by $(Q^2,P^2,Q\cdot P)$ the combinations (Q^TLQ,P^TLP,Q^TLP). Then for a fixed set of values of discrete T-duality invariants the degeneracy $d(Q,P)$, – or more precisely the sixth helicity trace B_6[57] – of a dyon carrying charges (Q,P) is given by a formula of the form:

$$d(Q,P)=(-1)^{Q\cdot P+1}\frac{1}{a_1a_2a_3}\int_{\mathcal{C}}d\check{\rho}\,d\check{\sigma}\,d\check{v}\,e^{-\pi i(\check{\rho}P^2+\check{\sigma}Q^2+2\check{v}Q\cdot P)}\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})}\,, \quad (3.2.1)$$

where $\check{\rho}\equiv\check{\rho}_1+i\check{\rho}_2$, $\check{\sigma}\equiv\check{\sigma}_1+i\check{\sigma}_2$ and $\check{v}\equiv\check{v}_1+i\check{v}_2$ are three complex variables, $\check{\Phi}$ is a function of $(\check{\rho},\check{\sigma},\check{v})$ which we shall refer to as the inverse of the dyon partition function, and $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labeled by $(\check{\rho},\check{\sigma},\check{v})$, given by

$$\begin{gathered}\check{\rho}_2=M_1,\quad \check{\sigma}_2=M_2,\quad \check{v}_2=M_3,\\ 0\le\check{\rho}_1\le a_1,\quad 0\le\check{\sigma}_1\le a_2,\quad 0\le\check{v}_1\le a_3\,.\end{gathered} \quad (3.2.2)$$

The periods a_1, a_2 and a_3 of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ are determined by the the quantization laws of Q^2, P^2 and $Q\cdot P$. M_1, M_2 and M_3 are large but fixed numbers. The choice of the M_i's depend on the domain of the asymptotic moduli space in which we want to compute $d(Q,P)$. As we move from one domain to another crossing the walls of marginal stability, $d(Q,P)$ changes. However this change is captured completely by a deformation of the contour labelled by (M_1,M_2,M_3) without any change in the partition function $\check{\Phi}$[41, 42]. A simple rule that expresses (M_1, M_2, M_3) in terms of the asymptotic moduli is[45]:

$$\begin{aligned} M_1 &= \Lambda \left(\frac{|\lambda|^2}{\lambda_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right), \\ M_2 &= \Lambda \left(\frac{1}{\lambda_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right), \\ M_3 &= -\Lambda \left(\frac{\lambda_1}{\lambda_2} + \frac{Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right), \end{aligned} \tag{3.2.3}$$

where Λ is a large positive number,

$$Q_R^2 = Q^T(M+L)Q, \quad P_R^2 = P^T(M+L)P, \quad Q_R \cdot P_R = Q^T(M+L)P\,, \tag{3.2.4}$$

$\lambda \equiv \lambda_1 + i\lambda_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the $r \times r$ symmetric matrix valued moduli field of the matter multiplet satisfying $MLM^T = L$.

A special point in the moduli space is the attractor point corresponding to the charges (Q, P). If we choose the asymptotic values of the moduli fields to be at this special point then all multi-centered black hole solutions are absent and the corresponding degeneracy formula captures the degeneracies of single centered black hole only[45]. This attractor point corresponds to the choice of (M, λ) for which

$$\begin{aligned} Q_R^2 = 2Q^2, \quad P_R^2 = 2P^2, \quad Q_R \cdot P_R &= 2Q \cdot P, \quad \lambda_2 = \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}, \\ \lambda_1 &= \frac{Q \cdot P}{P^2}\,. \end{aligned} \tag{3.2.5}$$

Substituting this into (3.2.3) we get

$$\begin{aligned} M_1 = 2\,\Lambda \frac{Q^2}{\sqrt{Q^2P^2 - (Q \cdot P)^2}} \quad , \quad M_2 = 2\,\Lambda \frac{P^2}{\sqrt{Q^2P^2 - (Q \cdot P)^2}}, \\ M_3 = -2\,\Lambda \frac{Q \cdot P}{\sqrt{Q^2P^2 - (Q \cdot P)^2}}\,. \end{aligned} \tag{3.2.6}$$

We can invert the Fourier integrals (3.2.1) by writing

$$d(Q,P) \;=\; (-1)^{Q\cdot P+1}\, g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q\cdot P\right), \tag{3.2.7}$$

where $g(m,n,p)$ are the coefficients of Fourier expansion of the function $1/\check{\Phi}(\check{\rho},\check{\sigma},\check{v})$:

$$\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})} = \sum_{m,n,p} g(m,n,p)\, e^{2\pi i(m\check{\rho}+n\check{\sigma}+p\check{v})}\,. \tag{3.2.8}$$

Different choices of (M_1, M_2, M_3) in (3.2.1) will correspond to different ways of expanding $1/\check{\Phi}$ and will lead to different $g(m,n,p)$. Conversely, for $d(Q,P)$ associated with a given domain of the asymptotic moduli space, if we define $g(m.n,p)$ via eq.(3.2.7), then the choice of (M_1, M_2, M_3) is determined by requiring that the series (3.2.8) is convergent for $(\check{\rho}_2, \check{\sigma}_2, \check{v}_2) = (M_1, M_2, M_3)$.

A special case on which we shall focus much of our attention is the $\mathcal{N} = 4$ supersymmetric string theory obtained by compactifying type IIB string theory on $K3\times T^2$ or equivalently heterotic string theory compactified on T^6. In this case the function $\check{\Phi}$ is given by the well known Igusa cusp form of weight 10:

$$\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) = \Phi_{10}(\check{\rho},\check{\sigma},\check{v}) = e^{2\pi i(\check{\rho}+\check{\sigma}+\check{v})} \prod_{\substack{k',l,j\in\mathbb{Z}\\ k',l\geq 0; j<0\\ \text{for } k'=l=0}} \left(1 - e^{2\pi i(\check{\sigma}k'+\check{\rho}l+\check{v}j)}\right)^{c(4lk'-j^2)}, \tag{3.2.9}$$

where $c(u)$ is defined via the equation

$$8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right] = \sum_{j,n\in\mathbb{Z}} c(4n-j^2)\, e^{2\pi i n\tau + 2\pi i j z}\,. \tag{3.2.10}$$

3.2.2 Asymptotic expansion and statistical entropy function

In order to compare the statistical entropy $S_{stat}(Q,P) \equiv \ln d(Q,P)$ with the black hole entropy we need to extract the behaviour of $S_{stat}(Q,P)$ for large charges. We shall now briefly review the strategy and the results. For details the reader is referred to [46].

1. Beginning with the expression for $d(Q,P)$ given in (3.2.1), we first de-

form the contour to small values of $(\check\rho_2, \check\sigma_2, \check v_2)$ (say of the order of 1/charge). In this case the contribution to S_{stat} from the deformed contour can be shown to be subleading, and hence the major contribution comes from the residue at the poles picked up by the contour during the deformation.

2. For any given pole, one of the three integrals in (3.2.1) can be done using residue theorem. The integration over the other two variables are carried out using the method of steepest descent. It turns out that in all known examples, the dominant contribution to S_{stat} computed using this procedure comes from the pole of the integrand i.e. zero of $\check\Phi$ at

$$\check\rho\check\sigma - \check v^2 + \check v = 0\,. \tag{3.2.11}$$

Furthermore near this pole $\check\Phi$ behaves as

$$\check\Phi(\check\rho, \check\sigma, \check v) \propto (2v - \rho - \sigma)^k\, v^2\, g(\rho)\, g(\sigma)\,, \tag{3.2.12}$$

where

$$\rho = \frac{\check\rho\check\sigma - \check v^2}{\check\sigma}, \qquad \sigma = \frac{\check\rho\check\sigma - (\check v - 1)^2}{\check\sigma}, \qquad v = \frac{\check\rho\check\sigma - \check v^2 + \check v}{\check\sigma}\,, \tag{3.2.13}$$

k is related to the rank r of the gauge group via the relation

$$r = 2k + 8\,, \tag{3.2.14}$$

and $g(\tau)$ is a known function which depends on the details of the theory. Typically it transforms as a modular function of weight $(k+2)$ under a certain subgroup of the $SL(2, \mathbb{Z})$ group. In the (ρ, σ, v) variables the pole at (3.2.11) is at $v = 0$. The constant of proportionality in (3.2.12) depends on the specific $\mathcal{N} = 4$ string theory we are considering, but can be calculated in any given theory.

3. Using the residue theorem the contribution to the integral (3.2.1) from

the pole at (3.2.11) can be brought to the form

$$e^{S_{stat}(Q,P)} \equiv d(Q,P) \simeq \int \frac{d^2\tau}{\tau_2^2} e^{-F(\vec{\tau})}, \tag{3.2.15}$$

where τ_1 and τ_2 are two complex variables, related to ρ and σ via

$$\rho \equiv \tau_1 + i\tau_2 \ , \ \sigma \equiv -\tau_1 + i\tau_2 \, , \tag{3.2.16}$$

and

$$\begin{aligned} F(\vec{\tau}) &= -\Big[\frac{\pi}{2\tau_2} |Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) \\ &\quad + \ln\Big\{ K_0 \Big(2(k+3) + \frac{\pi}{\tau_2}|Q - \tau P|^2 \Big) \Big\} \Big] \, , \\ K_0 &= constant \, . \end{aligned} \tag{3.2.17}$$

Even though τ_1 and τ_2 are complex, we have used the notation $\tau = \tau_1 + i\tau_2$, $\bar{\tau} = \tau_1 - i\tau_2$, $|\tau|^2 = \tau\bar{\tau}$, and $|Q - \tau P|^2 = (Q - \tau P)(Q - \bar{\tau} P)$. Note that $F(\vec{\tau})$ also depends on the charge vectors (Q, P), but we have not explicitly displayed these in its argument. The $\simeq$ in (3.2.15) denotes equality up to the (exponentially subleading) contributions from the other poles.

4. We can analyze the contribution to (3.2.13) using the saddle point method. To leading order the saddle point corresponds to the extremum of the first term in the right hand side of (3.2.17). This gives

$$\tau_1 = \frac{Q \cdot P}{P^2}, \qquad \tau_2 = \frac{\sqrt{Q^2 P^2 - (Q \cdot P)^2}}{P^2} \, . \tag{3.2.18}$$

Using (3.2.13), (3.2.16) we get

$$(\check{\rho}, \check{\sigma}, -\check{v}) = \frac{i}{2\sqrt{Q^2 P^2 - (Q\cdot P)^2}} (Q^2, P^2, Q \cdot P) - (0, 0, \frac{1}{2}) \, . \tag{3.2.19}$$

We can regard the result for $-S_{stat}$ as the extremal value of the 1PI effective action in the zero dimensional quantum field theory, with fields $\tau, \bar{\tau}$ (or equivalently τ_1, τ_2) and action $F(\vec{\tau}) - 2\ln\tau_2$. A manifestly duality invariant procedure for evaluating S_{stat} was given in [37] using background field method and Riemann normal coordinates. The final result of this analysis is that S_{stat} is given by

$$S_{stat} \simeq -\Gamma_B(\vec{\tau}_B) \qquad \text{at} \qquad \frac{\partial \Gamma_B(\vec{\tau}_B)}{\partial \vec{\tau}_B} = 0\,, \tag{3.2.20}$$

where $\Gamma_B(\vec{\tau}_B)$ is the sum of 1PI vacuum diagrams calculated with the action

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\tau_{B2})^n \xi_{i_1} \ldots \xi_{i_n} D_{i_1} \cdots D_{i_n} F(\vec{\tau}) \Big|_{\vec{\tau}=\vec{\tau}_B} - \ln \mathcal{J}(\vec{\xi})\,, \tag{3.2.21}$$

where

$$\mathcal{J}(\vec{\xi}) = \left[\frac{1}{|\xi|} \sinh |\xi| \right], \qquad |\xi| \equiv \sqrt{\bar{\xi}\xi}\,. \tag{3.2.22}$$

Here $\vec{\tau}_B$ is a fixed background value, ξ, $\bar{\xi}$ are zero dimensional quantum fields and

$$\begin{aligned} D_\tau(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) &= (\partial_\tau - im/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})), \\ D_{\bar{\tau}}(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) &= (\partial_{\bar{\tau}} + in/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau}))\,, \end{aligned} \tag{3.2.23}$$

for any arbitrary ordering of D_τ and $D_{\bar{\tau}}$ in $D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})$.

This finishes the required background for generating the asymptotic expansion of the statistical entropy to any given order in inverse powers of charges, – all we need is to compute $\Gamma_B(\tau_B)$ to the desired order and then find its value at the extremum. The function $-\Gamma_B(\tau_B)$ is called the statistical entropy function.

3.2.3 Exponentially suppressed corrections

In our analysis we shall also be interested in studying the exponentially subleading contribution to the statistical entropy. These come from picking up the residues at the other zeroes of $\check{\Phi}$. The details of the analysis has been reviewed in [46]; here we summarize the results for the special case of heterotic string theory on T^6[22]. In this case $k = 10$, $\check{\Phi}$ is given by the Siegel modular form Φ_{10}, and the periods (a_1, a_2, a_3) are all equal to 1. Φ_{10} has second order zeroes at

$$n_2(\check\sigma\check\rho - \check v^2) + j\check v + n_1\check\sigma - m_1\check\rho + m_2 = 0,$$
$$\text{for} \quad m_1, n_1, m_2, n_2 \in \mathbb{Z}, j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} \tag{3.2.24}$$

.

Since eqs.(3.2.24) are invariant under $(\vec m, \vec n, j) \to (-\vec m, -\vec n, -j)$, we can use this symmetry to set $n_2 \geq 0$. For any given $n_2 \geq 1$ we can use the symmetry of Φ_{10} under integer shifts in $(\check\rho, \check\sigma, \check v)$ to bring m_1, n_1 and j in the range

$$0 \leq n_1 \leq n_2 - 1, \qquad 0 \leq m_1 \leq n_2 - 1, \qquad 0 \leq j \leq 2n_2 - 1\,. \tag{3.2.25}$$

Using this symmetry we can fix (m_1, n_1, j) in this range, but then we must extend the integration range over (ρ_1, σ_1, v_1) to be over the whole real axes. For given n_2, m_1, n_1, j, the last equation in (3.2.24) then determines m_2 in terms of the other variables. This equation also forces j to be odd, and $m_1 n_1 + (j^2-1)/4$ to be an integer multiple of n_2. We can now evaluate the contribution from each of these poles using saddle point method. To leading order the location of the saddle point from the pole associated with a given set of values of m_i, n_i and j is given by[22, 46]

$$(\check\rho, \check\sigma, -\check v) = \frac{i}{2n_2\sqrt{Q^2P^2 - (Q\cdot P)^2}}(Q^2, P^2, Q\cdot P) - \frac{1}{n_2}(n_1, -m_1, \frac{j}{2})\,. \tag{3.2.26}$$

For $n_2 = 1$ we can choose $n_1 = m_1 = 0$, $j = 1$ and (3.2.26) reduces to (3.2.11).

Besides these there are also contributions from the poles corresponding to $n_2 = 0$. These are in fact the poles responsible for the jump in the degeneracy as we cross walls of marginal stability[41]. In particular for the wall associated with a decay of the form

$$(Q, P) \to (Q_1, P_1) + (Q_2, P_2)\,, \tag{3.2.27}$$

$$(Q_1, P_1) = (\alpha Q + \beta P, \gamma Q + \delta P), \qquad (Q_2, P_2) = (\delta Q - \beta P, -\gamma Q + \alpha P)\,, \tag{3.2.28}$$

$$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1\,, \tag{3.2.29}$$

the jump in the index is given by the residue at the pole at

$$\check{\rho}\gamma - \check{\sigma}\beta + \check{v}(\alpha - \delta) = 0\,. \tag{3.2.30}$$

Unlike the residues from the poles at (3.2.24), which grow as exponentials of quadratic powers of charges, the residues at the poles at (3.2.30) grow as exponentials of linear powers of charges. Thus one expects them to be suppressed compared to the contribution from all other poles of the form given in (3.2.24). Nevertheless we shall see that for small charges the residues at (3.2.30) give substantial subleading contribution to the statistical entropy.

3.2.4 Organising the Asymptotic Expansion

Consider the contour integral given in (3.2.1) with (M_1, M_2, M_3) given as in (3.2.3). In order to find the asymptotic expansion of this expression we need to deform the contour so that it passes through the saddle point. Since the integral is done over the real parts of $(\check{\rho}, \check{\sigma}, \check{v})$ keeping their imaginary parts fixed, we shall deform the contour by varying the imaginary parts $(\check{\rho}_2, \check{\sigma}_2, \check{v}_2)$ of $(\check{\rho}, \check{\sigma}, \check{v})$. For this we first note that in the $(\check{\rho}_2, \check{\sigma}_2, \check{v}_2)$ space, the point (M_1, M_2, M_3) given in (3.2.6) corresponding to the choice of the contour for single centered black holes, and the values of $(\check{\rho}_2, \check{\sigma}_2, \check{v}_2)$ given in (3.2.26) corresponding to various saddle points, lie along a straight line passing through the origin:

$$\frac{\check{\rho}_2}{Q^2} = \frac{\check{\sigma}_2}{P^2} = -\frac{\check{v}_2}{Q \cdot P}\,. \tag{3.2.31}$$

Thus we can first deform the contour from its initial position to the position (3.2.6), keeping $Im(\check{\rho}, \check{\sigma}, \check{v})$ large all through, and then deform it along a straight line towards the origin. In the first step we shall only cross the poles of the type given in (3.2.30). This picks up the contribution to the entropy from the multi-centered black holes which were present at the point in the moduli space where we are computing the entropy. In the second stage we pick up the contribution from all the saddle points with $n_2 \geq 1$, but do not cross any pole of the type given in (3.2.30). These can then be regarded as the contribution to the entropy of a pure single centered black hole. Thus we see that the complete contribution to single centered black hole entropy comes from residues at the poles (3.2.24) with $n_2 \geq 1$. This suggests that at least for finite values of charges where the jumps across the walls of marginal stability are not extremely small compared to the total index, the asymptotic expansion, based on the residues at the poles at (3.2.24) with $n_2 \geq 1$, is better suited for reproducing the entropy of single centered black holes than that of single and multi-centered black holes together. We shall see this explicitly in our numerical analysis.

3.3 Power Suppressed Corrections

In §3.2 we outlined a general procedure for computing the statistical entropy as an expansion in inverse powers of charges. In this section we shall use this method to compute the statistical entropy to order $1/q^2$ where q stands for a generic charge. For comparison we note that the leading correction to the entropy is quadratic in the charges. Contribution to S_{stat} up to order q^0 has been computed in [23, 34, 37].

We begin with the expression for $F(\vec{\tau})$ given in (3.2.17) and carry out the background field expansion as described in (3.2.21). For this we organise (3.2.21) as a sum of three terms

$$F(\vec{\tau}) - \ln \mathcal{J}(\vec{\xi}) = F_0 + F_1 + F_2 \qquad (3.3.32)$$

where

$$\begin{aligned}
F_0 &= -\frac{\pi}{2\tau_2}\,|Q-\tau P|^2, \\
F_1 &= \ln g(\tau) + \ln g(-\bar{\tau}) + (k+2)\ln(2\tau_2) - \ln \mathcal{J}(\vec{\xi}) \\
&\quad - \ln\left[K_0 \frac{\pi|Q-\tau P|^2}{\tau_2}\right], \\
F_2 &= -\ln\left[1 + \frac{2(k+3)\tau_2}{\pi|Q-\tau P|^2}\right],
\end{aligned} \qquad (3.3.33)$$

represent respectively the leading piece of order q^2, the $O(q^0)$ piece and all terms of the order $q^{-2n}, n \geq 1$. Since the loop expansion is an expansion in powers of q^{-2}, in order to carry out a systematic expansion in powers of q^{-2}

we need to regard F_0 as the tree level contribution, F_1 as the 1-loop contribution and F_2 as two and higher loop contributions. To compute Γ_B up to a certain order, we need to compute 1PI vacuum diagrams in the zero dimensional field theory with action $(F_0 + F_1 + F_2)$ up to that order regarding ξ as fundamental field. Thus for example in order to compute the contribution to Γ_B to order q^{-2} we need to include all one and two loop diagrams involving vertices from F_0, all one loop diagrams involving a single vertex of F_1 and the tree level contribution from F_0, F_1 and F_2.

To see more explicitly how the powers of q appear, we expand $F(\vec{\tau})$ in field variable ξ around the background point $\vec{\tau}_B$. We then identify the quadratic term in ξ in the leading action F_0 with the inverse propagator and all other terms (including quadratic terms in the expansion of F_1 and F_2) as vertices. Since F_0 is of order q^2, this gives a propagator of order q^{-2}. All vertices coming from F_0 are of order q^2, all vertices coming from F_1 are of order q^0 and the vertices coming from F_2 are of order q^{-2n} with $n \geq 1$. Let us now consider a 1PI vacuum diagram with V_n number of n-th order vertices coming from F_0. Since there are no external legs, we have $\sum_n nV_n/2$ propagators. Thus the contribution from this diagram goes as

$$q^{\sum_n (2-n)V_n} . \tag{3.3.34}$$

Similar counting works for vertices coming from F_1 and F_2, but every vertex coming from F_1 will carry an extra power of q^{-2} and every vertex coming from F_2 will carry two or more extra powers of q^{-2}. Thus an order q^{-2}

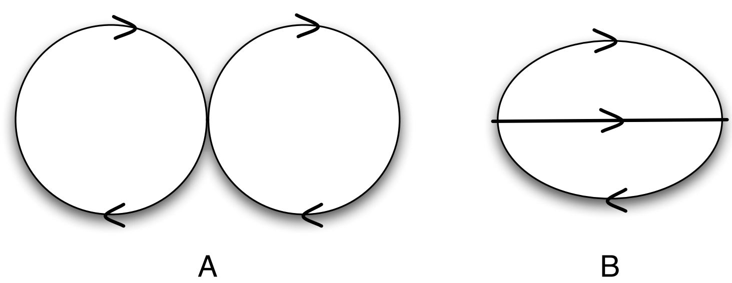

Figure 3.1: 2-loop graphs using the vertices from F_0.

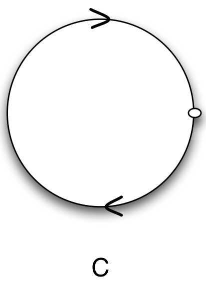

Figure 3.2: 1-loop graph using a 2-vertex from F_1.

contribution to the effective action can come from

$$(V_4 = 1, \quad V_n = 0 \quad \text{for} \quad n \neq 4) \quad \text{or} \quad (V_3 = 2, \quad V_n = 0 \quad \text{for} \quad n \neq 3)\,, \tag{3.3.35}$$

if all the vertices are from F_0, and

$$V_2 = 1, \quad V_n = 0 \quad \text{for} \quad n \neq 2\,, \tag{3.3.36}$$

if this single two point vertex is from F_1.[2] The possible diagrams associated with (3.3.35) have been shown in Fig.3.1 whereas the diagram associated with (3.3.36) have been shown in Fig.3.2. Finally the order q^{-2} contribution from F_2 is obtained by just adding the $F_2(\tau_B)$ term to $\Gamma_B(\tau_B)$.

The above analysis shows that in order to calculate the contribution to Γ_B up to order q^{-2}, we need to expand $F_0(\vec{\tau})$ to quartic order in $\vec{\xi}$, and $F_1(\vec{\tau})$ to quadratic order in $\vec{\xi}$. This is done with the help of (3.2.21), (3.2.23). We get[3]

$$\begin{aligned}
F_0(\vec{\tau}) &= F_0(\vec{\tau}_B) - \frac{i\pi}{4\tau_{B2}} \left\{ \xi (Q - \bar{\tau}_B P)^2 - \bar{\xi}(Q - \tau_B P)^2 \right\} \\
&- \frac{\pi}{4\tau_{B2}} |Q - \tau_B P|^2\, \bar{\xi}\xi + \frac{i\pi}{24\tau_{B2}} \left\{ (Q - \tau_B P)^2 \bar{\xi}^2 \xi - (Q - \bar{\tau}_B P)^2 \xi^2 \bar{\xi} \right\} \\
&- \frac{\pi}{48\tau_{B2}} |Q - \tau_B P|^2\, \bar{\xi}^2 \xi^2\,, \\
F_1(\vec{\tau}) &= F_1(\vec{\tau}_B) \\
&+ \tau_{B2} \left[\left\{ \frac{g'(\tau_B)}{g(\tau_B)} + \frac{k+2}{\tau_B - \bar{\tau}_B} + \frac{1}{\tau_B - \bar{\tau}_B} \frac{(Q - \bar{\tau}_B P)^2}{|Q - \tau_B P|^2} \right\} \xi + c.c. \right] \\
&- \left\{ \frac{k+4}{4} - \frac{(Q - \tau_B P)^2 (Q - \bar{\tau}_B P)^2}{4\,(|Q - \tau_B P|^2)^2} + \frac{1}{6} \right\} \xi \bar{\xi} + \mathcal{O}(\xi^2, \bar{\xi}^2).
\end{aligned} \tag{3.3.37}$$

[2]Note that F_0 does not give a two point vertex.
[3]Whenever a τ (τ_B) appears without a vector sign, it should be interpreted as $\tau_1 + i\tau_2$ ($\tau_{B1} + i\tau_{B2}$).

The quadratic term in the expansion of $F_0(\vec{\tau})$ gives the propagator

$$M^{\xi\bar{\xi}} = M^{\bar{\xi}\xi} = -\frac{4\tau_{B2}}{\pi|Q-\tau_B P|^2}, \qquad M^{\xi\xi} = M^{\bar{\xi}\bar{\xi}} = 0\,. \tag{3.3.38}$$

Using the vertices we can evaluate the order q^{-2} contribution to Γ_B shown in the three diagrams in Figs. 3.1 and 3.2. The results are

$$\begin{aligned} A &= -\frac{2\tau_{B2}}{3\pi|Q-\tau_B P|^2}, \\ B &= \frac{2\tau_{B2}(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{9\pi(|Q-\tau_B P|^2)^3}, \\ C &= \frac{2\tau_{B2}}{3\pi|Q-\tau_B P|^2} + \frac{(4+k)\tau_{B2}}{\pi|Q-\tau_B P|^2} - \frac{\tau_{B2}(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{\pi\,(|Q-\tau_B P|^2)^3}\,. \end{aligned} \tag{3.3.39}$$

Combining this with the order q^2 and q^0 contribution to Γ_B given in [37], the complete statistical entropy function goes as,

$$\begin{aligned} \Gamma_B(\vec{\tau}_B) &= F_0(\vec{\tau}_B) + F_1(\vec{\tau}_B) + F_2(\vec{\tau}_B) - \ln(\pi\,|M^{\xi\bar{\xi}}|) + A + B + C \\ &= \Gamma_0(\vec{\tau}_B) + \Gamma_1(\vec{\tau}_B) + \Gamma_2(\vec{\tau}_B) \\ \Gamma_0(\vec{\tau}_B) &= -\frac{\pi}{2\tau_{B2}}\,|Q-\tau_B P|^2, \\ \Gamma_1(\vec{\tau}_B) &= \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{B2}) - \ln(4\pi K_0) \\ \Gamma_2(\vec{\tau}_B) &= -\frac{\tau_{B2}}{\pi|Q-\tau_B P|^2}\left((k+2) + \frac{7}{9}\frac{(Q-\tau_B P)^2(Q-\bar{\tau}_B P)^2}{(|Q-\tau_B P|^2)^2}\right)\,. \end{aligned} \tag{3.3.40}$$

The last term in $\Gamma_2(\vec{\tau}_B)$ vanishes at the extremum of $\Gamma_0(\vec{\tau}_B)$ where

$$\tau_{B2} = \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}, \quad \tau_{B1} = \frac{Q\cdot P}{P^2} \tag{3.3.41}$$

We can therefore get rid of this term by doing a field redefinition. Using this we can write

$$\Gamma_2(\vec{\tau}_B) = -\frac{\tau_{B2}}{\pi|Q-\tau_B P|^2}(k+2)\,. \tag{3.3.42}$$

We now note that $\Gamma_2(\vec{\tau}_B)$ is independent of the modular form $g(\tau)$. This fact has some important implications for our result; we will come back to it at the end of this section.

We can now extremize $\Gamma_B(\vec{\tau}_B)$ given in (3.3.40) with respect to $\vec{\tau}_B$ to evaluate the black-hole entropy up to this order. For this it is enough to find the location of the extremum to order $1/q^2$. Let $\vec{\tau}_{(0)}$ be the extremum of $F_0(\vec{\tau}_B)$ given in (3.3.41). By extremizing $F_0 + F_1$ we can find the extremum to order $1/q^2$. We get

$$\tau = \tau_{(0)} + \frac{2\sqrt{Q^2P^2 - (Q\cdot P)^2}}{\pi(P^2)^2}\,\overline{\frac{\partial \Gamma_1}{\partial \tau}} + \mathcal{O}(1/q^4)\,, \qquad (3.3.43)$$

where the derivative of Γ_1 is taken at fixed $\bar{\tau}$. Substituting this in the argument of the Γ_i's we get

$$S_{stat} = -\Gamma_0 - \Gamma_1 - \Gamma_2 = S^{(0)} + S^{(1)} + S^{(2)}, \qquad (3.3.44)$$

where

$$\begin{aligned}
S^{(0)} &= \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} \\
S^{(1)} &= -\ln g(\tau_{(0)}) - \ln g(-\bar{\tau}_{(0)}) - (k+2)\ln(2\tau_{(0)2}) + \ln(4\pi K_0) \\
S^{(2)} &= \frac{2+k}{2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}} \\
&+ \left[\left(\frac{g'(\tau_{(0)})}{g(\tau_{(0)})} + \frac{k+2}{\tau_{(0)} - \bar{\tau}_{(0)}}\right)\left(\frac{g'(-\bar{\tau}_{(0)})}{g(-\bar{\tau}_{(0)})} + \frac{k+2}{\tau_{(0)} - \bar{\tau}_{(0)}}\right)\right]\frac{4\tau_{(0)2}^3}{\pi|Q - \tau_{(0)}P|^2}\,.
\end{aligned} \qquad (3.3.45)$$

For type IIB string theory compactified on $K3 \times T^2$, $k = 10$, $g(\tau) = \eta(\tau)^{24}$ and $4\pi K_0 = 1$. We have shown in table 3.1 the approximate statistical entropies $S^{(0)}_{stat} = S^{(0)}$ calculated with the 'tree level' statistical entropy function, $S^{(1)}_{stat} = S^{(0)} + S^{(1)}$ calculated with the 'tree level' plus 'one loop' statistical entropy function and $S^{(2)}_{stat} = S^{(0)} + S^{(1)} + S^{(2)}$ calculated with the 'tree level' plus 'one loop' plus 'two loop' statistical entropy function and compared the results with the exact statistical entropy S_{stat}. The exact results for $d(Q, P)$ are computed using a choice of contour for which only single centered black holes contribute to the index for $Q \cdot P > 0$ and both single and 2-centered black hole solutions contribute for $Q \cdot P < 0$. We clearly see that the asymptotic expansion has better agreement with the exact results when only single centered black holes are present, in accordance with our general argument.

Given the result for the statistical entropy to this order, one would like to see if this can be reproduced from the macroscopic calculation on the black hole side. So far black hole entropy calculation has been done for the leading supergravity action and a subset of the four derivative terms which include curvature squared contribution to the effective action[58–61]. The results of these two completely independent calculations match up to order q^0 and give us enough confidence on the expected equivalence of the statistical entropy and the black hole entropy. However there are many open issues. Even at the level of the four derivative terms, only a subset of the four derivative terms have been included in the analysis of the black hole entropy. Furthermore at this order the full 1PI effective action of string theory also contains non-local terms from integrating out the massless fermions and Wald's formula cannot even be applied in principle to take into account the effect of these terms. Recently a generalization of the Wald's formula for extremal black holes in the full quantum theory has been proposed[21] (see also [62, 63]). This will be discussed in more detail in §3.5 in the context of exponentially suppressed terms. However as far as the power law corrections are concerned, at present we do not have a complete calculation of the quantum entropy function for quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric theory even at the level of order q^0 terms. This prevents us from making a concrete statement on the agreement between the two entropies.[4]

Given that even at order q^0 we do not have a complete test of the equality between the microscopic and the macroscopic calculations, we cannot hope to have such a test for the order q^{-2} terms calculated here. However we can say a few words about the possible contributions on the macroscopic side which

[4]It was shown in [33] that the leading asymptotic expansion of the entropy to all orders in inverse powers of charges, associated with the pole at (3.2.11), is consistent with the OSV formula[64] after inclusion of certain additional measure factors. Refs.[65–67] independently derived the same measure factor in the semiclassical approximation by requiring that the entropy is invariant under duality transformations. Our goal is to derive a general formula for the entropy of an extremal black hole based on some principle (like AdS/CFT) from which the results of [33, 65–67] would follow. In particular if one can establish that the asymptotic expansion of the quantum entropy function reduces to the formula given in [33, 65–67], this will automatically prove that the quantum entropy function agrees with the statistical entropy to all orders in inverse powers of charges.

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S_{stat}^{(0)}$	$S_{stat}^{(1)}$	$S_{stat}^{(2)}$	D_1	D_2
2	2	0	50064	10.82	6.28	10.62	11.576	.2	-0.756
4	4	0	32861184	17.31	12.57	16.90	17.382	.41	-0.072
6	6	0	16193130552	23.51	18.85	23.19	23.506	.32	.004
8	8	0	7999169992704	29.71	25.13	29.47	29.71	.24	.000
10	10	0	4074192429737760	35.943	31.42	35.754	35.945	.189	-0.002
6	6	1	11232685725	23.14	18.59	22.88	23.15	.26	-0.01
6	6	2	4173501828	22.15	17.77	21.94	22.198	.21	-0.05
6	6	3	920577636	20.64	16.32	20.41	20.766	.23	-0.13
6	6	-1	11890608225	23.19	18.59	22.88	23.15	.31	.04
6	6	-2	2857656822	21.77	17.77	21.94	22.198	-0.17	-0.43
6	6	-3	2894345136	21.78	16.32	20.41	20.766	1.37	1.01

Table 3.1: Comparison of the exact statistical entropy to the tree level, one loop and two loop results obtained via the asymptotic expansion. In the last two columns D_1 is the difference of the exact result and the one loop result and D_2 is the difference of the exact result and the two loop result. We clearly see that for $Q \cdot P > 0$ where only single centered black holes contribute to S_{stat}, inclusion of the two loop results reduces the error, at least for large charges.

is needed to reproduce the order q^{-2} corrections to the statistical entropy. To this end we note that the order q^{-2} correction to the statistical entropy function $\Gamma_B(\vec{\tau}_B)$ given in (3.3.42) is manifestly invariant under continuous duality transformation

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad ad-bc=1, \quad a,b,c,d \in \mathbf{R}\,. \tag{3.3.46}$$

Now while comparing the statistical entropy function to the black hole entropy function, the parameters τ get identified with the near horizon axion-dilaton modulus λ in the heterotic description[23, 34, 37]. This suggests that if the required correction comes from a local correction to the 1PI action, then the corresponding term must be invariant under a continuous S-duality transformation. Furthermore since we are looking for a correction of order q^{-2}, we require the correction to the Lagrangian density to be a six derivative term. This puts a strong restriction on the type of contribution to the local Lagrangian density that can be responsible for such corrections. We have not been able to find a candidate Lagrangian density. The most straightforward method for constructing duality invariant terms using Riemann tensors constructed out of canonical Einstein metric does not work since all such terms vanish in the $AdS_2 \times S^2$ near horizon geometry and hence do not contribute to the entropy function to this order. This of course does not rule out the existence of duality invariant terms constructed out of other fields. The other possibility is that these contributions cannot be encoded in a local Lagrangian density, but come from the non-local contributions to the quantum entropy function arising from the path integral over string fields in the near horizon geometry. To this end we note that since the OSV formula reproduces the complete asymptotic expansion to all orders in q^{-2}, if we can derive the OSV formula from the quantum entropy function we shall automatically reproduce these corrections to the statistical entropy.

3.4 Exponentially Suppressed Corrections

In this section we shall analyze the exponentially suppressed contributions from the zeroes of Φ_{10} given in (3.2.24):

$$n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - m_1\check{\rho} + m_2 = 0\,, \tag{3.4.47}$$

with

$$m_1, n_1, m_2, n_2 \in \mathbb{Z}, j \in 2\,\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \tag{3.4.48}$$

For this we define

$$\check{\Omega} = \begin{pmatrix} \check{\rho} & \check{v} \\ \check{v} & \check{\sigma} \end{pmatrix}, \tag{3.4.49}$$

and look for a symplectic transformation of the form:

$$\begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \equiv \Omega = (A\check{\Omega} + B)(C\check{\Omega} + D)^{-1}\,, \tag{3.4.50}$$

such that

$$v = \frac{n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - m_1\check{\rho} + m_2}{\det(C\check{\Omega} + D)}\,. \tag{3.4.51}$$

Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a 4×4 symplectic matrix. In this case (3.4.47) gets mapped to $v = 0$. On the other hand the modular transformation law of Φ_{10} gives

$$\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v}) = \{\det(C\check{\Omega} + D)\}^{-k}\, \Phi_{10}(\rho, \sigma, v)\,, \qquad k = 10\,. \tag{3.4.52}$$

Thus the behaviour of $\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v})$ near the zero (3.4.47) is given by

$$\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v}) = -\{\det(C\check{\Omega} + D)\}^{-k}\, 4\pi^2\, v^2\, g(\rho)\, g(v) + \mathcal{O}(v^4) \;\;, g(\rho) = \eta(\rho)^{24}\,. \tag{3.4.53}$$

We can now substitute (3.4.53) into (3.2.1) (with $\check{\Phi}$ replaced by Φ_{10}) and evaluate the integral over $\check{v}$ using residue theorem. For this we need to regard (ρ, σ, v) appearing in (3.4.53) as functions of $(\check{\rho}, \check{\sigma}, \check{v})$ via eq.(3.4.50), (3.4.51).

The result is, up to a sign,

$$(-1)^{Q\cdot P} \int d\check\rho\, d\check\sigma\, e^{-\pi i(\check\rho P^2+\check\sigma Q^2+2\check v Q\cdot P)}\ \det(C\check\Omega+D)^{k+2}\,(2n_2\check v-j)^{-2}$$
$$\times g(\rho)^{-1}\,g(\sigma)^{-1}\,(Q\cdot P+\mathcal{O}(1))\,, \tag{3.4.54}$$

where $\check v$ and (ρ,σ) are to be regarded as functions of $(\check\rho,\check\sigma)$ via eqs.(3.4.47) and (3.4.50). The last factor in (3.4.54) proportional to $Q\cdot P$ comes from taking the derivative of the integrand other than the pole term with respect to $\check v$. We can now evaluate the $(\check\rho,\check\sigma)$ integral using the saddle point method. To leading order the location of the saddle point is obtained by extremizing the term in the exponent of (3.4.54) subject to the constraint (3.4.47). The result is given in eq.(3.2.26):

$$(\check\rho,\check\sigma,-\check v) = \frac{i}{2n_2\sqrt{Q^2P^2-(Q\cdot P)^2}}(Q^2,P^2,Q\cdot P) - \frac{1}{n_2}\,(n_1,-m_1,\frac{j}{2})\,. \tag{3.4.55}$$

The result of the integration over $(\check\rho,\check\sigma)$ can be expressed as

$$(-1)^{Q\cdot P}\Big[\exp\left(-\pi i(\check\rho P^2+\check\sigma Q^2+2\check v Q\cdot P)\right)\ \det(C\check\Omega+D)^{k+2}\,(2n_2\check v-j)^{-2}$$
$$\times\, g(\rho)^{-1}\,g(\sigma)^{-1}\,(Q\cdot P+\mathcal{O}(1))\,\Big((\det\Delta)^{-1/2}+\mathcal{O}(1)\Big)\Big]_{\rm saddle}\,, \tag{3.4.56}$$

where the subscript 'saddle' denotes that we need to set $(\check\rho,\check\sigma,\check v)$ to their saddle point values given in (3.4.55), and Δ is the 2×2 matrix:

$$\Delta = i\,Q\cdot P\begin{pmatrix} \partial^2\check v/\partial\check\rho^2 & \partial^2\check v/\partial\check\rho\partial\check\sigma \\ \partial^2\check v/\partial\check\rho\partial\check\sigma & \partial^2\check v/\partial\check\sigma^2 \end{pmatrix}\,. \tag{3.4.57}$$

In evaluating (3.4.57) we need to regard $\check v$ as a function of $(\check\rho,\check\sigma)$ via eq.(3.4.47). Explicit computation gives

$$\det\Delta = (Q\cdot P)^2\,n_2^2/(2n_2\check v-j)^4\,. \tag{3.4.58}$$

Substituting this and (3.4.55) into (3.4.56) gives

$$\frac{(-1)^{Q\cdot P}}{n_2}\exp\left(\frac{\pi\sqrt{Q^2P^2-(Q\cdot P)^2}}{n_2}\right)\exp\left[\frac{i\pi(n_1P^2-m_1Q^2+jQ\cdot P)}{n_2}\right]$$
$$\left[\det(C\check{\Omega}+D)^{k+2}\,g(\rho)^{-1}\,g(\sigma)^{-1}\,(1+\mathcal{O}(q^{-2}))\right]_{\text{saddle}} \tag{3.4.59}$$

where we have how fixed the overall sign by requiring that it agrees with the result of [46] for $(m_1, n_1, n_2, m_2, j) = (0, 0, 1, 0, 1)$.

In order to evaluate the factor $\det(C\check{\Omega}+D)^{k+2}\,g(\rho)^{-1}\,g(\sigma)^{-1}$ appearing in (3.4.59) explicitly, we need to find explicitly the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying (3.4.51). We shall do this explicitly for $n_2 = 2$. In this case there are six possible values of $(\vec{m}, \vec{n}, j)$ consistent with (3.2.25), (3.4.48). They are

$$\begin{aligned}(m_1, n_1, m_2, n_2, j) \quad = \quad & (0,0,0,2,1), (1,0,0,2,1), (0,1,0,2,1), \\ & (0,0,-1,2,3), (1,0,-1,2,3), (0,1,-1,2,3)\,.\end{aligned} \tag{3.4.60}$$

In each of these cases we can find appropriate matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying

(3.4.51). These transformations take the form:

$$\Omega = \begin{pmatrix} \frac{\check{\rho}}{(1-2\check{v})^2-4\check{\rho}\check{\sigma}} & \frac{-2\check{v}^2+\check{v}+2\check{\rho}\check{\sigma}}{(1-2\check{v})^2-4\check{\rho}\check{\sigma}} \\ \frac{-2\check{v}^2+\check{v}+2\check{\rho}\check{\sigma}}{(1-2\check{v})^2-4\check{\rho}\check{\sigma}} & \frac{\check{\sigma}}{(1-2\check{v})^2-4\check{\rho}\check{\sigma}} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \frac{\check{\rho}}{4(\check{v}-1)\check{v}+2\check{\rho}-4\check{\rho}\check{\sigma}+1} & \frac{-2\check{v}^2+\check{v}+\check{\rho}(2\check{\sigma}-1)}{4(\check{v}-1)\check{v}+2\check{\rho}-4\check{\rho}\check{\sigma}+1} \\ \frac{-2\check{v}^2+\check{v}+\check{\rho}(2\check{\sigma}-1)}{4(\check{v}-1)\check{v}+2\check{\rho}-4\check{\rho}\check{\sigma}+1} & \frac{2(\check{v}-1)\check{v}+\check{\rho}-2\check{\rho}\check{\sigma}+\check{\sigma}}{4(\check{v}-1)\check{v}+2\check{\rho}-4\check{\rho}\check{\sigma}+1} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \frac{-2(\check{v}-1)\check{v}+\check{\rho}+2\check{\rho}\check{\sigma}+\check{\sigma}}{(1-2\check{v})^2-2(2\check{\rho}+1)\check{\sigma}} & \frac{-2\check{v}^2+\check{v}+2\check{\rho}\check{\sigma}+\check{\sigma}}{(1-2\check{v})^2-2(2\check{\rho}+1)\check{\sigma}} \\ \frac{-2\check{v}^2+\check{v}+2\check{\rho}\check{\sigma}+\check{\sigma}}{(1-2\check{v})^2-2(2\check{\rho}+1)\check{\sigma}} & \frac{\check{\sigma}}{(1-2\check{v})^2-2(2\check{\rho}+1)\check{\sigma}} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \frac{\check{\rho}}{(\check{v}-1)^2-\check{\rho}\check{\sigma}} & \frac{1-\check{v}}{(\check{v}-1)^2-\check{\rho}\check{\sigma}}-2 \\ \frac{1-\check{v}}{(\check{v}-1)^2-\check{\rho}\check{\sigma}}-2 & \frac{\check{\sigma}}{(\check{v}-1)^2-\check{\rho}\check{\sigma}} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -\frac{(1-2\check{v})^2-4\check{\rho}\check{\sigma}}{-2\check{v}+\check{\rho}+\check{\sigma}+1} & -\frac{\check{v}(2\check{v}-3)+\check{\rho}-2\check{\rho}\check{\sigma}+1}{-2\check{v}+\check{\rho}+\check{\sigma}+1} \\ -\frac{\check{v}(2\check{v}-3)+\check{\rho}-2\check{\rho}\check{\sigma}+1}{-2\check{v}+\check{\rho}+\check{\sigma}+1} & -\frac{\check{v}^2-(\check{\rho}+1)\check{\sigma}}{-2\check{v}+\check{\rho}+\check{\sigma}+1} \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -\frac{\check{v}(3\check{v}-4)-\check{\rho}-3\check{\rho}\check{\sigma}-\check{\sigma}+1}{-2(\check{v}-1)\check{v}+\check{\rho}+2\check{\rho}\check{\sigma}+\check{\sigma}} & \frac{\check{v}-\check{\rho}-1}{-2(\check{v}-1)\check{v}+\check{\rho}+2\check{\rho}\check{\sigma}+\check{\sigma}}+1 \\ \frac{\check{v}-\check{\rho}-1}{-2(\check{v}-1)\check{v}+\check{\rho}+2\check{\rho}\check{\sigma}+\check{\sigma}}+1 & \frac{-2\check{\rho}-1}{-2(\check{v}-1)\check{v}+\check{\rho}+2\check{\rho}\check{\sigma}+\check{\sigma}}+2 \end{pmatrix}. \quad (3.4.61)$$

These transformations can be used to get ρ and σ in terms of $(Q^2, P^2, Q\cdot P)$ using (3.4.55). Substituting these into (3.4.59) and summing over the allowed values of (m_1, n_1, j) given in (3.4.60) we get the correction to $d(Q,P) = \exp(S_{stat})$ to this order. If we denote the resulting correction to $d(Q,P)$ by $\Delta d(Q,P)$, then the values of $\Delta d(Q,P)$ for different values of $(Q^2, P^2, Q\cdot P)$ have been shown in table 3.2.

3.5 Macroscopic Origin of the Exponentially Suppressed Corrections

We have seen that the corrections to the leading contribution to the statistical entropy are of two types, power suppressed corrections which arise from expansion about the saddle point associated with pole (3.2.11), and exponentially suppressed corrections associated with the contribution from the residues at the other poles (3.2.24). Given that we have not been able to reproduce even

Q^2	2	4	6	6	6	6
P^2	2	4	6	6	6	6
$Q\cdot P$	0	0	0	1	2	3
$\Delta d(Q,P)$	34.617	480.638	18537.1	20104.8	27652.3	0

Table 3.2: First exponentially suppressed contribution to $d(Q,P)$ and $S_{stat}(Q,P)$. Note that the correction vanishes accidentally for $Q\cdot P = Q^2/2 = P^2/2$ odd.

the power suppressed corrections from the macroscopic side, it may seem futile to attempt to understand the exponentially suppressed corrections. However we shall now argue that quantum entropy function may provide a natural mechanism for understanding the exponentially suppressed corrections.

We shall begin with a lightening review of the quantum entropy function. Let us consider an extremal black hole with an AdS_2 factor in the near horizon geometry. We shall regard string theory in this background as a two dimensional theory, treating all other directions as compact. The background fields describing the AdS_2 near horizon geometry has the form[68]

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{rt} = e_i, \quad \cdots \tag{3.5.62}$$

where $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ are the gauge field strengths associated with two dimensional gauge fields $A^{(i)}_\mu$, v and e_i are constants and $\cdots$ denotes near horizon values of other fields. Under euclidean continuation

$$t = -i\theta\,, \tag{3.5.63}$$

we have

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{r\theta} = -i\,e_i, \quad \cdots \tag{3.5.64}$$

Under a further coordinate change

$$r = \cosh\eta \ , \tag{3.5.65}$$

(3.5.64) takes the form

$$ds^2 = v \left(d\eta^2 + \sinh^2\eta \, d\theta^2 \right), \qquad F^{(i)}_{\theta\eta} = ie_i \sinh\eta, \qquad \cdots .$$

The metric is non-singular at the point $\eta = 0$ if we choose θ to have period 2π. Integrating the field strength we can get the form of the gauge field:

$$A^{(i)}_\mu dx^\mu = -i\, e_i\, (\cosh\eta - 1) d\theta = -i\, e_i\, (r-1) d\theta \, . \tag{3.5.66}$$

Note that the -1 factor inside the parenthesis is required to make the gauge fields non-singular at $\eta = 0$. In writing (3.5.66) we have chosen $A^{(i)}_\eta = 0$ gauge. If q_i denotes the charge of the black hole corresponding to the ith gauge field and $\mathcal{L}$ denotes the Lagrangian density evaluated in the near horizon geometry (3.5.66), then $\vec{q}$ and $\vec{e}$ are related as

$$q_i = \frac{\partial(v\mathcal{L})}{\partial e_i} \, . \tag{3.5.67}$$

Quantum entropy function is a proposal for computing the exact degeneracy of states of an extremal black hole. It is given by

$$d(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta \, A^{(i)}_\theta] \right\rangle^{finite}_{AdS_2} , \tag{3.5.68}$$

where $\langle \ \rangle_{AdS_2}$ denotes the unnormalized path integral over various fields of string theory on euclidean global AdS_2 described in (3.5.66) and $A^{(i)}_\theta$ denotes the component of the i-th gauge field along the boundary of AdS_2. The superscript '$finite$' refers to the finite part of the amplitude defined as follows. If we regularize the infra-red divergence by putting an explicit cut-off that regularizes the volume of AdS_2, then the amplitude has the form $e^{CL} \times$ a finite part where C is a constant and L is the length of the boundary of regulated AdS_2. We define the finite part as the one obtained by dropping the e^{CL} part. This equation gives a precise relation between the microscopic degeneracy and an appropriate partition function in the near horizon geometry of the black hole.

In defining the path integral over AdS_2 we need to put boundary conditions on various fields. We require that the asymptotic geometry coincides with (3.5.66). Special care is needed to fix the boundary condition on $A^{(i)}_\theta$. In the $A^{(i)}_\eta = 0$ gauge the Maxwell's equation around this background has two independent solutions near the boundary: $A^{(i)}_\theta =$ constant and $A^{(i)}_\theta \propto r$. Since the latter is the dominant mode we put boundary condition on the latter mode, allowing the constant mode of the gauge field to fluctuate. This corresponds to working with fixed asymptotic values of the electric fields, or equivalently fixed charges via eq.(3.5.67).

Let us now review how in the classical limit the quantum entropy function reduces to the exponential of the Wald entropy. For this we need to put an infra-red cut-off; this is done by restricting the coordinate r in the range $1 \leq r \leq r_0$. Then in the classical limit the quantum entropy function is given by the finite part of

$$\exp\left(-A_{bulk} - A_{boundary} - iq_i \oint A^{(i)}_\theta \, d\theta\right) , \tag{3.5.69}$$

where A_{bulk} and $A_{boundary}$ represent contributions from the bulk and the boundary terms in the classical action in the background (3.5.66). If $\mathcal{L}$ denotes the Lagrangian density of the two dimensional theory, then the bulk contribution to the action in the background (3.5.66) takes the form:

$$\begin{aligned} A_{bulk} &= -\int d^2x \sqrt{\det g}\, \mathcal{L} \\ &= -\int_0^{2\pi} d\theta \int_0^{\cosh^{-1} r_0} d\eta \, \sinh\eta \, v \, \mathcal{L} \\ &= -2\pi v \, \mathcal{L}\,(r_0 - 1) + \mathcal{O}(r_0^{-1}) \, . \end{aligned} \tag{3.5.70}$$

In going from the second to the third step in (3.5.70) we have used the fact that due to the $SO(2,1)$ invariance of the AdS_2 background, $\mathcal{L}$ must be independent of η and θ. In this parametrization the length L of the boundary is given by

$$L = \sqrt{v} \int_0^{2\pi} \sqrt{r_0^2 - 1}\, d\theta = 2\pi \sqrt{v}\, r_0 + \mathcal{O}(r_0^{-1}) \, . \tag{3.5.71}$$

The contribution from the last term in (3.5.69) can also be calculated easily using the expression for $A_\theta^{(i)}$ given in (3.5.66). We get

$$i q_i \oint A_\theta^{(i)} \, d\theta = 2\pi \, \vec{q} \cdot \vec{e} (r_0 - 1) \,. \tag{3.5.72}$$

Finally, the contribution from $A_{boundary}$ can be shown to have the form[21]

$$A_{boundary} = 2\pi r_0 \, K + \mathcal{O}(r_0^{-1}) \,, \tag{3.5.73}$$

for some constant K. This gives

$$\begin{aligned} &\exp\left(-A_{bulk} - A_{boundary} - i q_i \oint A_\theta^{(i)} \, d\theta\right) \\ &= \exp\left[2\pi(\vec{q} \cdot \vec{e} - v \, \mathcal{L})\right] \exp\left[-2\pi r_0 (\vec{q} \cdot \vec{e} - v \, \mathcal{L} + K) + \mathcal{O}(r_0^{-1})\right] \end{aligned} \tag{3.5.74}$$

Thus the quantum entropy function, given by the finite part of (3.5.74), takes the form

$$d(q) \simeq \exp\left[2\pi(\vec{q} \cdot \vec{e} - v \, \mathcal{L})\right] \,. \tag{3.5.75}$$

The right hand side of (3.5.75) is the exponential of the Wald entropy[20].[5] For the particular case of quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories the leading contribution to (3.5.75) has the form

$$d(q) \simeq \exp\left(\pi \sqrt{Q^2 P^2 - (Q \cdot P)^2}\right) \,. \tag{3.5.76}$$

Quantum corrections to (3.5.75) can be of two types. First of all we can have fluctuations of the string field around the AdS_2 background (3.5.64). We expect this to produce power law corrections, but not change the exponent in (3.5.76) which is related to the finite part of the action in the AdS_2 background. The other class of corrections could come from picking altogether different classical solutions with the same asymptotic field configuration as the one given in (3.5.64). These could have different actions and hence give contributions with different exponential factors. Thus such corrections are the

[5]For the special case of two derivative actions this has also been noted recently in [69].

ideal candidates for producing exponentially subleading corrections to the degeneracy.

Can we identify classical solutions which could produce the subleading corrections discussed in §3.4? To this end consider a $\mathbb{Z}_N$ quotient of the background (3.5.64) by the transformation

$$\theta \to \theta + \frac{2\pi}{N}\,. \tag{3.5.77}$$

If we denote by $(\widetilde{r}, \widetilde{\theta})$ the coordinates of this new space then the solution may be expressed as

$$ds^2 = v\left((\widetilde{r}^2 - 1)d\widetilde{\theta}^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2 - 1}\right), \quad F^{(i)}_{\widetilde{r}\widetilde{\theta}} = -i\, e_i, \quad \cdots, \qquad \widetilde{\theta} \equiv \widetilde{\theta} + \frac{2\pi}{N}\,. \tag{3.5.78}$$

Since $\widetilde{\theta}$ has a different period than θ, this does not manifestly have the same asymptotic form as the solution (3.5.64). Let us now make a change of coordinates

$$r = \widetilde{r}/N, \quad \theta = N\widetilde{\theta}\,. \tag{3.5.79}$$

In this coordinate system the new metric takes the form:

$$\begin{aligned} ds^2 &= v\left((r^2 - N^{-2})d\theta^2 + \frac{dr^2}{r^2 - N^{-2}}\right), \quad F^{(i)}_{r\theta} = -i\, e_i, \quad \cdots, \\ &\theta \equiv \theta + 2\pi\,. \end{aligned} \tag{3.5.80}$$

This has the same asymptotic behaviour as the original solution and hence is a potential saddle point that could contribute to the quantum entropy function. The action associated with this solution, with the cut-off $r \le r_0$, can be easily calculated. After removing the r_0 dependent piece we get the following classical contribution to the quantum entropy function[6]

$$\exp\left[2\pi(\vec{q}\cdot\vec{e} - v\,\mathcal{L})/N\right] = \exp\left(\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/N\right)\,. \tag{3.5.81}$$

[6] This is easiest to derive in the $(\widetilde{r}, \widetilde{\theta})$ coordinate system where the total action is $1/N$ times the action for the original AdS_2 background with r_0 replaced by $\widetilde{r}_0$. Since $\widetilde{r}_0 = Nr_0$, the terms linear in r_0 are the same as in the original AdS_2 background, whereas the r_0 independent term gets divided by N.

This has precisely the right form as the exponentially subleading contributions described in §3.4 if we identify N with the integer n_2 appearing there.

This however cannot be the complete story. From the form of the solution given in (3.5.78) it is clear that the the solution has a $\mathbb{Z}_N$ orbifold singularity of the type $\mathbf{R}^2/\mathbb{Z}_N$ at the origin $\widetilde{r} = 1$. This is *a priori* a singular configuration and it is not clear if this is an allowed configuration in string theory. We resolve this difficulty by accompanying the $\mathbb{Z}_N$ action by an internal $\mathbb{Z}_N$ transformation

$$\phi \to \phi - \frac{2\pi}{N}, \tag{3.5.82}$$

where ϕ is the azimuthal coordinate of the sphere S^2 that is also part of the near horizon geometry of the black hole. If ψ denotes the polar angle on S^2 then the orbifold group has fixed points at $(\widetilde{r} = 1, \psi = 0)$ and $(\widetilde{r} = 1, \psi = \pi)$. Thus the manifold is still singular but now the singularities are of the type $\mathbb{C}^2/\mathbb{Z}_N$, and these can certainly be resolved in string theory. Thus we conclude that the resulting configuration is non-singular. The classical action is not affected by the additional shifts in the ϕ coordinate and hence the contribution to the quantum entropy function continues to be given by (3.5.81).

There is however a new issue that we need to address. Now the identification $\theta \equiv \theta + 2\pi$ changes to

$$(\theta, \phi) \equiv \left(\theta + 2\pi, \phi - \frac{2\pi}{N}\right). \tag{3.5.83}$$

Thus one needs to check if this is consistent with the asymptotic boundary conditions imposed on various fields. To this end we note that if we denote by $\mathcal{A}_\mu$ the two dimensional gauge field arising from the ϕ translation isometry, then the twisted boundary condition (3.5.84) is equivalent to switching on a Wilson line of the form

$$\oint \mathcal{A}_\theta \, d\theta = \frac{2\pi}{N}. \tag{3.5.84}$$

Now as discussed earlier, for all gauge fields the boundary conditions fix the electric field, or equivalently the charge, but the zero modes of the gauge fields are allowed to fluctuate. Here the charge associated with the gauge field $\mathcal{A}_\mu$ is the angular momentum[70] which has been taken to be zero. But there is no constraint on the Wilson line $\oint \mathcal{A}_\theta \, d\theta$. Thus we are instructed to integrate over different possible values of this Wilson line, and in that process pick up contribution from the different saddle points given in (3.5.80). This shows that there is no conflict between the asymptotic boundary conditions and the twist described in (3.5.83).

Another issue that needs attention is integration over bosonic and fermionic zero modes associated with this solution. The near horizon geometry of the black hole has an $\mathcal{N} = 4$ superconformal algebra. The generators of this algebra are the $SL(2,R)$ generators L_0, $L_{\pm 1}$, the $SU(2)$ generators J^3, $J^{\pm}$ and the supersymmetry generators $G^{\pm\alpha}_{\pm\frac{1}{2}}$. with $\alpha = 1,2$. Of these $(L_1 - L_{-1})/2$ is the generator of rotation about the origin of AdS_2 and J^3 is the generator of rotation about the north pole of S^2. Since the orbifold action is generated by $(L_1 - L_{-1} - 2J^3)$, the quotient is not invariant under the full $\mathcal{N} = 4$ superconformal algebra; it is invariant only under a subalgebra that commutes with $(L_1 - L_{-1} - 2J^3)$. This subalgebra is generated by $L_1 - L_{-1}$, J^3, $G^{+\alpha}_{1/2} + G^{+\alpha}_{-1/2}$ and $G^{-\alpha}_{1/2} - G^{-\alpha}_{-1/2}$. The broken bosonic and fermionic generators leads to four bosonic and four fermionic zero modes of the solution. Of these the bosonic zero modes parametrize the coset $(SL(2,R)/U(1)) \times (SU(2)/U(1)) = AdS_2 \times S^2$. This is precisely the situation analyzed in [71].[7] Naively the integration over the bosonic zero modes will produce infinite result and the fermionic zero mode integrals vanish. But it was shown in [71] that we can regularize the integrals by adding to the action an extra term that does not affect the integral. The extra term lifts both the bosonic and the fermionic zero modes and as a result the path integral produces a finite result.

There are several other minor issues which need to be addressed. For type II string theory in flat space-time, the $\mathbb{Z}_N$ orbifold action described here generates an allowed configuration. Here we have an $AdS_2 \times S^2$ background instead of flat space. Hence the original analysis is not strictly valid. However since the orbifold fixed point is localized in $AdS_2 \times S^2$, it should not 'feel' the effect of the background geometry and continue to be an allowed configuration. What is not guaranteed is that the blow up modes which allow us to deform the configuration away from the orbifold point will remain flat directions. This is an important issue we need to address if we want to explore the

[7] The notation of [71] is slightly different; what we are calling $L_1 - L_{-1}$ was called L_0 in [71].

constant multiplying (3.5.81). We also need to explore if there can be any additional contribution to the action from the orbifold fixed point. We expect however that since the fixed point is localized at a point in $AdS_2 \times S^2$, to leading order such a contribution (if non-zero) will be independent of the background geometry of $AdS_2 \times S^2$. In particular it will not have a factor proportional to the size of $AdS_2 \times S^2$, and hence will at most give an order q^0 correction to the leading term $\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/N$ in the exponent of (3.5.81).

The analysis described above is independent of which kind of extremal black hole we are considering.[8] This suggests a universal pattern of the exponentially suppressed corrections to the entropy of all extremal black holes. If we denote by S_0 the leading contribution to the entropy then the exact degeneracy should contain subleading corrections of order $e^{S_0/N}$ for all $N \in \mathbb{Z}$, $N \geq 2$. It will be interesting to see if the exact degeneracy formulæ of extremal black holes in theories with less number of supersymmetries obey this structure.

[8]For higher dimensional black holes the near horizon geometry contains a (squashed) S^n factor instead of S^2. In this case we can choose a suitable embedding of the $\mathbb{Z}_N$ action inside the symmetry group of (squashed) S^n.

Chapter 4

Subleading Correction to Statistical Entropy for BMPV Black Hole

4.1 Introduction

Counting of 1/4 BPS dyonic states in four dimensional $\mathcal{N} = 4$ supersymmetric string theories has been studied in great detail in last few years[22, 34–36, 43, 47–49]. We now have a good understanding of the degeneracy formula, its moduli dependence and the wall crossing formulae. Large charge asymptotic expansion of these degeneracy formulae exactly capture the dyonic black hole entropy including certain subleading corrections due to higher derivative corrections to the supergravity.

Five dimensional spinning (BMPV) black holes[8] is a close cousin of the four dimensional dyonic black hole. These black holes were first constructed in [8], as a spinning generalization of [6]. These are charged, spinning, 5-dimensional black holes with constant dilation and constant moduli in type IIB theory on $K3 \times S^1$. The microscopic configurations of these black holes can be described as p-solitonic states (Dp- branes) in type IIB theory on $K3 \times S^1$, for $p = 1, 3, 5$. The states also carry certain momenta along the S^1 circle and angular momenta along the non compact directions. This microscopic description is much similar to that of the four dimensional dyonic black holes. In fact,when described in terms of D-branes, the BMPV black hole consists of D1-D5-p system, whereas the four dimensional dyonic black hole in addition has a KK monopole background. It is therefore natural to study BMPV black hole entropy in terms of the four dimensional dyon degeneracy formula without KK monopole contribution. A general degeneracy formula for D1-D5-p system is then easy to write down. However, we are interested in finding out subleading correction to the BMPV black hole entropy due to higher derivative terms in the effective action. Higher derivative correction to five dimensional black holes has been computed in [72] and their entropy has been computed [73]. In this paper we will take a different approach to this problem. Determination of subleading correction is done in a most effective fashion using the statistical entropy function and the effective action formalism. Using the statistical entropy function one can write down a one dimensional effective action, and using the Feynman diagram technique one can obtain systematic large charge asymptotic expansion of the statistical entropy. This method correctly reproduces subleading correction to the entropy of four dimensional 1/4 BPS dyonic black holes.

This particular feature of the statistical entropy function gives the motivation to compute similar subleading correction to the five dimensional BMPV black hole. The exactness of the statistical entropy (or the statistical entropy function) suggests that we can evaluate the entropy (or the entropy function) to any order.

The rests of the paper is divided into three sections. In the first section, we present a different form of the degeneracy function of the five dimensional BMPV black holes, based on the degeneracy of the four dimensional dyonic black holes. In the next section we discuss the first subleading $(O(Q^0))$ correction to the statistical entropy function and statistical entropy of these black holes. In the last section, we have some discussions on our results.

As this paper was being written a paper [74] appeared in the arXiv that discusses similar issues.

4.2 Degeneracy Function For 5-dimensional BMPV Black Holes

In this section, we rewrite the degeneracy function of the BMPV black holes in a different form. The microscopic description for this black hole is a particular D-brane configuration in type IIB theory compactified on $K3 \times S^1$. This contains Q_1 number of $D1$ branes, Q_5 number of D5 branes, $-n$ units of momenta along S^1 circle and angular momenta J1 and J2 along the non-compact spatial directions. This configuration, however does not contain any D3 branes. For extremal black holes, the corresponding microscopic configuration requires the modulus of the two angular momenta to be same $|J1| = |J2| = J$. The microscopic computation for the leading entropy was first done in [8]. We will write α' (inverse string tension) exact degeneracy function for this configuration, from the knowledge of the degeneracy function of 4-dimensional dyonic black holes in $\mathcal{N} = 4$ supersymmetric string theory. Here we will sketch in brief how the degeneracy function was obtained for these 4-dimensional black holes[37, 46].

4.2.1 Degeneracy Function of 4-dimensional Dyonic Black holes

Let us consider type IIB theory compactified on $K3 \times S^1 \times \tilde{S}^1$. Following a chain of duality transformations, one can look at the same theory as a heterotic string theory compactified on T^6. These theories have dyonic black-hole solutions. Let us consider a specific configuration in this compactified type IIB theory : Q_1 number of D1-branes wrapped along S^1, Q_5 number of D5-brane wrapped along $K3 \times S^1$, a single Kaluza-Klein monopole associated with $\tilde{S}^1$ circle,$-n$ units of momentum along S^1 direction and J units of angular momentum along $\tilde{S}^1$ direction. In the dual Heterotic picture, this represents dyonic black hole solutions. If we stay in a region of the moduli space where the type IIB theory is weakly coupled, the partition function of the entire system can be obtained by considering three weakly interacting sources:

1. the relative motion of the D1-brane in the plane of D5-brane, carrying certain momenta $-L$ along S^1 and J' $\tilde{S}^1$ directions,

2. the center of mass motion of D1-D5 system in the KK-monopole background carrying momenta $-l_0$ along S^1 and j_0 along $\tilde{S}^1$ directions,

3. excitations of the KK-monopole carrying $-l_0'$ momentum along S^1,

with $n = L + l_0 + l_0'$ and $J = J' + j_0$ being the sum of momenta along S^1 and $\tilde{S}^1$ directions respectively. Hence, in the weak coupling limit, the partition function $f(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ of the configuration can be expressed as,

$$\begin{aligned} f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) &= -\frac{1}{64} \left(\sum_{Q_1, L, J'} (-1)^{J'} d_{D1}(Q_1, L, J') e^{2\pi i (\tilde{\sigma} Q_1/N + \tilde{\rho} L + \tilde{v} J')} \right) \\ & \left(\sum_{l_0, j_0} (-1)^{j_0} d_{CM}(l_0, j_0) e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{v}} \right) \left(\sum_{l_0'} d_{KK}(l_0') e^{2\pi i l_0' \tilde{\rho}} \right) \\ &, \end{aligned} \tag{4.2.1}$$

where $d_{D1}(Q_1, L, J')$ is the degeneracy of source (1), $d_{CM}(l_0, j_0)$ is the degeneracy associated with source (2) and $d_{KK}(l_0')$ denotes the degeneracy associated with source (3). The factor of $1/64$ accounts for the fact that a single quarter BPS supermultiplet has 64 states. Evaluating these three pieces separately, the full partition function of the system looks like,

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = e^{-2\pi i (\tilde{\rho} + \tilde{v})} \prod_{\substack{k' \in \mathbb{Z} + r, l \in \mathbb{Z}, j \in 2\mathbb{Z} \\ k', l \geq 0, j < 0 \text{ for } k' = l = 0}} \left(1 - e^{2\pi i (\tilde{\sigma} k' + \tilde{\rho} l + \tilde{v} j)} \right)^{-c(4lk' - j^2)} . \tag{4.2.2}$$

Then we define the degeneracy function $\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ and degeneracy of states $d(\vec{Q}, \vec{P})$ as,

$$\begin{aligned} f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) &= \frac{e^{2\pi i \tilde{\sigma}}}{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}, \\ d(\vec{Q}, \vec{P}) &= (-1)^{Q \cdot P + 1} h \left(\frac{1}{2} Q^2, \frac{1}{2} P^2, Q \cdot P \right), \end{aligned} \tag{4.2.3}$$

where $(\vec{Q}, \vec{P})$ are the charge vectors carried by the black holes:

$$Q = \begin{pmatrix} 0 \\ -n \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} Q_5(Q_1 - Q_5) \\ -J \\ Q_5 \\ 0 \end{pmatrix}. \tag{4.2.4}$$

and $h(m,n,p)$ are the coefficients of Fourier expansion of the function $1/\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})$:

$$\frac{1}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})} = \sum_{m,n,p} g(m,n,p)\, e^{2\pi i(m\,\tilde{\rho}+n\,\tilde{\sigma}+p\,\tilde{v})}\,. \tag{4.2.5}$$

At this point it is worth noting from equations (4.2.2) and (4.2.5) that the power series gets a contribution $e^{-2\pi i\tilde{v}}(1-e^{-2\pi i\tilde{v}})^{-1}$ from $k' = l = 0$ term and one can expand the series either in $e^{-2\pi i\tilde{v}}$ or in $e^{2\pi i\tilde{v}}$. There is an ambiguity in the expansion, we will come back to this point in the last section .

To evaluate the degeneracy of a state associated with charges $(\vec{Q},\vec{P})$, we need to invert equation (4.2.5) as,

$$d(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\, e^{-\pi i(\tilde{\rho}Q^2+\tilde{\sigma}P^2+2\tilde{v}Q\cdot P)}\, \frac{1}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}\,, \tag{4.2.6}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labeled by $(\tilde{\rho},\tilde{\sigma},\tilde{v})$, given by

$$\begin{aligned} &\tilde{\rho}_2 = M_1, \quad \tilde{\sigma}_2 = M_2, \quad \tilde{v}_2 = -M_3, \\ &0 \le \tilde{\rho}_1 \le 1, \quad 0 \le \tilde{\sigma}_1 \le 1, \quad 0 \le \tilde{v}_1 \le 1\,. \end{aligned} \tag{4.2.7}$$

M_1, M_2 and M_3 are large but fixed positive numbers with $M_3 << M_1, M_2$. The choice of the M_i's is determined from the requirement that the Fourier expansion is convergent in the region of integration.

The $\mathcal{N} = 4$ supersymmetric string theories discussed above are invariant under $O(6,22,\mathbb{Z})$ T-duality and $SL(2,\mathbb{Z})$ S-duality symmetry. The T-duality invariants are given as,

$$Q^2 = 2n, \quad P^2 = 2Q_5(Q_1 - Q_5), \qquad Q\cdot P = J. \tag{4.2.8}$$

The function $\tilde{\Phi}$ actually behaves as a modular form of weight $k = 10$ under the S-duality group $SL(2,\mathbb{Z})$.

4.2.2 Degeneracy for BMPV Black Holes

Here we will compute the degeneracy function for BMPV black holes from our knowledge of the degeneracy function of 4-dimensional black holes we studied in the last section. Comparing with the four-dimensional black hole, we will treat the microscopic configuration of the BMPV black holes to be same as the one considered in 4-dimensional case except for the following changes. The radius of the $\tilde{S}^1$ circle is infinite and therefore the KK-monopole sector is replaced by $\mathbf{R^4}$. We will again work in a region of the moduli space where IIB theory is weakly coupled. The partition function of this configuration will only get contribution from the source (1) of section(2.1), i.e., the relative motion of the D1-branes in the plane of D5-branes. Hence, we have,

$$\begin{aligned} f_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v}) &= -\frac{1}{64}\left(\sum_{Q_1,L,J'}(-1)^{J'}\, d_{D1}(Q_1,L,J')e^{2\pi i(\tilde{\sigma}Q_1/N+\tilde{\rho}L+\tilde{v}J')}\right) \\ &= \prod_{\substack{k'\in\mathbb{Z},l\in\mathbb{Z},j\in2\mathbb{Z}\\ k'>0,l\geq 0}}\left(1-e^{2\pi i(\tilde{\sigma}k'+\tilde{\rho}l+\tilde{v}j)}\right)^{-c(4lk'-j^2)}. \end{aligned} \tag{4.2.9}$$

Following the steps given in equations (4.2.3), (4.2.5) and (4.2.6), we define the degeneracy function and degeneracy of states for the BMPV black hole. The degeneracy function is given as,

$$\tilde{\Phi}_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \frac{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}{G(\tilde{\rho},\tilde{v})}, \tag{4.2.10}$$

where,

$$\begin{aligned} G(\tilde{\rho},\tilde{v}) = 64e^{2\pi i(\tilde{\rho}+\tilde{v})}(1-e^{-2\pi i\tilde{v}})^2\prod_{n=1}^{\infty}(1 &- e^{2\pi i n\tilde{\rho}})^{20}(1-e^{2\pi i(n\tilde{\rho}+\tilde{v})})^2 \\ (1 &- e^{2\pi i(n\tilde{\rho}-\tilde{v})})^2. \end{aligned} \tag{4.2.11}$$

Here the function $G(\tilde{\rho},\tilde{v})$ basically captures the degeneracy of the KK-monopole sector and the center of mass motion of the D1-D5 system in KK-monopole background for four dimensional dyonic black holes.

4.3 Correction to The Statistical Entropy Function

Similar to the 4-dimensional black hole , we define the degeneracy of states for the BMPV black holes as,

$$d(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\, e^{-\pi i(\tilde{\rho}Q^2+\tilde{\sigma}P^2+2\tilde{v}Q\cdot P)}\,\frac{1}{\tilde{\Phi}_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v})}\,. \quad (4.3.12)$$

The statistical entropy for the system is then given as,

$$S_{stat} = \ln d(\vec{Q},\vec{P})\,. \quad (4.3.13)$$

One can evaluate the integral (4.3.12) by saddle point method and estimate the statistical entropy for the system. We will take a different approach to estimate the entropy. From the integral (4.3.12), we will first evaluate a function Γ^{stat} analogous to black hole entropy function. This function is called the statistical entropy function. The statistical entropy is then obtained as the value of this function at its extrema. This can be done by following two steps:

- The v integral is done by residue methods. The function $\Phi_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v})$ has a zero at

$$\tilde{\rho}\tilde{\sigma}-\tilde{v}^2+\tilde{v}=0. \quad (4.3.14)$$

  Near this pole the function Φ_{bmpv} behaves as,

$$\Phi_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = (2v-\rho-\sigma)^k\, v^2\,\frac{g(\rho)\,g(\sigma)}{\hat{G}(\rho,\sigma,v)}\,, \quad (4.3.15)$$

  where

$$\rho = \frac{\tilde{\rho}\tilde{\sigma}-\tilde{v}^2}{\tilde{\rho}}, \qquad \sigma = \frac{\tilde{\rho}\tilde{\sigma}-(\tilde{v}-1)^2}{\tilde{\rho}}, \qquad v = \frac{\tilde{\rho}\tilde{\sigma}-\tilde{v}^2+\tilde{v}}{\tilde{\rho}}\,, \quad (4.3.16)$$

  k is related to the rank r of the gauge group via the relation

$$r = 2k+8\,, \quad (4.3.17)$$

and $g(\tau)$ is a known function which depends on the details of the theory. Typically it transforms as a modular function of weight $(k+2)$ under a certain subgroup of the $SL(2,\mathbb{Z})$ group. In the (ρ,σ,v) variables the pole at (4.3.14) is at $v = 0$. Near this pole, the integrand looks like, $v^{-2}F(\rho,\sigma,v)\hat{G}(\rho,\sigma,v)$ where,

$$\begin{aligned} F(\rho,\sigma,v) &= \frac{(2v-\rho-v)^{(-k-3)}}{g(\rho)g(\sigma)} e^{\left[-i\pi\left(\frac{v^2-\rho\sigma}{2v-\rho-\sigma}P^2+\frac{Q^2}{2v-\rho\sigma}+\frac{2(v-\rho)}{2v-\rho\sigma}Q\cdot P\right)\right]}. \\ \hat{G}(\rho,\sigma,v) &= 64 e^{2\pi i\frac{1+v-\rho}{2v-\rho-\sigma}}(1-e^{-2\pi i\frac{v-\rho}{2v-\rho-\sigma}})^2 \\ & \prod_{n=1}^{\infty}(1-e^{2\pi i\frac{n}{2v-\rho-\sigma}})^{20}(1-e^{2\pi i\frac{n+(v-\rho)}{2v-\rho-\sigma}})^2(1-e^{2\pi i\frac{n-(v-\rho)}{2v-\rho-\sigma}})^2 \\ & . \end{aligned} \tag{4.3.18}$$

After doing the v integral using the above relation, (4.3.12) takes the form,

$$e^{S_{stat}(\vec{Q},\vec{P})} \equiv d(\vec{Q},\vec{P}) \simeq \int \frac{d^2\tau}{\tau_2^2}\, e^{-F_{bmpv}(\vec{\tau})}\,, \tag{4.3.19}$$

where τ_1 and τ_2 are two complex variables, related to ρ and σ via

$$\rho \equiv \tau_1 + i\tau_2\ ,\ \ \sigma \equiv \tau_1 - i\tau_2\,, \tag{4.3.20}$$

and the effective action F_{bmpv} is given as,

$$F_{bmpv}(\vec{\tau}) = F(\vec{\tau}) - \ln\hat{G}(\vec{\tau}) - \ln\left(1+\frac{f\hat{G}'}{\hat{G}f'}(\vec{\tau})\right) \tag{4.3.21}$$

where

$$\begin{aligned} F(\vec{\tau}) &= -\left[\frac{\pi}{2\tau_2}|Q-\tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2)\right. \\ &\qquad \left. +\ln\left\{K_0\left(2(k+3)+\frac{\pi}{\tau_2}|Q-\tau P|^2\right)\right\}\right], \\ K_0 &= constant\,. \end{aligned} \tag{4.3.22}$$

The function $F(\vec{\tau})$ is actually the effective action for 4-dimensional black holes. The function $\hat{G}(\vec{\tau})$ and $f(\vec{\tau})$ are same as $\hat{G}(\rho,\sigma,v)$ and $F(\rho,\sigma,v)$

in (4.3.18) respectively, evaluated at $v = 0$ and expressed as functions of $\vec{\tau}$. Here $'$ means derivative with respect to v evaluated at $v = 0$. We give the expressions for the function $\hat{G}(\vec{\tau})$ here for later use:

$$\begin{aligned} \hat{G}(\vec{\tau}) &= -64 e^{-\frac{\pi}{\tau_2}(1-\tau_1)}(1+e^{\frac{-\pi}{\tau_2}\tau_1})^2 \prod_{n=1}^{\infty}(1-e^{\frac{-n\pi}{\tau_2}})^{20}(1+e^{\frac{-\pi}{\tau_2}(n+\tau_1)})^2 \\ &\qquad (1+e^{\frac{-\pi}{\tau_2}(n+\tau_1)})^2. \end{aligned} \tag{4.3.23}$$

- Next we evaluate (4.3.19) by considering it to be a zero dimensional field theory with fields $\tau, \bar{\tau}$ (or equivalently τ_1, τ_2) and action $F_{bmpv}(\vec{\tau}) - 2\ln\tau_2$. We apply background field method technique to obtain the statistical entropy function. In this method, we do an asymptotic expansion of the action around a fixed background point $\vec{\tau}_B$, which is not the saddle point of the action. This expansion is valid for

$$Q^2 > 0 \qquad P^2 > 0 \qquad Q^2P^2 > (Q\cdot P)^2\,. \tag{4.3.24}$$

The statistical entropy function (to a certain order in charges) is then given as a sum of all 1PI vacuum diagrams (required to that order) in this zero dimensional field theory.

We now want to evaluate the four derivative, i.e., $O(Q^0)$ correction to the statistical entropy. The last term in (4.3.21) is of $O(Q^{-2n}, n \geq 1)$. Similarly, the last term in $F(\vec{\tau})$ also goes as $O(Q^{-2n}, n \geq 1)$. Hence, up to order Q^0, these terms will not contribute. The first term in $F(\vec{\tau})$ is $O(Q^2)$, while the second term of $F(\vec{\tau})$ and $F_{bmpv}(\vec{\tau})$ are $O(Q^0)$. Therefore the first term needs to be expanded up to one loop, whereas the other two terms are required at tree level.

Taking all these issues in to account, we find the statistical entropy function up to order Q^0 as,

$$\begin{aligned}\Gamma^{stat}_{bmpv}(\vec{\tau}_B) &= \Gamma_0(\vec{\tau}_B) + \Gamma_1(\vec{\tau}_B) - \ln \hat{G}(\vec{\tau}_B) \\ \Gamma_0(\vec{\tau}_B) &= \frac{\Pi}{2\tau_{B_2}}|Q - \tau_B P|^2 \sim \mathcal{O}(Q^2) \\ \Gamma_1(\vec{\tau}_B) &= \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k+2)\ln(2\tau_{2B}) - \ln(4\pi K_0) \sim \mathcal{O}(Q^0)\,.\end{aligned} \tag{4.3.25}$$

4.4 Correction to Statistical Entropy

The statistical entropy of the system can be obtained by extremizing the function Γ^{stat}_{bmpv} and evaluating it at its extrema. It is an straightforward exercise to check that for evaluating the entropy up to order Q^0, it is enough to compute Γ_{bmpv} at the extrema of Γ_0, given as,

$$(\tau_0)_1 = \frac{Q \cdot P}{P^2}, \qquad (\tau_0)_2 = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}. \tag{4.4.26}$$

Correction to this extrema due to Γ_1 will give $\mathcal{O}(Q^{-2})$ correction to the entropy. The expressions for the corrected entropy is,

$$S^{stat}_{bmpv} = \Gamma^{stat}_{bmpv}(\vec{\tau}_0)\,. \tag{4.4.27}$$

Here, we give the approximate statistical entropies $S^{(0)}_{stat} = S^{(0)}$ calculated using the 'tree level' statistical entropy function, $S^{(1)}_{stat} = S^{(0)} + S^{(1)}$ calculated using the 'tree level' and 'one loop' statistical entropy function in a tabular form.

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S^{(0)}_{stat}$	$S^{(1)}_{stat}$
2	2	0	5424	8.59	6.28	8.12
4	4	0	2540544	14.74	12.57	14.40
6	6	0	1254480000	20.95	18.85	20.69
6	6	1	991591800	20.71	18.59	20.46
6	6	2	483665920	20.00	17.77	19.76
6	6	-1	991591800	20.71	18.59	20.46
6	6	-2	483665920	20.00	17.77	19.76

We find that the asymptotic expansion of the statistical entropy is in good agreement with the exact entropy of the system. The agreement is better for higher values of charges. This is expected because asymptotic expansion accurate for large charges but starts deviating for small values of charges.

4.5 Degeneracy for More General 5D Black Holes

The above analysis can easily be generalized to all five-dimensional CHL black holes (for 4D CHL dyonic black holes see [46]). These are black holes in the theory obtained by compactifying Heterotic and type IIB string theory compactified on $\frac{T^4\times S^1}{\mathbb{Z}_N}$, where the $\mathbb{Z}_N$ group involves $\frac{1}{N}$ units of shift along the S^1 circle and an order N transformation on T^4. This transformation is chosen such that the theory preserves $\mathcal{N}$ =4 supersymmetry. The partition function of these dyons is given by,

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = e^{-2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{v})}\prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbf{Z}+\frac{r}{N},l\in\mathbf{Z},j\in2\mathbf{Z}+b\\ k',l\geq0,j<0\,\text{for}\,k'=l=0}} \left(1-e^{2\pi i(\tilde{\sigma}k'+\tilde{\rho}l+\tilde{v}j)}\right)^{-\sum_{s=0}^{N-1}e^{-\frac{2\pi isl}{N}}c_b^{(r,s)}(4lk'-j^2)} \qquad (4.5.28)$$

where, $c_b^{(r,s)}(4lk' - j^2)$ are some constants that can be obtained from the elliptic genus of the theory. Here also we can eliminate the contribution from the KK-monopole sector and get the degeneracy function for the generic five-dimensional black holes as,

$$\tilde{\Phi}_{bmpv}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \frac{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}{G(\tilde{\rho},\tilde{v})}, \tag{4.5.29}$$

where,

$$\begin{aligned} G(\tilde{\rho},\tilde{v}) &= 64 e^{2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{v})}(1-e^{-2\pi i\tilde{v}})^2 \prod_{n=1}^{\infty}(1-e^{2\pi i n\tilde{\rho}})^{-\sum_{s=0}^{N-1} e^{-\frac{2\pi i l s}{N}} c_0^{(0,s)}(0)} \\ &\quad (1-e^{2\pi i(n\tilde{\rho}+\tilde{v})})^{-\sum_{s=0}^{N-1} e^{-\frac{2\pi i l s}{N}} c_1^{(0,s)}(-1)} \\ &\quad (1-e^{2\pi i(n\tilde{\rho}-\tilde{v})})^{-\sum_{s=0}^{N-1} e^{-\frac{2\pi i l s}{N}} c_1^{(0,s)}(-1)}. \end{aligned} \tag{4.5.30}$$

With these modified expressions, one can proceed to compute first subleading correction to the entropy of these general black holes. For these orbifolded theories, the rank of the gauge group r reduces and accordingly the number k defined in (4.3.17) changes. Our previous analysis, corresponding to $r = 28$ and $k = 10$ goes through in all these cases. One can also produce an explicit chart for systematic corrections to statistical entropy as we have in the previous section for these black holes, while there are quantitative changes, qualitative behaviour remains the same.

4.6 Discussion

We studied the four-derivative ($O(Q^0)$) correction to the statistical entropy function and the statistical entropy by doing asymptotic expansion of the statistical entropy function. This expansion is valid in the limit (4.3.24), but is different from the Cardy limit (or Fareytail limit [55, 56]), in our case $Q^2(= n)$ and $P^2(Q_1Q_5)$ can be of same order whereas the Cardy limit corresponds to $n >> Q_1Q_5$.

We find that the exact statistical degeneracy computed around the saddle point $v = 0$, is independent of the sign of $Q \cdot P$. It is worthwhile to compare this with the four-dimensional black holes. In 4D case, the exact degeneracy changes as the sign of $Q \cdot P$ change. This jump in the degeneracy is related to the issue of walls of marginal stability as discussed in details in [41, 75]. As pointed out below (4.2.5), there is an extra zero at $\tilde{v} = 0$ in $\tilde{\Phi}$, compared to $\tilde{\Phi}_{bmpv}$. Because of this pole, there is an ambiguity in the Fourier expansion and we get the jump in degeneracy for two signs of $Q \cdot P$. Physically, it is related to the dynamics of the KK-monopole. However, this sector is absent in the 5D BMPV black holes. For this five dimensional black holes, we do not have any walls of marginal stability associated with this particular zero of the function $\tilde{\Phi}_{bmpv}$.

Part III

Hydrodynamics from Black Holes

Chapter 5

Hydrodynamics from AdS/CFT

5.1 Introduction

In chapter 1, we have discussed about the AdS/CFT conjecture and briefly mentioned its finite temperature version. We also discussed that the thermodynamic and hydrodynamic properties of boundary field theory can be realized from bulk theory. In this chapter, we will look into these issues in somewhat more detailed way.

Hydrodynamics is an effective theory, describing the dynamics of some field theory at large distances and time-scales. The equations of hydrodynamics assume that the fluid is in local thermodynamic equilibrium at each point in space, even though different thermodynamic quantities like energy and the charge densities of the fluid may vary in space. Fluid mechanics applies only when the length scales of variation of thermodynamic variables are large compared to the equilibration length scale of the fluid, namely mean free path l_{mfp}. Hydrodynamic description does not follow from any action principle rather it is normally formulated in the language of equations of motion. The reason for this is the presence of dissipation in thermal media. In the simplest case, the hydrodynamic equations are just the laws of conservation of energy and momentum (we are considering the fluid does not have any global charge or current),

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{5.1.1}$$

At any space time point the fluid is characterized by $d+1$ variables (in d dimensions), four velocities $u^\mu(x)$ (of fluid particles) and its temperature $T(x)$. All these variables are function of space and time. Since four velocities are time like they follow $u_\mu u^\mu = -1$, therefor total number of variables describing the fluid is d which is equal to the number of equations of motion. In hydrodynamics we express $T_{\mu\nu}$ through $T(x)$ and $u^\mu(x)$ through the so-called constitutive equations. Following the standard procedure of effective field theories, we expand in powers of spatial derivatives. To zeroth order, $T_{\mu\nu}$ is given by the familiar formula for ideal fluids,

$$T_{\mu\nu} = (e+p)u_\mu u_\nu + p g_{\mu\nu} \tag{5.1.2}$$

where e is energy density and p is pressure. At next order in derivative expansion fluid energymomentum tensor is given by (for conformal fluid, $T^\mu_\mu = 0$),

$$\begin{aligned} T_{\mu\nu} &= (e+p)u_\mu u_\nu + p g_{\mu\nu} - 2\eta\sigma_{\mu\nu} \\ \sigma_{\mu\nu} &= \frac{P^\alpha_\mu P^\beta_\nu}{2}\left[\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\nabla\cdot u\right] \end{aligned} \tag{5.1.3}$$

where $\sigma^{\mu\nu}$ is proportional to derivatives of $T(x)$ and $u^\mu(x)$ and is termed the dissipative part of $T^{\mu\nu}(x)$ and the coefficient η is called shear viscosity coefficient.

There are different approaches to compute the hydrodynamic quantities of the boundary theory namely the shear viscosity coefficient in the context of the AdS/CFT correspondence. The first approach was proposed by [76]. They compute the shear viscosity coefficient of boundary fluid using *Kubo* formula. In the next section we will discuss about this. During last decades there are lot of attempts to study the hydrodynamic properies of boundary fluid from the point of view of holography [76]-[115], and become an interesting issue of recent research. All these approaches correctly reproduce the universality of shear viscosity coefficient in Einstein gravity. In [120] the authors computed the shear viscosity coefficient of boundary gauge theory plasma from graviton's effective coupling. In the subsequent section, we will review this approach, which only deals with the Einstein gravity, without any higher derivative term added to it. In the next chapter 6, we have generalized this approach to higher derivative gravity. Chapter 6 and 7 are self contained, and experts can safely skip remaining sections of this chapter.

5.2 Viscosity from Kubo Formula

Let us consider field theory at long length and time scale. This can be described by hydrodynamic equations. Let us compute the two point function of energy momentum tensor $T_{\mu\nu}$ in this theory away from the equilibrium. For relativistic conformal viscous fluid, $T_{\mu\nu}$ is given as (1.2.12),

$$\begin{aligned} T_{\mu\nu} &= (e+p)u_\mu u_\nu + p g_{\mu\nu} - 2\eta\sigma_{\mu,\nu} \\ \sigma_{\mu\nu} &= \frac{P_\mu^\alpha P_\nu^\beta}{2}\left[\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\nabla\cdot u\right] \end{aligned} \tag{5.2.4}$$

here, we treat temperature $T(x)$ and four velocity $u_\mu(x)$ to be basic variables of the theory.

Let us consider a set of bosonic operators $O_\mu(x)$ in this field theory sourced by small $J_\mu(x)$. Hence, we can consider perturbation by linear response in J_μ. The average values of $O_\mu(x)$ is given as,

$$\langle O_\mu(x)\rangle = -\int G^R_{\mu\nu}(x-y)J_\nu(y), \tag{5.2.5}$$

where, G^R is the retarded Green's function given as,

$$G^R_{\mu\nu}(x-y) = -i\theta(x^0-y^0)\langle[O_\mu(x), O_\nu(y)]\rangle. \tag{5.2.6}$$

Now, the source for $T_{\mu\nu}$ is the metric $g_{\mu\nu}$. Thus, to determine the correlation function of $T_{\mu\nu}$, we can couple weak gravity with it and determine the average value of $T_{\mu\nu}$ after the source turned on. Evaluating this correlator at low momenta, we can extract hydrodynamic informations from it. let us consider the following homogeneous metric perturbation,

$$
\begin{aligned}
g_{ij}(t,\vec{x}) &= \delta_{ij} + h_{ij}(t), \qquad h_{ii} = 0, \qquad h_{ij} << 1 \\
g_{00}(t,\vec{x}) &= -1, \qquad g_{0i}(i,\vec{x}) = 0.
\end{aligned} \tag{5.2.7}
$$

using above relation (5.2.4), one can find the zero spatial momentum, low-frequency limit of retarded Green's function of $T_{\mu\nu}$ as,

$$
G^R_{xy,xy}(\omega,0) = \int dt d\vec{x} e^{-\omega t}\theta(t)\Big\langle [T_{xy}(t,\vec{x})T_{xy}(0,0)]\Big\rangle = -i\eta\omega + O(\omega^2). \tag{5.2.8}
$$

Thus, we can write the shear viscosity coefficient of the boundary fluid as,

$$
\eta = -\lim_{\omega\to 0}\frac{1}{\omega} Im G^R_{xy,xy}(\omega,0). \tag{5.2.9}
$$

This is the Kubo formula for shear viscosity coefficient.

Hence, we need to compute the real time retarded Green's function of EM tensor. Now, the usual prescription of AdS/CFT to compute boundary correlator is Euclidean ([12, 13]). In principle, some real time Green's function can be obtained by analytic continuation of the corresponding Euclidean ones. However, in many cases it is actually very difficult to get it. In particular the low frequency low momentum limit Green's function (which is interesting for hydrodynamics) is difficult to obtain from analytic continuation of Euclidean one. The difficulty here is , we need to analytically continue from a discrete set of points in Euclidean frequencies (the Matsubra frequencies) $\omega = 2\pi in$ (n integer) to real values of ω. The smallest value of the Matsubra frequency is quite large. Hence to get information in small ω limit that we are interested in, is quite difficult. The authors of ([76–78]) have done a detailed analysis of this difficulty and have given a prescription to compute the real time correlator. We will mention about their prescription in the next section.

5.3 Hydrodynamic limit in AdS/CFT and Membrane paradigm

In this section we briefly review the recent proposal of Iqbal ad Liu [120] relating the hydrodynamic limit of AdS/CFT to the membrane paradigm. Their proposal relates a generic transport coefficient of the boundary theory to some geometric quantities evaluated at the black hole horizon in the bulk. We will concentrate on the shear viscosity coefficient of the boundary fluid and confine ourselves at the level of linear response and low frequency limit of the strongly coupled gauge theory.

Membrane side

Let us start with classical black hole membrane paradigm, which says that the black hole has a fictitious fluid on its horizon. In general, the black hole action can be expressed as,

$$S_{eff} = S_{out} + S_{surf},$$

where S_{out} contains integration over space time out side the horizon and S_{surf} is the boundary term on the horizon. Physically, S_{surf} represents the effect of the horizon fluid on the spacetime. Let us consider a general black hole background,

$$dS^2 = g_{MN}dx^M dx^N \quad = \quad g_{rr}dr^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{5.3.10}$$

where M, N runs over $d+1$ dimensional bulk spacetime and μ, ν runs over d dimensional boundary spacetime. This black hole has a horizon at r_h and asymptotic boundary (where the dual gauge theory sits) at r_b. We assume $SO(3)$ invariance in the boundary spatial directions, that is all the metric components and the couplings in the theory are only function of r. We consider a small perturbation h_{xy} in the $SO(3)$ tensor sector of this metric. We use $\phi(r, x^\mu) = h^x_y$ as the off diagonal component of graviton and in the Fourier space, the perturbation looks like,

$$\phi(r, k_\mu) = \int \frac{d^d x}{(2\pi)^d}\, \phi(r, x^\mu)\, e^{ik_\mu x^\mu}, \qquad k_\mu = (-\omega, \vec{k})\,. \tag{5.3.11}$$

The action for this massless perturbation can be written as,

$$\begin{aligned} S_{out} &= -\int_{r>r_h} d^{d+1}x\sqrt{-g}\,\frac{1}{q(r)}\,(\nabla\phi)^2, \\ S_{surf} &= \int_{\Sigma} d^d x\sqrt{-\gamma}\left(\frac{\Pi(r_h,x)}{\sqrt{-\gamma}}\right)\phi(r_h,x) \end{aligned} \tag{5.3.12}$$

here, Π is the conjugate momentum to ϕ for the $r-$foliation and γ is the induced metric on the horizon. We can interprete S_{surf} as the effect of the membrane fluid on the spacetime and $\Pi_{mb} = (\frac{\Pi(r_h)}{\sqrt{\gamma}})$ as the "membrane $\phi-$charge".

Now, following membrane paradigm, the horizon is a regular place for the in-falling observer, hence, any physical deformation of the system has to satisfy the in-falling boundary condition. The in-falling boundary condition implies that near the horizon r_h,

- the deformation should behave as $\phi \sim (r - r_h)^{i\omega\beta}$, for some constant β and

- the solution should be a function of the non singular "Eddington-Finklestein" co-ordinate v defined as,

$$dv = dt + \sqrt{\frac{g_{rr}}{g_{tt}}}dr. \tag{5.3.13}$$

This implies near the horizon r_h, the deformation satisfies,

$$\partial_r\phi = \sqrt{\frac{g_{rr}}{g_{tt}}}\partial_t\phi\ . \tag{5.3.14}$$

The above equation 5.3.14 puts constraint on the constant β as,

$$\beta = \sqrt{\frac{g_{rr}(r-r_h)^2}{g_{tt}}}\Bigg|_{r_h}. \tag{5.3.15}$$

With equations 5.3.11,5.3.14 and some redefinition of the time, we can also express the membrane charge as,

$$\Pi_{mb} = -\frac{1}{q(r_h)}\partial_{\hat{t}}\phi(r_h). \tag{5.3.16}$$

As per our interpretation, Π_{mb} is the response of the membrane fluid induced by ϕ and using equation 5.3.11, in linear response, we define a shear viscosity coefficient for the membrane fluid as,

$$\Pi_{mb} = i\omega\eta_{mb}\phi$$
$$\eta_{mb} = \frac{1}{q(r_h)}. \qquad (5.3.17)$$

Boundary Side

With this much of analysis of the membrane fluid, we concentrate on the boundary side where the gauge theory lives. This is a interacting theory at finite temperature and behaves as a fluid at sufficiently long length scale or low energy. The real time (Lorentzian signature) finite temperature version of AdS/CFT correspondence allows to to compute various hydrodynamic quantities of this gauge theory at strong coupling by doing some supergravity calculations in the AdS space. Now using the Kubo formula, the shear viscosity of the boundary fluid is given as,

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d\vec{x}\, e^{i\omega t} \langle [T_{12}(x), T_{12}(0)] \rangle = -\lim_{\omega \to 0} \frac{1}{\omega} Im\, G^R(\omega, 0) \qquad (5.3.18)$$

Here, G^R is the retarded green function, the response of graviton to the boundary stress tensor. In [76],[77],[78], the authors have given a simple prescription to compute the boundary correllator 5.3.18 using the bulk field ϕ, the off diagonal component of the graviton. Their prescription requires to find a solution for the graviton which is infalling at the horizon and constant at the boundary. Then one compute the on- shell action with this solution and the retarded Green's function is related to the surface term of the on-shell action at the boundary. Taking $\phi(k_\mu, r) = f(k_\mu, r)\phi_0(k_\mu)$ with normalization $f(k_\mu, r_b) \to 1$,

$$S = -\sum_{r=r_h,r_b} \int \frac{d^d k}{(2\pi)^d} \phi_0(k_\mu)\, G(k_\mu, r)\, \phi_0(-k_\mu)$$
$$G^R(k_\mu) = \lim_{r \to r_b} 2G(k_\mu, r) = \lim_{r \to r_b} 2\frac{\sqrt{-g}g^{rr}}{q(r)} \partial_r f(k_\mu, r) \qquad (5.3.19)$$

For this profile of the graviton, we can evaluate its conjugate momenta Π, it readily gives us the relation,

$$G^R(k_\mu) = -\lim_{r\to r_b} \frac{\Pi(k_\mu, r)}{\phi(k_\mu, r)} \, . \tag{5.3.20}$$

Hence, the shear viscosity coefficient η can be written as,

$$\eta = \lim_{k_\mu \to 0} \lim_{r\to r_b} \frac{\Pi(k_\mu, r)}{i\omega\phi(k_\mu, r)}. \tag{5.3.21}$$

Important point to note is that, in the low frequency limit ($k_\mu \to 0$, $with\, \Pi$, $\omega\phi$ $fixed$), the flows of Π and $\omega\phi$ in the $r-$ direction are trivial. Hence, we can actually compute the shear viscosity coefficient of the boundary fluid at any constant $r-$slice and its value would be same. We compute it at the horizon and get,

$$\eta = \frac{1}{q(r_h)} \sqrt{\frac{-g}{g_{rr} g_{tt}}}\Bigg|_{r_h} = \frac{1}{q(r_h)} \frac{A}{V}$$

Comparing equations 5.3.22 and 5.3.17, we see that the viscosity coefficient of the boundary fluid is related to that of the membrane fluid and more importantly they are given as just the value of the inverse effective coupling of the transverse graviton evaluated at the horizon. To emphasize, equation 5.3.14 plays a crucial role in this equivalence. The AdS/CFT response of the graviton is almost same as that of the membrane except that the membrane now has to sit in the boundary. The in-falling boundary condition of the graviton field in AdS/CFT is precisely the regularity condition 5.3.14 of the membrane paradigm. In the low frequency limit, we can place a fictitious membrane at each constant r and define the transport coefficient as $\eta(r)$. Since the flow is trivial in this limit, $\eta(r)$ actually comes out to be a constant, $\frac{1}{q(r0)}$.

In the next two chapter we have generalized this idea to higher derivative gravity. The chapters are self contained.

Chapter 6

Higher Derivative Corrections to Shear Viscosity coefficient From Graviton's Effective Coupling

6.1 Introduction

The AdS/CFT correspondence is a powerful tool to study different properties of strongly coupled gauge theory in terms of dual (super) gravity theory in AdS space. In low frequency limit the boundary field theory can be described by hydrodynamics. In this limit different transport coefficients like shear viscosity, diffusion constant, thermal and electrical conductivity of strongly coupled boundary fluid have been computed in the context of AdS/CFT (see [76] - [115]).

In [76], the authors evaluated the shear viscosity coefficient of boundary fluid using Kubo formula. This formula relates the shear viscosity to two point function of energy momentum tensor in zero frequency limit. On the other hand from field operator correspondence of the AdS/CFT conjecture we know that energy momentum tensor of boundary field theory is sourced by bulk graviton excitations. Therefore in the context of AdS/CFT, to calculate thermal two point correlation function of field theory energy momentum tensor we need to add small perturbations to the bulk metric. In [76], the authors considered graviton excitations polarized parallel to the black brane (*i.e.* xy components are turned on) and moving transverse to it. When one sends the gravitons to the brane, there is a probability that it will be absorbed by the brane. They calculated the absorption coefficient and showed that it is related to two point functions of energy momentum tensor of boundary fluid.

To calculate the absorption coefficients, one needs to solve the wave equation for transverse gravitons. In presence of any higher derivative terms in the bulk action the solution may be technically difficult in general [116, 117, 125]. Recently there is a proposal that the shear viscosity of strongly coupled boundary gauge theory plasma is related to the effective coupling of graviton calculated at the black hole horizon [118, 119]. In [120], using membrane paradigm, the authors have confirmed that at the level of linear response the low frequency limit of strongly coupled boundary field theory at finite temperature is determined by the horizon geometry of its gravity dual. They have proved that generic boundary theory transport coefficients can be expressed in terms of geometric quantities evaluated at the horizon[1]. In particular, they have found that the shear viscosity coefficient is given by transverse graviton coupling computed at the horizon. The novelty of this result is that one does not need to solve the equation of motion for the graviton to calculate the thermal Green function. From graviton's action one can easily read off the coupling constant and hence determine the shear viscosity coefficient.

To find the effective coupling of gravitons one has to find the general action. This can be achieved in the following way. Consider the Einstein-Hilbert action with negative cosmological constant

$$I = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\,(R+12)\ . \tag{6.1.1}$$

The equation of motion obtained from this action has a black hole solution. We denote this background solution by $g^{(0)}_{\mu\nu}$. Now we consider fluctuation about

[1]See [121] also.

this spacetime in xy (for example) direction[2],

$$g_{xy} = g_{xy}^{(0)} + \epsilon\, h_{xy}(r,x) = g_{xy}^{(0)}\,(1 + \epsilon\, \Phi(r,x)) \ . \tag{6.1.2}$$

Then substituting the metric with fluctuation in the action (6.1.1) and keeping terms up to order ϵ^2 we get the action for graviton. The form of this action is,

$$S \sim \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left(a(r)\phi'(r,k)\phi'(r,-k) + b(r)\phi(r,k)\phi(r,-k) \right) \tag{6.1.3}$$

where,

$$\phi(r,k) = \int \frac{d^4x}{(2\pi)^4} e^{-ik.x} \Phi(r,x) \, , \tag{6.1.4}$$

$k = \{-\omega, \vec{k}\}$ and $'\,'\,'$ denotes derivative with respect to r. The effective coupling is related to the coefficient of ϕ'^2 *i.e.* a (we have reviewed this calculation in section 6.2).

This gives the correct viscosity coefficient for the Einstein-Hilbert gravity. But it is not obvious how to generalize this approach for higher derivative case. The proof given in [120] was based on the canonical form (6.1.3) of graviton's action. In presence of arbitrary higher derivative terms in the bulk, the general action for the perturbation h_{xy} does not have the above form (6.1.3). Rather it will have more than two derivative (with respect to r) terms. [120, 122] have considered Gauss-Bonnet term in the bulk action. In general, presence of $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ terms in the bulk result terms like ϕ''^2 and $\phi'\phi''$ in the action for h_{xy}. For Gauss-Bonnet combinations these terms get canceled and the general action still has the form (6.1.3).

In this paper we have considered generic higher derivatives terms in the bulk Lagrangian. We have given a procedure to construct an effective action S_{eff} for transverse graviton of the form (6.1.3) in presence of any higher derivative terms in the bulk. The details of the construction is given in section (6.3). Our construction ensures that in low frequency limit, the calculations of retarded Green function (imaginary part) using either effective action or original action are same. Therefore following the similar argument given in [120], we can relate the shear viscosity coefficient of the boundary fluid with the hori-

[2]Notations: x denotes the boundary coordinates. $x = \{t, \vec{x}\}$.

zon value of the effective coupling obtained from S_{eff} (section 6.4). In section (6.5) we have also discussed how membrane fluid captures the properties of boundary fluid in low frequency limit in generic higher derivative gravity. We have checked our procedure for two cases:

- General four derivative terms, (section (6.6))
- $Weyl^4$ term which arises in type II string theory (section (6.7)).

In both examples we get exact agreement between our results and the results that already exist in the literature [116, 117, 125]. Hence we conclude that:

The shear viscosity coefficient of the boundary fluid is given by the horizon value of the effective coupling of transverse graviton obtained from its effective action in presence of arbitrary higher derivative terms in the bulk.

6.2 Shear Viscosity from Effective Coupling

In this section we briefly review how to calculate the shear viscosity coefficient of the boundary fluid from the effective coupling constant of transverse graviton in Einstein-Hilbert gravity.

We first fix the background spacetime. We start with the following Einstein-Hilbert action in AdS space.

$$I = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\,(R+12)\ . \tag{6.2.5}$$

Here we have taken the radius of the AdS space 1. The background spacetime is given by the following metric[3]

$$ds^2 = -h_t(r)dt^2 + \frac{dr^2}{h_r(r)} + \frac{1}{r}d\vec{x}^2 \tag{6.2.6}$$

where,

$$h_t(r) = \frac{1-r^2}{r}\ . \tag{6.2.7}$$

[3]We are working in a coordinate frame where asymptotic boundary is at $r > 0$.

and

$$h_r(r) = 4r^2(1-r^2)\ . \tag{6.2.8}$$

The black hole has horizon at $r_0 = 1$ and the temperature of this black hole is given by,

$$T = \frac{1}{\pi}\ . \tag{6.2.9}$$

We consider the following metric perturbation,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r,x) = g_{xy}^{(0)}(1+\epsilon\Phi(r,x)) \tag{6.2.10}$$

where ϵ is an order counting parameter. We consider terms up to order ϵ^2 in the action of $\Phi(r,x)$. The action (in momentum space) is given by,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d\omega d^3\vec{k}}{(2\pi)^4}dr\Big[\mathcal{A}_{1,1}(r)\phi'(r,-k)\phi'(r,k) \\ &\qquad +\mathcal{A}_{1,0}(r,k)\phi(r,-k)\phi'(r,k) + \mathcal{A}_{0.0}(r,k)\phi(r,k)\phi(r,-k)\Big] \end{aligned} \tag{6.2.11}$$

where, $\mathcal{A}_{i,j}(r,k)$ are functions of r and k and $\phi(r,k)$ is given by (6.1.4). Up to some total derivative the action (6.2.11) can be written as[4],

$$S = \frac{1}{16\pi G_5}\int \frac{d\omega d^3\vec{k}}{(2\pi)^4}dr\left(\mathcal{A}_1^{(0)}(r)\phi'(r,-k)\phi'(r,k) + \mathcal{A}_0^{(0)}(r,k)\phi(r,k)\phi(r,-k)\right) \tag{6.2.12}$$

where,

$$\mathcal{A}_1^{(0)}(r) = \frac{r^2-1}{r} \tag{6.2.13}$$

and

$$\mathcal{A}_0^{(0)}(r,k) = \frac{\omega^2}{4r^2(1-r^2)}\ . \tag{6.2.14}$$

This can be viewed as an action for minimally coupled scalar field $\phi(r,k)$ with

[4]Though throughout this paper we have written the four vector k, but in practice we have worked in $\vec{k}\to 0$ limit. In all the expressions we have dropped the terms proportional to $\vec{k}$ or its power.

effective coupling given by,

$$K_{\text{eff}}(r) = \frac{1}{16\pi G_5} \frac{\mathcal{A}^{(0)}_{1(r)}}{\sqrt{-g^{(0)} g^{rr}}}. \tag{6.2.15}$$

Therefore according to [120, 122] the effective coupling K_{eff} calculated at the horizon r_0 gives the shear viscosity coefficient of boundary fluid,

$$\begin{aligned} \eta &= r_0^{-\frac{3}{2}}(-2K_{\text{eff}}(r_0)) \\ &= \frac{1}{16\pi G_5}\,. \end{aligned} \tag{6.2.16}$$

6.3 The Effective Action

Having understood the above procedure to determine the shear viscosity coefficient from the effective coupling of transverse graviton it is tempting to generalize this method for any higher derivative gravity. As we discussed in the introduction, the first problem one faces is that the action for transverse graviton no more has the canonical form (6.2.11). For generic 'n' derivative gravity theory the action can have terms with (and up to) 'n' derivatives of $\Phi(r, x)$. Therefore, from that action it is not very clear how to determine the effective coupling. In this section we try to address this issue.

We construct an effective action which is of form (6.2.12) with different coefficients capturing higher derivative effects. We determine these two coefficients by claiming that the equation of motion for $\phi(r, k)$ coming from these two actions (general action and effective action) are same up to first order in perturbation expansion (in coefficient of higher derivative term). Once we determine the effective action for transverse graviton in canonical form then we can extract the effective coupling from the coefficient of $\phi'(r, k)\phi'(r, -k)$ term in the action. Needless to say, our method is perturbatively correct.

6.3.1 The General Action and Equation of Motion

Let us start with a generic $'n'$ derivative term in the action with coefficient μ. We study this system perturbatively and all our expressions are valid up to order μ. The action is given by,

$$S = \frac{1}{16\pi G_5} \int d^5x \left(R + 12 + \mu\, \mathcal{R}^{(n)} \right) \quad (6.3.17)$$

where, $\mathcal{R}^{(n)}$ is any n derivative Lagrangian. The metric in general is given by (assuming planar symmetry),

$$ds^2 = -(h_t(r) + \mu\, h_t^{(n)}(r))dt^2 + \frac{dr^2}{h_r(r) + \mu\, h_t^{(n)}(r)} + \frac{1}{r}(1 + \mu\, h_s^{(n)}(r))d\vec{x}^2 \quad (6.3.18)$$

where $h_t^{(n)}, h_r^{(n)}$ and $h_s^{(n)}$ are higher derivative corrections to the metric.

Substituting the background metric with fluctuations in the action (6.3.17) (we call it general action or original action) for the scalar field $\phi(r,k)$ we get,

$$S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p,q=0}^{n} \mathcal{A}_{p,q}(r,k)\phi^{(p)}(r,-k)\phi^{(q)}(r,k) \quad (6.3.19)$$

where, $\phi^{(p)}(r,k)$ denotes the p^{th} derivative of the field $\phi(r,k)$ with respect to r and $p+q \leq n$. The coefficients $\mathcal{A}_{p,q}(r,k)$ in general depends on the coupling constant μ. $\mathcal{A}_{p,q}$ with $p+q \geq 3$ are proportional to μ and vanishes in $\mu \to 0$ limit , since the terms $\phi^{(p)}\phi^{(q)}$ with $p+q \geq 3$ appears as an effect of higher derivative terms in the action (6.3.17). Up to some total derivative terms, the general action (6.3.19) can also be written as,

$$\begin{aligned} S &= \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{n/2} \mathcal{A}_p(r,k)\phi^{(p)}(r,-k)\phi^{(p)}(r,k), \qquad n \quad \text{even} \\ &= \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{(n-1)/2} \mathcal{A}_p(r,k)\phi^{(p)}(r,-k)\phi^{(p)}(r,k), \quad n \quad \text{odd}\,. \end{aligned} \quad (6.3.20)$$

The equation of motion for the scalar field $\phi(r,k)$ is given by,

$$\begin{aligned} &\sum_{p=0}^{n/2} \left(-\frac{d}{dr}\right)^p \frac{\partial\mathcal{L}(\{\phi^{(m)}\})}{\partial\phi^{(p)}(r,k)} = 0, \qquad n \quad \text{even} \\ &\sum_{p=0}^{(n-1)/2} \left(-\frac{d}{dr}\right)^p \frac{\partial\mathcal{L}(\{\phi^{(m)}\})}{\partial\phi^{(p)}(r,k)} = 0, \qquad n \quad \text{odd} \end{aligned} \quad (6.3.21)$$

where $\mathcal{L}(\{\phi^{(m)}\})$ is given by

$$\mathcal{L}(\{\phi^{(m)}\}) = \sum_p \mathcal{A}_p(r,k)\phi^{(p)}(r,-k)\phi^{(p)}(r,k) \ . \qquad (6.3.22)$$

We analyze the general action for the scalar field $\phi(r,k)$ and their equation of motion perturbatively and write an effective action for the field $\phi(r,k)$.

The generic form of the equation of motion (varying the general action) upto order μ is given by,

$$\mathcal{A}_0(r,k)\phi(r,k) - \mathcal{A}_1^{'}(r,k)\phi'(r,k) - \mathcal{A}_1(r,k)\phi''(r,k) = \mu\,\hat{\mathcal{F}}(\{\phi^{(p)}\}) + \mathcal{O}(\mu^2) \qquad (6.3.23)$$

where $\hat{\mathcal{F}}(\{\phi^{(p)}\})$ is some linear function of double and higher derivatives of $\phi(r,k)$, coming from two or higher derivative terms in action (6.3.19). The zeroth order ($\mu \to 0$) equation of motion is given by,

$$\mathcal{A}_0^{(0)}(r,k)\phi(r,k) - \mathcal{A}_1^{'(0)}(r,k)\phi'(r,k) - \mathcal{A}_1^{(0)}(r,k)\phi''(r,k) = 0 \qquad (6.3.24)$$

where, $\mathcal{A}_p^{(0)}$ is the value of $\mathcal{A}_p$ at $\mu \to 0$. From this equation we can write $\phi''(r,k)$ in terms of $\phi'(r,k)$ and $\phi(r,k)$ in $\mu \to 0$ limit.

$$\phi''(r,k) = \frac{\mathcal{A}_0^{(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)}\phi(r,k) - \frac{\mathcal{A}_1^{'(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)}\phi'(r,k) \ . \qquad (6.3.25)$$

Then the full equation of motion can be written in the following way,

$$\begin{aligned} &\mathcal{A}_0^{(0)}(r,k)\phi(r,k) - \mathcal{A}_1^{'(0)}(r,k)\phi'(r,k) - \mathcal{A}_1^{(0)}(r,k)\phi''(r,k) \\ &\quad = \mu\,\tilde{\mathcal{F}}(\phi(r,k),\phi'(r,k),\phi''(r,k),...) + \mathcal{O}(\mu^2) \ . \end{aligned} \qquad (6.3.26)$$

Since the right hand side of equation (6.3.26) is proportional to μ, we can replace the $\phi''(r,k)$ and other higher (greater than 2) derivatives of $\phi(r,k)$ by its leading order value (6.3.25). Therefore up to order μ the equation of motion for ϕ is given by,

$$\begin{aligned}
&\mathcal{A}_0^{(0)}(r,k)\phi(r,k) - \mathcal{A}_1^{'(0)}(r,k)\phi'(r,k) - \mathcal{A}_1^{(0)}(r,k)\phi''(r,k) \\
&\quad = \mu\,\mathcal{F}(\phi(r,k),\phi'(r,k)) + \mathcal{O}(\mu^2) \\
&\quad = \mu(\mathcal{F}_1\phi'(r,k) + \mathcal{F}_0\phi(r,k)) + \mathcal{O}(\mu^2) \qquad (6.3.27)
\end{aligned}$$

where $\mathcal{F}_0$ and $\mathcal{F}_1$ are some function of r. This is the perturbative equation of motion for the scalar field $\phi(r,k)$ obtained from the general action (6.3.19).

6.3.2 Strategy to Find The Effective Action

In this subsection we describe the strategy to write an effective action for the field $\phi(r,k)$ which has form (6.2.12) with different functions. The prescription is following.

- **(a)** We demand the equation of motion for $\phi(r,k)$ obtained from the original action and the effective action are same upto order μ. This will fix the coefficients of ϕ'^2 and ϕ^2 terms in effective action.

  Let us start with the following form of the effective action.

$$\begin{aligned}
S_{\text{eff}} &= \frac{1}{16\pi G_5}\int \frac{d\omega d^3\vec{k}}{(2\pi)^4} dr \Big[(\mathcal{A}_1^{(0)}(r,k) + \mu\mathcal{B}_1(r,k))\phi'(r,-k)\phi'(r,k) \\
&\qquad + (\mathcal{A}_0^{(0)}(r,k) + \mu\mathcal{B}_0(r,k))\phi(r,k)\phi(r,-k)\Big]. \qquad (6.3.28)
\end{aligned}$$

  The functions $\mathcal{B}_0$ and $\mathcal{B}_1$ are yet to be determined. We determine these functions by claiming that the equation of motion for the scalar field $\phi(r,k)$ obtained from this effective action is same as (6.3.27) up to order μ. The equation of motion for $\phi(r,k)$ from the effective action is given by,

$$\begin{aligned}
\mathcal{A}_0^{(0)}(r,k)\phi(r,k) &- \mathcal{A}_1^{'(0)}(r,k)\phi'(r,k) - \mathcal{A}_1^{(0)}(r,k)\phi''(r,k) \\
&= \mu\left(\mathcal{B}_1'(r,k) - \frac{\mathcal{A}_1^{'(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)}\mathcal{B}_1(r,k)\right)\phi'(r,k) \\
&\quad + \mu\left(\mathcal{B}_1(r,k)\frac{\mathcal{A}_0^{(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)} - \mathcal{B}_0(r,k)\right)\phi(r,k) + \mathcal{O}(\mu^2) \\
&. \qquad (6.3.29)
\end{aligned}$$

Therefore comparing with (6.3.27) we get,

$$\mathcal{B}_1'(r,k) - \frac{\mathcal{A}_1'^{(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)}\mathcal{B}_1(r,k) - \mathcal{F}_1(r,k) = 0 \qquad (6.3.30)$$

and

$$\mathcal{B}_0(r,k) = \mathcal{B}_1(r,k)\frac{\mathcal{A}_0^{(0)}(r,k)}{\mathcal{A}_1^{(0)}(r,k)} - \mathcal{F}_0(r,k)\,. \qquad (6.3.31)$$

The solutions are given by,

$$\begin{aligned} \mathcal{B}_1(r,k) &= \mathcal{A}_1^{(0)}(r,k)\int dr\frac{\mathcal{F}_1(r,k)}{\mathcal{A}_1^{(0)}(r,k)} + \kappa\mathcal{A}_1^{(0)}(r,k) \\ &= \tilde{\mathcal{B}}_1(r,k) + \kappa\mathcal{A}_1^{(0)}(r,k) \end{aligned} \qquad (6.3.32)$$

and

$$\mathcal{B}_0 = \tilde{\mathcal{B}}_0(r,k) + \kappa\mathcal{A}_0^{(0)} \qquad (6.3.33)$$

for some constant κ. We need to fix this constant.

- **(b)** Condition **(a)** can not fix the overall normalization factor of the effective action. In particular we can multiply it by $(1+\mu\Gamma)$ (for some constant Γ) and still get the same equation of motion. Considering this normalization, the effective action is given by,

$$\begin{aligned} S_{\text{eff}} &= \frac{1+\mu\,\Gamma}{16\pi G_5}\int\frac{d\omega d^3\vec{k}}{(2\pi)^4}dr\Big[(\mathcal{A}_1^{(0)}(r,k)+\mu\mathcal{B}_1(r,k))\phi'(r,-k)\phi'(r,k) \\ &\quad +(\mathcal{A}_0^{(0)}(r,k)+\mu\mathcal{B}_0(r,k))\phi(r,k)\phi(r,-k)\Big]\,. \end{aligned} \qquad (6.3.34)$$

Substituting the values of $\mathcal{B}$'s (6.3.32) and (6.3.33) we get,

$$\begin{aligned} S_{\text{eff}} &= (1+\mu(\Gamma+\kappa))S^{(0)} + \mu\int dr\Big[\tilde{\mathcal{B}}_1(r,k)\phi'(r,-k)\phi'(r,k) \\ &\quad +\tilde{\mathcal{B}}_0(r,k)\phi(r,-k)\phi(r,k)\Big] \end{aligned} \qquad (6.3.35)$$

where $S^{(0)}$ is the effective action at $\mu\to 0$ limit. This implies that the integration constant κ can be absorbed in the overall normalization constant Γ. Henceforth we will denote this combination as Γ.

Our prescription is to take Γ to be **zero** from the following observation.

– The shear viscosity coefficient of boundary fluid is related to the imaginary part of retarded Green function in low frequency limit. The retarded Green function $G^R_{xy,xy}(k)$ is defined in the following way. The on-shell action for graviton can be written as a surface term,

$$S \sim \int \frac{d^4k}{(2\pi)^4} \phi_0(k)\, \mathcal{G}_{xy,xy}(k,r)\, \phi_0(-k) \qquad (6.3.36)$$

where $\phi_0(k)$ is the boundary value of $\phi(r,k)$ and $G^R_{xy,xy}$ is given by,

$$G^R_{xy,xy}(k) = \lim_{r\to 0} 2\mathcal{G}_{xy,xy}(k,r) \qquad (6.3.37)$$

and shear viscosity coefficient is given by[5],

$$\eta = \lim_{\omega\to 0}\left[\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(k)\right] \qquad (computed\ \ on-shell)\,. \qquad (6.3.38)$$

– Now it turns out that the imaginary part of this retarded Green function obtained from the original action and effective action are same upto the normalization constant Γ in presence of generic higher derivative terms in the bulk action. Therefore it is quite natural to take Γ to be *zero* as it ensures that starting from the effective action also one can get same shear viscosity using Kubo machinery. To show that the above statement is true we do not need to know the full solution for ϕ, in other words to find the difference between the two Green functions one does not need to calculate the Green functions explicitly. Assuming the following general form of solution for ϕ

$$\phi \sim (1-r^2)^{-i\omega\beta}\,(1+i\omega\beta\mu\xi(r)) \qquad (6.3.39)$$

it can be shown generically. In appendix B we have given the proof.

– Because of the canonical form of the effective action, it follows from the argument in [120] and the statement above, that the shear vis-

[5] To calculate this number one has to know the exact solution, *i.e.* the form of ξ and the value of β in (6.3.39).

cosity coefficient of boundary fluid is given by the horizon value of the effective coupling obtained from the effective action in presence of any higher derivative terms in the bulk action. We discuss elaborately on this point in section (6.4).

- **(c)** After getting the effective action for $\phi(r,k)$, the effective coupling is given by,

$$K_{\text{eff}}(r) = \frac{1}{16\pi G_5} \frac{\mathcal{A}_1^{(0)}(r,k) + \mu \mathcal{B}_1(r,k)}{\sqrt{-g} g^{rr}} \tag{6.3.40}$$

where g^{rr} is the $'rr'$ component of the inverse perturbed metric and $\sqrt{-g}$ is the determinant of the perturbed metric. Hence the shear viscosity coefficient is given by,

$$\eta = r_0^{-\frac{3}{2}} (-2K_{\text{eff}}(r = r_0)) \tag{6.3.41}$$

where r_0 is the corrected horizon radius.

To summaries, we have obtained a well defined procedure to find the correction (up to order μ) to the coefficient of shear viscosity of the boundary fluid in presence of general higher derivative terms in the action.

6.4 Flow from Boundary to Horizon

Following [120], let us define the following linear response function

$$\bar{\chi}(r,k) = \frac{\Pi(r,k)}{i\omega\phi(r,k)} \tag{6.4.42}$$

where $\Pi(r,k)$ is conjugate momentum of the scalar field ϕ (with respect to a foliation in the r direction),

$$\begin{aligned} \Pi(r,k) &= \left(\mathcal{A}_1^{(0)}(r,k) + \mu \mathcal{B}_1(r,k)\right) \phi'(r,-k) \\ &= \tilde{K}_{\text{eff}}(r) \sqrt{-g^{(0)}} g^{(0)^{rr}} \partial_r \phi \end{aligned} \tag{6.4.43}$$

where $\tilde{K}_{\text{eff}}(r) = 16\pi G_5 K_{\text{eff}}(r)$. Now we will show, using the equation of motion, that the function $\Pi(r,k)$ and the combination $\omega\phi(r,k)$ is independent of the radial coordinate r in $k \to 0$ limit. The equation of motion is given by,

$$\frac{d}{dr}\left[\left(\mathcal{A}_1^{(0)}(r,k) + \mu\mathcal{B}_1(r,k)\right)\phi'(r,k)\right] = \left(\mathcal{A}_0^{(0)}(r,k) + \mu\mathcal{B}_0(r,k)\right)\phi(r,k)$$
$$\frac{d}{dr}\left[\Pi(r,k)\right] = \left(\mathcal{A}_0^{(0)}(r,k) + \mu\mathcal{B}_0(r,k)\right)\phi(r,k) \quad (6.4.44)$$

Since $\mathcal{A}_0^{(0)} \sim \omega^2$, therefore it follows from (6.4.44) and (6.4.43) that, in $\mu \to 0$ limit $\Pi(r,k)$ and $\omega\phi(r,k)$ are independent of r. But this is true even in $\mu \neq 0$ case. To understand this we note that, function $\mathcal{A}_0$ in (6.3.20) is proportional to ω^2 in general[6]. Therefore it follows from (6.3.25), (6.3.27) and (6.3.31) that $\mathcal{B}_0$ is also proportional to ω^2. Hence, in presence of higher derivative terms also it follows from (6.4.43) and (6.4.44) that the function $\Pi(r,k)$ and $\omega\phi(r,k)$ are independent of radial direction r in low frequency limit.

Therefore this response function calculated at the asymptotic boundary and at the horizon gives the same result and is equal to the shear viscosity coefficient. One can calculate the function $\bar{\chi}$ and it turns out that,

$$\begin{aligned} \bar{\chi}(r=0, k\to 0) &= \frac{\mathrm{Im}G^{R\mathrm{eff}}_{xy,xy}}{i\omega}, \\ \bar{\chi}(r=r_0, k\to 0) &= -\frac{r_0^{-3/2}}{8\pi G_5}\left.\frac{\mathcal{A}_1^{(0)}(r,k)+\mu\mathcal{B}_1(r,k)}{\sqrt{-g}g^{rr}}\right|_{r_0} = r_0^{-\frac{3}{2}}\left(-2K_{\mathrm{eff}}(r_0)\right). \end{aligned} \quad (6.4.45)$$

Thus, shear viscosity coefficient of boundary fluid is related to horizon value of graviton's effective coupling obtained from the effective action.

6.5 Membrane Fluid in Higher Derivative Gravity

The **UV/IR** connection tells us that the boundary field theory physics in low frequency limit should be governed by the near horizon geometry of its gravity dual. In [120], the authors have established a connection between horizon membrane fluid and boundary fluid in linear response approximation. They

[6] In general when we write action (6.3.20) action (6.3.19) we get some terms like $\omega^2\phi^2 + Z(r)\phi^2$. The function $Z(r)$ is zero when background equation of motion is satisfied. We have explicitly checked this for two, four and eight derivative case.

considered a mass less scalar field (with action given in (6.2.12)) outside the horizon and studied the response of the membrane fluid to this bulk scalar field. They defined a *membrane charge* Π_{mb} which is equal to the conjugate momentum of the scalar field ϕ (with respect to a foliation in the r direction) at the horizon. Imposing regularity condition on the scalar field at the horizon they interpreted the membrane charge Π_{mb} as a response of the horizon fluid to the scalar field. Considering the scalar field ϕ to be bulk graviton excitation (h_x^y), Π_{mb} gives the shear viscosity of the membrane (horizon) fluid which is also equal to horizon value of the effective coupling of graviton. In this way, they proved that the shear viscosity of boundary fluid is related to that of membrane fluid.

In higher derivative gravity, since the canonical form of the action (6.2.12) breaks down, it is not very obvious how to define the membrane charge Π_{mb}. Instead of the original action if we consider the effective action (6.3.28) for graviton then it is possible to write the membrane action perturbatively and define the membrane charge (Π_{mb}) in higher derivative gravity. As if the membrane fluid is sensitive to the effective action S_{eff} in higher derivative gravity.

Following [120] we can write the membrane action and charge in the following way (in momentum space)

$$S_{\text{mb}} = \int_{\Sigma} \frac{d^4k}{(2\pi)^4} \sqrt{-\sigma} \left(\frac{\Pi(r_0, k)}{\sqrt{-\sigma}} \phi(r_0, -k) \right) \tag{6.5.46}$$

where $\sigma_{\mu\nu}$ is the induced metric on the membrane and $\Pi(r, k)$ is given by (6.4.43) and the membrane charge is given by,

$$\Pi_{\text{mb}} = \frac{\Pi(r_0, k)}{\sqrt{-\sigma}} = -\tilde{K}_{\text{eff}}(r_0) \sqrt{g^{(0)rr}} \partial_r \phi(r, k) \big|_{r_0} . \tag{6.5.47}$$

Imposing the in-falling wave boundary condition on ϕ, it can be shown that the membrane charge Π_{mb} is the response of the horizon fluid to the bulk graviton excitation and the membrane fluid transport coefficient is given by,

$$\eta_{\text{mb}} = \tilde{K}_{\text{eff}}(r_0) . \tag{6.5.48}$$

Hence, we see that even in higher derivative gravity the shear viscosity coefficient of boundary fluid is captured by the membrane fluid.

6.6 Four Derivative Lagrangian

In this section we apply our effective action approach to calculate the correction to the shear viscosity in presence of general four derivative terms in the action. The four derivative bulk action we consider is of the following form

$$S = \frac{1}{16\pi G_5}\int d^5x \left[R + 12 + \mu\left(c_1 R^2 + c_2 R_{ab}R^{ab} + c_3 R_{abcd}R^{abcd}\right)\right] \tag{6.6.49}$$

with constant c_1, c_2 and c_3. The background metric is given by,

$$ds^2 = -\frac{f(r)}{r}dt^2 + \frac{dr^2}{4r^2 f(r)} + \frac{1}{r}d\vec{x}^2 \tag{6.6.50}$$

where,

$$f(r) = 1 - r^2 + \frac{\mu}{3}(4(5c_1 + c_2) + 2c_3) + 2\mu c_3 r^4 \ . \tag{6.6.51}$$

The position of the horizon is given by,

$$f(r_0) = 0 \tag{6.6.52}$$

which implies that,

$$r_0 = 1 + \frac{2}{3}(5c_1 + c_2 + 2c_3)\mu + \mathcal{O}(\mu^2) \ . \tag{6.6.53}$$

The temperature of this black hole is given by,

$$T = \frac{1}{\pi} + \frac{(5c_1 + c_2 - 7c_3)\mu}{3\pi} + O\left(\mu^2\right) \ . \tag{6.6.54}$$

In this coordinate frame the boundary metric is given by,

$$ds_4^2 = (-f(0)dt^2 + d\vec{x}^2) \tag{6.6.55}$$

which is not Minkowskian. Therefore we rescale our time coordinate to make the boundary metric Minkowskian. We replace,

$$t \to \frac{t}{\sqrt{f(0)}} \tag{6.6.56}$$

in the metric (6.6.50). The rescaled metric is,

$$ds^2 = -\frac{f(r)}{f(0)r}dt^2 + \frac{dr^2}{4r^2 f(r)} + \frac{1}{r}d\vec{x}^2 \,. \tag{6.6.57}$$

This is our background metric and we consider fluctuation around this.

6.6.1 The General Action

In this theory, the general action for the scalar field $\phi(r,k)$ is given by,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Big[A_1^{GB}(r,k)\phi(r,k)\phi(r,-k) + A_2^{GB}(r,k)\phi'(r,k)\phi'(r,-k) \\ &\quad + A_3^{GB}(r,k)\phi''(r,k)\phi''(r,-k) + A_4^{GB}(r,k)\phi(r,k)\phi'(r,-k) \\ &\quad + A_5^{GB}(r,k)\phi(r,k)\phi''(r,-k) + A_6^{GB}(r,k)\phi'(r,k)\phi''(r,-k)\Big] \end{aligned} \tag{6.6.58}$$

where the expressions for A_i^{GB}s are given in appendix C. Up to some total derivative terms this action can be written as,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Big[\mathcal{A}_0^{GB}\phi(r,k)\phi(r,-k) + \mathcal{A}_1^{GB}\phi'(r,k)\phi'(r,-k) \\ &\quad + \mathcal{A}_2^{GB}\phi''(r,k)\phi''(r,-k)\Big] \end{aligned} \tag{6.6.59}$$

where,

$$\begin{aligned} \mathcal{A}_0^{GB} &= A_1^{GB}(r,k) - \frac{A_4^{\prime GB}(r,k)}{2} + \frac{A_5^{\prime\prime GB}(r,k)}{2} \\ \mathcal{A}_1^{GB} &= A_2^{GB}(r,k) - A_5^{GB}(r,k) - \frac{A_6^{\prime GB}(r,k)}{2} \\ \mathcal{A}_2^{GB} &= A_3^{GB}(r,k)\,. \end{aligned} \tag{6.6.60}$$

6.6.2 The Effective Action and Shear Viscosity

Following the general discussion of section (6.3) we write the effective action for the scalar field,

$$S_{\text{eff}}^{GB} = \frac{(1+\Gamma\mu)}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\Big[(\mathcal{A}_1^{(0)}(r,k)+\mu\mathcal{B}_1^{GB}(r,k))\phi'(r,-k)\phi'(r,k) + (\mathcal{A}_0^{(0)}(r,k)+\mu\mathcal{B}_0^{GB}(r,k))\phi(r,k)\phi(r,-k)\Big] . \quad (6.6.61)$$

To evaluate the functions $\mathcal{B}_1^{GB}$ and $\mathcal{B}_0^{GB}$ and to fix the normalization constant Γ, we follow the strategy given in section (6.3.2). Comparing the equation of motion for $\phi(r,k)$ from two actions we get the function $\mathcal{B}_1^{GB}$ and $\mathcal{B}_0^{GB}$ of the following form,

$$\mathcal{B}_0^{GB} = \frac{\omega^2}{12r^2(1-r^2)^2}\Big(10(11r^2-13)c_1+(22r^2-26)c_2+(11-25r^2+6r^4)c_3\Big)$$
$$\mathcal{B}_1^{GB} = \frac{1}{3r}\Big((110-130r^2)c_1+(22-26r^2)c_2-(13-23r^2+18r^4)c_3\Big) . \quad (6.6.62)$$

The normalization constant $\Gamma = 0$ (appendix B).

Now we can calculate the effective coupling using the formula (6.3.40). It turns out to be,

$$K_{\text{eff}}(r) = \frac{1}{16\pi G_5}\left(-\frac{1}{2}+\left(4(5c_1+c_2)-2(1-r^2)c_3\right)\mu\right) . \quad (6.6.63)$$

Therefore the shear viscosity is given by,

$$\begin{aligned}\eta &= \frac{1}{r_0^{3/2}}\left(-2K_{\text{eff}}(r_0)\right)\\ &= \frac{1}{16\pi G_5}\frac{1}{r_0^{3/2}}\left(1-8(5c_1+c_2)\mu\right)\\ &= \frac{1}{16\pi G_5}\left(1-9\mu\,(5c_1+c_2)-2\mu\, c_3\right) . \end{aligned} \quad (6.6.64)$$

This result is in agreement with [116, 117, 126].

6.7 String Theory Correction to Shear Viscosity

In this section we apply the effective action approach for eight derivative terms in the Lagrangian. We consider the well known $Weyl^4$ term. This term appears in type II string theory. The five dimensional bulk action is given by,

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + 12 + \mu W^{(4)} \right) \tag{6.7.65}$$

where,

$$W^{(4)} = C^{hmnk} C_{pmnq} C_h{}^{rsp} C^q{}_{rsk} + \frac{1}{2} C^{hkmn} C_{pqmn} C_h{}^{rsp} C^q{}_{rsk} \tag{6.7.66}$$

and the weyl tensors C_{abcd} are given by,

$$C_{abcd} = R_{abcd} + \frac{1}{3}(g_{ad}R_{cb} + g_{bc}R_{ad} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{12}(g_{ac}g_{bd} - g_{ad}g_{cb})R \,. \tag{6.7.67}$$

The background metric is given by [127, 130],

$$\begin{aligned} ds^2 &= -\frac{(1-r^2)}{r}\left(1 + 45\mu r^6 - 75\mu r^4 - 75\mu r^2\right) dt^2 \\ &+ \frac{1}{4(1-r^2)r^2}\left(1 - 285\mu r^6 + 75\mu r^4 + 75\mu r^2\right) dr^2 + \frac{1}{r} d\vec{x}^2 \,. \end{aligned} \tag{6.7.68}$$

The temperature of this black hole is given by,

$$T = \frac{1}{\pi}\left(1 + 15\mu\right) \,. \tag{6.7.69}$$

The horizon is located at $r_0 = 1$.

6.7.1 The General Action

Putting the perturbed metric in (6.7.65) we get the general action for the scalar field $\phi(r,k)$,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Big[A_1^W(r,k)\phi(r,k)\phi(r,-k) + A_2^W(r,k)\phi'(r,k)\phi'(r,-k) \\ &\quad + A_3^W(r,k)\phi''(r,k)\phi''(r,-k) + A_4^W(r,k)\phi(r,k)\phi'(r,-k) \\ &\quad + A_5^W(r,k)\phi(r,k)\phi''(r,-k) + A_6^W(r,k)\phi'(r,k)\phi''(r,-k)\Big] . \end{aligned} \tag{6.7.70}$$

The coefficients A_i^Ws are given in appendix (D). Like four derivative case, up to some total derivative terms this action can be written as,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Big[\mathcal{A}_0^W\phi(r,k)\phi(r,-k) + \mathcal{A}_1^W\phi'(r,k)\phi'(r,-k) \\ &\quad + \mathcal{A}_2^W\phi''(r,k)\phi''(r,-k)\Big] \end{aligned} \tag{6.7.71}$$

where,

$$\begin{aligned} \mathcal{A}_0^W &= A_1^W(r,k) - \frac{A_4'^W(r,k)}{2} + \frac{A_5''^W(r,k)}{2} \\ \mathcal{A}_1^W &= A_2^W(r,k) - A_5^W(r,k) - \frac{A_6'^W(r,k)}{2} \\ \mathcal{A}_2^W &= A_3^W(r,k) . \end{aligned} \tag{6.7.72}$$

6.7.2 The Effective Action and Shear Viscosity

We write the effective action for the scalar field in the following way,

$$\begin{aligned} S_{\text{eff}}^W &= \frac{(1+\Gamma\mu)}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\Big[(\mathcal{A}_1^{(0)}(r,k) + \mu\mathcal{B}_1^W(r,k))\phi'(r,-k)\phi'(r,k) \\ &\quad + (\mathcal{A}_0^{(0)}(r,k) + \mu\mathcal{B}_0^W(r,k))\phi(r,k)\phi(r,-k)\Big] . \end{aligned} \tag{6.7.73}$$

The functions $\mathcal{B}_0^W$ and $\mathcal{B}_1^W$ are given by,

$$\mathcal{B}_0^W(r,k) = -\frac{\omega^2\left(663r^6 - 573r^4 + 75r^2\right)}{4r^2\left(r^2-1\right)} \tag{6.7.74}$$

$$\mathcal{B}_1^W(r,k) = \frac{\left(r^2-1\right)\left(129r^6 + 141r^4 - 75r^2\right)}{r} \,. \tag{6.7.75}$$

The normalization constant $\Gamma = 0$ (Appendix B).

The effective coupling constant is given by (6.3.40),

$$\begin{aligned} K_{\text{eff}}(r) &= \frac{1}{16\pi G_5}\frac{\mathcal{A}_1^{(0)}(r,k) + \mu\,\mathcal{B}_1^W(r,k)}{\sqrt{-g}g^{rr}} \\ &= \frac{1}{16\pi G_5}\left(-\frac{1}{2}\left(1 + 36\mu\, r^4(6-r^2)\right)\right) \,. \end{aligned} \tag{6.7.76}$$

Therefore the shear viscosity is given by,

$$\begin{aligned} \eta &= r_0^{-\frac{3}{2}}(-2K_{\text{eff}}(r_0)) \\ &= \frac{1}{16\pi G_5}\left(1 + 180\,\mu\right), \quad (r_0 = 1) \end{aligned} \tag{6.7.77}$$

and shear viscosity to entropy density ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + 120\,\mu\right) \tag{6.7.78}$$

where entropy density s is given by [127, 130],

$$s = \frac{1}{4G_5}\left(1 + 60\,\mu\right) \,. \tag{6.7.79}$$

These results agree with the one in the literature.

6.8 Discussion

We have found a procedure to construct an effective action for transverse graviton in canonical form in presence of any higher derivative terms in bulk and showed that the horizon value of the effective coupling obtained from the effective action gives the shear viscosity coefficient of boundary fluid. Our results are valid upto first order in μ. We discussed two non trivial examples to check the method. We have considered four derivative and eight derivative ($Weyl^4$) Lagrangian and calculated the correction to the shear viscosity using our method. We found complete agreement between our result and the results obtained using other methods.

Since the equation of motion for scalar field $\phi(r,k)$ obtained from effective and original actions are same, these two actions should be related by some field re-definition. If one finds such field re-definition then the normalization of the effective action will be fixed automatically.

In [118] the authors have proposed a formula for shear viscosity for generalized higher derivative gravity in terms of some geometric quantity evaluated at the event horizon (like Wald's formula for entropy). Though their proposal gives correct results for Einstein-Hilbert and Gauss-Bonnet action but unfortunately we are unable to get the correct result for $Weyl^4$ term. We find the shear viscosity coefficient for $Weyl^4$ term (using their proposal)

$$\eta = \frac{1}{16\pi G_5}\left(1+20\mu\right) \tag{6.8.80}$$

which implies,

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1-40\mu\right) \ . \tag{6.8.81}$$

In this paper we have concentrated on a particular transport coefficient, namely the shear viscosity coefficient. But the other transport coefficients like electrical and thermal conductivity of boundary fluid can also be captured in terms of membrane fluid. It would also be interesting to study these other transport coefficients in the context of higher derivative gravity.

Chapter 7

Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity

7.1 Introduction and Summary

One of the current interests, in the context of AdS/CFT, is to investigate different properties of quark-gluon plasma (**QGP**) created at the Relativistic Heavy Ion Collider (**RHIC**). The temperature of the gas of quarks and gluons produced at RHIC is approximately $170 MeV$ which is very close to the confinement temperature of QCD. Therefore, at this high temperature they are not in the weakly coupled regime of QCD. In fact near the transition temperature the gas of quarks and gluons belongs to the non-perturbative realm of QCD, where one can not apply the result of perturbative QFT to study their properties. Different kinetic coefficients of this strongly coupled plasma is not possible to calculate with the usual set up of perturbative QCD. The AdS/CFT correspondence [11, 12, 16], at this point, appears as a technically powerful tool to deal with strongly coupled (*conformal*) field theory in terms of weakly coupled (super)-gravity theory in **AdS** space. The AdS/CFT can be an approximate representation of QCD only at high enough temperature since QCD does not have any conformal invariance (β function is not *zero*). However, we assume that the QCD plasma is well described by some conformal field theory which has a gravity dual.

The first success in this direction came from the holographic calculation of shear viscosity coefficient of conformal gauge theory plasma in the context of AdS/CFT [76]. Other transport coefficients of dual gauge theory have also been calculated in the AdS/CFT framework. The literatures are listed in the previous chapter.. In this paper we will concentrate on an interesting conformally invariant measurable parameter of gauge theory plasma, namely, shear viscosity to entropy density ratio ($\frac{\eta}{s}$). The primary motivation for this particular ratio is following. A large class of gauge theories with gravity dual have $\frac{\eta}{s} = \frac{1}{4\pi}$ which is in a good agreement with RHIC data.

In [78] Kovtun, Son and Starinets have conjectured that the ratio $\frac{\eta}{s}$ has a lower bound (**KSS** bound)

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \tag{7.1.1}$$

for all relativistic quantum field theories at finite temperature and the inequality is saturated by theories with gravity dual *i.e.* without any higher derivative corrections. The leading α' correction coming from type II string theory is R^4 term. Presence of R^4 term in the action increases the value of $\frac{\eta}{s}$ beyond $\frac{1}{4\pi}$ [125]. But the story is different when one considers four derivative terms in the bulk action. These terms appear in Heterotic string theory. It has been shown in [116, 117] that four derivative terms actually decreases the value of $\frac{\eta}{s}$ bellow the lower bound. In [116], authors proposed an example of string theory model where the conjectured bound is violated.

An explicit and more detailed investigation on violation of **KSS** bound has been studied in [99] in the context of four derivative gravity. A generic four derivative action can have three terms : $\mathbf{Riemann^2}$, $\mathbf{Ricci^2}$ and $\mathbf{R^2}$($\mathbf{R}$ is Ricci scalar). Second and third term can be removed by field re-definition. Therefore we are left with two independent parameters: coefficients of $\mathbf{Riemann^2}$ and (dimension less) radius of AdS space. [99] found relations between these two parameters in gravity side and two parameters in the boundary theory, namely the central charges c and a. Hence $\frac{\eta}{s}$ can be expressed in terms of these two central charges. Therefore they argued that the violation of **KSS** bound depends on these two central charges of boundary conformal field theory. First of all the central charges should satisfy two conditions: $c \sim a \gg 1$ and $|c - a|/c \ll 1$ and then the bound is violated when $c - a > 0$.

Though it is possible to determine these two parameters in the bulk action and hence $\frac{\eta}{s}$ in terms of two central charges of boundary theory in four derivative case but in a generic higher derivative gravity it is not obvious how to express $\frac{\eta}{s}$ in terms of independent boundary parameters. For example, in this paper we consider generic six derivative terms in bulk. These six derivative terms do not appear in any super-string (type IIA or IIB) or heterotic theory but they can arise in bosonic string theory [100]. Therefore it is quite interesting to study the effects of these terms on the hydrodynamic behavior of boundary gauge theory plasma, in particular on the ratio $\frac{\eta}{s}$. Needless to mention, the gauge theory plasma is not super-symmetric in this case. There can be total ten possible six derivative terms with different coefficients in bulk Lagrangian. We call those coefficients (or terms) **"ambiguous"** which can be removed from the effective action by some field re-definition and other coefficients (or terms) which can not be removed by any field re-definition we refer them **"unambiguous"**. It is possible to show that among ten different terms eight of them can be removed after a suitable field re-definition [101]. Therefore the bulk theory has two unambiguous (six derivative) coefficients (we denote them by α_1 and α_2). If we assume that the effective bulk theory has a dual field theory description then different parameters of boundary conformal field theory, which capture its aggregate properties, should be able to fix the unambiguous couplings of dual gravity theory. In other words, all the unambiguous coefficients of bulk theory can be expressed in terms of physical boundary parameters. For example in [102] authors found that a combination of α_1 and α_2 (namely $\alpha_1 + \alpha_2/2$) is related to a coefficient (we denote it by τ_4) in field theory which appears in correlation of energy one point function (three point function of energy momentum tensor). We discussed about this in brief details in section [7.6]. Similarly the other unambiguous coefficient(s) (α_1 or α_2 or their combination) of six derivative terms can also be fixed in terms of other boundary parameters[1]. Therefore any measurable quantities of boundary theory, for example shear viscosity to entropy density ratio, when calculated holograph-

[1]For example, it can appear in four point correlation function of energy momentum tensor [101]. However its expression in terms of boundary physical parameters is yet to compute.

ically should be expressed in terms of unambiguous coefficients in the bulk theory or boundary parameters.

We calculate the ratio $\frac{\eta}{s}$ for generic six derivative terms. It turns out that the ratio depends on two ambiguous coefficients (we call them α_3 and α_4). In section [7.2] we have discussed these in details. The *apparent* dependence on ambiguous coefficients in physical quantities is an artifact of our choice of starting Lagrangian. One could start with a Lagrangian where all the ambiguous coefficients are set to zero. In that case, shear viscosity coefficient, entropy density and their ratio would be independent of these ambiguous coefficients. However, for being more explicit we start with the most generic Lagrangian and find that the physical quantities like η, s and $\frac{\eta}{s}$ depend on some ambiguous coefficients. Therefore it seems to be puzzling how to express these quantities completely in terms of boundary parameters. In this paper we show that it is still possible to express η, s and $\frac{\eta}{s}$ in terms two central charges a and c and other two unambiguous coefficients[2] α_1 and α_2. Our final results are

$$\eta = 8\pi^3 c\, T^3\left[1+\frac{1}{4}\frac{c-a}{c}-\frac{1}{8}\left(\frac{c-a}{c}\right)^2-\frac{180}{\lambda}(2\alpha_1+\alpha_2)\right]+\mathcal{O}(\lambda^{-3/2})\,, \tag{7.1.2}$$

$$s = 32\pi^4 c\, T^3\left[1+\frac{5}{4}\frac{c-a}{c}+\frac{3}{8}\left(\frac{c-a}{c}\right)^2+\frac{12}{\lambda}(2\alpha_1+\alpha_2)\right]+\mathcal{O}(\lambda^{-3/2}) \tag{7.1.3}$$

and

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1-\frac{c-a}{c}+\frac{3}{4}\left(\frac{c-a}{c}\right)^2-\frac{192}{\lambda}(2\alpha_1+\alpha_2)\right]+\mathcal{O}(\lambda^{-3/2})\,, \tag{7.1.4}$$

where T is the temperature and λ is the 't Hooft coupling.

We obtain this result in the following way. Since six derivative terms appear with coefficient α'^2 where four derivative terms are proportional to α', therefore to make all the expressions correct up to order α'^2, we need to con-

[2]We assume that the "unambiguous" coefficients of higher derivative gravity can be fixed by boundary parameters.

sider the effect of four derivative terms to order α'^2 also. As we mentioned earlier at order α', the coefficients of $\mathbf{R^2}$ and $\mathbf{Ricci^2}$ terms (β_1 and β_3 respectively) are ambiguous, they can be removed by field re-definition [101]. In fact they do not appear in the expression of $\frac{\eta}{s}$ at order α'. But these two ambiguous coefficients appear at order α'^2 (see section [7.2]). Therefore the ratio $\frac{\eta}{s}$ depends on three unambiguous coefficients β_2 (at order α'), α_1 and α_2 (at order α'^2) and four ambiguous coefficients $\beta_1, \beta_3, \alpha_3$ and α_4 at order α'^2. Then we calculate two central charges a and c for six derivative gravity. We consider a particular combination of these central charges, namely $\frac{c-a}{c}$. It turns out that the combinations of ambiguous coefficients, which appear in the expression of $\frac{\eta}{s}$, the same combination appears in $\frac{c-a}{c}$. Therefore one can remove all ambiguous coefficients in terms of this particular combination of central charges a and c.

Let us summarize the main results of this paper.

- In section [7.2] we consider the most general six derivative action. There can be total ten independent invariants. We identify the ambiguous and unambiguous coefficients of this generic action. We find that it is possible to drop six ambiguous terms from the action on which $\frac{\eta}{s}$ does not depend. We also consider the effect of four derivative terms to order α'^2.
- In section [7.3] we calculate the perturbed background metric up to order α'^2.
- In section [7.4] we compute the ratio $\frac{\eta}{s}$ using effective action approach of [124].
- In section [7.5] we calculate the central charges a and c for six derivative gravity.
- Finally in section [7.6] we write the expression for η, s and $\frac{\eta}{s}$ in terms of central charges and two unambiguous parameters of bulk Lagrangian. We also discuss how to relate the unambiguous coefficients of bulk theory to the physical boundary parameters following [102].
- In appendix [E] and [F] we present the expressions for A_i's and $\mathcal{B}$'s respectively which appear in section [7.4].
- We also calculate shear viscosity coefficient using Kubo formula as a

check of our effective action calculation. In appendix [5.3.18] we outline the calculations.

- In appendix [H] we calculate leading r dependence of Riemann and Ricci tensors which appear in section [7.5].

7.2 The Field Re-definition and $\frac{\eta}{s}$

In this section we discuss the most general six derivative terms in the bulk Lagrangian and their effects on shear viscosity to entropy density ratio. Generic six derivative terms can be constructed out of Riemann tensor, Ricci tensors and curvature scalar terms or their covariant derivatives. There are five possible dimension-6 invariants which do not involve Ricci tensors or curvature scalars,

$$\begin{aligned} I_1 &= R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu}, \qquad I_2 = R^{\mu\nu}{}_{\rho\sigma}R^{\rho\tau}{}_{\lambda\mu}R^{\sigma}{}_{\tau}{}^{\lambda}{}_{\nu}, \\ I_3 &= R^{\alpha\nu}{}_{\mu\beta}R^{\beta\gamma}{}_{\nu\lambda}R^{\lambda\mu}{}_{\gamma\alpha}, \qquad I_4 = R_{\mu\nu\alpha\beta}R^{\mu\alpha}{}_{\gamma\delta}R^{\nu\beta\gamma\delta}, \\ &\qquad I_5 = R_{\mu\nu\alpha\beta}\mathcal{D}^2 R^{\mu\nu\alpha\beta}\,. \end{aligned} \tag{7.2.5}$$

They satisfy the following relations,

$$I_3 = I_2 - \frac{1}{4}I_1, \quad I_4 = \frac{1}{2}I_1, \quad I_5 = -I_1 - 4I_2\,. \tag{7.2.6}$$

Hence only two of them are independent. We will choose these two invariants to be I_1 and I_2.

Now consider the most general action containing all possible independent curvature invariants

$$\mathcal{I} = \int d^5x\sqrt{-g}\,\mathcal{L} \tag{7.2.7}$$

where

$$\begin{aligned}\mathcal{L} &= a_0 R - 2\Lambda + \alpha'\Big(\beta_1 R^2 + \beta_2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta_3 R_{\mu\nu}R^{\mu\nu}\Big) \\ &+ \alpha'^2\Big(\alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}R^{\mu\nu} + \alpha_4 R R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &+ \alpha_5 R_{\mu\nu\rho\lambda}R^{\nu\lambda}R^{\mu\rho} + \alpha_6 R_{\mu\nu}R^{\nu\lambda}R^\mu{}_\lambda + \alpha_7 R_{\mu\nu}\mathcal{D}^2 R^{\mu\nu} + \alpha_8 R R_{\mu\nu}R^{\mu\nu} \\ &\quad +\alpha_9 R^3 + \alpha_{10} R\mathcal{D}^2 R\Big) + \mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{7.2.8}$$

However, this action is ambiguous up to a field re-definition. It has been shown in [101] that under the following field re-definition

$$\begin{aligned} g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} &+ \alpha'(d_1 g_{\mu\nu}R + d_2 R_{\mu\nu}) \\ &+ \alpha'^2(d_3 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + d_4 g_{\mu\nu}R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} + d_5 R_{\mu\alpha\beta\nu}R^{\alpha\beta} \\ &+ d_6 R_{\mu\lambda}R^\lambda_\nu + d_7\mathcal{D}^2 R_{\mu\nu} + d_8 g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + d_9 g_{\mu\nu}R^2 \\ &+ d_{10}g_{\mu\nu}\mathcal{D}^2 R) + \mathcal{O}(\alpha'^3) \end{aligned} \tag{7.2.9}$$

the coefficients a_0, β_2, α_1 and α_2 in the Lagrangian (7.2.8) remain invariant and all other coefficients changes. This is because it is not possible to generate any higher rank tensor from a lower rank tensor in (7.2.9). For example one can not get $\mathbf{Riemann^2}$ term from any **Ricci** term at order α' and similarly any $\mathbf{Riemann^3}$ term can not be generated from any $\mathbf{Ricci^2}$, $\mathbf{Riemann^2}$ or $\mathbf{Ricci}\cdot\mathbf{Riemann}$ terms at order α'^2. Therefore the coefficients β_2, α_1 and α_2 are unambiguous. By proper choice of $d_1, ..., d_{10}$ one can set any desired values to the coefficients β_1, β_3 and $\alpha_3, ..., \alpha_{10}$, for example we can set all of them to zero. These are the ambiguous coefficients. Setting all ambiguous coefficients to zero the action (7.2.7) becomes,

$$\sqrt{-g}\,\mathcal{L} \;\to\; \sqrt{-g}\Big(\tilde{a}_0 R - 2\Lambda + \alpha'\,\beta_2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha'^2\Big(\alpha_1 I_1 + \alpha_2 I_2\Big)\Big) \tag{7.2.10}$$

with some different $\tilde{a}_0$ which is related to a_0 and other ambiguous parameters[3]. The action (7.2.10) and (7.2.7) are equivalent up to a field re-definition.

[3] $\tilde{a}_0$ gets contribution from $\sqrt{-g}$.

Any physical quantity like entropy, shear viscosity or their ratio calculated either from action (7.2.10) or (7.2.7) turns out to be same after using the relation between a_0 and $\tilde{a}_0$. That is, these quantities are field re-definition invariant.

We calculate $\frac{\eta}{s}$ for generic six derivative action and find that the ratio depends on some ambiguous coefficients in (7.2.7). Before we start calculating $\frac{\eta}{s}$ for the generic action (7.2.7) we can use the following logic to understand that among ten ambiguous parameters six of them never appear in the expression of $\frac{\eta}{s}$[4]. Therefore we can drop those terms at the beginning to simplify our life[5]. Let us now find out those terms in the action on which $\frac{\eta}{s}$ does not depend.

Consider the following Lagrangian,

$$\begin{aligned}\tilde{\mathcal{L}} &= a_0 R - 2\Lambda + \alpha'^2 \Big(\alpha_5 R_{\mu\nu\rho\lambda} R^{\nu\lambda} R^{\mu\rho} + \alpha_6 R_{\mu\nu} R^{\nu\lambda} R^{\mu}{}_{\lambda} + \alpha_7 R_{\mu\nu} \mathcal{D}^2 R^{\mu\nu} \\ &\qquad + \alpha_8 R R_{\mu\nu} R^{\mu\nu} + \alpha_9 R^3 + \alpha_{10} R \mathcal{D}^2 R \Big) + \mathcal{O}(\alpha'^3)\end{aligned} \tag{7.2.11}$$

and following field re-definition,

$$\begin{aligned} g_{\mu\nu} &\to g_{\mu\nu} + \alpha'^2 \big(d_5 R_{\mu\alpha\beta\nu} R^{\alpha\beta} + d_6 R_{\mu\lambda} R^{\lambda}_{\nu} \\ &\qquad + d_7 \mathcal{D}^2 R_{\mu\nu} + d_8 g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + d_9 g_{\mu\nu} R^2 + d_{10} g_{\mu\nu} \mathcal{D}^2 R \big) + \mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{7.2.12}$$

With proper choice of $d_5, d_6 ... d_{10}$ one can check that the resultant Lagrangian becomes,

$$\sqrt{-g}\tilde{\mathcal{L}} \to \sqrt{-g}(\tilde{a}_0 R - 2\Lambda)\,. \tag{7.2.13}$$

Also under the field re-definition (7.2.12) the metric scales in the following way,

$$g_{\mu\nu} \to \mathcal{C}(\alpha') g_{\mu\nu} \tag{7.2.14}$$

where,

$$\mathcal{C}(\alpha') = 1 + \alpha'^2(-16 d_5 + 16 d_6 + 80 d_8 + 400 d_9)\,. \tag{7.2.15}$$

[4]Though these coefficients may arise in the individual expressions of η and s. Since we are interested in $\frac{\eta}{s}$ we drop these terms. However the final expressions (7.1.2 and 7.1.3) for η and s remain unchanged even if we consider these terms.

[5]Other ambiguous terms can not be dropped using this logic.

Here we have used the leading equation of motion $R_{\mu\nu} = -4g_{\mu\nu}$. The scaling in (7.2.14) does not change the temperature of the background spacetime and hence the diffusion pole calculated from action (7.2.13) gives the standard result $D = \frac{1}{4\pi T}$, where D is diffusion constant and T is temperature. Thus the ratio $\frac{\eta}{s}$ turns out to be $\frac{1}{4\pi}$ for action (7.2.11). Therefore we see that shear viscosity to entropy density ratio does not depend on $\alpha_5, \alpha_6 \cdots \alpha_{10}$ up to order α'^2.

One important thing to notice here is that the ratio $\frac{\eta}{s}$ does not depend on β_1 and β_3 up to order α' [116, 117]. One can consider the following field re-definition

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \alpha'(d_1 g_{\mu\nu} R + d_2 R_{\mu\nu}) \tag{7.2.16}$$

and get rid off the terms $\beta_1 R^2$ and $\beta_3 R^2_{\mu\nu}$ with proper choice of d_1 and d_2. The new metric is same as the original metric up to some constant scaling factor *to order α'* (substituting the leading equation of motion at order α'). Therefore one can argue that $\frac{\eta}{s}$ is independent of β_1 and β_3 up to order α'. But this is not true when we consider terms to order α'^2. We can not only substitute the leading order equation of motion in (7.2.16) when we are interested in α'^2 order. We have to consider equations of motion to order α'. The equations of motion to order α' is given by [108],

$$\begin{aligned} R_{\mu\nu} &= -4g_{\mu\nu} + \frac{\alpha'}{3} L^{(2)} g_{\mu\nu} - 2\alpha' L^{(2)}_{\mu\nu} - \alpha'(\beta_3 + 4\beta_2)\mathcal{D}^2 R_{\mu\nu} \\ &\quad + \frac{2\alpha'}{3}(3\beta_1 + \beta_3 + \beta_2) g_{\mu\nu} \mathcal{D}^2 R + \alpha'(2\beta_1 + \beta_3 + 2\beta_2) D_{\mu\nu} R \\ &\quad + \mathcal{O}(\alpha'^2) \, . \end{aligned} \tag{7.2.17}$$

Substituting this equations of motion in (7.2.16) we get,

$$\begin{aligned} &g_{\mu\nu} \rightarrow g_{\mu\nu} - 4\alpha'(5d_1 + d_2) g_{\mu\nu} \\ &+\alpha'^2 \left[\frac{d_2 - d_1}{3}(400\beta_1 + 80\beta_3) - 2d_2(16(2\beta_2 + \beta_3) - 32\beta_2 + 80\beta_1)\right] g_{\mu\nu} \\ &+\alpha'^2 \beta_2 \left[\frac{d_2 - d_1}{3} R^2_{\mu\nu\rho\sigma} g_{\mu\nu} - 2d_2 R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma}\right] \, . \end{aligned} \tag{7.2.18}$$

Therefore we see that the new metric is proportional to the original metric (with constant proportionality factor) at order α' but not at order α'^2 when

$\beta_2 \neq 0$. Hence $\frac{\eta}{s}$ may not be independent of β_1 and β_3 at order α'^2. It can have terms like $\beta_1\beta_2$, $\beta_2\beta_3$ and β_2^2 at order α'^2.

7.3 The Perturbed Background Metric

In this section we will find the perturbative solution to Einstein equations in presence of six derivative terms in the action up to order α'^2. We write the basic equation of motions and mention how to solve these equations up to order α'^2. We will start with the following five dimensional action with negative cosmological constant $\Lambda = -6$.

$$\begin{aligned}\mathcal{I} &= \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big[R - 2\Lambda + \alpha'\Big(\beta_1 R^2 + \beta_2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta_3 R_{\mu\nu}R^{\mu\nu}\Big) \\ &\quad +\alpha'^2\Big(\alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 R_{\mu\alpha\beta\gamma}R_{\nu}^{\ \alpha\beta\gamma}R^{\mu\nu} + \alpha_4 R R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big)\Big] . \qquad (7.3.19)\end{aligned}$$

We take the leading value of of AdS radius is 1.

We consider the following metric ansatz (assuming planer symmetry of the spacetime),

$$ds^2 = -\rho^2 e^{2A(\rho)+8B(\rho)}dt^2 + \rho^2 e^{2B(\rho)}d\rho^2 + \rho^2 d\vec{x}^2 \, . \qquad (7.3.20)$$

Substituting this metric in the (7.3.19) we get,

$$\mathcal{I} = \frac{1}{16\pi G_5}\int_{\rho_0}^{\infty} d\rho\left[\mathfrak{L}^{(2)} + \alpha'\mathfrak{L}^{(4)} + \alpha'^2\mathfrak{L}^{(6)}\right] \qquad \text{where,} \qquad (7.3.21)$$

$$\begin{aligned}\mathfrak{L}^{(2)} &= \sqrt{-g}(R+12) \\ &= 12\rho^5 e^{A(\rho)+5B(\rho)} - 2\rho(2+\rho A'(\rho))e^{A(\rho)+3B(\rho)} \\ &\quad -2\frac{d}{d\rho}\left[(A'(\rho)+4B'(\rho))\rho^3 e^{A(\rho)+3B(\rho)}\right] \qquad (7.3.22)\end{aligned}$$

and $\mathfrak{L}^{(4)}$ and $\mathfrak{L}^{(6)}$ are four and six derivative terms in the Lagrangian evaluated on the metric ansatz. The Euler-Lagrange equations which follow from this action is given by,

$$\begin{aligned}
\frac{\partial \mathfrak{L}^{(2)}}{\partial A(\rho)} - \frac{d}{d\rho}\frac{\partial \mathfrak{L}^{(2)}}{\partial A'(\rho)} &= -\alpha' \left(\frac{\partial \mathfrak{L}^{(4)}}{\partial A(\rho)} - \frac{d}{d\rho}\frac{\partial \mathfrak{L}^{(4)}}{\partial A'(\rho)} + \frac{d^2}{d\rho^2}\frac{\partial \mathfrak{L}^{(4)}}{\partial A''(\rho)} \right) \\
&\quad -\alpha'^2 \left(\frac{\partial \mathfrak{L}^{(6)}}{\partial A(\rho)} - \frac{d}{d\rho}\frac{\partial \mathfrak{L}^{(6)}}{\partial A'(\rho)} + \frac{d^2}{d\rho^2}\frac{\partial \mathfrak{L}^{(6)}}{\partial A''(\rho)} \right) \\
\frac{\partial \mathfrak{L}^{(2)}}{\partial B(\rho)} &= -\alpha' \left(\frac{\partial \mathfrak{L}^{(4)}}{\partial B(\rho)} - \frac{d}{d\rho}\frac{\partial \mathfrak{L}^{(4)}}{\partial B'(\rho)} + \frac{d^2}{d\rho^2}\frac{\partial \mathfrak{L}^{(4)}}{\partial B''(\rho)} \right) \\
&\quad -\alpha'^2 \left(\frac{\partial \mathfrak{L}^{(6)}}{\partial B(\rho)} - \frac{d}{d\rho}\frac{\partial \mathfrak{L}^{(6)}}{\partial B'(\rho)} + \frac{d^2}{d\rho^2}\frac{\partial \mathfrak{L}^{(6)}}{\partial B''(\rho)} \right) .
\end{aligned} \tag{7.3.23}$$

We solve this equation perturbatively to find $A(\rho)$ and $B(\rho)$. First we solve this equations up to order α'. We use leading order solutions for A and B on the right hand side. The order α' terms on the right hand side will act as a source terms and we solve the equations to find corrected A and B in presence of these source terms. There are two integration constants when we solve this equations. We choose these two integration constants (to order α') such a way that the corrected (black hole)solution has horizon at $\rho = 1$ and the boundary ($\rho \to \infty$) metric is Minkowskian.

After getting the metric up to order α' we now solve A and B to order α'^2. We substitute the solutions for A and B (corrected up to order α') on the right hand side of equation (7.3.23) and get the solution for A and B to order α'^2. We again choose the integration constants in order to set the black hole horizon radius at $\rho = 1$ and the boundary metric to be Minkowskian.

The solution is given by (after changing the coordinate $\rho \to \frac{1}{\sqrt{r}}$),

$$ds^2 = f(r)dt^2 + \frac{g(r)}{4r^3}dr^2 + \frac{1}{r}d\vec{x}^2 \tag{7.3.24}$$

where $f(r)$ and $g(r)$ are given by,

$$\begin{aligned} f(r) &= r - \frac{1}{r} - 2r(r^2-1)\beta_2\alpha' \\ &+ \frac{1}{3}r(r^2-1)\Big(12(2r^2-31)\alpha_1 + (48r^2-33)\alpha_2 + 24(2r^2+3)\alpha_3 \\ &- 24(12r^2+7)\alpha_4 + 4\beta_2(-22\beta_1 + 48r^2\beta_1 + 149\beta_2 - 42r^2\beta_2 + 34\beta_3)\Big)\alpha'^2 \end{aligned} \tag{7.3.25}$$

and

$$\begin{aligned} g(r) &= \frac{r}{1-r^2} + \frac{2r(10\beta_1 + (1-3r^2)\beta_2 + 2\beta_3)\alpha'}{3(r^2-1)} \\ &+ \frac{r}{9(r^2-1)}\Big(12(1-93r^2+240r^4)\alpha_1 + 9(1-11r^2-2r^4)\alpha_2 \\ &-24(1-9r^2+36r^4)\alpha_3 + 24(5-21r^2+126r^4)\alpha_4 + 400\beta_1^2 \\ &+16(5-9r^2-126r^4)\beta_1\beta_2 + 4(1+450r^2-927r^4)\beta_2^2 + 160\beta_1\beta_3 \\ &+16(1+27r^2-90r^4)\beta_2\beta_3 + 16\beta_3^2\Big)\alpha'^2 \,. \end{aligned} \tag{7.3.26}$$

This is the background metric corrected up to order α'^2. Also the black brane temperature is given by,

$$\begin{aligned} T &= \frac{1}{\pi} + \frac{10\beta_1 - 5\beta_2 + 2\beta_3}{3\pi}\alpha' \\ &+ \frac{1}{18\pi}\Big(732\alpha_1 - 63\alpha_2 - 312\alpha_3 + 1272\alpha_4 + 700\beta_1^2 \\ &-1948\beta_1\beta_2 - 605\beta_2^2 + 280\beta_1\beta_3 - 620\beta_2\beta_3 + 28\beta_3^2\Big)\alpha'^2 \,. \end{aligned} \tag{7.3.27}$$

7.4 The Effective Action and Shear Viscosity

To calculate six derivative correction to the shear viscosity coefficient we need to find the quadratic action for transverse graviton moving in background spacetime (7.3.24). We consider the following metric perturbation,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r,x) = g_{xy}^{(0)}(1+\epsilon\Phi(r,x)) \tag{7.4.28}$$

where ϵ is an order counting parameter. We consider terms up to order ϵ^2 in the action of $\Phi(r,x)$. The action (in momentum space) is given by,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int\frac{d^4k}{(2\pi)^4}dr\Big[A_1(r,k)\phi(r,k)\phi(r,-k) + A_2(r,k)\phi'(r,k)\phi'(r,-k) \\ &\quad + A_3(r,k)\phi''(r,k)\phi''(r,-k) + A_4(r,k)\phi(r,k)\phi'(r,-k) \\ &\quad + A_5(r,k)\phi(r,k)\phi''(r,-k) + A_6(r,k)\phi'(r,k)\phi''(r,-k)\Big] \end{aligned} \tag{7.4.29}$$

where the expressions for A_is are given in appendix [E] and $\phi(r,k)$ is given by,

$$\phi(r,k) = \int\frac{d^4x}{(2\pi)^4}e^{-ik.x}\Phi(r,x)\,, \tag{7.4.30}$$

$k = \{-\omega,\vec{k}\}$ and ' $'$ ' denotes derivative with respect to r. Up to some total derivative terms this action can be written as,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int\frac{d^4k}{(2\pi)^4}dr\Big[\mathcal{A}_0\phi(r,k)\phi(r,-k) + \mathcal{A}_1\phi'(r,k)\phi'(r,-k) \\ &\quad + \mathcal{A}_2\phi''(r,k)\phi''(r,-k)\Big] \end{aligned} \tag{7.4.31}$$

where,

$$\begin{aligned} \mathcal{A}_0 &= A_1(r,k) - \frac{A_4'(r,k)}{2} + \frac{A_5''(r,k)}{2} \\ \mathcal{A}_1 &= A_2(r,k) - A_5(r,k) - \frac{A_6'(r,k)}{2} \\ \mathcal{A}_2 &= A_3(r,k)\,. \end{aligned} \tag{7.4.32}$$

This action does not have the canonical form. Therefore to obtain the shear viscosity coefficients from this action we follow the prescription given in

[124]. We write the effective action for the scalar field,

$$S_{\text{eff}} = \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\Big[(\mathcal{A}_1^{(0)}(r,k)+\alpha'\mathcal{B}_1^{(0)}(r,k)+\alpha'^2\mathcal{B}_1^{(1)}(r,k))\phi'(r,-k)\phi'(r,k) + (\mathcal{A}_0^{(0)}(r,k)+\alpha'\mathcal{B}_0^{(0)}(r,k)+\alpha'^2\mathcal{B}_0^{(1)}(r,k))\phi(r,k)\phi(r,-k)\Big]. \tag{7.4.33}$$

where,

$$\mathcal{A}_1^{(0)}(r) = \frac{r^2-1}{r} \tag{7.4.34}$$

and

$$\mathcal{A}_0^{(0)}(r,k) = \frac{\omega^2}{4r^2(1-r^2)}. \tag{7.4.35}$$

To evaluate the functions $\mathcal{B}_1^{(0)}, \mathcal{B}_1^{(1)}, \mathcal{B}_0^{(0)}$, and $\mathcal{B}_0^{(1)}$, we demand the equations of motion obtained from action (7.4.31) and (7.4.33) are same at order α' and order α'^2 separately. Comparing the equations of motion for $\phi(r,k)$ from two actions at order α' and α'^2 we get the function $\mathcal{B}_1$'s and $\mathcal{B}_0$'s. Explicit expression for $\mathcal{B}_0$'s and $\mathcal{B}_1$'s are given in appendix [F].

The effective coupling K_{eff} of transverse graviton is given by,

$$\begin{aligned}16\pi G_5 K_{\text{eff}}(r) &= \frac{(\mathcal{A}_1^{(0)}(r,k)+\alpha'\mathcal{B}_1^{(0)}(r,k)+\alpha'^2\mathcal{B}_1^{(1)}(r,k))}{\sqrt{-g}g^{rr}} \\ &= -\frac{1}{2} + \big(20\beta_1 + 2(r^2-1)\beta_2 + 4\beta_3\big)\alpha' \\ &\quad + \frac{1}{6}\Big[-36(r^4-22r^2-3)\alpha_1 + 9(45r^4+18r^2-7)\alpha_2 \\ &\quad + 8\big(3(7r^4-2r^2-1)\alpha_3 - 3(9r^4+10r^2-5)\alpha_4 + 237\beta_2^2 r^4 \\ &\quad + 18\beta_1\beta_2 r^4 + 66\beta_2\beta_3 r^4 - 188\beta_2^2 r^2 + 10\beta_1\beta_2 r^2 - 46\beta_2\beta_3 r^2 \\ &\quad + 100\beta_1^2 - \beta_2^2 + 4\beta_3^2 + 40\beta_1\beta_3\big)\Big]\alpha'^2 .\end{aligned} \tag{7.4.36}$$

The shear viscosity coefficient is determined by the following expression,

$$\begin{aligned}\eta &= \frac{1}{r_0^{3/2}}\left(-2K_{\text{eff}}(r_0)\right) \\ &= \frac{1}{16\pi G_5} - \frac{(5\beta_1+\beta_3)\alpha'}{2\pi G_5} \\ &\quad -\frac{\alpha'^2}{6\pi G_5}\Big[108\alpha_1+63\alpha_2+12\alpha_3-42\alpha_4+100\beta_1^2+28\beta_2\beta_1 \\ &\qquad +40\beta_3\beta_1+48\beta_2^2+4\beta_3^2+20\beta_2\beta_3\Big] \qquad (7.4.37)\end{aligned}$$

where r_0 is the position of horizon and in our parametrization $r_0 = 1$.

7.4.1 Shear Viscosity to Entropy Density Ratio

One can calculate entropy density using Wald's formula [129, 130]. Order α'^2 correction to entropy density s turns out to be,

$$\begin{aligned}s = &\frac{1}{4G_5} - \frac{2(5\beta_1-\beta_2+\beta_3)\alpha'}{G_5} \\ &+\frac{\left(36\alpha_1+27\alpha_2-36\alpha_4-4\left(50\beta_1^2+4\beta_2\beta_1+20\beta_3\beta_1+26\beta_2^2+2\beta_3^2+8\beta_2\beta_3\right)\right)\alpha'^2}{3G_5}\,. \qquad (7.4.38)\end{aligned}$$

Then we find shear viscosity to entropy density ratio is given by,

$$\begin{aligned}\frac{\eta}{s} &= \frac{1}{4\pi} - \frac{2\beta_2\alpha'}{\pi} \\ &\quad -\frac{(252\alpha_1+153\alpha_2+24\alpha_3-120\alpha_4+56\beta_2(5\beta_1-\beta_2+\beta_3))\alpha'^2}{3\pi}\,. \qquad (7.4.39)\end{aligned}$$

Thus we see that the ratio $\frac{\eta}{s}$ depends on ambiguous coefficients $\beta_1, \beta_3, \alpha_3$ and α_4 at order α'^2. But, we will show in the next section that we can get rid of these ambiguous coefficients and express the result in terms of physical boundary parameters. To be explicit, we calculate six derivative corrections to central charges a and c and show that it is possible to express $\frac{\eta}{s}$ in terms of these central charges and unambiguous coefficients α_1 and α_2, which can be fixed by other physical boundary parameters.

7.5 Conformal Anomaly in Six-derivative Gravity

So far we have computed shear viscosity to entropy density ratio for some gauge theory plasma whose gravity dual is governed by six derivative Lagrangian given by (7.3.19). In this section we compute the six derivative corrections to central charges a and c of this dual field theory. The holographic procedure to compute conformal anomaly form two derivative gravity has been given in [105] and later it has been generalized to four derivative gravity in [106, 107]. We will follow the same approach and carry on the analysis for six derivative terms in the action.

First we assume (can be easily checked) that the gravity theory has a AdS solution even in presence of the higher derivative terms in the action. The metric, the curvature tensors and the scalar are given as,

$$ds^2 = G^{(0)}_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{4r^2} dr^2 + \sum_{i=1}^{d} \frac{\eta_{ij}}{r} dx^i dx^j \tag{7.5.40}$$

and,

$$R^{(0)} = -\frac{d(d+1)}{L^2},\ R^{(0)}_{\mu\nu} = -\frac{d}{L^2} G^{(0)}_{\mu\nu},\ R^{(0)}_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(G^{(0)}_{\mu\rho} G^{(0)}_{\nu\sigma} - G^{(0)}_{\mu\sigma} G^{(0)}_{\nu\rho}). \tag{7.5.41}$$

Here, L is the corrected AdS radius given in (7.5.62) and $L = 1$ when there is no higher derivative terms present in the action. d is the dimension of boundary space-time. One can obtain the equation of motion for the action (7.3.19) following [108, 128]. The terms in the equations of motion containing covariant derivatives of the curvature tensors vanish for the above background (7.5.40). The equation finally reduces to,

$$\begin{aligned}(d-1)L^4 - \frac{12L^6}{d} &= \alpha' L^2 \Big(\beta_1 d(d+1)(d-3) + 2\beta_2(d-3) + \beta_3 d(d-3) \Big) \\ &\quad -\alpha'^2 (d-5) \Big(4\alpha_1 + \alpha_2(d-1) - 2\alpha_3 d + 2\alpha_4 d(d+1) \Big). \end{aligned}\tag{7.5.42}$$

As the AdS metric has a second order pole at infinity, it only induces a conformal equivalence class $[g^{ij}_{(0)}]$ of metrics on the boundary. Following Gauge-Gravity correspondence, the boundary field theory effective action in large N limit is,

$$W_{FT}(g_{(0)}) = S_{grav}(g; g_{(0)}), \tag{7.5.43}$$

where $S_{grav}(g; g_{(0)})$ is the gravity action evaluated on classical (AdS) configuration which approaches a representative boundary metric $g_{(0)}$. Now, for computing conformal anomaly, we consider the following fluctuation around (7.5.40),

$$\begin{aligned} ds^2 = G_{\mu\nu}dx^\mu dx^\nu &= \frac{L^2}{4r^2}dr^2 + \sum_{i=1}^{d} \frac{g_{ij}}{r} dx^i dx^j \qquad \text{with,} \\ g_{ij} &= g_{(0)ij} + r g_{(1)ij} + r^2 g_{(2)ij} + r^2(\ln r) h_{(2)ij} + \cdots \end{aligned} \tag{7.5.44}$$

Here, $g_{(0)}$ is the representative boundary metric and $h_{(2)}$ is traceless with respect to $g_{(0)}$. The determinant of the full metric (7.5.44) can be written as,

$$\begin{aligned} \sqrt{-G} &= \frac{L}{2} r^{-\frac{d}{2}-1} \sqrt{-g_{(0)}} \Big[1 + \frac{r}{2}\mathrm{Tr}[g_{(1)}] \\ &\quad + r^2 \big(\frac{1}{2}\mathrm{Tr}[g_{(2)}] - \frac{1}{4}\mathrm{Tr}[(g^2_{(1)})] \\ &\quad + \frac{1}{8}(\mathrm{Tr}[g_{(1)}])^2\big) \Big] + \mathcal{O}(r^3) \, . \end{aligned} \tag{7.5.45}$$

For computing the conformal anomaly of the boundary field theory, we need to evaluate all the terms in the bulk action (7.3.19) in terms of $(g_{(0)}, g_{(1)}, g_{(2)})$. Then, we regard $g_{(0)}$ as independent field on the boundary and solve $g_{(1)}$ in terms $g_{(0)}$. As we will see, the term involving $g_{(2)}$ will vanish on-shell (7.5.42). To regularize the infrared divergences of the on-shell action, we introduce a cutoff ϵ restricting the range of r integral as $r \geq \epsilon$. Then the on-shell action can be written as,

$$\begin{aligned} S &= S_0(g_{(0)})\epsilon^{-\frac{d}{2}} + S_1(g_{(0)}, g_{(1)})\epsilon^{-\frac{d}{2}-1} \\ &\quad + \cdots\cdots + S_{\ln} \ln[\epsilon] + S_{\frac{d}{2}} + \mathcal{O}(\epsilon^{\frac{1}{2}}) \, . \end{aligned} \tag{7.5.46}$$

Then, the conformal anomaly $\mathcal{T}$ of the boundary field theory is given as,

$$S_{\ln} = -\frac{1}{2}\int d^d x \sqrt{g_{(0)}}\,\mathcal{T}\ . \tag{7.5.47}$$

We want to find $\mathcal{T}$ for $d = 4$. The expressions for $\mathcal{T}$ for four derivatives terms in Lagrangian are given in [106, 107]. Here we present the computation for six-derivative terms only. The generic structure of any term in the action has the following structure

$$\frac{1}{2r^{\frac{d}{2}+1}}\sqrt{g_{(0)}}(\mathcal{X}_1 + \mathcal{X}_2 r + \mathcal{X}_3 r^2 + \cdots), \tag{7.5.48}$$

where, $(\mathcal{X}_1, \mathcal{X}_2, \cdots)$ are some functions of $(g_{(0)}, g_{(1)}, g_{(2)}, \cdots)$. Since we are looking for the term $S_{\ln}$ in (7.5.46), we only need the terms of order $\mathcal{O}(\frac{1}{r})$ in (7.5.48). Hence, it is enough for us to terminate the expansion in (7.5.48) at $\mathcal{O}(r^2)$ for $d = 4$. Therefore the coefficient $\mathcal{X}_3$ will finally contribute to the anomaly $\mathcal{T}$. As we will see, this knowledge will help us to pre-eliminate certain terms in our calculation.

We will summarize our main results for four six derivative terms in the Lagrangian (7.3.19). We follow the following notations:

$$\begin{aligned}
\mathrm{r}^{(0)^i}_{\ jkl} &\rightarrow \text{Riemann tensor constructed out of } g_{(0)}.\\
\mathrm{r}^{(0)}_{ij} &\rightarrow \text{Ricci tensor constructed out of } g_{(0)}.\\
\mathrm{rim}^{(0)^2} = \mathrm{r}^{(0)}_{ijkl}\mathrm{r}^{(0)ijkl}&, \quad \mathrm{ric}^{(0)^2} = \mathrm{r}^{(0)ij}\mathrm{r}^{(0)}_{ij}.\\
\mathrm{r}^{(0)} &= g^{ij}_{(0)}\mathrm{r}^{(0)}{}_{ij}\ .
\end{aligned}$$

- $T_1 = R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu}$:

  Here, (μ, ν) indices run over full five dimensional space-time. One can split the indices in $(r; i, j)$, where (i, j) runs over four dimensional boundary space time. From the leading $r-$dependence of the curvature tensors (appendix [H]), it is easy to see that only two combinations $R^{ij}_{kl} R^{kl}_{mn} R^{mn}_{ij}$ and $R^{ir}_{jr} R^{jr}_{kr} R^{kr}_{ir}$ will contribute to $S_{\ln}$. The leading $r-$dependence of other possible combinations starts from r^3 and hence they do not contribute to anomaly. The expansions of T_1 is[6],

$$
\begin{aligned}
T_1 &= R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} \\
&= R^{ij}_{kl}R^{kl}_{mn}R^{mn}_{ij} + 8\,R^{ir}_{jr}R^{jr}_{kr}R^{kr}_{ir} \\
&= -4d(d+1) + 12r\Big[\mathrm{r}^{(0)} + 2(d-1)\mathrm{Tr}[g_{(1)}]\Big] \\
&\quad + r^2\Big[-6\,\mathrm{rim}^{(0)^2} - 60\mathrm{r}^{(0)^{ij}}g_{(1)_{ij}} + 48(d-3)\mathrm{Tr}[g_{(2)}] \\
&\qquad + 12(9-4d)\mathrm{Tr}[(g_{(1)})^2] - 36(\mathrm{Tr}[g_{(1)}])^2\Big] + \mathcal{O}(r^3)\,.
\end{aligned}
\tag{7.5.49}
$$

- $T_2 = R^{\mu\nu}_{\rho\sigma}R^{\rho\tau}_{\lambda\mu}R^{\sigma\ \lambda}_{\ \tau\ \nu}$:

Similarly, for T_2, only $R^{ij}_{kl}R^{km}_{ni}R^{l\ \ n}_{\ m\ j}$ and $R^{ir}_{jr}R^{jk}_{li}R^{r\ l}_{\ k\ r}$ contribute to the anomaly. The expansion is,

$$
\begin{aligned}
T_2 &= R^{\mu\nu}_{\rho\sigma}R^{\rho\tau}_{\lambda\mu}R^{\sigma\ \lambda}_{\ \tau\ \nu} \\
&= R^{ij}_{kl}R^{km}_{ni}R^{l\ \ n}_{\ m\ j} + 3R^{ir}_{jr}R^{jk}_{li}R^{r\ l}_{\ k\ r} \\
&= d(1-d^2) + 3(d-1) + r\Big[\mathrm{r}^{(0)} + 2(d-1)\mathrm{Tr}[g_{(1)}]\Big] \\
&\quad + r^2\Big[-3\,\mathrm{ric}^{(0)^2} + \frac{3}{2}\,\mathrm{rim}^{(0)^2} + 9(2-d)\mathrm{r}^{(0)^{ij}}g_{(1)_{ij}} - 6\mathrm{r}^{(0)}\mathrm{Tr}[g_{(1)}] \\
&\qquad + 12(d-1)(3-d)\mathrm{Tr}[g_{(2)}] + (-9d^2+39d-39)\mathrm{Tr}[(g_{(1)})^2] \\
&\qquad + 3(7-4d)(\mathrm{Tr}[g_{(1)}])^2\Big] + \mathcal{O}(r^3)\,.
\end{aligned}
\tag{7.5.50}
$$

- $T_3 = R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\nu\beta}R^{\beta}_{\mu}$:

For T_3, three combinations contribute. They are $R^{ij}_{kl}R^{kl}_{jm}R^{m}_{i}$, $R^{ri}_{rj}R^{rj}_{ir}R^{r}_{r}$ and

[6]We have set $L=1$ for these expansion. We will put back L later by dimensional analysis.

$R^{ri}_{rj}R^{rj}_{mr}R^{m}_{i}$. The expansion is,

$$\begin{aligned}
T_3 &= R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\nu\beta}R^{\beta}_{\mu} \\
&= R^{ij}_{kl}R^{kl}_{jm}R^{m}_{i} + 2\left(R^{ri}_{rj}R^{rj}_{ir}R^{r}_{r} + R^{ri}_{rj}R^{rj}_{mr}R^{m}_{i}\right) \\
&= 2d^2(d+1) - 6dr\left[\mathrm{r}^{(0)} + 2(d-1)\mathrm{Tr}[g_{(1)}]\right] \\
&\quad +r^2\Big[4\,\mathrm{ric}^{(0)^2} + d\,\mathrm{rim}^{(0)^2} + 2(11d-8)\mathrm{r}^{(0)^{ij}}g_{(1)_{ij}} + 8\mathrm{r}^{(0)}\mathrm{Tr}[g_{(1)}] \\
&\quad\quad +24d(3-d)\mathrm{Tr}[g_{(2)}] + (20d^2-54d+16)\mathrm{Tr}[(g_{(1)})^2] \\
&\quad\quad +2(11d-8)(\mathrm{Tr}[g_{(1)}])^2\Big] + \mathcal{O}(r^3)\,. \qquad (7.5.51)
\end{aligned}$$

- $T_4 = RR^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$:

For this term we only need to find contraction of two Riemann tensors. The expansion is,

$$\begin{aligned}
T_4 &= RR^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu} \\
&= R(RR^{ij}_{kl}R^{kl}_{ij} + 4RR^{ir}_{jr}R^{jr}_{ir}) \\
&= -2d^2(1+d)^2 + 6rd(1+d)\left[\mathrm{r}_{(0)} + 2(d-1)\mathrm{Tr}g_{(1)}\right] \\
&\quad +r^2\Big[-4\mathrm{r}^2_{(0)} - d(1+d)\mathrm{rim}^2_{(0)} - 14d(1+d)\mathrm{r}^{(0)^{ij}}g_{(1)_{ij}} \\
&\quad\quad -16(d-1)\mathrm{r}^{(0)}\mathrm{Tr}[g_{(1)}] + 24d(d-3)(d+1)\mathrm{Tr}[g_{(2)}] \\
&\quad\quad -2d(1+d)(8d-19)\mathrm{Tr}[(g_{(1)})^2] \\
&\quad\quad -2(13d^2-11d+8)(\mathrm{Tr}[g_{(1)}])^2\Big] + \mathcal{O}(r^3)\,. \qquad (7.5.52)
\end{aligned}$$

Substituting all these expressions and the expressions for order α' terms in (7.3.19), we get,

$$\begin{aligned}
S_{\mathrm{ln}} &= -\frac{1}{2}\int dx^4\sqrt{g_{(0)}}\Big[\left(t_1\,\mathrm{r}^{(0)^2} + t_2\,\mathrm{ric}^{(0)^2} + t_3\,\mathrm{rim}^{(0)^2}\right) \\
&+ A\,\mathrm{r}^{(0)^{ij}}g_{(1)_{ij}} + B\,\mathrm{r}^{(0)}\mathrm{Tr}[g_{(1)}] + C\,\mathrm{Tr}[(g_{(1)})^2] + D\,(\mathrm{Tr}[g_{(1)}])^2 + E\,\mathrm{Tr}[g_{(2)}]\Big]. \qquad (7.5.53)
\end{aligned}$$

where,

$$
\begin{aligned}
t_1 &= L\beta_1 - \frac{4}{L}\alpha_4, \\
t_2 &= L\beta_3 - \frac{3}{L}\alpha_2 + \frac{4}{L}\alpha_3 \\
t_3 &= L\beta_2 - \frac{6}{L}\alpha_1 + \frac{3}{2L}\alpha_2 + \frac{4}{L}\alpha_3 - \frac{20}{L}\alpha_4
\end{aligned}
\tag{7.5.54}
$$

and

$$
\begin{aligned}
A &= -\frac{L}{16G_5\pi} + \alpha'\left(\frac{5\beta_1}{2G_5L\pi} + \frac{3\beta_2}{4G_5L\pi} + \frac{3\beta_3}{4G_5L\pi}\right) \\
&\quad -\alpha'^2\left(\frac{15\alpha_1}{4G_5L^3\pi} + \frac{9\alpha_2}{8G_5L^3\pi} - \frac{9\alpha_3}{2G_5L^3\pi} + \frac{35\alpha_4}{2G_5L^3\pi}\right)
\end{aligned}
\tag{7.5.55}
$$

$$
\begin{aligned}
B &= \frac{L}{32G_5\pi} - \alpha'\left(\frac{\beta_1}{2G_5L\pi} + \frac{\beta_2}{8G_5L\pi} + \frac{\beta_3}{8G_5L\pi}\right) \\
&\quad +\alpha'^2\left(\frac{3\alpha_1}{8G_5L^3\pi} - \frac{3\alpha_2}{32G_5L^3\pi} - \frac{\alpha_3}{4G_5L^3\pi} + \frac{3\alpha_4}{4G_5L^3\pi}\right)
\end{aligned}
\tag{7.5.56}
$$

$$
\begin{aligned}
C &= \frac{1}{8G_5\pi L} - \frac{3L}{16G_5\pi} + \alpha'\left(\frac{5\beta_1}{4G_5L^3\pi} + \frac{5\beta_2}{8G_5L^3\pi} + \frac{\beta_3}{2G_5L^3\pi}\right) \\
&\quad -\alpha'^2\left(\frac{4\alpha_1}{G_5L^5\pi} + \frac{3\alpha_2}{4G_5L^5\pi} - \frac{5\alpha_3}{G_5L^5\pi} + \frac{20\alpha_4}{G_5L^5\pi}\right)
\end{aligned}
\tag{7.5.57}
$$

$$
\begin{aligned}
D &= -\frac{1}{32G_5\pi L} + \frac{3L}{32G_5\pi} + \alpha'\left(\frac{3\beta_1}{8G_5L^3\pi} + \frac{\beta_2}{16G_5L^3\pi} + \frac{\beta_3}{8G_5L^3\pi}\right) \\
&\quad -\alpha'^2\left(\frac{5\alpha_1}{8G_5L^5\pi} + \frac{69\alpha_2}{32G_5L^5\pi} - \frac{5\alpha_3}{4G_5L^5\pi} + \frac{21\alpha_4}{4G_5L^5\pi}\right)
\end{aligned}
\tag{7.5.58}
$$

$$
\begin{aligned}
E &= \frac{1}{16G_5\pi}\left[-\frac{6}{L} + 6L + \alpha'\left(\frac{40\beta_1}{L^3} + \frac{4\beta_2}{L^3} + \frac{8\beta_3}{L^3}\right)\right. \\
&\quad \left. +\alpha'^2\left(\frac{8\alpha_1}{L^5} + \frac{6\alpha_2}{L^5} - \frac{16\alpha_3}{L^5} + \frac{80\alpha_4}{L^5}\right)\right].
\end{aligned}
\tag{7.5.59}
$$

It is easy to see that $\mathrm{Tr}[g_{(2)}]$ term vanishes when the equation of motion (7.5.42) is satisfied. The equation and the solution for $g_{(1)}$ are given by

$$A r^{ij}_{(0)} + B g^{ij}_{(0)} r_{(0)} r + 2C g^{ik}_{(0)} g^{jl}_{(0)} g_{(1)kl} + 2D g^{ij}_{(0)} g^{kl}_{(0)} g_{(1)kl} = 0 \qquad (7.5.60)$$

and

$$g_{(1)ij} = -\frac{A}{2C} r_{(0)ij} + \frac{AD - BC}{2C(C+4D)} r_{(0)} g_{(0)ij}. \qquad (7.5.61)$$

We can also rearrange equation (7.5.42) to write the corrected AdS radius as (for $d = 4$),

$$\begin{aligned} L &= 1 - \frac{1}{3}\alpha'(10\beta_1 - \beta_2 - 2\beta_3) + \frac{1}{18}\alpha'^2(-12\alpha_1 - 9\alpha_2 + 24\alpha_3 - 120\alpha_4 \\ &\quad -500\beta_1^2 - 5\beta_2^2 - 20\beta_3^2 - 100\beta_1\beta_2 - 200\beta_1\beta_3 - 20\beta_2\beta_3)\,. \end{aligned} \qquad (7.5.62)$$

Substituting all these expression in (7.5.53), we get the conformal anomaly as,

$$\begin{aligned} \mathcal{T} &= -aE_4 - cI_4 \\ &= -a(r^2_{(0)} - 4\mathrm{ric}^2_{(0)} + \mathrm{rim}^2_{(0)}) + c(\frac{1}{3}r^2_{(0)} - 2\mathrm{ric}^2_{(0)} + \mathrm{rim}^2_{(0)}), \end{aligned} \qquad (7.5.63)$$

where the coefficients a and c are given as,

$$\begin{aligned} a &= \frac{1}{128G_5\pi} - \alpha'\frac{5(10\beta_1 + \beta_2 + 2\beta_3)}{128(G_5\pi)} \\ &\quad +\alpha'^2\frac{60\alpha_1 + 153\alpha_2 + 5\left((10\beta_1 + \beta_2 + 2\beta_3)^2 - 24\alpha_3 + 120\alpha_4\right)}{768G_5\pi} \end{aligned} \qquad (7.5.64)$$

and

$$\begin{aligned} c &= \frac{1}{128G_5\pi} - \alpha'\frac{(50\beta_1 - 3\beta_2 + 10\beta_3)}{128G_5\pi} \\ &\quad +\alpha'^2\frac{(500\beta_1^2 - 60\beta_2\beta_1 + 200\beta_3\beta_1 - 11\beta_2^2 + 20\beta_3^2 - 12\beta_2\beta_3)}{768G_5\pi} \\ &\quad -\alpha'^2\frac{(228\alpha_1 - 225\alpha_2 - 72\alpha_3 + 360\alpha_4)}{768G_5\pi}\,. \end{aligned} \qquad (7.5.65)$$

7.6 η, s and $\frac{\eta}{s}$

It is interesting to compute the following combination,

$$\frac{c-a}{c} = 8\alpha'\beta_2 + \frac{4}{3}\alpha'^2(-36\alpha_1 + 9\alpha_2 + 4(6\alpha_3 - 30\alpha_4 + \beta_2(70\beta_1 - 5\beta_2 + 14\beta_3)))\,. \tag{7.6.66}$$

From the above relation (7.6.66) and (7.3.27), (7.4.37), (7.4.38) and (7.4.39), one can see that the the ambiguous coefficients $(\beta_1, \beta_3, \alpha_3, \alpha_4)$ appear in s, η and $\frac{\eta}{s}$ and $\frac{c-a}{c}$ in such a way that one can replace them in terms of this combination of central charges. Hence, we can rewrite η, s and $\frac{\eta}{s}$ as,

$$\eta = 8\pi^3 c\, T^3\left[1 + \frac{1}{4}\frac{c-a}{c} - \frac{1}{8}\left(\frac{c-a}{c}\right)^2 - 180\alpha'^2(2\alpha_1+\alpha_2)\right] + \mathcal{O}(\alpha'^3)\,, \tag{7.6.67}$$

$$s = 32\pi^4 c\, T^3\left[1 + \frac{5}{4}\frac{c-a}{c} + \frac{3}{8}\left(\frac{c-a}{c}\right)^2 + 12\alpha'^2(2\alpha_1+\alpha_2)\right] + \mathcal{O}(\alpha'^3) \tag{7.6.68}$$

and

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{c-a}{c} + \frac{3}{4}\left(\frac{c-a}{c}\right)^2 - 192\alpha'^2(2\alpha_1+\alpha_2)\right] + \mathcal{O}(\alpha'^3)\,. \tag{7.6.69}$$

These are the main results of this paper. Here, we have been able to rewrite shear viscosity η, entropy density s and the ratio $\frac{\eta}{s}$ in terms of central charges c and a of boundary field theory and two other unambiguous parameters α_1 and α_2.

In [102] the authors considered energy correlation function which is quantum expectation value of a product of energy flux operators on the state produced by the localized operator insertion,

$$\langle \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle \equiv \frac{\langle 0|\mathcal{O}^\dagger \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle} \tag{7.6.70}$$

where $\mathcal{O}$ is the operator creating the localized state and $\theta_1\cdots\theta_n$ are the (angular)positions of the calorimeters which measures the total energy per unit angle deposited at each of these angles. In particular they considered energy one point function $\langle\mathcal{E}(\theta)\rangle$ when states are created by stress tensor. This energy one point function is basically three point correlation function of CFT stress tensors. The most general expression for this energy one point function is

$$\langle \mathcal{E}(\theta) \rangle = \frac{\langle 0 | \epsilon^*_{ij} T_{ij} \mathcal{E}(\theta) \epsilon_{lk} T_{lk} | 0 \rangle}{\langle 0 | \epsilon^*_{ij} T_{ij} \epsilon_{lk} T_{lk} | 0 \rangle} = \frac{q^0}{4\pi} \left[1 + \tau_2 \left(\frac{\epsilon^*_{ij} \epsilon_{il} n_j n_l}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{1}{3} \right) + \tau_4 \left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{2}{15} \right) \right] \qquad (7.6.71)$$

where ϵ_{ij} is symmetric polarization tensor and θ is the angle between the point on S^2, labeled by n_i.

There are two undetermined parameters τ_2 and τ_4. In [102], it has been shown that these two parameters can be related to the coefficients multiply higher order gravity correction. When the dual gravity theory is governed by Einstein-Hilbert action (no higher derivative terms) then these two parameters turn out to be zero. In higher derivative bosonic theory when one considers terms like

$$\tau_2 \sim \alpha' \beta_2 + \mathcal{O}(\alpha'^2) \qquad \text{and} \qquad \tau_4 \sim \alpha'^2 f(\alpha_1, \alpha_2) \,,$$

where, f are some linear functions in α_1 and α_2 ($\sim 2\alpha_1 + \alpha_2$). τ_2 is also related to central charges a and c of the theory ($\tau_2 \sim (c-a)/c$). Hence β_2 is fixed in terms of central charges (at order α') [106, 107] and f is fixed in terms of τ_4 at order α'^2. Since all physical quantities depend on a particular combination $2\alpha_1 + \alpha_2$ of unambiguous coefficients therefore we can completely fix them in terms of CFT parameters c, a and τ_4.

Thus we see that the physical measurable quantities η, s and $\frac{\eta}{s}$ of boundary field theory are finally independent of ambiguous parameters and completely depend on physical boundary parameters.

Part IV

Appendices

Appendix A

Normalization and Sign Conventions

In this appendix we shall describe the various normalization and sign conventions we use during our analysis. We begin by describing the ten dimensional action of type IIB string theory that appears in thecfirst description:

$$\begin{aligned} S &= \frac{1}{(2\pi)^7(\alpha')^4}\int d^{10}x\,\sqrt{-g}\Big[e^{-2\Phi}\Big(R+4\partial_M\Phi\partial^M\Phi-\frac{1}{2\cdot 3!}H_{MNP}H^{MNP}\Big) \\ &\quad -\frac{1}{2}F^{(1)}_M F^{(1)M}-\frac{1}{2\cdot 3!}\widetilde{F}^{(3)}_{MNP}\widetilde{F}^{(3)MNP}-\frac{1}{4\cdot 5!}\widetilde{F}^{(5)}_{M_1\cdots M_5}\widetilde{F}^{(5)M_1\cdots M_5}\Big] \\ &\quad +\frac{1}{2(2\pi)^7(\alpha')^4}\int C^{(4)}\wedge\widetilde{F}^{(3)}\wedge H\,, \end{aligned} \tag{A-1}$$

where

$$\begin{gathered} H=dB,\quad F^{(1)}=dC^{(0)},\quad F^{(3)}=dC^{(2)},\quad F^{(5)}=dC^{(4)} \\ \widetilde{F}^{(3)}=F^{(3)}-C^{(0)}H,\quad \widetilde{F}^{(5)}=F^{(5)}-\frac{1}{2}C^{(2)}\wedge H+\frac{1}{2}B\wedge F^{(3)}\,, \end{gathered} \tag{A-2}$$

g_{MN} denotes the string metric, B_{MN} denotes the NSNS 2-form fields, Φ denotes the dilaton and $C^{(k)}$ denotes the RR k-form field. The field strengths $dC^{(k)}$ are subject to the relations $*dC^{(k)}=(-1)^{k(k-1)/2}dC^{(8-k)}+\cdots$ where $*$ denotes Hodge dual taken with respect to the string metric and $\cdots$ denote terms quadratic and higher order in the fields. For $k=4$ this gives a constraint on $C^{(4)}$ whereas for $k>4$ this defines the field $C^{(k)}$. In computing the Hodge dual in the first description we shall use the convention that on $S^1\times\widetilde{S}^1\times\mathbf{R}^{3,1}$ we have $\epsilon^{ty\psi r\theta\phi}>0$ where r, θ, ϕ and t denote the spherical polar coordinates and the time coordinate of the (3+1) dimensional non-compact space-time and

y and ψ denote coordinates of S^1 and $\widetilde{S}^1$ respectively, each normalized to have period $2\pi\sqrt{\alpha'}$. Inside $K3$ we use the standard volume form on K3 to define the ϵ tensor. Our normalization conventions are consistent with that of [24].

The moduli space of $K3$ with NSNS 2-form fields switched on, is labelled by elements of the coset $SO(4,20;\mathbb{Z})\backslash SO(4,20)/(SO(4)\times SO(20))$. These elements may be parametrized by a symmetric SO(4,20) matrix $\widetilde{M}$ and we choose the coordinate system on this coset in such a way that the identity matrix represents a $K3$ of volume $(2\pi\sqrt{\alpha'})^4$ in string metric, with the NSNS 2-form fields set to zero.

The low energy effective action of heterotic string theory on T^4 that appears in the second description has the form:

$$\frac{1}{(2\pi)^3(\alpha')^2}\int d^6x\,\sqrt{-\det g}\,e^{-2\Phi}\Big[R+4\partial_\alpha\Phi\partial^\alpha\Phi-\frac{1}{2\cdot 3!}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} -\frac{1}{8}Tr(\partial_\alpha\widetilde{M}\widetilde{L}\partial^\alpha\widetilde{M}\widetilde{L})-\mathcal{F}^{(a)}_{\alpha\beta}(\widetilde{L}\widetilde{M}\widetilde{L})_{ab}\mathcal{F}^{(b)\alpha\beta}\Big] \tag{A-3}$$

where $\widetilde{L}$ is a fixed 24×24 matrix with 4 positive and 20 negative eigenvalues, $\widetilde{M}$ is a 24×24 symmetric matrix valued scalar field satisfying $\widetilde{M}\widetilde{L}\widetilde{M}=\widetilde{L}$, and $\mathcal{F}^{(a)}_{\alpha\beta}$ for $1\le a\le 24$, $0\le\alpha,\beta\le 5$ are the field strengths associated with 24 U(1) gauge fields $\mathcal{A}^{(a)}_\alpha$ obtained by heterotic string compactification on T^4. The fields $g_{\alpha\beta}$, $B_{\alpha\beta}$ and Φ are the string metric, NSNS 2-form field and the six dimensional dilaton of the heterotic string theory and should be distinguished from those appearing in (A-1). Upon further compactification on $\widehat{S}^1\times S^1$ labelled by $x^4\equiv\chi$ and $x^5\equiv y$, both normalized to have period $2\pi\sqrt{\alpha'}$, we get four more gauge fields $A^{(i)}_\mu$ $(1\le i\le 4, 0\le\mu,\nu\le 3)$ and a 4×4 symmetric matrix valued scalar field $\bar{M}$ defined via the relations:

$$\begin{aligned}
&\widehat{G}_{mn}\equiv g_{mn},\quad \widehat{B}_{mn}\equiv B_{mn}\,,\qquad m,n=4,5\,,\\
&\bar{M}=\begin{pmatrix}\widehat{G}^{-1} & \widehat{G}^{-1}\widehat{B}\\ -\widehat{B}\widehat{G}^{-1} & \widehat{G}-\widehat{B}\widehat{G}^{-1}\widehat{B}\end{pmatrix}\\
&A^{(m-3)}_\mu=\frac{1}{2}(\widehat{G}^{-1})^{mn}G^{(10)}_{m\mu},\quad A^{(m-1)}_\mu=\frac{1}{2}B^{(10)}_{m\mu}-\widehat{B}_{mn}A^{(m-3)}_\mu,\\
&\quad 4\le m,n\le 5,\quad 0\le\mu,\nu\le 3\,.
\end{aligned}\tag{A-4}$$

For simplicity we have set the Wilson lines of the gauge fields $\mathcal{A}_\alpha^{(a)}$ along S^1 and $\widehat{S}^1$ to zero. In the $\alpha' = 16$ unit the electric and magnetic charges $(k_3, \cdots k_6, l_3, \cdots l_6)$ appearing in eq.(2.2.3) are related to the asymptotic values of the gauge field strengths $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ via the relations[61]

$$(\bar{L}\bar{M}\bar{L})_{ij} F^{(j)}_{rt}\bigg|_\infty = \frac{k_{i+2}}{r^2}\,, \qquad \bar{L}_{ij}\, F^{(j)}_{\theta\phi}\bigg|_\infty = l_{i+2}\sin\theta\,, \qquad \bar{L} \equiv \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}. \tag{A-5}$$

The other charges $\widehat{Q}$, k_1, k_2 and $\widehat{P}$, l_1, l_2 appearing in (2.2.3) can be related to the asymptotic values of the gauge field strengths $\mathcal{F}^{(a)}_{rt}$ and $\mathcal{F}^{(a)}_{\theta\phi}$ in a similar manner.

The chain of duality transformations taking us from the first to the second description are chosen so that at the linearized level the first S-duality transformation of IIB acts as $C^{(2)} \to B$, $B \to -C^{(2)}$, and the next $R \to 1/R$ duality transformations of $\widetilde{S}^1$ acts as $g_{\psi\mu} \to -B_{\chi\mu}$, $B_{\psi\mu} \to -g_{\chi\mu}$ together with appropriate transformations on the various RR gauge fields. The final string string duality transformation acts via a Hodge duality transformation in six dimensions on the NS sector 3-form field strength with $\epsilon^{t\chi yr\theta\phi} > 0$, and maps various four dimensional gauge fields arising from various components of the RR sector fields to the 24 gauge fields in heterotic string theory on T^4.

Finally, we use the following convention for the signs of the charges carried by various branes in the first description. If $F^{(3)} \equiv dC^{(2)}$ denotes the RR 3-form field strength, then asymptotically a D1-brane along S^1 will carry positive $F^{(3)}_{yrt}$, a D5-brane along $\widetilde{S}^1 \times K3$ will carry positive $F^{(3)}_{\theta y\phi}$, a D1-brane along $\widetilde{S}^1$ will carry positive $F^{(3)}_{\psi rt}$ and a D5-brane along $S^1 \times K3$ will carry negative $F^{(3)}_{\theta\psi\phi}$. The same convention is followed for fundamental string and NS 5-brane with $F^{(3)}$ replaced by the NSNS 3-form field strength $H = dB$. A state carrying positive momentum along S^1 or $\widetilde{S}^1$ is defined to be the one which produces positive $\partial_r g_{yt}$ or $\partial_r g_{\psi t}$, and a positively charged Kaluza-Klein monopole associated with the circle S^1 or $\widetilde{S}^1$ is defined to be the one that carries positive $\partial_\theta g_{y\phi}$ or $\partial_\theta g_{\psi\phi}$ asymptotically. Note that in this convention the asymptotic configuration for $F^{(7)} \equiv dC^{(6)}$ around a D5-brane wrapped on $K3 \times S^1$ or $K3 \times \widetilde{S}^1$ will have negative $F^{(7)}_{(K3)yrt}$ or $F^{(7)}_{(K3)\psi rt}$, with the subscript $_{(K3)}$ denoting components of $F^{(7)}$ along the volume form of $K3$.

The same conventions are followed for the signs of the charges carried by various states in the second description, with the coordinate ψ of $\widetilde{S}^1$ replaced by the coordinate χ of $\widehat{S}^1$.

Appendix B

Fixing the Normalization Constant

In this appendix we fix the normalization constant Γ. We consider a general class of action for ϕ which appears when the higher derivative terms are made of different contraction of Ricci tensor, Riemann tensor, Weyl tensor, Ricci scalar etc. or their different powers. Since, all these tensors involve two derivatives of metric they can only have terms like $\partial_a \partial_b \phi(r,x)$ and its lower derivatives. Therefor the most generic quadratic (in $\phi(r,x)$, in linear response theory) action for this kind of higher derivative gravity has the following form (in momentum space)[1]

$$\begin{aligned} S &= \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \Big[\mathrm{a1}(r)\phi(r)^2 + \mathrm{a2}(r)\phi'(r)^2 + \mathrm{a4}(r)\phi(r)\phi'(r) \\ &\quad + \mu\, \mathrm{a6}(r)\phi''(r)\phi'(r) + \mu\, \mathrm{a3}(r)\phi''(r)^2 + \mathrm{a5}(r)\phi(r)\phi''(r) \Big] \end{aligned} \tag{B-1}$$

where,

$$\begin{aligned} a1(r) &= \frac{-8r^2 + \omega^2 r + 8}{4r^3 - 4r^5} + \mu\, \mathrm{f2}(r) \\ a2(r) &= -3r + \frac{3}{r} + \mu\, \mathrm{h2}(r) \\ a4(r) &= -\frac{6}{r^2} - 2 + \mu\, \mathrm{g2}(r) \\ a5(r) &= -4r + \frac{4}{r} + \mu\, \mathrm{j2}(r) \end{aligned} \tag{B-2}$$

[1] In all the expressions we have omitted k dependence of ϕ.

and $a3(r), a6(r), j2(r), g2(r), h2(r)$ and $f2(r)$ depends on higher derivative terms in the action. Now let us write the effective Lagrangian as follows,

$$\begin{aligned} S_{\text{eff}} &= \frac{1+\mu\Gamma}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Bigg[\frac{4r\left(r^2-1\right)^2\phi'(r)^2-\omega^2\phi(r)^2}{4r^2\left(r^2-1\right)} \\ &\qquad +\mu\Big(\mathrm{b2}(r)\phi(r)^2+\ \mathrm{b1}(r)\phi'(r)^2\Big)\Bigg]\,. \end{aligned} \tag{B-3}$$

From condition **(a)** of section (6.3) the solutions for $b1$ and $b2$ are given by,

$$\begin{aligned} b1(r) &= \frac{1}{2r\left(r^2-1\right)^2}\big(\left(-4r^3-12r+\omega^2\right)\mathrm{a3}(r) \\ &\quad +\left(r^2-1\right)\left(2\kappa r^4-\mathrm{a6}'(r)r^3-4\kappa r^2+2\mathrm{a3}'(r)r^2\right. \\ &\quad \left.+2\left(r^2-1\right)\mathrm{h2}(r)r-2\left(r^2-1\right)\mathrm{j2}(r)r+\mathrm{a6}'(r)r+2\kappa+2\mathrm{a3}'(r)\right)\big) \end{aligned} \tag{B-4}$$

$$\begin{aligned} b2(r) &= -\frac{1}{16r^2\left(r^2-1\right)^4}\big(\left(\omega^4+144r^3\omega^2\right)\mathrm{a3}(r) \\ &\quad +4\left(r^2-1\right)\left(-4r^2\mathrm{f2}(r)\left(r^2-1\right)^3+\left(2r^2\mathrm{g2}'(r)\left(r^2-1\right)^2\right.\right. \\ &\quad +\left(\omega^2\kappa-2r^2\left(r^2-1\right)\mathrm{j2}''(r)\right)\left(r^2-1\right) \\ &\quad \left.\left.+r\omega^2\mathrm{a3}''(r)\right)\left(r^2-1\right)+\left(1-11r^2\right)\omega^2\mathrm{a3}'(r)\right)\big)\,. \end{aligned} \tag{B-5}$$

The boundary term coming from the original action (after adding *Gibbons-Hawking* boundary terms) are given by,

$$
\begin{aligned}
S^{\mathcal{B}} \;=\; & \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\Bigg[-\frac{\phi(r)^2}{r^2}+\phi(r)^2+r\phi'(r)\phi(r)-\frac{\phi'(r)\phi(r)}{r} \\
& +\mu\Bigg(\frac{1}{2}\mathrm{g2}(r)\phi(r)^2-\frac{1}{2}\mathrm{j2}'(r)\phi(r)^2 \\
& +\mathrm{h2}(r)\phi'(r)\phi(r)-\mathrm{j2}(r)\phi'(r)\phi(r)-\frac{1}{2}\mathrm{a6}'(r)\phi'(r)\phi(r) \\
& +\frac{\mathrm{a3}'(r)\left(\phi(r)\omega^2+4\left(r^4-1\right)\phi'(r)\right)\phi(r)}{4r\left(r^2-1\right)^2} \\
& -\frac{\mathrm{a3}(r)\left(6r\phi(r)\omega^2+\left(r^2-1\right)\left(8r^3+24r-\omega^2\right)\phi'(r)\right)\phi(r)}{4r\left(r^2-1\right)^3} \\
& -\frac{\mathrm{a3}(r)\phi'(r)\left(\phi(r)\omega^2+4\left(r^4-1\right)\phi'(r)\right)}{4r\left(r^2-1\right)^2} \\
& +\mathrm{a3}(r)\phi'(r)\left(-\frac{\phi(r)\omega^2}{2r\left(r^2-1\right)^2}-\frac{\left(r^4-1\right)\phi'(r)}{r\left(r^2-1\right)^2}\right)\Bigg)\Bigg].
\end{aligned}
\tag{B-6}
$$

And the boundary terms coming from the effective action are given by,

$$
\begin{aligned}
S^{\mathcal{B}}_{\text{eff}} \;=\; & \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\Bigg[\left(r-\frac{1}{r}\right)\phi(r)\phi'(r) \\
& +\frac{\mu}{2r\left(r^2-1\right)^2}\Bigg(\phi(r)(2\Gamma\left(r^2-1\right)^3+(-\mathrm{a6}'(r)r^3 \\
& +2\mathrm{a3}'(r)r^2+2\left(r^2-1\right)\mathrm{h2}(r)r-2\left(r^2-1\right)\mathrm{j2}(r)r \\
& +\mathrm{a6}'(r)r+2\mathrm{a3}'(r))\left(r^2-1\right) \\
& +\left(-4r^3-12r+\omega^2\right)\mathrm{a3}(r))\phi'(r)\Bigg)\Bigg].
\end{aligned}
\tag{B-7}
$$

Let the form of the solution of ϕ is given by,

$$\left(1-r^2\right)^{i\beta\omega}\left(1+i\beta\omega\mu F(r)\right) \tag{B-8}$$

with

$$F(0)=0. \tag{B-9}$$

The imaginary part of the retarded Green function for original action is given by,

$$\frac{1}{\omega}\text{Im}[G^{R(\text{original})}_{xy,xy}] = \lim_{r\to 0}\Bigg[-2\beta + \frac{1}{r\left(r^2-1\right)^3}\Big(\mu\text{fi}(4\left(r^2+3\right)\text{a3}(r)r^2 + \left(r^2-1\right)$$
$$\left(F'(r)r^6 + \text{a6}'(r)r^4 - 3F'(r)r^4 - 2\text{a3}'(r)r^3 - 2\left(r^2-1\right)\text{h2}(r)r^2\right.$$
$$\left.+2\left(r^2-1\right)\text{j2}(r)r^2 - \text{a6}'(r)r^2 + 3F'(r)r^2 - 2\text{a3}'(r)r - F'(r)))\Big)\Bigg] \tag{B-10}$$

and imaginary part of the retarded Green function for effective action is given by

$$\begin{aligned}\frac{1}{\omega}\text{Im}[G^{R(\text{effective})}_{xy,xy}] \quad = \quad \lim_{r\to 0}\Bigg[& -2\beta - \mu\Bigg(\frac{1}{\left(r^2-1\right)^3}\Big(\left(r^2-1\right) \\ & (2\Gamma r^4 - \text{a6}'(r)r^3 - 4\Gamma r^2 + 2\text{a3}'(\text{r})\text{r}^2 \\ & +2\left(r^2-1\right)\text{h2}(r)r - 2\left(r^2-1\right)\text{j2}(r)r + \text{a6}'(r)r \\ & +2\Gamma + 2\text{a3}'(r)) - 4r\left(r^2+3\right)\text{a3}(r)\Big) - rF'(r) \\ & + \frac{F'(r)}{r}\Bigg)\beta\Bigg] \,.\end{aligned} \tag{B-11}$$

Therefore, in low frequency limit the difference between the imaginary part of retarded Green function coming from this two boundary terms are given by ,

$$\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}\left[G^{R(\text{original})}_{xy,xy}\right] - \frac{1}{\omega}\text{Im}\left[G^{R(\text{effective})}_{xy,xy}\right] \quad = \quad 2\mu\,\beta\,\Gamma\,. \tag{B-12}$$

Therefore for this general class of theory,

$$\Gamma = 0\,. \tag{B-13}$$

The other kind of higher derivative theory one can consider is covariant derivatives acting on curvature tensors. In that case one can have a more general action like (6.3.19). For this kind of action the boundary terms one gets are of the form $\phi^{(n)}\phi^{(p)}$ (here $\phi^{(n)}$ means n-th derivative of ϕ with respect to r). Using the form of ϕ given in (6.3.39) it can be shown that except

$\phi^{(n)}\phi$ kind of terms, other boundary terms do not contribute in low frequency limit. For example, if we consider $C_n\phi^{(n)^2}$ term in the original action, the the relevant boundary term which will contribute in low frequency limit is $(-1)^{(n+1)}(C_n\phi^{(n)})^{(n-1)}\phi$. One can check that though we need to add *Gibbons-Hawking* terms to make the variation of the action well defined but most of them are zero in low frequency limit. We have checked it for few nontrivial terms like, $\phi^{(3)^2}$ and $\phi^{(4)^2}$ and Γ turns out to be zero. But we expect it is true in general.

Appendix C

Expressions for A^{GB}

$$A_1^{GB}(r,k) = \frac{8r^2 - \omega^2 r - 8}{4r^3\left(r^2-1\right)} - \frac{1}{(12(r^3(r^2-1)^2))}\Big[(10c_1[88r^4 - 11\omega^2 r^3 - 176r^2 + 13\omega^2 r + 88] + c_3[144r^8 - 288r^6 + 66\omega^2 r^5 + 232r^4 + 25\omega^2 r^3 - 4(3\omega^4 + 44)r^2 + 13\omega^2 r + 88] + c_2[176r^4 - 22\omega^2 r^3 - (3\omega^4 + 352)r^2 + 26\omega^2 r + 176])\mu\Big] + O(\mu^2) \tag{C-1}$$

$$A_2^{GB}(r,k) = -\frac{3(r^2-1)}{r} + \frac{(10c_1(13r^2-11) + 2c_2(2r^4 + 17r^2 - 9) + c_3(34r^4 + 9r^2 - 8\omega^2 r - 3))\mu}{r} + O(\mu^2) \tag{C-2}$$

$$A_3^{GB}(r,k) = 4(c_2 + 4c_3)r\left(r^2-1\right)^2\mu + O\left(\mu^2\right) \tag{C-3}$$

$$A_4^{GB}(r,k) = -\frac{2(r^2+3)}{r^2} + \frac{1}{3r^2(r^2-1)}(2(10c_1(13r^4 + 20r^2 - 33) + c_2(26r^4 + 3\omega^2 r^3 + 40r^2 + 3\omega^2 r - 66) + c_3(90r^6 - 89r^4 + 30\omega^2 r^3 + 32r^2 + 6\omega^2 r - 33))\mu) + O\left(\mu^2\right) \tag{C-4}$$

$$A_5^{GB}(r,k) = -\frac{4(r^2-1)}{r}$$
$$+\frac{2(20c_1(13r^2-11)+2c_3(18r^4+r^2-11)+c_2(52r^2+3\omega^2 r-44))\mu}{3r}+O\left(\mu^2\right) \tag{C-5}$$

$$A_6^{GB}(r,k) = 8\left(r^2-1\right)\left(c_2 r^2+4c_3 r^2+c_2\right)\mu+O\left(\mu^2\right)\ . \tag{C-6}$$

Appendix D

Expressions for A^W

$$
\begin{aligned}
A_1^W &= \frac{8r^2 - \omega^2 r - 8}{4r^3\left(r^2-1\right)} \\
&+ \frac{\left(-360r^9 - 240r^7 + 129\omega^2 r^6 + 1560r^5 - 300\omega^2 r^4 + 8\left(\omega^4 - 120\right) r^3 + 75\omega^2\right)\mu}{4\left(r^2-1\right)^2} \\
&\qquad + O\left(\mu^2\right) \\
A_2^W &= -\frac{3\left(r^2-1\right)}{r} + r\left(-419r^6 + 668r^4 - 24\omega^2 r^3 + 8r^2 - 225\right)\mu + O\left(\mu^2\right) \\
A_3^W &= 32r^5\left(r^2-1\right)^2 \mu + O\left(\mu^2\right) \\
A_4^W &= -\frac{2\left(r^2+3\right)}{r^2} \\
&\quad -\frac{2\left(2045r^8 - 4185r^6 - 26\omega^2 r^5 + 2140r^4 - 2\omega^2 r^3 + 75r^2 - 75\right)\mu}{r^2-1} + O\left(\mu^2\right) \\
A_5^W &= -\frac{4\left(r^2-1\right)}{r} - 4\left(r\left(145r^6 - 220r^4 + 2\omega^2 r^3 + 75\right)\right)\mu + O\left(\mu^2\right) \\
A_6^W &= 32r^4\left(2r^4 - 3r^2 + 1\right)\mu + O\left(\mu^2\right) \; . \qquad \text{(D-1)}
\end{aligned}
$$

Appendix E

Expressions for A_i's

Expressions for A_i's in $k \to 0$ limit are given by,

$$
\begin{aligned}
&A_1(r) = -\frac{1}{9r^3}\big(\big(2592\beta_2^2 r^6 + 6048\alpha_3 r^6 - 27648\alpha_4 r^6 + 18432\beta_1\beta_2 r^6 \\
&+5760\beta_2\beta_3 r^6 - 4788\beta_2^2 r^4 - 3168\alpha_3 r^4 + 13392\alpha_4 r^4 - 9288\beta_1\beta_2 r^4 - 3816\beta_2\beta_3 r^4 \\
&+6500\beta_1^2 + 65\beta_2^2 + 260\beta_3^2 - 12\left(816r^6 - 558r^4 + 19\right)\alpha_1 - \left(2880r^6 - 1134r^4 - 171\right)\alpha_2 \\
&+456\alpha_3 - 2280\alpha_4 + 1300\beta_1\beta_2 + 2600\beta_1\beta_3 + 260\beta_2\beta_3\big)\alpha'^2\big) \\
&-\frac{\left(36\beta_2 r^4 + 220\beta_1 + 22\beta_2 + 44\beta_3\right)\alpha'}{3r^3} + \frac{2}{r^3} \\
&A_2(r) = \frac{1}{6r}\big(\big(2736\beta_2^2 r^6 + 1152\alpha_3 r^6 - 3456\alpha_4 r^6 + 2304\beta_1\beta_2 r^6 + 1248\beta_2\beta_3 r^6 - 6996\beta_2^2 r^4 \\
&+48\beta_3^2 r^4 - 432\alpha_3 r^4 + 1104\alpha_4 r^4 - 1416\beta_1\beta_2 r^4 + 240\beta_1\beta_3 r^4 - 2448\beta_2\beta_3 r^4 \\
&+6500\beta_1^2 r^2 + 3701\beta_2^2 r^2 356\beta_3^2 r^2 + 984\alpha_3 r^2 - 2808\alpha_4 r^2 + 1852\beta_1\beta_2 r^2 \\
&+3080\beta_1\beta_3 r^2 + 1436\beta_2\beta_3 r^2 - 6500\beta_1^2 - 17\beta_2^2 - 212\beta_3^2 - 12\big(312r^6 \\
&-318r^4 + 193r^2 + 5\big)\alpha_1 + 9\left(24r^6 + 58r^4 - 69r^2 + 19\right)\alpha_2 - 168\alpha_3 \\
&+1320\alpha_4 - 820\beta_1\beta_2 - 2360\beta_1\beta_3 - 140\beta_2\beta_3\big)\alpha'^2\big) \\
&+\frac{\left(110\left(r^2-1\right)\beta_1 + \left(34r^4 + r^2 - 3\right)\beta_2 + 2\left(2r^4 + 15r^2 - 9\right)\beta_3\right)\alpha'}{r} - 3r + \frac{3}{r} \\
&A_3(r) = \alpha' r\left(r^2-1\right)^2\left(4\beta_2 + \beta_3\right) - 4\alpha'^2 r\left(r^2-1\right)^2\big(16\beta_2^2 r^2 - 4\alpha_3 r^2 \\
&+4\beta_2\beta_3 r^2 - 4\beta_2^2 - 2\beta_3^2 + 24\left(r^2+1\right)\alpha_1 + 3\left(r^2-1\right)\alpha_2 - 20\alpha_3 + 80\alpha_4 \\
&-40\beta_1\beta_2 - 10\beta_1\beta_3 - 9\beta_2\beta_3\big)
\end{aligned}
\tag{E-1}
$$

$$
\begin{aligned}
A_4(r) &= \frac{1}{9r^2}\Big(\big(-72\big(372\alpha_1-2\big(241\beta_2^2+96\beta_1\beta_2+100\beta_3\beta_2+49\alpha_3-144\alpha_4\big)\big)r^6\\
&+\big(2106\alpha_2+36\big(-1049\beta_2^2-138\beta_1\beta_2-338\beta_3\beta_2+642\alpha_1-104\alpha_3+132\alpha_4\big)\big)r^4\\
&+\big(6500\beta_1^2+1372\beta_2\beta_1+2600\beta_3\beta_1+3557\beta_2^2+260\beta_3^2-2028\alpha_1-477\alpha_2+600\alpha_3\\
&-1848\alpha_4+1196\beta_2\beta_3\big)r^2+19500\beta_1^2+195\beta_2^2+780\beta_3^2-684\alpha_1-513\alpha_2+1368\alpha_3\\
&-6840\alpha_4+3900\beta_1\beta_2+7800\beta_1\beta_3+780\beta_2\beta_3\big)\alpha'^2\Big)\\
&+\frac{2\big(110\big(r^2+3\big)\beta_1+\big(90r^4-7r^2+33\big)\beta_2+22\big(r^2+3\big)\beta_3\big)\alpha'}{3r^2}-\frac{6}{r^2}-2\\
A_5(r) &= \frac{2\big(r^2-1\big)}{9r}\big(3600\beta_2^2r^4+1008\alpha_3r^4-3456\alpha_4r^4+2304\beta_1\beta_2r^4+1728\beta_2\beta_3r^4\\
&-3492\beta_2^2r^2-144\alpha_3r^2-432\alpha_4r^2-72\beta_1\beta_2r^2-936\beta_2\beta_3r^2+6500\beta_1^2+65\beta_2^2\\
&+260\beta_3^2-12\big(264r^4-150r^2+19\big)\alpha_1+9\big(16r^4+34r^2-19\big)\alpha_2+456\alpha_3\\
&-2280\alpha_4+1300\beta_1\beta_2+2600\beta_1\beta_3+260\beta_2\beta_3\big)\alpha'^2\\
&+\frac{4\big(r^2-1\big)\big(110\beta_1+\big(18r^2+11\big)\beta_2+22\beta_3\big)\alpha'}{3r}-4r+\frac{4}{r}\\
A_6(r) &= 8\big(r^2-1\big)\big(-\big(24\beta_2^2+8\beta_3\beta_2+24\alpha_1-3\alpha_2-6\alpha_3\big)r^4+\big(12\beta_2^2+40\beta_1\beta_2\\
&+9\beta_3\beta_2+2\beta_3^2-24\alpha_1+6\alpha_2+22\alpha_3-80\alpha_4\\
&+10\beta_1\beta_3\big)r^2+2\beta_3^2-3\alpha_2+4\alpha_3+10\beta_1\beta_3+\beta_2\beta_3\big)\alpha'^2\\
&+8\big(r^2-1\big)\big((4\beta_2+\beta_3)r^2+\beta_3\big)\alpha'\,. \qquad \text{(E-2)}
\end{aligned}
$$

Appendix F

Expressions for $\mathcal{B}_0$ and $\mathcal{B}_1$

$$
\begin{aligned}
\mathcal{B}_0^{(0)} &= \frac{\left(130\beta_1\omega^2 + 6r^2\beta_2\omega^2 - 11\beta_2\omega^2 + 26\beta_3\omega^2\right)}{12r^2\left(r^2-1\right)} \\
\mathcal{B}_0^{(1)} &= \frac{1}{72r^2\left(r^2-1\right)}\left(\left(-4248\beta_2^2\omega^2r^4 + 12240\alpha_1\omega^2r^4 + 5148\alpha_2\omega^2r^4 - 432\alpha_3\omega^2r^4\right.\right. \\
&+864\alpha_4\omega^2r^4 - 576\beta_1\beta_2\omega^2r^4 - 1152\beta_2\beta_3\omega^2r^4 + 5916\beta_2^2\omega^2r^2 - 6984\alpha_1\omega^2r^2 \\
&-1818\alpha_2\omega^2r^2 + 864\alpha_3\omega^2r^2 - 2448\alpha_4\omega^2r^2 + 1272\beta_1\beta_2\omega^2r^2 + 1752\beta_2\beta_3\omega^2r^2 \\
&+2900\beta_1^2\omega^2 - 19\beta_2^2\omega^2 + 116\beta_3^2\omega^2 + 660\alpha_1\omega^2 - 369\alpha_2\omega^2 - 168\alpha_3\omega^2 + 840\alpha_4\omega^2 \\
&\left.\left.+100\beta_1\beta_2\omega^2 + 1160\beta_1\beta_3\omega^2 + 20\beta_2\beta_3\omega^2\right)\right)
\end{aligned}
\tag{F-1}
$$

$$
\begin{aligned}
\mathcal{B}_1^{(0)} &= -\frac{\left(r^2-1\right)\left(18\beta_2r^2 + 110\beta_1 - 13\beta_2 + 22\beta_3\right)}{3r} \\
\mathcal{B}_1^{(1)} &= \frac{1}{18r}\left(\left(r^2-1\right)\left(-15408\beta_2^2r^4 + 3168\alpha_1r^4 - 2304\alpha_2r^4 - 1728\alpha_3r^4 + 3456\alpha_4r^4\right.\right. \\
&-2304\beta_1\beta_2r^4 - 4608\beta_2\beta_3r^4 + 12420\beta_2^2r^2 - 6984\alpha_1r^2 - 1170\alpha_2r^2 + 720\alpha_3r^2 \\
&+432\alpha_4r^2 + 72\beta_1\beta_2r^2 + 3240\beta_2\beta_3r^2 - 6500\beta_1^2 + 79\beta_2^2 - 260\beta_3^2 - 636\alpha_1 + 387\alpha_2 \\
&\left.\left.+120\alpha_3 - 600\alpha_4 + 140\beta_1\beta_2 - 2600\beta_1\beta_3 + 28\beta_2\beta_3\right)\alpha'\right)\,.
\end{aligned}
\tag{F-2}
$$

Appendix G

Shear Viscosity from Kubo's Formula

The shear viscosity coefficient of boundary fluid is related to the imaginary part of retarded Green function in low frequency limit. The retarded Green function $G^R_{xy,xy}(k)$ is defined in the following way. The on-shell action for graviton can be written as a surface term,

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\,\phi_0(k)\,\mathcal{G}_{xy,xy}(k,r)\,\phi_0(-k)\Big|_{r=0} \\ &= \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\,\mathcal{F}_{xy,xy}(k) \end{aligned} \tag{G-1}$$

where $\phi_0(k)$ is the boundary value of $\phi(r,k)$ and $G^R_{xy,xy}$ is given by,

$$G^R_{xy,xy}(k) = \lim_{r\to 0} 2\mathcal{G}_{xy,xy}(k,r) \tag{G-2}$$

and shear viscosity coefficient is given by,

$$\eta = \lim_{\omega\to 0}\left[\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(k)\right]. \tag{G-3}$$

To calculate this number one has to know the exact solution, *i.e.* the form of $\phi(r,k)$. The solution for $\phi(r,k)$ up to order α'^2 is given by,

$$\begin{aligned} \phi(r,k) &= 1 - i\beta\omega\log\left(1-r^2\right) - 6i\alpha'\beta\beta_2\omega r^2 + 2i\alpha'^2\beta\big(-223\beta_2^2 r^2 - 24\alpha_3 r^2 + 48\alpha_4 r^2 \\ &\quad -32\beta_1\beta_2 r^2 - 64\beta_2\beta_3 r^2 - 70\beta_2^2 + 2(22r^2-53)\alpha_1 - (32r^2+\frac{193}{2})\alpha_2 \\ &\quad -28\alpha_3 + 108\alpha_4 - 172\beta_1\beta_2 - 60\beta_2\beta_3\big)\omega r^2 \end{aligned} \tag{G-4}$$

where,

$$\beta = \sqrt{-\frac{g_{rr}}{g_{tt}}(1-r)^2}\,. \tag{G-5}$$

With this solution we calculate $\mathcal{F}_{xy,xy}(k)$ after adding proper $Gibbons - Hawking$ boundary terms to the action (7.4.29). Then we find shear viscosity coefficient η from imaginary part of $\mathcal{F}_{xy,xy}(k)$ following equation (G-3). It turns out that,

$$\begin{aligned}\eta &= \frac{1}{16\pi G_5} - \frac{(5\beta_1+\beta_3)\alpha'}{2\pi G_5}\\ &\quad - \frac{\left(108\alpha_1+63\alpha_2+12\alpha_3-42\alpha_4+100\beta_1^2\right)\alpha'^2}{6\pi G_5}\\ &\quad + \frac{\left(28\beta_2\beta_1+40\beta_3\beta_1+48\beta_2^2+4\beta_3^2+20\beta_2\beta_3\right)\alpha'^2}{6\pi G_5}\,.\end{aligned} \tag{G-6}$$

Appendix H

Leading $r-$Dependence of Curvature Tensors

In this appendix, we give the $r-$dependence of various Riemann and Ricci tensors. As discussed in section (7.5) below equation (7.5.48), while computing the four and six derivative terms, we need to keep those terms up to order r^2. If for some combinations, the leading $r-$dependence starts from order r^3, they will not contribute to anomaly.

$$\begin{aligned} R_{ijkl} &= r^{-2}[g_{(0)_{il}}g_{(0)_{jk}} - g_{(0)_{ik}}g_{(0)_{jl}}] \\ &+r^{-1}r^{(0)}_{ijkl} \\ &+[g_{(0)_{ik}}(g_{(2)}+h_{(2)})_{jl} + g_{(0)_{jl}}(g_{(2)}+h_{(2)})_{ik} \\ &-g_{(0)_{il}}(g_{(2)}+h_{(2)})_{jk} - g_{(0)_{jk}}(g_{(2)}+h_{(2)})_{il}] + [\nabla^0_k\delta\Gamma_{ijl} - \nabla^{(0)}_l\delta\Gamma_{ijk}] \\ &+[g_{(2)_{in}}r^{(0)n}_{\ \ jkl}] + \mathcal{O}(r)\ . \end{aligned} \tag{H-1}$$

$$R_{rijk} = r^{-1}[-\frac{1}{2}(\nabla_j g_{(2)_{ik}} - \nabla_k g(2)_{ij})] + \mathcal{O}(1)\ . \tag{H-2}$$

$$\begin{aligned} R^r_{\ irj} &= r^{-1}[-g(0)_{ij}] \\ &+ r^0[-g_{(0)_{ij}}] \\ &+ r^{+1}[-5(g_{(2)}+h_{(2)})_{ij} + (g_{(2)})^2_{ij}] + \mathcal{O}(r^2)\ , \end{aligned} \tag{H-3}$$

$$\begin{aligned}
R^{ij}{}_{kl} &= \mathcal{O}(1) & R^{ir}{}_{kr} &= \mathcal{O}(1) \\
R^{ir}{}_{kl} &= \mathcal{O}(r^2) & R^{ij}{}_{kr} &= \mathcal{O}(r) \\
R^{r}_{r} &= \mathcal{O}(1) & R^{i}_{r} &= \mathcal{O}(r) \\
R^{r}_{i} &= \mathcal{O}(r^2) & R^{i}_{j} &= \mathcal{O}(1)\,. \qquad \text{(H-4)}
\end{aligned}$$

Bibliography

[1] S. W. Hawking and Ellis, "The large scale structure of space time" Cambridge University Press (1973).

[2] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973).

[3] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," Phys. Rev. D **9**, 3292 (1974).

[4] S.W. Hawking, "Generalized second law of thermodynamics in black hole physics," Commun. Math. Phys. 43 (1975).

[5] B. Zwiebach, "A First Course in String Theory," Cambridge University Press (2004).

[6] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," Phys. Lett. B **379**, 99 (1996) [arXiv:hep-th/9601029].

[7] A. Sen, "Extremal black holes and elementary string states," Mod. Phys. Lett. A **10**, 2081 (1995) [arXiv:hep-th/9504147].

[8] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. B **391**, 93 (1997) [arXiv:hep-th/9602065].

[9] A. Sen, "How does a fundamental string stretch its horizon?," JHEP **0505**, 059 (2005) [arXiv:hep-th/0411255].

[10] A. Dabholkar, "Exact counting of black hole microstates," Phys. Rev. Lett. **94**, 241301 (2005) [arXiv:hep-th/0409148].

[11] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[12] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[13] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[14] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[15] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.
O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[16] S. S. Gubser, I. R. Klebanov and A. W. Peet, "Entropy and Temperature of Black 3-Branes," Phys. Rev. D **54**, 3915 (1996) [arXiv:hep-th/9602135].

[17] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, "Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **534**, 202 (1998) [arXiv:hep-th/9805156].

[18] A. Fotopoulos and T. R. Taylor, "Comment on two-loop free energy in N = 4 supersymmetric Yang-Mills theory at finite temperature," Phys. Rev. D **59**, 061701 (1999) [arXiv:hep-th/9811224].

[19] R. M. Wald, "Black hole entropy in the Noether charge," Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].

[20] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," JHEP **0509**, 038 (2005) [arXiv:hep-th/0506177].

[21] A. Sen, "Quantum Entropy Function from AdS(2)/CFT(1) Correspondence," arXiv:0809.3304 [hep-th].

[22] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, "Counting dyons in N = 4 string theory," Nucl. Phys. B **484**, 543 (1997) [arXiv:hep-th/9607026].

[23] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy," JHEP **0412**, 075 (2004) [arXiv:hep-th/0412287].

[24] R. C. Myers, "Dielectric-branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[25] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N = 4 string theory," arXiv:hep-th/0505094.

[26] D. Brill, Phys. Rev. B133 (1964) 845.

[27] C. N. Pope, "Axial Vector Anomalies And The Index Theorem In Charged Schwarzschild And Taub - Nut Spaces," Nucl. Phys. B **141**, 432 (1978).

[28] A. Sen, "Kaluza-Klein dyons in string theory," Phys. Rev. Lett. **79**, 1619 (1997) [arXiv:hep-th/9705212].

[29] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[30] D. Gaiotto, "Re-recounting dyons in N = 4 string theory," arXiv:hep-th/0506249.

[31] N. Nekrasov and A. S. Schwarz, "Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory," Commun. Math. Phys. **198**, 689 (1998) [arXiv:hep-th/9802068].

[32] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," Commun. Math. Phys. **185**, 197 (1997) [arXiv:hep-th/9608096].

[33] D. Shih and X. Yin, "Exact black hole degeneracies and the topological string," arXiv:hep-th/0508174.

[34] D. P. Jatkar and A. Sen, "Dyon spectrum in CHL models," JHEP **0604**, 018 (2006) [arXiv:hep-th/0510147].

[35] J. R. David, D. P. Jatkar and A. Sen, "Product representation of dyon partition function in CHL models," JHEP **0606**, 064 (2006) [arXiv:hep-th/0602254].

[36] A. Dabholkar and S. Nampuri, "Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift," arXiv:hep-th/0603066.

[37] J. R. David and A. Sen, "CHL dyons and statistical entropy function from D1-D5 system," arXiv:hep-th/0605210.

[38] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in N = 4 supersymmetric type II string theories," arXiv:hep-th/0607155.

[39] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in generic N = 4 supersymmetric Z(N) orbifolds," arXiv:hep-th/0609109.

[40] A. Dabholkar and D. Gaiotto, "Spectrum of CHL dyons from genus-two partition function," arXiv:hep-th/0612011.

[41] A. Sen, "Walls of marginal stability and dyon spectrum in N = 4 supersymmetric string theories," arXiv:hep-th/0702141.

[42] A. Dabholkar, D. Gaiotto and S. Nampuri, "Comments on the spectrum of CHL dyons," arXiv:hep-th/0702150.

[43] N. Banerjee, D. P. Jatkar and A. Sen, "Adding Charges to N=4 Dyons," arXiv:0705.1433 [hep-th].

[44] A. Sen, "Two Centered Black Holes and N=4 Dyon Spectrum," arXiv:0705.3874 [hep-th].

[45] M. C. N. Cheng and E. Verlinde, "Dying Dyons Don't Count," arXiv:0706.2363 [hep-th].

[46] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," arXiv:0708.1270 [hep-th].

[47] S. Banerjee, A. Sen and Y. K. Srivastava, "Generalities of Quarter BPS Dyon Partition Function and Dyons of Torsion Two," JHEP **0805**, 101 (2008) [arXiv:0802.0544 [hep-th]].

[48] S. Banerjee, A. Sen and Y. K. Srivastava, "Partition Functions of Torsion ¿1 Dyons in Heterotic String Theory on T^6," JHEP **0805**, 098 (2008) [arXiv:0802.1556 [hep-th]].

[49] A. Dabholkar, J. Gomes and S. Murthy, "Counting all dyons in N =4 string theory," arXiv:0803.2692 [hep-th].

[50] M. C. N. Cheng and E. P. Verlinde, "Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra," arXiv:0806.2337 [hep-th].

[51] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody Algebras from CHL dyons," arXiv:0807.4451 [hep-th].

[52] S. Banerjee, A. Sen and Y. K. Srivastava, "Genus Two Surface and Quarter BPS Dyons: The Contour Prescription," arXiv:0808.1746 [hep-th].

[53] M. C. N. Cheng and A. Dabholkar, "Borcherds-Kac-Moody Symmetry of N=4 Dyons," arXiv:0809.4258 [hep-th].

[54] J. R. David, "On the dyon partition function in N=2 theories," JHEP **0802**, 025 (2008) [arXiv:0711.1971 [hep-th]].

[55] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, "A black hole farey tail," arXiv:hep-th/0005003.

[56] J. Manschot and G. W. Moore, "A Modern Farey Tail," arXiv:0712.0573 [hep-th].

[57] A. Gregori, E. Kiritsis, C. Kounnas, N. A. Obers, P. M. Petropoulos and B. Pioline, "R**2 corrections and non-perturbative dualities of N = 4 string ground states," Nucl. Phys. B **510**, 423 (1998) [arXiv:hep-th/9708062].

[58] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Corrections to macroscopic supersymmetric black-hole entropy," Phys. Lett. B **451**, 309 (1999) [arXiv:hep-th/9812082].

[59] G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes,” Nucl. Phys. B **567**, 87 (2000) [arXiv:hep-th/9906094].

[60] T. Mohaupt, “Black hole entropy, special geometry and strings,” Fortsch. Phys. **49**, 3 (2001) [arXiv:hep-th/0007195].

[61] A. Sen, “Entropy function for heterotic black holes,” JHEP **0603**, 008 (2006) [arXiv:hep-th/0508042].

[62] A. Sen, “Entropy Function and AdS_2/CFT_1 Correspondence,” arXiv:0805.0095v4 [hep-th].

[63] R. K. Gupta and A. Sen, “Ads(3)/CFT(2) to Ads(2)/CFT(1),” arXiv:0806.0053 [hep-th].

[64] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,” arXiv:hep-th/0405146.

[65] B. de Wit, talk at Strings 2005

[66] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, “Black hole partition functions and duality,” arXiv:hep-th/0601108.

[67] G. L. Cardoso, B. de Wit and S. Mahapatra, “Subleading and non-holomorphic corrections to N=2 BPS black hole entropy,” arXiv:0808.2627 [hep-th].

[68] J. M. Maldacena, J. Michelson and A. Strominger, “Anti-de Sitter fragmentation,” JHEP **9902**, 011 (1999) [arXiv:hep-th/9812073].

[69] A. Castro, D. Grumiller, F. Larsen and R. McNees, “Holographic Description of AdS_2 Black Holes,” arXiv:0809.4264 [hep-th].

[70] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, “Rotating attractors,” JHEP **0610**, 058 (2006) [arXiv:hep-th/0606244].

[71] C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Strominger and X. Yin, “Why Z(BH) = —Z(top)—**2,” arXiv:hep-th/0608021.

[72] M. Guica, L. Huang, W. W. Li and A. Strominger, “R**2 corrections for 5D black holes and rings,” JHEP **0610**, 036 (2006) [arXiv:hep-th/0505188].

[73] A. Castro, J. L. Davis, P. Kraus and F. Larsen, “5D Black Holes and Strings with Higher Derivatives,” JHEP **0706**, 007 (2007) [arXiv:hep-th/0703087].

[74] A. Castro and S. Murthy, “Corrections to the statistical entropy of five dimensional black holes,” arXiv:0807.0237 [hep-th].

[75] A. Sen, "Wall Crossing Formula for N=4 Dyons: A Macroscopic Derivation," JHEP **0807**, 078 (2008) [arXiv:0803.3857 [hep-th]].

[76] G. Policastro, D. T. Son and A. O. Starinets, "The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma," Phys. Rev. Lett. **87**, 081601 (2001) [arXiv:hep-th/0104066].

[77] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007) [arXiv:0704.0240 [hep-th]].

[78] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231].

[79] P. K. Kovtun and A. O. Starinets, "Quasinormal modes and holography," Phys. Rev. D **72**, 086009 (2005) [arXiv:hep-th/0506184].

[80] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051].

[81] C. P. Herzog and D. T. Son, "Schwinger-Keldysh propagators from AdS/CFT correspondence," JHEP **0303**, 046 (2003) [arXiv:hep-th/0212072].

[82] R. A. Janik and R. B. Peschanski, "Asymptotic perfect fluid dynamics as a consequence of AdS/CFT," Phys. Rev. D **73**, 045013 (2006) [arXiv:hep-th/0512162].
R. A. Janik and R. B. Peschanski, "Gauge / gravity duality and thermalization of a boost-invariant perfect fluid," Phys. Rev. D **74**, 046007 (2006) [arXiv:hep-th/0606149].

[83] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]].

[84] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi and S. R. Wadia, "Forced Fluid Dynamics from Gravity," arXiv:0806.0006 [hep-th]. = ARXIV:0806.0006;

[85] M. Haack and A. Yarom, "Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT," arXiv:0806.4602 [hep-th].

[86] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812**, 116 (2008) [arXiv:0809.4272 [hep-th]].

[87] M. Van Raamsdonk, "Black Hole Dynamics From Atmospheric Science, `arXiv:0802.3224 [hep-th]`.

[88] R. Loganayagam, "Entropy Current in Conformal Hydrodynamics," `arXiv:0801.3701 [hep-th]`.

[89] M. P. Heller, P. Surowka, R. Loganayagam, M. Spalinski and S. E. Vazquez, "On a consistent AdS/CFT description of boost-invariant plasma," arXiv:0805.3774 [hep-th].

[90] I. P. Neupane and N. Dadhich, "Higher Curvature Gravity: Entropy Bound and Causality Violation," arXiv:0808.1919 [hep-th].

[91] J. J. Friess, S. S. Gubser, G. Michalogiorgakis, and S. S. Pufu, "Expanding plasmas and quasinormal modes of anti-de Sitter black holes," *JHEP* **04** (2007) 080, `hep-th/0611005`.

[92] C. P. Herzog, "The hydrodynamics of M-theory," *JHEP* **12** (2002) 026, `hep-th/0210126`.

[93] G. Policastro, D. T. Son, and A. O. Starinets, "From AdS/CFT correspondence to hydrodynamics. II: Sound waves," *JHEP* **12** (2002) 054, `hep-th/0210220`.

[94] R. K. Gupta and A. Mukhopadhyay, "On the universal hydrodynamics of strongly coupled CFTs with gravity duals," arXiv:0810.4851 [hep-th].

[95] C. P. Herzog, "The sound of M-theory," *Phys. Rev.* **D68** (2003) 024013, `hep-th/0302086`.

[96] A. Buchel, R. C. Myers, M. F. Paulos and A. Sinha, "Universal holographic hydrodynamics at finite coupling," arXiv:0808.1837 [hep-th].

[97] A. Buchel, R. C. Myers and A. Sinha, "Beyond eta/s = 1/4pi," arXiv:0812.2521 [hep-th].

[98] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," arXiv:0809.2596 [hep-th].

[99] A. Buchel, R. C. Myers and A. Sinha, "Beyond eta/s = 1/4pi," JHEP **0903**, 084 (2009) [arXiv:0812.2521 [hep-th]].

[100] D. J. Gross and E. Witten, "Superstring Modifications Of Einstein's Equations," Nucl. Phys. B **277**, 1 (1986).
M. T. Grisaru, A. E. M. van de Ven and D. Zanon, "Four Loop Beta Function For The N=1 And N=2 Supersymmetric Nonlinear Sigma Model In Two-Dimensions," Phys. Lett. B **173**, 423 (1986).
D. J. Gross and J. H. Sloan, "The Quartic Effective Action for the Heterotic String," Nucl. Phys. B **291**, 41 (1987).

M. B. Green, M. Gutperle and P. Vanhove, "One loop in eleven dimensions," Phys. Lett. B **409**, 177 (1997) [arXiv:hep-th/9706175].
F. Moura, "Type II and heterotic one loop string effective actions in four dimensions," JHEP **0706**, 052 (2007) [arXiv:hep-th/0703026].

[101] R. R. Metsaev and A. A. Tseytlin, "CURVATURE CUBED TERMS IN STRING THEORY EFFECTIVE ACTIONS," Phys. Lett. B **185**, 52 (1987).

[102] D. M. Hofman and J. Maldacena, "Conformal collider physics: Energy and charge correlations," JHEP **0805**, 012 (2008) [arXiv:0803.1467 [hep-th]].

[103] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, "Fluid dynamics of R-charged black holes," arXiv:0809.2488 [hep-th].

[104] R. C. Myers, M. F. Paulos and A. Sinha, "Quantum corrections to eta/s," arXiv:0806.2156 [hep-th].

[105] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[106] S. Nojiri and S. D. Odintsov, "On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence," Int. J. Mod. Phys. A **15**, 413 (2000) [arXiv:hep-th/9903033].

[107] M. Blau, K. S. Narain and E. Gava, "On subleading contributions to the AdS/CFT trace anomaly," JHEP **9909**, 018 (1999) [arXiv:hep-th/9904179].

[108] N. Deruelle and J. Madore, "On the quasi-linearity of the Einstein-'Gauss-Bonnet' gravity field equations," arXiv:gr-qc/0305004.

[109] A. Buchel and M. Paulos, "Second order hydrodynamics of a CFT plasma from boost invariant expansion," arXiv:0808.1601 [hep-th].

[110] E. Shuryak, "Why does the quark gluon plasma at RHIC behave as a nearly ideal fluid?," *Prog. Part. Nucl. Phys.* **53** (2004) 273–303, `hep-ph/0312227`.

[111] E. V. Shuryak, "What RHIC experiments and theory tell us about properties of quark-gluon plasma?," *Nucl. Phys.* **A750** (2005) 64–83, `hep-ph/0405066`.

[112] E. V. Shuryak, "Strongly coupled quark-gluon plasma: The status report," `hep-ph/0608177`.

[113] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, "Relativistic viscous hydrodynamics, conformal invariance, and holography," JHEP **0804**, 100 (2008) [arXiv:0712.2451 [hep-th]].

[114] M. Natsuume and T. Okamura, "Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality," Phys. Rev. D **77**, 066014 (2008) [Erratum-ibid. D **78**, 089902 (2008)] [arXiv:0712.2916 [hep-th]].

[115] S. S. Gubser and A. Yarom, "Linearized hydrodynamics from probe-sources in the gauge-string duality," arXiv:0803.0081 [hep-th].

[116] Y. Kats and P. Petrov, "Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory," arXiv:0712.0743 [hep-th].

[117] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, "Viscosity Bound Violation in Higher Derivative Gravity," Phys. Rev. D **77**, 126006 (2008) [arXiv:0712.0805 [hep-th]].

[118] R. Brustein and A. J. M. Medved, "The ratio of shear viscosity to entropy density in generalized theories of gravity," arXiv:0808.3498 [hep-th].

[119] R. Brustein and A. J. M. Medved, "The shear diffusion coefficient for generalized theories of gravity," arXiv:0810.2193 [hep-th].

[120] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," arXiv:0809.3808 [hep-th].

[121] P. Kovtun, D. T. Son and A. O. Starinets, "Holography and hydrodynamics: Diffusion on stretched horizons," JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].

[122] R. G. Cai, Z. Y. Nie and Y. W. Sun, "Shear Viscosity from Effective Couplings of Gravitons," Phys. Rev. D **78**, 126007 (2008) [arXiv:0811.1665 [hep-th]].

[123] R. G. Cai, Z. Y. Nie, N. Ohta and Y. W. Sun, "Shear Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling," arXiv:0901.1421 [hep-th].

[124] N. Banerjee and S. Dutta, "Higher Derivative Corrections to Shear Viscosity from Graviton's Effective Coupling," JHEP **0903**, 116 (2009) [arXiv:0901.3848 [hep-th]].

[125] A. Buchel, "Shear viscosity of boost invariant plasma at finite coupling," Nucl. Phys. B **802**, 281 (2008) [arXiv:0801.4421 [hep-th]].
A. Buchel, "Resolving disagreement for eta/s in a CFT plasma at finite coupling," Nucl. Phys. B **803**, 166 (2008) [arXiv:0805.2683 [hep-th]].

[126] S. Dutta, "Higher Derivative Corrections to Locally Black Brane Metrics," JHEP **0805**, 082 (2008) [arXiv:0804.2453 [hep-th]].

[127] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, "Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **534**, 202 (1998) [arXiv:hep-th/9805156].

[128] I. Jack, D. R. T. Jones and D. A. Ross, "On The Relationship Between String Low-Energy Effective Actions And O (Alpha-Prime**3) Sigma Model Beta Functions," Nucl. Phys. B **307**, 130 (1988).

[129] R. M. Wald, "Black hole entropy is the Noether charge," Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].

[130] S. Dutta and R. Gopakumar, "On Euclidean and noetherian entropies in AdS space," Phys. Rev. D **74**, 044007 (2006) [arXiv:hep-th/0604070].
D. Astefanesei, N. Banerjee and S. Dutta, "(Un)attractor black holes in higher derivative AdS gravity," JHEP **0811**, 070 (2008) [arXiv:0806.1334 [hep-th]].

Index

AdS/CFT correspondence, 4, 5, 9, 10, 13, 14, 51, 91

black hole, 5, 8, 35, 36, 68, 79, 92
  BPS, 6
  dyonic, 7, 73, 77
  entropy, 5, 9, 51
  four dimensional, 7
  thermodynamics, 4
black hole, 3, 4, 6, 14, 35–37, 45, 46, 50–52, 59, 60, 63–65, 67–70, 72–74, 77, 78, 92, 95, 123
  BMPV, 67, 72
  BPS, 51, 61
  dyonic, 8, 67
  entropy, 4, 5, 9, 37, 40, 51
  entropy function, 73
  entropy of extremal, 36
  extremal, 35, 51, 58, 59, 65, 69
  five dimensional, 6, 79
  five dimensional, 67
  microstates, 4, 9
  multi centered, 36, 39, 45
  single centered, 50
  single centered, 36, 45
  temperature, 105, 108
  thermodynamics, 9

dyon, 7, 8, 19, 22, 30, 32, 36–38, 67–69, 73, 77
  CHL, 77

entropy function
  quantum, 63
  statistical, 50, 68
entropy function, 35, 53, 68, 75
  black hole, 53
  quantum, 9, 36, 37, 51, 58, 59, 61, 62
  quantum , 53
  statistical, 43, 49, 50, 53, 68, 73, 75, 76, 78

fluid, 13, 94, 103–105, 111
  boundary, 14, 91–94, 96, 101–105, 110, 111
  relativistic viscous conformal, 13
  viscous, 13

horizon, 3, 8, 11, 13, 14, 92–96, 102–105, 108, 110, 111, 123, 127
  area, 3, 8
  near, 36, 37, 53, 58–60, 63–65, 103

stretched, 8
hydrodynamics, 5, 9, 13, 14, 91, 115

string, 5, 7, 24, 141
coupling, 5, 10, 29, 31–33, 36
Dirichlet, 21, 24
duality, 20, 141
entropy, 5, 7
field, 61
fields, 53
fundamental winding, 21
magnetically charged, 24
metric, 33, 139, 140
states, 7, 8, 20
tension, 5, 69
type IIB, 21, 24, 26
string theory
heterotic, 20
type IIA, 20
type IIB, 20
string theory, 4–7, 9, 11–13, 35, 36, 38, 41, 51, 58, 59, 63, 77, 114
bosonic, 115
closed, 4, 9
heterotic, 20, 40, 43, 69, 114, 140, 141
open, 4, 7
supersymmetric, 8, 19, 36–38, 40, 61, 67, 69, 71
type II, 19
type IIB, 139
type II, 19, 64, 94, 108, 114
type IIB, 4, 9, 10, 20, 21, 24, 25, 27–29, 31, 40, 50
superstring, 6, 9