On Tannakian Fundamental Group Schemes, and on Real Parabolic Vector Bundles over a Real Curve

By

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Certificate

This is to certify that the Ph.D. thesis titled "**On Tannakian Fundamental Group Schemes, and on Real Parabolic Vector Bundles over a Real Curve**" by **Sanjaykumar Hansraj Amrutiya** is a record of bonafide research work done under my supervision. It is further certified that the thesis represents independent and original work by the candidate.

Thesis Supervisor: ______________________________

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Declaration

The author hereby declares that the work in the thesis titled "**On Tannakian Fundamental Group Schemes, and on Real Parabolic Vector Bundles over a Real Curve**," submitted for Ph.D. degree to the **Homi Bhabha National Institute** has been carried out at Harish-Chandra Research Institute under the supervision of Professor N. Raghavendra. Wherever contributions of others are involved, every effort has been made to indicate that clearly, with due reference to the literature. The author attests that the work is original and has not been submitted in part or full for any degree or diploma to any other institute or university.

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Abstract

This thesis is divided into two parts.

In the first part, we consider Frobenius-finite vector bundles on an arbitrary pointed smooth projective variety (X, x) over a perfect field k of prime characteristic. Using these bundles, we construct a neutral Tannakian category $\mathcal{C}_F(X)$ over k. By a well-known theorem, there is an affine group scheme over k associated to $\mathcal{C}_F(X)$. This group scheme will be denoted by $\pi_F(X,\, x)$, and is called the F–fundamental group scheme of (X, x). We study the properties of this affine group scheme.

In the second part, we define parabolic structures on real vector bundles over a real curve. Let (X, σ_X) be a real algebraic curve, and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be an integer. We construct an N-fold cyclic cover $p : Y \longrightarrow X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ of p, and $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$. The main theorem in this part gives an equivalence between the category of real parabolic vector bundles over X with parabolic structure over S, all of whose weights are integral multiples of $1/N$, and the category of Γ–equivariant real vector bundles over Y.

SYNOPSIS

1 Introduction

This thesis is divided into two parts. In the first part, we consider Frobenius-finite vector bundles on any pointed smooth projective variety (X, x) over a perfect field k of prime characteristic. Using these bundles, we construct a neutral Tannakian category $\mathcal{C}_F(X)$ over k. By a well-known theorem, there is a affine group scheme over k associated to $\mathcal{C}_F(X)$. This group scheme will be denoted by $\pi_F(X,\, x)$ and is called the F–fundamental group scheme of (X, x). We study the properties of this affine group scheme.

In the second part, we define real parabolic vector bundles over a real curve (X, σ_X). Let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be an integer. We construct an N-fold cyclic cover $p : Y \longrightarrow X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ of p, and $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$. We established the equivalence between the category of real parabolic vector bundles over X with real parabolic structure over S having the property that all the weights are integral multiples of $1/N$, and the category of Γ–equivariant real vector bundles over Y.

The summary of my thesis work is given in Section 2 and Section 3. Section 2 is based on the paper [AB10]. Section 3 is based on unpublished work of mine, which is being prepared into a paper for publication.

2 Tannakian Fundamental Group Schemes

The étale fundamental group of a scheme has been introduced and its properties studied by Grothendieck (see [Mu67] and [SGA1]). For a scheme X, we denote by $\pi_1(X,\, x)$ the étale fundamental group of X with base point x. In [No76], M. V. Nori introduced the fundamental group scheme for proper integral scheme defined over a perfect field. The fundamental group scheme defined by Nori is a natural generalization of the étale fundamental group. It is defined using a neutral Tannakian category of certain vector bundles. Let k be a field, and let $\mathbf{Vect}(k)$ denote the category of finite dimensional vector spaces over a field k.

Definition 2.1 A neutral Tannakian category over k is a rigid k–linear abelian tensor category $\mathcal{C}$ equipped with an exact faithful k–linear tensor functor $\omega : \mathcal{C} \longrightarrow \mathbf{Vect}(k)$ into the category of finite dimensional k–vector spaces. The functor ω is called a (neutral) fibre functor.

It is a well-known theorem due to Savedra, that any neutral Tannakian category over a field k determines an affine group scheme defined over k. Suppose that k is a perfect field.

2.1 Étale Fundamental Group scheme Let X be a proper integral scheme over k endowed with a rational point $x \in X(k)$.

Definition 2.2 A vector bundle E over X is called *étale trivializable* if there exists a finite étale covering $\psi : Y \longrightarrow X$ such that the pull-back ψ^*E is trivial.

Consider the neutral Tannakian category defined by all étale trivializable vector bundles E over X The fiber functor for this neutral Tannakian category sends E to its fiber $E(x)$ over x. We denote this category by $\mathcal{C}^{\text{ét}}(X)$. The corresponding affine group scheme is called the *étale fundamental group scheme* and is denoted by $\pi^{\text{ét}}(X, x)$.

2.1 The Nori Fundamental Group scheme For a proper integral scheme X defined over a perfect field k endowed with a rational point $x \in X(k)$, Nori defined in [No76] and [No82] the fundamental group scheme $\pi^N(X, x)$ over k.

Definition 2.3 A vector bundle E over X is said to be *Nori–semistable* if for every non-constant morphism $f : C \longrightarrow X$ with C a smooth projective curve, the pull-back $f^*E \longrightarrow C$ is semi-stable of degree zero.

Let $\mathbf{Ns}(X)$ denote the full subcategory of Vect(X) of all Nori–semistable vector bundles over X. Recall that a vector bundle E over X is essentially finite if there is a finite group scheme G and a principal G–bundle $\pi : P \longrightarrow X$ such that π^*E is trivial. Let $\mathcal{C}^N(X)$ be the category of all essentially finite vector bundles over X. Let $\omega_x : \mathcal{C}^N(X) \longrightarrow \mathbf{Vect}(k)$ be a fibre functor which sends an essentially finite vector bundle E to its fibre $E(x)$ over x. Then $\mathcal{C}^N(X)$ defines a neutral Tannakian category over k. The corresponding affine group scheme over k is called *the Nori fundamental group scheme* of X with base point x.

2.2 The F–fundamental Group Scheme Let k be a perfect field of prime characteristic p. For any k-scheme X, the absolute Frobenius morphism $F_X : X \longrightarrow X$ is defined as follows: F_X is the identity map on the topological space of X, and $F_X^\sharp : \mathcal{O}_X \longrightarrow \mathcal{O}_X$ by the pth power map $f \mapsto f^p$.

Let X be a smooth n-dimensional projective variety over k with a very ample divisor H. If E is a torsion-free sheaf on X then one can define its slope by setting

$$\mu(E) = \frac{c_1(E)\cdot H^{n-1}}{\mathrm{rk}(E)},$$

where $\mathrm{rk}(E)$ is the rank of E. Then E is *H-semistable* if for any nonzero subsheaf $F \subset E$ we have $\mu(F) \leq \mu(E)$.

We say that E is strongly H-semistable if for all $r \geq 0$ the pull back $(F_X^r)^*E$ is H-semistable.

Let E be a vector bundle over X. For any polynomial $g(t) = \sum_{i=0}^m n_i t^i$ with $n_i \in \mathbb{N}$, define

$$\widetilde{g}(E) := \bigoplus_{i=0}^{m}((F_X^i)^*E)^{\oplus n_i},$$

where F_X^0 is the identity morphism of X.

Definition 2.4 A vector bundle E over X is called *Frobenius–finite* if there are two distinct polynomials f and g of the above type such that $\widetilde{f}(E)$ is isomorphic to $\widetilde{g}(E)$.

The following proposition allow us to construct a neutral Tannakian category using Frobenius–finite vector bundles over X.

Proposition 2.5 *Any Frobenius–finite vector bundle over X is Nori–semistale.*

Let $\mathrm{TFF}(X)$ denote the collection of all finite tensor products of Frobenius–finite vector bundles over X. Consider the full subcategory, denoted by $\mathcal{C}_F(X)$, of the category $\mathbf{Ns}(X)$ whose objects are all Nori–semistable subquotients of finite direct sum of elements of $\mathrm{TFF}(X)$.

We define the operation $\otimes$ on $\mathcal{C}_F(X)$ to be the usual tensor product of vector bundles. Fix a k–rational point $x \in X$. Let $\omega_x : \mathcal{C}_F(X) \longrightarrow \mathbf{Vect}(k)$ be the fibre functor that sends a vector bundle E in $\mathcal{C}_F(X)$ to its fibre $E(x)$ over x. The quadruple $(\mathcal{C}_F(X), \otimes, \omega_x, \mathcal{O}_X)$ defines a neutral Tannakian category over k.

Definition 2.6 The *F–fundamental group–scheme* of X with the base point x is the group–scheme associated to the neutral Tannakian category

$$(\mathcal{C}_F(X), \otimes, T_x, \mathcal{O}_X)\,.$$

This group–scheme will be denoted by $\pi_F(X,\, x)$.

2.3 Some Properties of the F–fundamental Group Scheme First we have the following

Proposition 2.7 *If $\varphi : X \longrightarrow Y$ is a flat surjective morphism of smooth projective varieties over a field k with $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$. Then the induced homomorphism $\widehat{\varphi} : \pi_F(X, x) \longrightarrow \pi_F(Y, \varphi(x))$ is faithfully flat surjection.*

2.3.1 Base Change Let k' be an algebraically closed extension of an algebraically closed field k of prime characteristic p. Let X be a smooth projective k–variety, and let $X_{k'}$ denote the base change $X \times_k \operatorname{Spec} k'$. Then there is a canonical faithfully flat homomorphism $\pi_F(X_{k'}, x') \longrightarrow \pi_F(X, x) \times_k k'$. In general, this homomorphism is not an isomorphism.

2.3.2 Relation with the S–fundamental Group Scheme For a smooth projective curve defined over k, the S–fundamental group–scheme was introduced in [BPS06]. In [La09], this was generalized to the smooth projective varieties defined over an algebraically closed field.

Let $\mathrm{Vect}_0^S(X)$ denote the full subcategory of the category of coherent sheaves on X whose objects are all strongly H–semistable reflexive sheaves with $c_1(E) \cdot H^{n-1} = 0$ and $c_2(E) \cdot H^{n-2} = 0$. By [La09, Theorem 4.1], a strongly H–semistable reflexive sheaf with $c_1(E) \cdot H^{n-1} = 0$ and $c_2(E) \cdot H^{n-2} = 0$ is locally free. The category $\mathrm{Vect}_0^S(X)$ does not depend on the choice of H ([La09, Proposition 4.5]). Fix a k–rational point $x \in X$ and define the fiber functor $\omega_x : \mathrm{Vect}_0^S(X) \longrightarrow \mathbf{Vect}(k)$ by sending E to its fiber $E(x)$. Then $(\mathrm{Vect}_0^S(X), \otimes, \omega_x, \mathcal{O}_X)$ is a neutral Tannakian category.

Definition 2.8 The affine k–group scheme Tannaka dual to the neutral Tannakian category $(\mathrm{Vect}_0^S(X), \otimes, \omega_x, \mathcal{O}_X)$ is denoted by $\pi^S(X, x)$ and it is called the *S–fundamental group scheme* of X with base point x.

The following proposition gives the relation between the S–fundamental group scheme and the F–fundamental group scheme.

Proposition 2.9 *There exists a natural faithfully flat homomorphism*

$$\pi^S(X, x) \longrightarrow \pi_F(X, x).$$

2.3.3 Product Formula Let k be an algebraically closed field of prime characteristic p. Let X and Y be two smooth projective varieties over k. Fix k–rational points x_0 and y_0 of X and Y respectively. Let $p : X \times_k Y \longrightarrow X$ and $q : X \times_k Y \longrightarrow Y$ be the natural projections. Then we get a canonical homomorphism $\psi : \pi_F(X \times_k Y, x_0 \times y_0) \longrightarrow \pi_F(X, x_0) \times \pi_F(Y, y_0)$ of affine group schemes.

Theorem 2.10 *The canonical homomorphism* $\psi : \pi_F(X \times_k Y, x_0 \times y_0) \longrightarrow \pi_F(X, x_0) \times \pi_F(Y, y_0)$ *is an isomorphism.*

2.3.4 Behavior under Étale Morphism We study the bahavior of the F–fundamental group scheme under the finite étale morphism. In this regard, we have the following

Proposition 2.12 *Let* $\psi : Y \longrightarrow X$ *be a finite étale morphism of smooth projective varieties over* k. *Then the induced homomorphism* $\pi_F(Y, y) \longrightarrow \pi_F(X, \psi(y))$ *is closed immersion.*

2.3.5 The Case of Finite Field Let k be a finite field. Let X be a smooth projective variety defined over k. We will assume that X admits a k–rational point. Fix a k–rational point x_0 of X. Then there is a natural faithfully flat homomorphism $\pi_F(X, x_0) \longrightarrow \pi^{\text{ét}}(X, x_0)$ which is not an isomorphism in general. We also get a natural faithfully flat homomorphism $\pi_F(X, x_0) \longrightarrow \pi^N(X, x_0)$.

2 Real Parabolic Vector Bundles over a Real Curve

In this part of the thesis, we study real parabolic vector bundles over a real curve.

3.1 Real Vector Bundles over a Real Curve By a real curve, we mean a pair (X, σ), where X is a compact Riemann surface and σ is an anti-holomorphic involution on X. Let $\sigma_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ be the conjugate map $z \mapsto \bar{z}$.

Proposition 3.1 *A continuous involution* $\sigma : X \longrightarrow X$ *on a Riemann surface* X *is an anti-holomorphic involution if and only if for every open subset* U *of* X*, the map* $\tilde{\sigma} = \tilde{\sigma}_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(\sigma(U))$ *defined by* $f \mapsto \sigma_{\mathbb{C}} \circ f \circ \sigma$ *is an isomorphism of rings.*

Let (X, σ_X) be a real curve. A *real holomorphic vector bundle* $\pi : E \longrightarrow X$ is a holomorphic vector bundle, together with an anti-holomorphic involution σ^E of E such that $\pi \circ \sigma^E = \sigma_X \circ \pi$ and for all $x \in X$, the map $\sigma^E|_{E(x)} : E(x) \longrightarrow E(\sigma(x))$ is $\mathbb{C}$-antilinear: $\sigma^E(\lambda \cdot \eta) = \bar{\lambda} \cdot \sigma^E(\eta)$, for all $\lambda \in \mathbb{C}$ and all $\eta \in$

$E(x)$. A homomorphism between two real bundles (E, σ^E) and $(E', \sigma^{E'})$ is a homomorphism $f : E \longrightarrow E'$ of holomorphic vector bundles over X such that $f \circ \sigma^E = \sigma^{E'} \circ f$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define an $\mathcal{O}_X$-module $\mathcal{F}^\sigma$ as follows. For any open subset U of X, $\mathcal{F}^\sigma(U) = \mathcal{F}(\sigma(U))$, and for every $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}^\sigma(U)$, $f \cdot s = \tilde{\sigma}_U(f)s$. Note that $(\mathcal{F}^\sigma)^\sigma = \mathcal{F}$.

Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Define $\varphi^\sigma : \mathcal{F}^\sigma \longrightarrow \mathcal{G}^\sigma$ as follows: For every open subset U of X, $\varphi^\sigma_U : \mathcal{F}^\sigma(U) \longrightarrow \mathcal{G}^\sigma(U)$ is defined to be $\varphi_{\sigma(U)}$. Then φ^σ is a homomorphism of $\mathcal{O}_X$-modules.

A real structure on an $\mathcal{O}_X$–module $\mathcal{F}$ is an $\mathcal{O}_X$-module homomorphism $\sigma^{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}^\sigma$ such that $(\sigma^{\mathcal{F}})^\sigma \circ \sigma^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}}$. By a *real $\mathcal{O}_X$–module*, we mean a pair $(\mathcal{F}, \sigma^{\mathcal{F}})$, where $\mathcal{F}$ is an $\mathcal{O}_X$–module and $\sigma^{\mathcal{F}}$ is a real structure on an $\mathcal{O}_X$–module $\mathcal{F}$.

3.2 Real Parabolic Vector Bundles Let (E, σ^E) be a real vector bundle over a real curve (X, σ_X). Let S be a finite subset of X such that $\sigma_X(S) = S$.

By *real quasi-parabolic structure* on (E, σ^E) over S, we mean for each $x \in S$, there is a strictly decreasing flag

$$E(x) = F^1E(x) \supset F^2E(x) \supset \cdots \supset F^{k_x}E(x) \supset F^{k_x+1}E(x) = 0$$

of linear subspaces in $E(x)$ satisfying the following property:

- σ^E preserve the flags, i.e., $\sigma^E_x(F^iE(x)) = F^iE(\sigma_X(x))$.

A *real parabolic structure* on (E, σ^E) over S is a real quasi-parabolic structure on (E, σ^E) over S as above, together with a sequence of real numbers $0 \leq \alpha^x_1 < \cdots < \alpha^x_{k_x} < 1$, which are called weights corresponding to the subspaces $(F^1E(x), F^2E(x), \ldots, F^{k_x}E(x))$, with the following property:

- the weights over x and $\sigma_X(x)$ are same.

Given two real parabolic bundles (E_1, σ^{E_1}) and (E_2, σ^{E_2}) over (X, σ_X), a *real parabolic morphism* is a homomorphism $\psi : (E_1, \sigma^{E_1}) \longrightarrow (E_2, \sigma^{E_2})$ of real vector bundles which respects the real parabolic structures, i.e., for each real parabolic point x with the real parabolic structures on E_l at x for $l = 1, 2$ given by

$$E_l(x) = F^1E_l(x) \supset F^2E_l(x) \supset \ldots F^{k_x}E_l(x) \supset 0,$$

$$0 \leq \alpha_1^l < \alpha_2^l < \dots \alpha_{k_x}^l < 1,$$

we require that $\psi(x)$ satisfies $\alpha_i^1 > \alpha_j^2 \implies \psi(x)(F^i E_1(x)) \subseteq F^{j+1}E_2(x)$. An isomorphism $\psi : (E_1, \sigma^{E_1}) \longrightarrow (E_2, \sigma^{E_2})$ is said to be an isomorphism of real parabolic bundles if ψ and ψ^{-1} are real parabolic morphisms.

We denote by $\mathbf{RP}(X)$ the category whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S, and morphisms are real parabolic morphisms.

Equivariant real vector bundles Let (Y, σ_Y) be a real curve. Let G be a finite group acting holomorphically and effectively on Y with the property that $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$ for all $g \in G$ and $y \in Y$.

Definition 3.2 A *G–equivariant real vector bundle* on (Y, σ_Y) consists of the following data: a real holomorphic vector bundle (W, σ^W) on (Y, σ_Y), and a lift of the natural action of G on Y to W such that

(a) the bundle projection $\pi : W \longrightarrow Y$ is G-equivariant;

(b) if $y \in Y$ and $g \in G$, the map $W_y \longrightarrow W_{g \cdot y}$, given by $v \mapsto g \cdot v$ is linear isomorphism.

(c) the following diagram

$$\begin{tikzcd} G \times W \arrow[r] \arrow[d, "(\mathrm{inv}, \sigma^W)"'] & W \arrow[d, "\sigma^W"] \\ G \times W \arrow[r] & W \end{tikzcd}$$

commutes, where $\mathrm{inv} : G \longrightarrow G$ is an inverse map $g \mapsto g^{-1}$.

Let (X, σ_X) be a real curve, and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be an integer. We construct an N-fold cyclic cover $p : Y \longrightarrow X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ of p, and $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$, where σ_Y is the anti-holomorphic involution on Y. Let $\mathbf{RP}(X, N)$ denote the full subcategory of $\mathbf{RP}(X)$ whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S, all of whose weights are integral multiples of $1/N$. Let $\mathbf{RE}_{\boldsymbol{\Gamma}}(Y)$ denote the category whose objects are all Γ–equivariant real vector bundles on $(Y,\ \sigma_Y)$ and morphisms are morphisms of Γ–equivariant real vector bundles. The following is the main theorem of this part of the thesis:

Theorem 3.3 *There is a canonical functor* $\Psi : \mathbf{RE}_{\mathbf{\Gamma}}(Y) \longrightarrow \mathbf{RP}(X, N)$ *which is an equivalence of categories.*

List of Publications and Preprints:

1. Sanjay Amrutiya, Indranil Biswas *On the F-fundamental group scheme*, Bull. Sci. math. 134 (2010) 461-474.

2. Sanjay Amrutiya, *On real parabolic vector bundles over a real curve*, Preprint.

Dedicated to my family

Contents

Conventions and Notations

If $\mathcal{C}$ is a category, and if X and Y are objects in $\mathcal{C}$, then the set of morphisms from X into Y will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, or by $\mathrm{Hom}(X, Y)$ when no confusion is likely to occur. If $X = Y$, then the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is denoted by $\mathrm{End}_{\mathcal{C}}(X)$ or $\mathrm{End}(X)$. Furthermore, $X \cong_{\mathcal{C}} Y$ or $X \cong Y$ means that X is isomorphic to Y. If $\mathcal{C}$ is a category, then $\mathcal{C}^{\mathrm{op}}$ will denote the *opposite category of* $\mathcal{C}$, i.e., $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, for every X, Y in $\mathcal{C}^{\mathrm{op}}$.

We will assume that all rings are commutative unless otherwise mentioned. For a field k, all k–algebras are assumed to be commutative unless otherwise mentioned.

We will denote by $\mathbb{C}$ (respectively, $\mathbb{R}$) the field of complex numbers (respectively, real numbers). The ring of integers is denoted by $\mathbb{Z}$, and the set of non-negative integers is denoted by $\mathbb{N}$. The open unit disc in $\mathbb{C}$ with centre at the origin is denoted by $\mathbb{D}$. If n is a non-negative integer, then $\tau_n : \mathbb{C} \longrightarrow \mathbb{C}$ denotes the function $t \longrightarrow t^n$. We denote its restriction to any subset of $\mathbb{C}$ also by τ_n.

A *multi-index of length* n, where $n \in \mathbb{N}$, is an element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{Z}^n$. We define the *weight* of such a multi-index α by $\tilde{\alpha} = \alpha + 1 + \cdots + \alpha_n$. We write $\alpha \geq 0$ or $\alpha \in \mathbb{N}^n$ if $\alpha_i \geq 0$ for all $i = 1, \ldots, n$. If $\alpha \geq 0$ and $\alpha \neq 0$, we write $\alpha > 0$.

Chapter 1

Introduction

In this thesis we study two problems on vector bundles. The first problem is to study the properties of an affine group scheme defined using Frobenius–finite vector bundles on a pointed smooth projective variety (X, x) over a perfect field of prime characteristic. The other problem is to study an equivariant description of real parabolic vector bundles over a real curve.

Tannakian fundamental group schemes

In Chapter 2, we recall some basic definitions and results from the theory of neutral Tannakian categories over a field k.

In Chapter 3, we consider Frobenius-finite vector bundles on any pointed smooth projective variety (X, x) over a perfect field k of prime characteristic. Using these bundles, in Section 3.4.1, we construct a neutral Tannakian category $\mathcal{C}_F(X)$ over k. By a well-known theorem, associated to $\mathcal{C}_F(X)$, there is an affine group scheme over k. This group scheme will be denoted by $\pi_F(X,\, x)$ and is called the F–fundamental group scheme of (X, x). In Section 3.4.2, we study various properties of the F–fundamental group scheme.

Real parabolic vector bundles over a real curve

The notion of parabolic vector bundles over a compact Riemann surface was introduced by C. S. Seshadri [Se77] and their moduli studied in [MS80]. In Chapter 4, we study equivariant description of real parabolic vector bundles over a real curve.

By a real curve, we mean a pair (X, σ_X), where X is a compact Riemann surface and σ_X is an anti-holomorphic involution on X. A *real holomorphic* *vector bundle* $E \longrightarrow X$ is a holomorphic vector bundle, together with an anti-holomorphic involution σ^E of E such that $\pi \circ \sigma^E = \sigma_X \circ \pi$ and for all $x \in X$, the map $\sigma^E|_{E(x)} : E(x) \longrightarrow E(\sigma(x))$ is $\mathbb{C}$-antilinear: $\sigma^E(\lambda \cdot \eta) = \bar{\lambda} \cdot \sigma^E(\eta)$, for all $\lambda \in \mathbb{C}$ and all $\eta \in E(x)$.

In Section 4.2, we define real parabolic vector bundles over a real curve. Let (X, σ_X), and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be an integer. In Section 4.3, we construct an N-fold cyclic cover $p : Y \longrightarrow X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ of p, and $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$. Let $\mathbf{RP}(X)$ be the category whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S, and morphisms are real parabolic morphisms. Let $\mathbf{RP}(X, N)$ denote the full subcategory of $\mathbf{RP}(X)$ whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S whose weights are integral multiple of $1/N$. Let $\mathbf{RE_\Gamma}(Y)$ denote the category whose objects are all Γ–equivariant real vector bundles on $(Y,\ \sigma_Y)$ and morphisms are morphisms of Γ–equivariant real vector bundles. In Section 4.4, we established an equivalence between the category $\mathbf{RE_\Gamma}(Y)$ and the category $\mathbf{RP}(X, N)$.

A more detailed introduction to the thesis, including precise definitions, results and statements of the theorems, is given in the Synopsis (p. **i**).

Chapter 2

Basics on Tannakian Categories

In this chapter, we recall some basic definitions and results from the theory of Tannakian categories. It is a well-known theorem due to Saavedra that any neutral Tannakian category over a field k is equivalent to the category of finite dimensional representations of some affine group scheme G defined over k.

In Section 2.1, we recall some basic definitions and results about affine group scheme defined over a field k. In Section 2.2, we recall some basic definitions from the theory of tensor categories. In Section 2.3, we recall the definition of neutral Tannakian category and state the main theorem of the theory of neutral Tannakian categories (Theorem 2.3.3).

2.1 Affine Group Schemes and Hopf Algebras

In this section, we explain the correspondence between affine group schemes and Hopf algebras. In Subsection 2.1.1, we show that a representation of an affine group scheme corresponds to a comodule over a Hopf algebra. Results and proofs in this section are mainly from [Wa79].

Let k be a field. Let $\mathbf{Alg}_k$ denote the category of k–algebras, and let $\mathbf{Grp}$ denote the category of groups.

Definition 2.1.1 We say that a functor G from the category $\mathbf{Alg}_k$ to the category $\mathbf{Grp}$ is an *affine group scheme over* k if it is representable by some k–algebra A. We call A the *coordinate ring of* G, and denote it by $k[G]$.

Remark 2.1.2 In general a group scheme over k can be defined as a group object in the category of k–schemes, i.e. a k–scheme G together with k–morphisms

$m : G \times G \longrightarrow G$ (multiplication), $e : \mathrm{Spec}(k) \longrightarrow G$ (unit) and $i : G \longrightarrow G$ (inverse) subject to the usual group axioms. These morphisms induce a group structure on the set $G(S) := \mathrm{Hom}_k(S, G)$ of k–morphisms into G for each k–scheme S. Therefore, the contravariant functor $S \mapsto \mathrm{Hom}_k(S, G)$ on the category of k–schemes represented by G is in fact group-valued. Restricting it to the full subcategory of affine k–schemes we obtain a contravariant functor $\mathrm{Spec}(R) \mapsto \mathrm{Hom}_k(\mathrm{Spec}(R), G)$. Since the contravariant functor $R \mapsto \mathrm{Spec}(R)$ induces an isomorphism of the category of affine schemes with the opposite category of commutative rings with unit, it follows that when $G = \mathrm{Spec}(A)$ is itself affine, then this is none but the above functor.

Let G be an affine group scheme over k. The coordinate ring $k[G]$ of an affine group scheme G carries additional structure coming from the group operations. To see this, note first that the functor $G \times G$ given by

$$R \mapsto G(R) \times G(R)$$

is representable by the tensor product $A \otimes_k A$ in view of the functorial isomorphism

$$\mathrm{Hom}_{\mathbf{Alg}_k}(k[G], R) \times \mathrm{Hom}_{\mathbf{Alg}_k}(k[G], R) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Alg}_k}(k[G] \otimes_k k[G], R)$$

induced by $(\varphi, \psi) \mapsto \varphi \otimes \psi$ (the inverse map is given by $\lambda \mapsto (a \mapsto \lambda(a \otimes 1), a \mapsto \lambda(1 \otimes a))$), and the functor

$$R \mapsto \{1\}$$

is representable by k. Therefore, by the Yoneda Lemma, the morphism of functors

$$m : G \times G \longrightarrow G,$$

given by multiplication is induced by a k–algebra homomorphism

$$\Delta : k[G] \longrightarrow k[G] \otimes_k k[G] .$$

Similarly, morphism of functors

$$e : \{1\} \longrightarrow G$$

given by the identity element is induced by a k–algebra homomorphism

$$\varepsilon \,:\, k[G] \longrightarrow k,$$

and the morphism of functors

$$i \,:\, G \longrightarrow G$$

is induced by a k–algebra homomorphism

$$S \,:\, k[G] \longrightarrow k[G]\,.$$

The homomorphisms Δ, ε and S are called the *comultiplication*, *counit*, and *coinverse* (or *antipode*), of $k[G]$, respectively. The following diagrams commute:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
k[G] \otimes_k k[G] \otimes_k k[G] & k[G] \otimes_k k[G] \arrow[l, "1\otimes\Delta"'] \\
k[G] \otimes_k k[G] \arrow[u, "\Delta\otimes 1"] & k[G] \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "e\times 1"] \arrow[d, "1\times e"'] \arrow[dr, "\simeq"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
k[G] & k[G] \otimes_k k[G] \arrow[l, "1\otimes\varepsilon"'] \\
k[G] \otimes_k k[G] \arrow[u, "\varepsilon\otimes 1"] & k[G] \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "\simeq"]
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "e\times i"] \arrow[d, "i\times e"'] \arrow[dr, "c"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
k[G] & k[G] \otimes_k k[G] \arrow[l, "m'\circ(1\otimes S)"'] \\
k[G] \otimes_k k[G] \arrow[u, "m'\circ(S\otimes 1)"] & k[G] \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "\gamma"]
\end{tikzcd}$$

In the last two diagrams c is the constant map $G \longrightarrow \{1\}$, γ the composite $k[G] \longrightarrow k \longrightarrow k[G]$ and m' ; $k[G] \otimes_k k[G] \longrightarrow k[G]$ the algebra multiplication.

The above diagrams indicate the translation of the associativity, unit and inverse axioms for groups on the lef-hand side to the corresponding compatibility conditions on $k[G]$ on the right-hand side. They are called the *coassociativity,* *counit* and *antipode (or coinverse) axioms*, respectively.

Definition 2.1.3 A k–algebra (not necessarily commutative) equipped with the k–algebra maps Δ, ε and S above and satisfying the coassociativity, counit and coinverse axioms is called *Hopf algebra*.

Hopf algebras associated with affine group schemes are always commutative.

Remark 2.1.4 In calculations, it is often useful to write down the Hopf algebra axioms explicitly for concrete elements. For instance, if we write

$$\Delta(a) = \sum a_i \otimes b_i$$

for the comultiplication map, then the counit axiom says that

$$a = \sum \varepsilon(a_i) b_i = \sum a_i \varepsilon(b_i)$$

and the antipode axiom says that

$$\varepsilon(a) = \sum S(a_i) b_i = \sum a_i S(b_i).$$

Proposition 2.1.5 *The functor $A \mapsto \operatorname{Spec}(A)$ defines an anti-equivalence between the category of commutative Hopf algebras over k and the category of affine group scheme over k.*

Example 2.1.1 The functor $R \mapsto \mathbb{G}_a(R)$ mapping a k–algebra R to its underlying additive group R^+ is an affine group scheme with coordinate ring $k[x]$, in view of the functorial isomorphism $R^+ \cong \operatorname{Hom}_k(k[x], R)$. The comultiplication map on $k[x]$ is given by $\Delta(x) = 1 \otimes x + x \otimes 1$, the counit is the zero map and the antipode is induced by $x \mapsto -x$.

Example 2.1.2 The functor $R \mapsto \mathbb{G}_m(R)$ mapping a k–algebra R to the group $R^\times$ of invertible elements is an affine group scheme with coordinate ring $k[x, x^{-1}]$, because an invertible element in R corresponds to a k–algebra homomorphism $k[x, x^{-1}] \longrightarrow R$. The comultiplication map on $k[x, x^{-1}]$ is given by $\Delta(x) = x \otimes x$, the counit sends x to 1 and the antipode is induced by $x \mapsto x^{-1}$.

Example 2.1.3 More generally, the functor that maps a k–algebra R to the group $\mathrm{GL}_n(R)$ of invertible matrices with entries in R is an affine group scheme.

To find its co-ordinate ring A, notice that an $n \times n$ matrix M over R is invertible if and only if $\det M$ is invertible in R. This allows us to recover A as the quotient of the polynomial ring in n^2+1 variables $k[x_{11}, \ldots, x_{nn}, x]$ by the ideal generated by $\det(x_{ij})x - 1$. The isomorphism $\mathrm{GL}_n(R) \cong \mathrm{Hom}_{\mathbf{Alg}_k}(A, R)$ is induced by sending a matrix $M = (m_{ij})$ to the homomorphism given by $x_{ij} \mapsto m_{ij}$, $x \mapsto \det(m_{ij})^{-1}$. The comultiplication is induced by $x_{ij} \mapsto \sum_k x_{ik} \otimes x_{kj}$, the counit sends x_{ij} to δ_{ij} (Kronecker delta), and the antipode comes from the formula for the inverse matrix.

Definition 2.1.6 Let A be a finite dimensional k–algebra. We say that A is *separable* if $A \otimes \bar{k}$ is reduced.

Recall that if k is perfect field, then a finite k–algebra A is separable if and only if A is reduced.

Definition 2.1.7 Let G be an affine group scheme defined over a field k. We say that G is finite if $k[G]$ is a finitely generated k–module.

Definition 2.1.8 A finite group scheme G over k is called *étale* if $k[G]$ is separable k–algebra.

The proof of the following result can be found in [Wa79, p. 86-87].

Theorem 2.1.9 *All finite group schemes in characteristic zero are étale.*

Example 2.1.4 Let Γ be a finite group. Let A denote the k–algebra k^Γ of functions from Γ to k. Then A is a product of copies of k indexed by the elements of Γ. More precisely, let e_σ be the function that is 1 on σ and 0 on the remaining elements of Γ. The e_σ's are a complete system of orthogonal idempotents for A:

$$e_\sigma^2 = e_\sigma, \quad e_\sigma e_\tau = 0 \quad \text{for} \quad \sigma \neq \tau, \quad \sum e_\sigma = 1.$$

The maps

$$\Delta(e_\rho) = \sum_{\rho = \sigma\tau} (e_\sigma \otimes e_\tau), \quad \varepsilon(e_\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{otherwise} \end{cases} \quad S(e_\sigma) = e_{\sigma^{-1}}.$$

define a Hopf-algebra structure on A. Let $(\Gamma)_k$ be the associated affine group scheme, so that

$$(\Gamma)_k(R) = \mathrm{Hom}_{\mathbf{Alg}_k}(A, R).$$

If R has no idempotents other than 0 or 1, then a k–algebra homomorphism $A \longrightarrow R$ must send one e_σ to 1 and the remainder to 0; therefore, $(\Gamma)_k(R) \simeq \Gamma$, and it is easy to check that the group structure provided by the maps Δ, ε, S is the given one. For this reason, $(\Gamma)_k$ is called the *constant group scheme* defined by Γ.

Remark 2.1.10 Let G be a finite étale group scheme over k. Let k_s denote the separable closure of k. By [Wa79, Theorem 6.4, p. 49], the category of finite étale k–group schemes is equivalent to the category of finite groups Γ carrying a continuous $\mathrm{Gal}(k_s/k)$–action. In this equivalence, the constant group scheme $(\Gamma)_k$ corresponds to Γ with trivial Galois action. A finite group scheme G is étale if and only if $G \times_k k_s$ is constant group scheme.

By ignoring the k–algebra structure on a Hopf algebra, we obtain the following more general notion.

Definition 2.1.11 A coalgebra C over k is a k–vector space equipped with a comultiplication $\Delta : C \longrightarrow C \otimes_k C$ and a counit $\varepsilon : C \longrightarrow k$ subject to the coassociativity and counit axioms.

In this definition, the map Δ and ε are assumed to be only k–linear maps. Coalgebra over k forms a category: morphisms are defined to be k–linear maps compatible with the k–coalgebra structure.

We now define right comodules over a coalgebra by dualizing the notion of left modules over a k–algebra B. Note that to give a unitary left B–module is to give a k–vector space V together with a k–linear multiplication $l : B \otimes_k V \longrightarrow V$ so that the following diagrams commute:

$$\begin{tikzcd}
B \otimes_k B \otimes_k V \arrow[r, "1\otimes l"] \arrow[d, "m'\otimes 1"'] & B \otimes_k V \arrow[d, "l"] \\
B \otimes_k V \arrow[r, "l"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes_k V \arrow[r, "i\otimes 1"] \arrow[dr, "\simeq"'] & B \otimes_k V \arrow[d, "l"] \\
 & V
\end{tikzcd}$$

where $i : k \longrightarrow B$ is the natural map sending 1 to the unit element of B.

Definition 2.1.12 Let C be a coalgebra over a field k. A right C–comodule is a k–vector space V together with a k–linear map $\rho : V \longrightarrow V \otimes_k C$ such that the diagrams

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes_k C \arrow[d, "1\otimes\Delta"] \\ V \otimes_k C \arrow[r, "\rho\otimes 1"'] & V \otimes_k C \otimes_k C \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "\simeq"'] & V \otimes_k C \arrow[d, "1\otimes\varepsilon"] \\ & V \otimes_k k \end{tikzcd} \tag{2.1}$$

Remark 2.1.13 We can write out the comodule axioms explicitly on elements as follows. Assume ρ is given by $\rho(v) = \sum v_i \otimes a_i$, $\rho(v_i) = \sum v_{ij} \otimes c_j$ and furthermore $\Delta(a_i) = \sum a_{ik} \otimes b_k$. Here v, v_i, v_{ij} are in V and the other elements lie in C. Then the commutativity of the first diagram is described by the equality

$$\sum_{i,k} v_i \otimes a_{ik} \otimes b_k = \sum_{i,j} v_{ij} \otimes c_j \otimes a_i. \tag{2.2}$$

The second diagram reads

$$\sum_i \varepsilon(a_i) v_i = v. \tag{2.3}$$

Definition 2.1.14 A *subcoalgebra* of a coalgebra C is defined as a k–subspace $B \subset C$ with the property that $\Delta(B) \subset B \otimes_k B$. The restrictions of Δ and ε then turn B into a coalgebra over k. One defines a *subcomodule* of an C–comodule V as a k–subspace $W \subset V$ with the property that $\rho(W) \subset W \otimes_k C$.

A subcoalgebra $B \subset C$ is also naturally a subcomodule of C considered as a right comodule over itself. Subcomodules and subcoalgebras enjoy the following basic finiteness property.

Proposition 2.1.15 *Let C be a k–coalgebra and (V, ρ) a comodule over C. Any finite subset of V is contained in a subcomodule of V having finite dimension over k. In particular, any finite subset of C is contained in a subcoalgebra of C having finite dimension over k.*

Proof. Let $\{a_i\}$ be a basis for C over k. If v is in the finite subset, write

$$\rho(v) = \sum_i v_i \otimes a_i \quad \text{(finite sum)}. \tag{2.4}$$

The k–space generated by the v and v_i is a subcomodule of V. For, from the

(2.4) we have

$$(\rho \otimes \mathrm{id}_A)(\rho(v)) = \sum_i \rho(v_i) \otimes a_i.$$

On the other hand, by the first comodule axiom, we have

$$(\rho \otimes \mathrm{id}_A)(\rho(v)) = \sum_i v_i \otimes \Delta(a_i).$$

Writing $\Delta(a_i) = \sum_{j,k} \lambda_{ijk}(a_j \otimes a_k)$, we obtain

$$\sum_k \rho(v_k) \otimes a_k = \sum_i v_i \otimes \sum_{j,k} \lambda_{ijk}(a_j \otimes a_k).$$

Since $\{a_i\}$ is a basis for C over k, it follows that

$$\rho(v_k) = \sum_i v_i \otimes \sum_{j,k} \lambda_{ijk} a_j$$

for all k. This proves that k–space generated by the v and v_i is a subcomodule of V. Note that by the k–linearity of $\rho : V \longrightarrow V \otimes_k C$, k–linear span of finitely many subcomodules of V is again a subcomodule. □

Corollary 2.1.16 *Any comodule (V, ρ) over C is direct limit of its finite dimensional subcomodules.*

2.1.1 Representations and Comodules

Let G be an affine group scheme over k, and let V be a vector space over k (not necessarily finite dimensional).

Definition 2.1.17 A *linear representation* of G on V is a natural homomorphism

$$\Phi : G(R) \longrightarrow \mathrm{Aut}_R(V \otimes_k R).$$

In other words, for each k–algebra R, we have an action

$$G(R) \times (V \otimes_k R) \longrightarrow (V \otimes_k R)$$

of $G(R)$ on $(V \otimes_k R)$ in which each $g \in G(R)$ acts R–linearly, and for each homomorphism of k–algebra $R \longrightarrow S$, the following diagram

$$\begin{tikzcd}
G(R) \times (V \otimes_k R) \arrow[r] \arrow[d] & (V \otimes_k R) \arrow[d] \\
G(S) \times (V \otimes_k S) \arrow[r] & (V \otimes_k S)
\end{tikzcd}$$

commutes.

Let Φ be a linear representation of G on V. Given a homomorphism $\alpha : R \longrightarrow S$ and an element $g \in G(R)$ mapping to h in $G(S)$, we get a diagram:

$$\begin{tikzcd}
V \otimes_k R \arrow[r, "\Phi(g)"] \arrow[d, "1\otimes\alpha"'] & V \otimes_k R \arrow[d, "1\otimes\alpha"] \\
V \otimes_k S \arrow[r, "\Phi(h)"'] & V \otimes_k S
\end{tikzcd}$$

Now let $g \in G(R) = \mathrm{Hom}_k(A, R)$. Then $g : A \longrightarrow R$ sends the "universal" element $1_A \in G(A) = \mathrm{Hom}_k(A, A)$ to g, and so the picture becomes the bottom part of the following diagram:

$$\begin{tikzcd}
V = V \otimes_k k \arrow[d] \arrow[dr, "\rho := \Phi(1_A)|_{V\otimes_k k}"] & \\
V \otimes_k A \arrow[r, "\Phi(1_A)", "A-\text{linear}"'] \arrow[d, "1\otimes g"'] & V \otimes_k A \arrow[d, "1\otimes g"] \\
V \otimes_k R \arrow[r, "\Phi(g)", "R-\text{linear}"'] & V \otimes_k R
\end{tikzcd}$$

In particular, we see that Φ defines a k–linear map $\rho := \Phi(1_A)|_V : V \longrightarrow V \otimes_k A$.

Moreover, it is clear from the diagram that ρ determines Φ, because $\Phi(1_A)$ is the unique A–linear extension of ρ to $V \otimes_k A$, and $\Phi(g)$ is the unique R–linear extension of $\Phi(1_A)$ to $V \otimes_k R$.

Conversely, suppose we have a k–linear map $\rho : V \longrightarrow V \otimes_k A$. Then the diagram shows that we get a natural map

$$\Phi : G(R) \longrightarrow \mathrm{Aut}_R(V \otimes_k R),$$

namely, given $g : A \longrightarrow R$, $\Phi(g)$ is the unique R–linear map making the following diagram

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d] & V \otimes_k A \arrow[d, "1\otimes g"] \\ V \otimes_k R \arrow[r, "\Phi(g)"'] & V \otimes_k R \end{tikzcd}$$

commute.

These maps will be homomorphisms if and only if the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes_k A \arrow[d, "1\otimes\Delta"] \\ V \otimes_k A \arrow[r, "\rho\otimes 1"'] & V \otimes_k A \otimes_k A \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "\simeq"'] & V \otimes_k A \arrow[d, "1\otimes\varepsilon"] \\ & V \otimes_k k \end{tikzcd} \tag{2.5}$$

For, we must have $\Phi(1_{G(R)}) = 1_{V\otimes_k R}$. By definition, $1_{G(R)} = (A \xrightarrow{\varepsilon} k \longrightarrow R)$ as an element of $\mathrm{Hom}_k(A, R)$, and so the following diagram must commute:

$$\begin{tikzcd} V \arrow[r, "\rho", "k-\text{linear}"'] \arrow[d] & V \otimes_k A \arrow[d, "1\otimes\varepsilon"] \\ V \otimes_k k \arrow[r] \arrow[d] & V \otimes_k k \arrow[d] \\ V \otimes_k R \arrow[r, "1_{V\otimes_k R}"'] & V \otimes_k R \end{tikzcd}$$

This means that the upper part of the diagram must commute with the map $V \otimes_k k \longrightarrow V \otimes_k k$ being the identity map, which is the second of the diagrams (2.5).

Similarly, the first diagram in (2.5) commutes if and only if the formula $\Phi(gh) = \Phi(g) \circ \Phi(h)$ holds.

For, by definition, gh is the composition

$$A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{(g,h)} R$$

and so $\Phi(gh)$ is the extension of

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{1\otimes\Delta} V \otimes_k A \otimes_k A \xrightarrow{1\otimes(g,h)} V \otimes_k R \tag{2.6}$$

to $V \otimes_k R$.

On the other hand, $\Phi(g) \circ \Phi(h)$ is given by

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{1\otimes h} V \otimes_k R \xrightarrow{\rho\otimes 1} V \otimes_k A \otimes_k R \xrightarrow{1\otimes(g,1)} V \otimes_k R \tag{2.7}$$

which is equal to

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{\rho\otimes 1_A} V \otimes_k A \otimes_k A \xrightarrow{1\otimes(g,h)} V \otimes_k R \tag{2.8}$$

Now (2.6) and (2.8) agree for all g, h if and only if the first diagram in (2.5) commutes. Therefore, we have

Proposition 2.1.18 *Let G be an affine group scheme over k with corresponding Hopf algebra A, and let V be a k–vector space. To give a linear representation of G on V is canonically equivalent to giving an A–comodule structure on V.*

Example 2.1.5 For any Hopf algebra A over k, the map $\Delta : A \longrightarrow A \otimes_k A$ is a comodule structure on A. The corresponding representation of A is called the *regular representation.*

Remark 2.1.19

1. An element g of $G(R) = \mathrm{Hom}_k(A, R)$ acts on $v \in V \otimes_k R$ according to the rule:
$$g \cdot v = ((1_V, g) \circ \rho)(v)$$

2. Recall that a k–subspace W of an A–comodule V is a subcomodule if $\rho(W) \subset W \otimes_k A$. Then, W itself is an A–comodule, and the linear representation of G on W defined by this comodule structure is the restriction of that on V.

2.2 Tensor Categories

We begin this section from the definition of tensor category. In Subsections 2.2.2, 2.2.4, 2.2.5 we recall definitions of invertible objects, tensor functors and morphisms of tensor functors. The main reference for this section is [DM82] (see also [Sa72], [Sz09]).

2.2.1 Tensor Categories and Tensor Functors

Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ a functor. An *associativity constraint* for $(\mathcal{C}, \otimes)$ is an isomorphism Φ of functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ given on a triple (X, Y, Z) of objects by

$$\Phi_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

such that the diagram

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\Phi\otimes 1"] \arrow[d, "\Phi"'] & & ((X \otimes Y) \otimes Z) \otimes W \arrow[d, "\Phi"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[dr, "1\otimes\Phi"'] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[dl, "\Phi"] \\
& X \otimes (Y \otimes (Z \otimes W)) &
\end{tikzcd}$$

commutes for each four-tuple (X, Y, Z, W) of objects in $\mathcal{C}$. The commutativity of the above diagram is usually referred to as the *pentagon axiom*.

A *commutativity constraint* for $(\mathcal{C}, \otimes)$ is an isomorphism Ψ of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ given on a pair (X, Y) of objects by

$$\Psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

such that $\Psi_{X,Y} \circ \Psi_{Y,X} = \mathbf{1}_{X \otimes Y}$ for all objects X, Y.

An associativity constraint Φ and a commutativity constraint Ψ are compatible if, for all objects X, Y, Z, the diagram

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[dl, "1\otimes\Psi"'] \arrow[dr, "\Phi"] & \\
X \otimes (Z \otimes Y) \arrow[d, "\Phi"'] & & (X \otimes Y) \otimes Z \arrow[d, "\Psi"] \\
(X \otimes Z) \otimes Y \arrow[dr, "\Psi\otimes 1"'] & & Z \otimes (X \otimes Y) \arrow[dl, "\Phi"] \\
& (Z \otimes X) \otimes Y &
\end{tikzcd}$$

commutes. This compatibility is called the *hexagon axiom.*

A pair $(\mathbb{1}, \iota)$ consisting of an object $\mathbb{1}$ of $\mathcal{C}$ and an isomorphism $\iota : \mathbb{1} \longrightarrow \mathbb{1}\otimes\mathbb{1}$ is an *identity object* of $(\mathcal{C}, \otimes)$ if the functor $\mathcal{C} \longrightarrow \mathcal{C}$ given by $X \mapsto \mathbb{1} \otimes X$ is an equivalence of categories.

Proposition 2.2.1 *Let $(\mathbb{1}, \iota)$ be an identity object for $(\mathcal{C}, \otimes)$. Then*

(1) *there exists a unique functorial isomorphism*

$$l_X : X \longrightarrow \mathbb{1} \otimes X$$

such that $l_{\mathbb{1}}$ is ι and the diagrams

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l"] \arrow[d, equal] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\Phi"] \\
X \otimes Y \arrow[r, "l\otimes 1"'] & (\mathbb{1} \otimes X) \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "l\otimes 1"] \arrow[d, "1\otimes l"'] & (\mathbb{1} \otimes X) \otimes Y \arrow[d, "\Psi\otimes 1"] \\
X \otimes (\mathbb{1} \otimes Y) \arrow[r, "\Phi"'] & (X \otimes \mathbb{1}) \otimes Y
\end{tikzcd}$$

are commutative.

(2) *if $(\mathbb{1}', \iota')$ is an another identity object of $(\mathcal{C}, \otimes)$ then there is a unique iso-*

morphism $a : \mathbb{1} \xrightarrow{\simeq} \mathbb{1}'$ *making the following diagram*

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\iota"] \arrow[d, "a"'] & \mathbb{1} \otimes \mathbb{1} \arrow[d, "a\otimes a"] \\
\mathbb{1}' \arrow[r, "\iota'"] & \mathbb{1}' \otimes \mathbb{1}'
\end{tikzcd}$$

commute.

Proof.

(1) Since $X \mapsto \mathbb{1} \otimes X$ is an equivalence of categories, to define l_X, it suffices to define $1 \otimes l_X : \mathbb{1} \otimes X \longrightarrow \mathbb{1} \otimes (\mathbb{1} \otimes X)$; this we take to be

$$\mathbb{1} \otimes X \xrightarrow{\iota\otimes 1} (\mathbb{1} \otimes \mathbb{1}) \otimes X \xrightarrow{\Phi} \mathbb{1} \otimes (\mathbb{1} \otimes X).$$

(2) The map

$$\mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1}' \otimes \mathbb{1} \xrightarrow{\Psi} \mathbb{1} \otimes \mathbb{1}' \xrightarrow{l_{\mathbb{1}'}^{-1}} \mathbb{1}'$$

has the required properties.

□

Remark 2.2.2 The functorial isomorphism $r_X \doteq \Psi_{\mathbb{1},X} \circ l_X : X \longrightarrow X \otimes \mathbb{1}$ has analogous properties to l_X.

Definition 2.2.3 A *tensor category* is a system $(\mathcal{C}, \otimes, \Phi, \Psi)$, where $\mathcal{C}$ is a category, Φ is an associativity constraint satisfying the pentagon axiom, and Ψ is a commutativity constraint compatible with Φ, together with an identity element $(\mathbb{1}, \iota)$.

2.2.2 Invertible objects

Let $(\mathcal{C}, \otimes)$ be a tensor category. An object L of $\mathcal{C}$ is *invertible* if the functor $X \longrightarrow L \otimes X : \mathcal{C} \longrightarrow \mathcal{C}$ is an equivalence of categories.

Thus, if L is invertible, there exists an L' such that $L \otimes L' \simeq \mathbb{1}$; the converse assertion is also true. An *inverse of* L is a pair (L^{-1}, δ), where $\delta : L \otimes L^{-1} \xrightarrow{\simeq} \mathbb{1}$. Note that this definition is symmetric: (L, δ) is an inverse of L^{-1}. If (L_1, δ_1) and

(L_2, δ_2) are both inverse of L, then there exists a unique isomorphism $\alpha : L_1 \longrightarrow L_2$ such that

$$\delta_1 = \delta_2 \circ (1 \otimes \alpha) : L \otimes L_1 \longrightarrow L \otimes L_2 \longrightarrow \mathbb{1}.$$

2.2.3 Internal Hom

Let $(\mathcal{C}, \otimes)$ be a tensor category. If the functor $T \mapsto \mathrm{Hom}(T \otimes X, Y) : \mathcal{C}^0 \longrightarrow \mathbf{Set}$ is representable, then we denote by $\underline{\mathrm{Hom}}(X, Y)$ the representing object and by $\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \longrightarrow Y$ the morphism corresponding to $\mathbf{1}_{\underline{\mathrm{Hom}}(X,Y)}$.

Thus, to any element $g \in \mathrm{Hom}(T \otimes X, Y)$ corresponds a unique $f \in \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y))$ such that $\mathrm{ev} \circ (f \otimes \mathrm{id}) = g$;

$$\begin{tikzcd} T \arrow[d, dotted] \\ \underline{\mathrm{Hom}}(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T \otimes X \arrow[d, "f\otimes\mathrm{id}"'] \arrow[dr, "g"] & \\ \underline{\mathrm{Hom}}(X,Y) \otimes X \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd} \tag{2.9}$$

Note that $\underline{\mathrm{Hom}}(\mathbb{1}, Y) = Y$ and

$$\mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(X, Y)) = \mathrm{Hom}(\mathbb{1} \otimes X, Y) = \mathrm{Hom}(X, Y) \tag{2.10}$$

The *dual* $X^\vee$ of an object X is defined to be $\underline{\mathrm{Hom}}(X, \mathbb{1})$. Therefore, there is a map $\mathrm{ev}_X : X^\vee \otimes X \longrightarrow \mathbb{1}$ inducing a functorial isomorphism

$$\mathrm{Hom}(T, X^\vee) \xrightarrow{\simeq} \mathrm{Hom}(T \otimes X, \mathbb{1}) \tag{2.11}$$

Remark 2.2.4 The dual of an object X is an object $X^\vee$ together with the morphisms $\varepsilon : X^\vee \otimes X \longrightarrow \mathbb{1}$ and $\delta : \mathbb{1} \longrightarrow X \otimes X^\vee$ such that the following diagrams

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "\cong"'] & X \otimes \mathbb{1} \\ \mathbb{1} \otimes X \arrow[r, "\delta\otimes 1"'] & X \otimes X^\vee \otimes X \arrow[u, "1\otimes\varepsilon"'] \end{tikzcd} \qquad \begin{tikzcd} X^\vee \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathbb{1} \otimes X^\vee \\ X^\vee \otimes \mathbb{1} \arrow[r, "1\otimes\delta"'] & X^\vee \otimes X \otimes X^\vee \arrow[u, "\varepsilon\otimes 1"'] \end{tikzcd} \tag{2.12}$$

commutes.

A dual of an object X is uniquely determined up to isomorphism. If we fix one of the maps ε or δ, then this isomorphism is unique.

To verify this, fix X and ε. It then suffices to show that $X^\vee$ represents the contravariant functor $Z \mapsto \mathrm{Hom}(X \otimes Z, \mathbb{1})$, that is, it suffices to give a functorial bijection

$$\mathrm{Hom}(X \otimes Z, \mathbb{1}) \longrightarrow \mathrm{Hom}(Z, X^\vee)\,.$$

Let $\varphi \in \mathrm{Hom}(X \otimes Z, \mathbb{1})$. Define $\nu_\varphi : Z \longrightarrow X^\vee$ to be the composite

$$Z \xrightarrow{\simeq} \mathbb{1} \otimes Z \xrightarrow{\delta\otimes 1} (X^\vee \otimes X) \otimes Z \xrightarrow{1_{X^\vee}\otimes\varphi} X^\vee \otimes \mathbb{1} \xrightarrow{\simeq} X^\vee\,.$$

Then the map $\varphi \mapsto \nu_\varphi$ is the required bijection. Its inverse is given by $\nu \mapsto \varphi_\nu$, where φ_ν is the composite

$$X \otimes Z \xrightarrow{\mathbf{1}_X\otimes\nu} X \otimes X^\vee \xrightarrow{\varepsilon} \mathbb{1}\,.$$

Let $i_X : X \longrightarrow X^{\vee\vee}$ be the map corresponding in (2.11) to $\mathrm{ev}_X \circ \Psi : X \otimes X^\vee \longrightarrow \mathbb{1}$. If i_X is an isomorphism, then X is said to be *reflexive.* If X has an inverse $(X^{-1}, \delta : X^{-1} \otimes X \xrightarrow{\cong} \mathbb{1})$, then X is reflexive and δ determines an isomorphism $X^{-1} \xrightarrow{\cong} X^\vee$ as in (2.9).

For any pair $(X_i)_{i\in X}$ and $(Y_i)_{i\in X}$ of finite families of objects, there is a morphism

$$\otimes_{i\in I}\, \underline{\mathrm{Hom}}(X_i, Y_i) \longrightarrow \underline{\mathrm{Hom}}(\otimes_{i\in I} X_i, \otimes_{i\in I} Y_i) \tag{2.13}$$

corresponding to

$$\Big(\otimes_{i\in I}\, \underline{\mathrm{Hom}}(X_i, Y_i)\Big) \otimes \Big(\otimes_{i\in I}\, X_i\Big) \xrightarrow{\simeq} \otimes_{i\in I}\Big(\underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i\Big) \xrightarrow{\mathrm{ev}} \otimes_{i\in I} Y_i$$

in (2.9).

In particular, there are morphisms

$$\otimes_{i\in I}\, X_i^\vee \longrightarrow \Big(\otimes_{i\in I}\, X_i\Big)^\vee \tag{2.14}$$

and

$$X^\vee \otimes Y \longrightarrow \underline{\mathrm{Hom}}(X, Y) \tag{2.15}$$

obtained from (2.13).

Definition 2.2.5 A tensor category $(\mathcal{C}, \otimes)$ is *rigid* if $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects X and Y, the maps (2.13) are isomorphism for all finite families of objects, and all objects of $\mathcal{C}$ are reflexive.

If $(\mathcal{C}, \otimes)$ is a rigid tensor category, then the map $X \mapsto X^\vee$ can be made into a contravariant functor: to $f : X \longrightarrow Y$, we associate the unique map ${}^t f : Y^\vee \longrightarrow X^\vee$ defined as the composite

$$Y^\vee \xrightarrow{\simeq} Y^\vee \otimes \mathbb{1} \xrightarrow{1\otimes\delta} Y^\vee \otimes X \otimes X^\vee \xrightarrow{1\otimes f\otimes 1} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\varepsilon\otimes 1} \mathbb{1} \otimes X^\vee \xrightarrow{\simeq} X^\vee$$

such that the diagram

$$\begin{tikzcd} Y^\vee \otimes X \arrow[r, "{}^t f\otimes 1"] \arrow[d, "1\otimes f"'] & X^\vee \otimes X \arrow[d, "\mathrm{ev}_X"] \\ Y^\vee \otimes Y \arrow[r, "\mathrm{ev}_Y"'] & \mathbb{1} \end{tikzcd}$$

commutes.

The map $f \mapsto {}^t f$ induces a bijection

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(Y^\vee, X^\vee).$$

Remark 2.2.6 Recall that there is an isomorphism

$$\mathrm{Hom}(X, Y) \xrightarrow{\simeq} \mathrm{Hom}(Y^\vee, X^\vee) \ ; \ f \mapsto {}^t f.$$

There is also a canonical isomorphism

$$\underline{\mathrm{Hom}}(X, Y) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(Y^\vee, X^\vee)$$

namely

$$\underline{\mathrm{Hom}}(X, Y) \xrightarrow[\simeq]{2.15} X^\vee \otimes Y \xrightarrow{\simeq} X^\vee \otimes Y^{\vee\vee} \xrightarrow{\simeq} Y^{\vee\vee} \otimes X^\vee \xrightarrow[\simeq]{2.15} \underline{\mathrm{Hom}}(Y^\vee, X^\vee).$$

More generally, for a rigid tensor category $(\mathcal{C}, \otimes)$, the functor

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C};\ X \mapsto X^\vee,\ f \mapsto {}^t f$$

is an equivalence of categories as its composite with itself is isomorphic to the identity functor. It is even an equivalence of tensor categories in the sense of 2.2.10. Note that $\mathcal{C}^{\mathrm{op}}$ has an obvious tensor structure for which $\otimes X_i^{\mathrm{op}} = (\otimes X_i)^{\mathrm{op}}$.

2.2.4 Tensor Functors

Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be tensor categories.

Definition 2.2.7 A tensor functor $(\mathcal{C}, \otimes) \longrightarrow (\mathcal{C}', \otimes')$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ together with an isomorphism c of functors from $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}'$ given on a pair (X, Y) of objects of $\mathcal{C}$ by a morphism

$$c_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\simeq} F(X \otimes Y)$$

with the following properties:

(a) for a triple (X, Y, Z) of objects of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[r, "1\otimes' c"] \arrow[d, "\Phi'"'] & F(X) \otimes' F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\Phi)"] \\ (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "c\otimes' 1"'] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "c"'] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

is commutative;

(b) for a pair (X, Y) of objects of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \otimes' F(Y) \arrow[r, "c"] \arrow[d, "\Psi'"'] & F(X \otimes Y) \arrow[d, "F(\Psi)"] \\ F(Y) \otimes' F(X) \arrow[r, "c"'] & F(Y \otimes X) \end{tikzcd}$$

is commutative;

(c) if (U, u) is an identity object of $\mathcal{C}$, then $(F(U), F(u))$ is an identity object of $\mathcal{C}'$.

The conditions $(a), (b), (c)$ imply that for any finite family $(X_i)_{i \in I}$ of objects of $\mathcal{C}$, c gives rise to a well-defined isomorphism

$$c : \otimes'_{i \in I} F(X_i) \xrightarrow{\simeq} F\Big(\otimes_{i \in I} X_i \Big).$$

In particular, (F, c) maps invertible objects to invertible objects. Also, the morphism

$$F(\mathrm{ev}) : F(\underline{\mathrm{Hom}}(X, Y) \otimes F(X)) \longrightarrow F(Y)$$

gives rise to morphisms

$$F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(F(X), F(Y))$$

and

$$F_X : F(X^\vee) \longrightarrow F(X)^\vee.$$

Proposition 2.2.8 *Let $(F, c) : \mathcal{C} \longrightarrow \mathcal{C}'$ be a tensor functor. If $\mathcal{C}$ and $\mathcal{C}'$ are rigid, then $F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(F(X), F(Y))$ is an isomorphism for all pairs (X, Y) of objects of $\mathcal{C}$.*

Proof. It suffices to show that F preserves duality, but this is obvious from the following characterization of the dual of an object X in $\mathcal{C}$:

The dual of an object X in $\mathcal{C}$ is a pair $(X, Y \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1})$, for which there exists a morphism $\delta : \mathbb{1} \longrightarrow X \otimes Y$ such that

$$X = \mathbb{1} \otimes X \xrightarrow{\delta\otimes 1} (X \otimes Y) \otimes X = X \otimes (Y \otimes X) \xrightarrow{1\otimes\mathrm{ev}} X,$$

and the same map with X and Y interchanged, are identity maps. $\square$

2.2.5 Morphisms of Tensor Functors

Definition 2.2.9 Let (F, c) and (G, d) be tensor functor from $\mathcal{C}$ to $\mathcal{C}'$; a *morphism of tensor functors* $(F, c) \longrightarrow (G, d)$ is a morphism of functors $\lambda : F \longrightarrow G$ such that, for all finite families $(X_i)_{i\in I}$ of objects in $\mathcal{C}$, the diagram

$$\begin{tikzcd} \otimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\otimes\lambda_{X_i}"'] & F\Big(\otimes_{i\in I} X_i\Big) \arrow[d, "\lambda"] \\ \otimes_{i\in I} G(X_i) \arrow[r, "d"'] & G\Big(\otimes_{i\in I} X_i\Big) \end{tikzcd} \tag{2.16}$$

Note that it is enough to require that the above diagram commutes for $I = \{1, 2\}$. For I the empty set, (2.16) becomes,

$$\begin{tikzcd} \mathbb{1}' \arrow[r, "\simeq"] \arrow[d, equal] & F(\mathbb{1}) \arrow[d, "\lambda_{\mathbb{1}}"] \\ \mathbb{1}' \arrow[r, "\simeq"'] & G(\mathbb{1}) \end{tikzcd} \tag{2.17}$$

in which the horizontal maps are the unique isomorphisms compatible with the structure of $\mathbb{1}'$, $F(\mathbb{1})$ and $G(\mathbb{1})$ as identity objects of $\mathcal{C}'$. In particular, when diagram (2.17) commutes, the morphism $\lambda_{\mathbb{1}}$ is an isomorphism.

We write $\mathrm{Hom}^{\otimes}(F, F')$ for the set of morphisms of tensor functors from (F, c) to (F', c').

Definition 2.2.10 A tensor functor $(F, c) : \mathcal{C} \longrightarrow \mathcal{C}'$ is a *tensor equivalence* (or an equivalence of tensor categories) if $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is an equivalence of categories.

The above definition is justified by the following proposition.

Proposition 2.2.11 *Let $(F, c) : \mathcal{C} \longrightarrow \mathcal{C}'$ be a tensor equivalence; then there is a tensor functor $(F', c') : \mathcal{C}' \longrightarrow \mathcal{C}$ and an isomorphism of functors $F' \circ F \xrightarrow{\simeq} \mathbf{1}_{\mathcal{C}}$ and $F \circ F' \xrightarrow{\simeq} \mathbf{1}_{\mathcal{C}'}$, commuting with tensor product (i.e., isomorphism of tensor functors).*

Proof. See [Sa72, I4.4]. □

Proposition 2.2.12 *Let (F, c) and (G, d) be tensor functors from $\mathcal{C}$ to $\mathcal{C}'$. If $\mathcal{C}$ and $\mathcal{C}'$ are rigid, then any morphism of tensor functors $\lambda : F \longrightarrow G$ is an isomorphism.*

Proof. The morphism $\mu : G \longrightarrow F$, making the diagrams

$$\begin{tikzcd} F(X^\vee) \arrow[r, "\lambda_{X^\vee}"] \arrow[d, "\simeq"'] & G(X^\vee) \arrow[d, "\simeq"] \\ F(X)^\vee \arrow[r, "{}^t\mu_X"'] & G(X)^\vee \end{tikzcd}$$

commutative for all $X \in \mathrm{Obj}(\mathcal{C})$, is an inverse for λ. □

Remark 2.2.13 For any field k and k–algebra R, there is a canonical tensor functor

$$\Phi_R : \mathbf{Vect}(k) \longrightarrow \mathbf{Modfg}_R,$$

defined by $V \mapsto V \otimes_k R$, where $\mathbf{Modfg}_R$ is the category of finitely generated R–modules. If (F, c) is a tensor functor $\mathcal{C} \longrightarrow \mathbf{Vect}(k)$, then we can define set-valued functors $\mathbf{End}^{\otimes}(F)$ on the category of k–algebras by

$$\mathbf{End}^{\otimes}(F)(R) = \mathrm{Hom}^{\otimes}(\Phi_R \circ F, \Phi_R \circ F) \tag{2.18}$$

Thus, we get a functor $\mathbf{End}^{\otimes}(F) : \mathbf{Alg}_k \longrightarrow \mathbf{Set}$.

Tensor Subcategories

Definition 2.2.14 Let $\mathcal{C}'$ be a strictly full subcategory of a tensor category $\mathcal{C}$. We say $\mathcal{C}'$ is a *tensor subcategory* of $\mathcal{C}$ if it is closed under finite tensor products (equivalently, if it contains an identity object of $\mathcal{C}$ and if $X_1 \otimes X_2 \in \mathrm{Ob}(\mathcal{C}')$ whenever $X_1, X_2 \in \mathrm{Ob}(\mathcal{C}')$).

A tensor subcategory of a rigid tensor category is said to be a *rigid tensor subcategory* if it contains $X^{\vee}$ whenever it contains X.

A tensor subcategory of a tensor category becomes a tensor category under the induced tensor structure. Similarly, a rigid tensor subcategory of a rigid tensor category becomes a rigid tensor category.

Abelian Tensor Categories

Definition 2.2.15 An *additive (resp. abelian) tensor category* is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

When $(\mathcal{C}, \otimes)$ is an abelian, then we say that a family $\mathcal{X} = (X_i)_{i \in I}$ of objects of $\mathcal{C}$ is a *tensor generating family* for $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to a subquotient of $P\big((X_i)_{i\in I}\big)$ for some $P \in \mathbb{Z}[T_i]_{i \in I}$ with non-negative coefficients. Note that for every polynomial $P \in \mathbb{Z}[T_i]_{i\in I}$, there exists a finite subset $\{i_1, \ldots, i_n\}$ of I such that

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T_{i_1}^{\alpha_1} \cdots T_{i_n}^{\alpha_n},$$

where $a_\alpha \in \mathbb{Z}$ and $a_\alpha = 0$ for all but finitely many α. If $P \in \mathbb{Z}[T_i]_{i\in I}$ is a polynomial with non-negative integral coefficients, then

$$P\big((X_i)_{i\in I}\big) = \bigoplus_{\alpha\in\mathbb{N}^n} \big(X_{i_1}^{\otimes\alpha_1} \otimes \cdots \otimes X_{i_n}^{\otimes\alpha_n}\big)^{\oplus a_\alpha}.$$

2.3 Neutral Tannakian Categories

In this section, we recall the definition of neutral Tannakian category and state the main theorem due to Saavedra. Let k be a field, and let $\mathbf{Vect}(k)$ be the category of finite dimensional vector spaces over k.

Definition 2.3.1 A category $\mathcal{C}$ is said to be *k–linear* if the set of morphisms between two arbitrary objects of $\mathcal{C}$ is a k–vector space and for objects X, Y and Z in $\mathcal{C}$, the composite map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

is k–bilinear with respect to the k–vector space structures on the Hom-sets involved.

Definition 2.3.2 A neutral Tannakian category over k is a rigid k–linear abelian tensor category $\mathcal{C}$ equipped with an exact faithful k–linear tensor functor $\omega : \mathcal{C} \longrightarrow \mathbf{Vect}(k)$ into the category of finite dimensional k–vector spaces. The functor ω is called a neutral fibre functor.

Let $\mathbf{Aut}^{\otimes}(\omega)$ be the group-valued functor defined on the category of k–algebras by sending R to the set of R–linear tensor functor isomorphisms. By Proposition 2.2.12, the natural morphism $\mathbf{Aut}^{\otimes}(\omega) \longrightarrow \mathbf{End}^{\otimes}(\omega)$ of functors is an isomorphism (see, Remark 2.2.13).

Given an affine group scheme G, the category $\mathbf{Rep}_k(G)$ of finite dimensional representations of G with its usual tensor structure and the forgetful functor $\omega_G : \mathbf{Rep}_k(G) \longrightarrow \mathbf{Vect}(k)$ as fibre functor is a neutral Tannakian category. Conversely, we have:

Theorem 2.3.3 *Let $(\mathcal{C}, \otimes, \mathbb{1}, \omega)$ be a neutral Tannakian category over k. Then*

(a) *the functor $G = \mathbf{Aut}^{\otimes}(\omega)$ of k–algebras is an affine group scheme;*

(b) *there is an equivalence* $\omega' : \mathcal{C} \longrightarrow \mathbf{Rep}_k(G)$ *of tensor categories such that the following diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\omega'"] \arrow[dr, "\omega"'] & \mathbf{Rep}_k(G) \arrow[d, "\omega_G"] \\ & \mathbf{Vect}(k) \end{tikzcd}$$

commutes.

The affine group scheme determined by the Theorem 2.3.3 is called the *Tannakian fundamental group scheme* of $(\mathcal{C}, \otimes, \mathbb{1}, \omega)$.

Proposition 2.3.4 *Let* $f : G \longrightarrow G'$ *be a homomorphism of affine group schemes and* $\omega^f : \mathbf{Rep}_k(G') \longrightarrow \mathbf{Rep}_k(G)$ *the corresponding tensor functor.*

(1) *The homomorphism* f *is faithfully flat if and only if* ω^f *is fully faithful and every subobject of* $\omega^f(X')$ $(X' \in \mathbf{Rep}_k(G'))$ *is isomorphic to the image of a subobject of* X'.

(2) *The homomorphism* f *is closed immersion if and only if every object of* $\mathbf{Rep}_k(G)$ *is isomorphic to a subquotient of* $\omega^f(X')$ *for some* $X' \in \mathbf{Rep}_k(G')$.

Chapter 3

Tannakian Fundamental Group Schemes

In this chapter we consider Frobenius–finite vector bundles on an arbitrary pointed smooth projective variety (X, x) over a perfect field k of prime characteristic (see Section 3.4), and study some properties of the corresponding affine group scheme, which we called the F–fundamental group scheme of X with base point x over k (see Section 3.4.2).

3.1 Some Preliminaries

In this section we recall the definition of the Frobenius morphism, and the notions of semistability and strong semistability. In Subsection 3.1.3, we recall the definition of principal bundles and its functorial version. We also recall some basic results about finite étale morphisms.

3.1.1 Frobenius and Étale Morphisms

Let k be a field of prime characteristic p. For any k-scheme Y, the absolute Frobenius morphism $F_Y : Y \longrightarrow Y$ is defined as follows: F_Y is the identity map on the topological space of X, and $F_Y^\sharp : \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ by the pth power map $f \mapsto f^p$. For any morphism $f : Y \longrightarrow X$ of schemes over k, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d, "f"'] & Y \arrow[d, "f"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

Let $Y^{(p)}$ denote the fibered product $Y \times_X X$, where the second factor X endowed with the structure of an X–scheme via $F_X : X \longrightarrow X$. We will endow $Y^{(p)}$ with the structure of an X–scheme given by the second projection $q : Y \times_X X \longrightarrow X$. Let us denote the first projection by $\varphi : Y^{(p)} \longrightarrow Y$. Then, there exists a unique morphism of X–schemes $F_{Y/X} : Y \longrightarrow Y^{(p)}$ making the following diagram commutative:

$$\begin{tikzcd} Y \arrow[dr, "F_{Y/X}"] \arrow[drr, bend left, "F_Y"] \arrow[ddr, bend right, "f"'] & & \\ & Y^{(p)} \arrow[r, "\varphi"] \arrow[d, "q"] & Y \arrow[d, "f"] \\ & X \arrow[r, "F_X"'] & X \end{tikzcd} \tag{3.1}$$

We call the above morphism of X–schemes $F_{Y/X} : Y \longrightarrow Y^{(p)}$ the *relative Frobenius*. Let $X = \mathrm{Spec}(k)$, where k is perfect field. Then $F_k : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k)$ and $\varphi : Y^{(p)} \longrightarrow Y$ are isomorphisms. Let $F_X^n : X \longrightarrow X$ denotes the composition of n absolute Frobenius morphisms of X.

Remark 3.1.1 If $\pi : X \longrightarrow \mathrm{Spec}(k)$ is a scheme over k, then $F_X : X \longrightarrow X$ is not k-linear. On the contrary, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, "F_k"'] & \mathrm{Spec}(k) \end{tikzcd}$$

with the Frobenius morphism F of $\mathrm{Spec}(k)$ (which corresponds to the pth power map of k to itself). We define a new scheme over k, which we denote by X_p, to be the same scheme X, but with structural morphism $F \circ \pi$. Thus, k acts on $\mathcal{O}_{X_p}$ via pth powers. Then, F_X becomes a k-linear morphism $F' : X_p \longrightarrow X$. We call this the *$k$-linear Frobenius morphism.*

Étale Morphisms

We first recall the definition of étale morphism. For more details of the results stated here, see [Mi80, III, §4] and [Sz09, Chap. 5].

Definition 3.1.2 A morphism $f : X \longrightarrow Y$ of schemes is flat if for all points x of X, the induced map $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is flat.

The following proposition is immediate from the usual properties of flat modules.

Proposition 3.1.3 (1) *An open immersion is flat.*

(2) *The composite of two flat morphisms is flat.*

(3) *Any base extension of a flat morphism is flat.*

Remark 3.1.4 Recall that a morphism $f : X \longrightarrow Y$ is *faithfully flat* if it is flat and surjective. Any flat morphism that is locally of finite type is open. If $f : X \longrightarrow Y$ is finite and flat, then it is both open and closed; thus, if Y is connected, then f is surjective and hence faithfully flat.

Definition 3.1.5 A morphism $f : X \longrightarrow Y$ that is locally of finite-type is said to be *unramified* at $x \in X$ if $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{X,x}$ and $k(x)$ is a finite separable extension of $k(y)$, where $y = f(x)$.

If f is unramified at all $x \in X$, then it is said to be *unramified morphism.* The next proposition allows us an alternative definition of unramified in terms of differentials.

Proposition 3.1.6 *Let X and Y be locally Noetherian schemes, and let $f : X \longrightarrow Y$ be a morphism that is locally of finite-type. Then the following conditions are equivalent:*

(1) *f is unramified.*

(2) *The sheaf $\Omega^1_{X/Y}$ is zero.*

(3) *The diagonal morphism* $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ *is an open immersion.*

Proposition 3.1.7 (1) *Any immersion is unramified.*

(2) *The composite of two unramified morphisms is unramified.*

(3) *Any base extension of an unramified morphism is unramified.*

Proof. Assertions (1) and (2) are immediate from the definition. Let $f : X \longrightarrow S$ be an unramified morphism and let $g : Y \longrightarrow S$ be a morphism of locally Noetherian schemes. Then for the base change $h : X \times_S Y \longrightarrow Y$, we have

$$\Omega^1_{(X \times_S Y)/Y} \cong h'^* \Omega^1_{X/S},$$

where $h' : X \times_S Y \longrightarrow X$ is the first projection ([Ha77, Proposition 8.10]). From Proposition 3.1.6, we conclude that $h : X \times_S Y \longrightarrow Y$ is an unramified morphism. □

Definition 3.1.8 A morphism of schemes $f : X \longrightarrow Y$ is said to be *étale* at $x \in X$, if it is flat and unramified at x. A morphism f is said to be an *étale morphism*, if it is étale at all the points of X. A *finite étale cover* is a surjective finite étale morphism.

Proposition 3.1.9 (1) *Any open immersion is étale.*

(2) *The composite of two étale morphisms is étale.*

(3) *Any base extension of an étale morphism is étale.*

(4) *If* $f : X \longrightarrow S$ *and* $g : Y \longrightarrow S$ *are two morphisms such that* $f \circ g$ *is étale and* f *is unramified, then* g *is étale.*

Proof. Assertions (1), (2) and (3) hold with 'étale' replaced by 'flat' (see Proposition 3.1.3) or by 'unramified' (see Proposition 3.1.7). For (4), consider the diagram with Cartesian squares:

$$\begin{tikzcd}
X \arrow[r, "\Delta_{X/S}"] & X \times_S X & \\
Y \arrow[u, "g"] \arrow[r, "\Gamma_g"'] & Y \times_S X \arrow[u, "g \times \mathbf{1}_X"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "f \circ g"'] & S
\end{tikzcd}$$

Since $f \circ g$ is étale, pr_2 is étale by (3). Since $\Delta_{X/S}$ is an open immersion as f is unramified, Γ_g is étale by (3). Therefore, $g = \mathrm{pr}_2 \circ \Gamma_g$ is étale by (2). □

Definition 3.1.10 A morphism $f : Y \longrightarrow X$ of schemes over k is called *radicial* if it is injective and all the residue field extensions $k(f(y)) \longrightarrow k(y)$ ($y \in Y$) are purely inseparable.

Proposition 3.1.11 *Let $f : X \longrightarrow Y$ be a morphism of schemes. The following conditions are equivalent:*

(1) *f is radicial.*

(2) *For any field K, the map of K–points $f(K) : X(K) \longrightarrow Y(K)$ is injective.*

(3) (Universal injectivity) *For any morphism $Y' \longrightarrow Y$, the morphism $f_{(Y')}$: $X \times_Y Y' \longrightarrow Y'$ is injective.*

(4) (Geometric injectivity) *For any field K and any morphism* $\mathrm{Spec}(K) \longrightarrow Y$, *the morphism $f_K : X \times_Y \mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(K)$ is injective.*

Proof. Assume (1) and for any field K, let $u_1, u_2 \in X(K)$ be such that $f \circ u_1 = f \circ u_2$. Since f is injective, $x = \mathrm{Im}(u_1) = \mathrm{Im}(u_2)$. Hence, u_1, u_2 corresponds to $k(f(x))$–homomorphisms $k(x) \rightrightarrows K$. Since $k(x)/k(f(x))$ is purely inseparable, $u_1 = u_2$ and hence (2) holds.

Conversely, assume (2) and suppose $k(x)/k(f(x))$ is not purely inseparable for some $x \in X$. Then there exist two different $k(f(x))$–homomorphisms of $k(x)$ into some field K. Let $u_1, u_2 : \mathrm{Spec}(K) \rightrightarrows X$ be the corresponding morphisms. Then $f \circ u_1 = f \circ u_2$, but $u_1 \neq u_2$, a contradiction. Suppose $f(x_1) = f(x_2) = y$ for distinct points $x_1, x_2 \in X$. Then there exists a field K and two $k(y)$–homomorphisms $k(x_1) \longrightarrow K$ and $k(x_2) \longrightarrow K$. Let $u_1, u_2 : \mathrm{Spec}(K) \rightrightarrows X$

be the corresponding morphisms. Then $f \circ u_1 = f \circ u_2$, but $u_1 \neq u_2$. Therefore, (1) holds.

Assume (2). Then the following diagram

$$\begin{tikzcd}
(X \times_Y Y')(K) \arrow[r, equal] \arrow[d] & X(K) \times_{Y(K)} Y'(K) \arrow[d] \\
Y'(K) \arrow[r, equal] & Y'(K)
\end{tikzcd}$$

shows that $f_{Y'}$ also satisfies (2). So by (2) $\Rightarrow$ (1), $f_{Y'}$ is injective and (3) holds. The implication (3) $\Rightarrow$ (4) is trivial.

Assume (4) and for any field K, let $u_1, u_2 \in X(K)$ be such that $f \circ u_1 = f \circ u_2$. Then u_1 and u_2 gives rise to sections $u_1', u_2' : \operatorname{Spec}(K) \rightrightarrows X \times_Y \operatorname{Spec}(K)$.

$$\begin{tikzcd}
X \times_Y \operatorname{Spec}(K) \arrow[r, "q"] \arrow[d, "p"'] & \operatorname{Spec}(K) \arrow[dl, shift right, "u_2"'] \arrow[dl, shift left, "u_1"] \arrow[d, "f\circ u_1 = f \circ u_2"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Since q is injective, $X \times_Y \operatorname{Spec}(K)$ consists of a single point. It follows that $u_1' = u_2'$, and hence $u_1 = u_2$. Thus, (2) holds and proof is complete. □

Proposition 3.1.12 (1) *Any immersion is radicial.*

(2) *The composition of radicial morphisms is radicial.*

(3) *Any base extension of a radicial morphism is radicial.*

(4) *If $f \circ g$ is radicial, then g is radicial.*

Proof. This is immediate from the Proposition 3.1.11. □

Proposition 3.1.13 *Let $f : Y \longrightarrow X$ be an étale morphism of k–schemes. Then the relative Frobenius $F_{Y/X} : Y \longrightarrow Y^{(p)}$ is an isomorphism.*

Proof. Consider the diagram (3.1). Since the base change of étale morphism is étale, the morphism q in (3.1) is étale. Since $q \circ F_{Y/X} = f$ is étale, by (4) of Proposition 3.1.9, $F_{Y/X}$ is étale. Since F_Y is universally injective and $F_Y = \varphi \circ F_{Y/X}$, by (4) of Proposition 3.1.12, $F_{Y/X}$ is universally injective. By [Mu67, Lemma 7.2.1.1], $F_{Y/X}$ is an open immersion. Since $F_{Y/X}$ is homeomorphism, it follows that $F_{Y/X}$ is an isomorphism. $\square$

Given a morphism of schemes $\varphi : X \longrightarrow S$, we define $\mathrm{Aut}(X/S)$ to be the group of scheme automorphisms of X preserving φ. For a geometric point $\bar{s} : \mathrm{Spec}(\xi) \longrightarrow S$ there is a natural left action of $\mathrm{Aut}(X/S)$ on the geometric fibre $X_{\bar{s}} = X \times_S \mathrm{Spec}(\xi)$ coming by base change from its action on X. For a connected finite étale cover $\varphi : X \longrightarrow S$, the action of $\mathrm{Aut}(X/S)$ on each geometric fibre is faithful and hence $\mathrm{Aut}(X/S)$ is finite.

Definition 3.1.14 A connected finite étale cover $\varphi : X \longrightarrow S$ is said to be *Galois* if $\mathrm{Aut}(X/S)$ acts transitively on geometric fibres.

Recall that every finite separable field extension can be embedded in a finite Galois extension and there is a smallest such extension, the Galois closure. Analogously, every finite étale cover is the image of a Galois cover. More precisely, we have

Proposition 3.1.15 *Let $\varphi : X \longrightarrow S$ be a connected finite étale cover. There is a morphism $\psi : Y \longrightarrow X$ such that $\varphi \circ \psi : Y \longrightarrow S$ is a finite étale Galois cover, and moreover every S–morphism from a Galois cover to X factors through $Y \longrightarrow S$.*

3.1.2 Semistability

Let X be a smooth n-dimensional projective variety over k with a very ample divisor H. If E is a torsion-free coherent sheaf on X then one can define its slope by setting

$$\mu(E) = \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rk}(E)},$$

where $\mathrm{rk}(E)$ is the rank of E. Then E is *H-semistable* if for any nonzero subsheaf $F \subset E$ we have $\mu(F) \leq \mu(E)$.

Let us recall that every torsion free sheaf E on X has a unique Harder–Narasimhan filtration, that is, a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

in which all quotients E_i/E_{i-1} are H–semistable sheaves and

$$\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E/E_{k-1}).$$

We set $\mu_{\max}(E) = \mu(E_1)$ and $\mu_{\min}(E) = \mu(E/E_{k-1})$.

Definition 3.1.16 We say that E is strongly H-semistable if for all $r \geq 0$ the pull back $(F_X^r)^*E$ is H-semistable.

In [Gie73] D. Gieseker showed for every $g \geq 2$ an example of a sequence $\{E_m\}_{m\in\mathbb{N}}$ of rank 2 vector bundles on a curve C of genus g such that $E = E_1$ is not semistable and $E_m = F^*E_{m+1}$ for every $m \geq 1$. Then for large m the bundles E_m are semistable but not strongly semistable.

3.1.3 Principal Bundles

In this section we recall a description of principal bundles over a scheme X defined over k using the language of categories and functors. For the details see [No76]. Let G be an affine group scheme defined over k.

Definition 3.1.17 A principal G–bundle over a scheme X is a surjective flat affine morphism $j : P \longrightarrow X$ together with a group action $\sigma : P \times_k G \longrightarrow P$ such that the map $(p_1, \sigma) : P \times_k G \longrightarrow P \times_X P$ is an isomorphism.

Let $E_G \longrightarrow X$ be a principal G–bundle over X. Then, for every object

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

in $\mathbf{Rep}_k(G)$, we can construct the associated vector bundle $E_\rho := E_G \times^G V$ over X. Here $E_G \times^G V = (E_G \times V)/G$ with the action of any $g \in G$ sending any $(z,\, v) \in E_G \times V$ to $(zg,\, \rho(g^{-1})v)$. This defines a functor

$$E_G \times^G \bullet : \mathbf{Rep}_k(G) \longrightarrow \mathrm{Vect}(X).$$

Nori proved in [No82, Section 2.2] that this functor determines the principal G–bundle E_G in the following sense.

Theorem 3.1.18 [No82, Lemma 2.3, Proposition 2.4] Let $F : \mathbf{Rep}_k(G) \longrightarrow \mathrm{Vect}(X)$ be a functor satisfying the following:

(a) F is a k–additive exact functor;

(b) $F \circ \widehat{\otimes} = \otimes \circ (F \times F)$;

(c1) the functor F preserves commutativity, in other words, if c is the canonical isomorphism of $V \widehat{\otimes} W$ with $W \widehat{\otimes} V$ in $\mathbf{Rep}_k(G)$, then $F(c)$ is the canonical isomorphism of the corresponding vector bundles;

(c2) the functor F preserves associativity;

(c3) the vector bundle $F(\mathbb{1})$ is the trivial line bundle $\mathcal{O}_X$ on X;

(d) for any $V \in \mathbf{Rep}_k(G)$ of dimension n, the vector bundle $F(V)$ is of rank n.

Then there exists a principal G–bundle $E_G \longrightarrow X$ (unique up to an isomorphism) such that F is isomorphic to $E_G \times^G \bullet$.

3.2 Étale Fundamental Group Scheme

Let k be a perfect field, and let X be a proper integral scheme over k endowed with a rational point $x \in X(k)$.

Definition 3.2.1 A vector bundle E over X is called *étale trivializable* if there exists a finite étale covering $\psi : Y \longrightarrow X$ such that the pull-back $\psi^* E$ is trivializable.

Consider the neutral Tannakian category defined by all étale trivializable vector bundles E over X The fiber functor for this neutral Tannakian category sends E to its fiber $E(x)$ over x. We denote this category by $\mathcal{C}^{\text{ét}}(X)$. The corresponding affine group scheme is called the *étale fundamental group scheme* and is denoted by $\pi^{\text{ét}}(X, x)$. When k is algebraically closed field, the étale fundamental group $\pi_1(X, x)$ is canonically isomorphic to the group of k–valued points of the étale fundamental group scheme $\pi^{\text{ét}}(X, x)$.

3.3 The Nori Fundamental Group Scheme

For a proper integral scheme X defined over a perfect field k endowed with a rational point $x \in X(k)$, Nori defined in [No76] and [No82] the fundamental group scheme $\pi^N(X, x)$ over k. Let $\mathbf{Vect}(X)$ be the category of vector bundles over X.

Definition 3.3.1 A vector bundle E over X is said to be *Nori–semistable* if for every non-constant morphism $f : C \longrightarrow X$ with C a smooth projective curve, the pull-back $f^*E \longrightarrow C$ is semi-stable of degree zero.

Let $\mathbf{Ns}(X)$ denote the full subcategory of $\mathbf{Vect(X)}$ whose objects are all Nori–semistable vector bundles over X.

Lemma 3.3.2 [No76, p. 37, Lemma 3.6] *The category* $\mathbf{Ns}(X)$ *is an abelian category.*

For any vector bundle E over X and for any polynomial f with non-negative integer coefficients, we define

$$f(E) := \bigoplus_{i=0}^{n} (E^{\otimes i})^{\oplus a_i},$$

where $f(t) = \sum_{i=0}^{n} a_i t^i$ with $a_i \in \mathbb{N}$, $\forall\, i \in \{1, 2, \dots, n\}$.

Definition 3.3.3 A vector bundle E over X is said to be *finite* if there are two distinct polynomials f and g with non-negative integer coefficients such that $f(E)$ is isomorphic to $g(E)$.

Definition 3.3.4 A vector bundle E over X is said to be *essentially finite* if it is a Nori–semistable subquotient of a finite vector bundle over X.

Let $\mathcal{C}^N(X)$ be the full subcategory of $\mathbf{Ns}(X)$ whose objects are the essentially finite vector bundles over X. Let

$$\omega_x : \mathcal{C}^N(X) \longrightarrow \mathbf{Vect}(k)$$

be the functor which sends an essentially finite vector bundle E to its fibre $E(x)$ over x. With the usual tensor product of vector bundles, the quadruple $(\mathcal{C}^N(X), \otimes, \omega_x, \mathcal{O}_X)$ is a neutral Tannakian category over k. By Theorem 2.3.3, it defines an affine group scheme over k, which is denoted by $\pi^N(X, x)$, and called the *Nori fundamental group scheme* of X over k with base point x.

The following is a useful characterization of essentially finite vector bundles:

Proposition 3.3.5 [No76] *A vector bundle E over X is essentially finite vector bundle if and only if there exists a principal G–bundle $\psi : P \longrightarrow X$, with G a finite group scheme, such that $\psi^* E$ is trivial vector bundle over P.*

Corollary 3.3.6 *There is a natural faithfully flat homomorphism $\pi^N(X, x) \longrightarrow \pi^{\text{ét}}(X, x)$.*

Proof. Let E be an étale trivializable vector bundle over X. Then there exists a finite étale Galois covering $\psi : Y \longrightarrow X$ with Galois group Γ such that $\psi^* E$ is trivial vector bundle over Y. Note that $\psi : Y \longrightarrow X$ can be viewed as a principal $(\Gamma)_k$–bundle, where $(\Gamma)_k$ is a finite constant group scheme. By Proposition 3.3.5 it follows that E is essentially finite vector bundle. By Proposition 2.3.4(1), there is a natural faithfully flat homomorphism $\pi^N(X, x) \longrightarrow \pi^{\text{ét}}(X, x)$. □

3.4 The F–fundamental Group Scheme

3.4.1 Construction of the F–fundamental Group Scheme

Let k be a perfect field of prime characteristic p. Let X be a smooth projective variety defined over k. Let $F_X : X \longrightarrow X$ denote the absolute Frobenius morphism. For any integer $m \geq 1$, let

$$F_X^m := \overbrace{F_X \circ \cdots \circ F_X}^{m\text{–times}} : X \longrightarrow X$$

be the m–fold iteration of the morphism F_X. We define F_X^0 to be the identity morphism of X.

For any polynomial f with non-negative integral coefficients, and for any vector bundle E over X, we define

$$\widetilde{f}(E) := \bigoplus_{i=0}^{m} ((F_X^i)^* E)^{\oplus n_i} \,, \tag{3.2}$$

where $f(t) = \sum_{i=0}^m n_i t^i$ with $n_i \in \mathbb{N} \; \forall \; i \in \{1, 2, \cdots, m\}$.

Definition 3.4.1 A vector bundle E over X is said to be *Frobenius–finite* if there exist two distinct polynomials f and g with non-negative integral coefficients such that $\widetilde{f}(E)$ is isomorphic to $\widetilde{g}(E)$.

Let $\mathrm{FF}(X)$ denote the set of Frobenius–finite vector bundles over X. For any vector bundle E over X, let $\mathrm{I}(E)$ denote the set of all indecomposable components of $\{(F_X^n)^*E\}_{n\geq 0}$.

Lemma 3.4.2 *If E is a Frobenius–finite vector bundle over X, then $I(E)$ is a finite set.*

Proof. For $n \geq 0$, let $\mathrm{I}_n(E)$ denote the set of indecomposable components of $(F_X^n)^*E$. By the Krull-Remak-Schmidt theorem, $\mathrm{I}_n(E)$ is finite. Note that $\mathrm{I}(E) = \cup_{n\geq 0}\mathrm{I}_n(E)$. Since E is Frobenius–finite, there exist two distinct polynomials f and g as in the Definition 3.4.1 such that $\widetilde{f}(E)$ is isomorphic to $\widetilde{g}(E)$. Let $\mathrm{IV}(X)$ be the free abelian group generated by isomorphism classes of indecomposable vector bundles on X. Now, $\widetilde{h}(E) = 0$ in $\mathrm{IV}(X)$, where $h = f - g$. Let $d = \deg(h)$. Then, each indecomposable direct summand of $(F_X^d)^*E$ must be a direct summand of $(F_X^j)^*E$ for some $j < d$. Applying the above to the polynomials $(f-g)x^i$ for $i > 0$ and using induction on i, we see that each element of $\mathrm{I}(E)$ can be represented by a direct summand of $(F_X^j)^*E$ for some $j < d$. $\square$

Proposition 3.4.3 *Let C be a smooth projective curve. Then, any Frobenius–finite vector bundle over C is strongly semistable of degree zero.*

Proof. Let E be a Frobenius–finite vector bundle over C. Then by Lemma 3.4.2, $\mathrm{I}(E)$ is a finite set. Note that for any vector bundle $V \longrightarrow C$, we have

$$\deg((F_C^n)^*V) = p^n \cdot \deg(V), \tag{3.3}$$

where p is the characteristic of k. Therefore, the fact that $\mathrm{I}(E)$ is a finite set implies that $\deg(E) \leq 0$. Since the dual $E^\vee$ is also a Frobenius-finite vector bundle, we conclude that $\deg(E) = 0$.

Let F be any subbundle of $(F_C^r)^*E$ $(r \geq 0)$. Then $(F_C^n)^*F$ is subbundle of $(F_C^{n+r})^*E$. Since $I(E)$ is a finite set, from (3.3) it follows that $\deg(F) \leq 0$. Hence $(F_C^r)^*E$ is semistable. $\square$

Let $h : X \longrightarrow Y$ be a morphism. Then the pull-back of any Frobenius–finite vector bundle E over Y is also a Frobenius–finite vector bundle over X. Therefore, Proposition 3.4.3 has the following corollary.

Corollary 3.4.4 *Any Frobenius-finite vector bundle over a smooth projective variety X is Nori–semistable.*

It has been proved in [No76, p. 37, Lemma 3.6] that the category $\mathbf{Ns}(X)$ of all Nori–semistable vector bundles over X is abelian.

Let $\mathrm{TFF}(X)$ denote the collection of all tensor products of Frobenius–finite vector bundles over X. Consider the full subcategory, denoted by $\mathcal{C}_F(X)$, of the category $\mathbf{Ns}(X)$ whose objects are: vector bundles $E \in \mathbf{Ns}(X)$ for which there exist $F_i \in \mathrm{TFF}(X)$, $1 \leq i \leq m$ $(m \geq 0)$ and

$$E_1 \subseteq E_2 \subseteq \bigoplus_{i=1}^{m} F_i \text{ with } E_1,\ E_2 \in \mathbf{Ns}(X)$$

such that $E \cong E_2/E_1$.

Proposition 3.4.5 *The category $\mathcal{C}_F(X)$ defines a neutral Tannakian category over k.*

Proof. The category $\mathcal{C}_F(X)$ is an abelian category. For, let $f : E \longrightarrow F$ be a morphism in $\mathcal{C}_F(X)$. Since $\mathcal{C}_F(X)$ is a full subcategory of an abelian category $\mathbf{Ns}(X)$, both $\mathrm{Ker} f$ and $\mathrm{Coker} f$ lie in $\mathbf{Ns}(X)$. By the construction of $\mathcal{C}_F(X)$, both $\mathrm{Ker} f$ and $\mathrm{Coker} f$ are in $\mathcal{C}_F(X)$. Hence, $\mathcal{C}_F(X)$ is an abelian k–category.

We define the tensor operation $\otimes$ on $\mathcal{C}_F(X)$ to be the usual tensor product of vector bundles. We need to check that $\mathcal{C}_F(X)$ is closed under the tensor product operation. If E and F are objects of $\mathcal{C}_F(X)$, then there exist $V_1, V_2, W_1, W_2 \in \mathbf{Ns}(X)$, $E_i, F_j \in \mathrm{TFF}(X)$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$, such that

$$V_1 \subseteq V_2 \subseteq \bigoplus_{i=1}^{n} E_i, \quad W_1 \subseteq W_2 \subseteq \bigoplus_{j=1}^{m} F_j$$

and $E \cong V_2/V_1$ and $F \cong W_2/W_1$. Then, $E \otimes F$ is a quotient of $V_2 \otimes W_2$, i.e., $E \otimes F \cong V_2 \otimes W_2/K$. By definition of $\mathrm{TFF}(X)$, the bundle $(\bigoplus_{i=1}^{n} E_i) \otimes (\bigoplus_{j=1}^{m} F_j)$ is again a finite direct sum of elements of $\mathrm{TFF}(X)$. Since $V_2, W_2 \in \mathbf{Ns}(X)$, it follows that $V_2 \otimes W_2 \in \mathbf{Ns}(X)$. As $\deg(K) + \deg(E \otimes F) = \deg(V_2 \otimes W_2)$, this implies that $\deg(K) = 0$. This implies that $K \in \mathbf{Ns}(X)$. Therefore, $E \otimes F$ is an object of $\mathcal{C}_F(X)$. The pentagon and hexagon axioms follow from the properties of tensor product of vector bundles. The trivial vector bundle $\mathcal{O}_X$ is an object of $\mathcal{C}_F(X)$. It is a unit object for the tensor product.

That $\mathcal{C}_F(X)$ is rigid follows from the fact that $\mathrm{TFF}(X)$ is closed under taking dual of vector bundles. For, if $E \in \mathcal{C}_F(X)$ is a Nori-semistable subbundle of a direct sum of finitely many $E_i \in \mathrm{TFF}(X)$, then the dual $E^\vee$ is a quotient of the direct sum of duals $E_i^\vee$, which also belong to $\mathrm{TFF}(X)$, hence $E^\vee$ is an object of $\mathcal{C}_F(X)$. If $E \cong V_2/V_1$ with $V_1 \subseteq V_2 \subseteq \bigoplus_{i=1}^n E_i$, dualizing the exact sequence $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow E$, we get that $E^\vee = \mathrm{Ker}(V_2^\vee \longrightarrow V_1^\vee)$ is an object of $\mathcal{C}_F(X)$, since $\mathcal{C}_F(X)$ is an abelian category.

Now, we will define a fibre functor on $\mathcal{C}_F(X)$. Let $\mathrm{Vect}(k)$ denote the category of finite dimensional vector spaces over a field k. Fix a k–rational point $x \in X$. We define ω_x to be the functor

$$\omega_x : \mathcal{C}_F(X) \longrightarrow \mathrm{Vect}(k) \tag{3.4}$$

that sends a vector bundle E in $\mathcal{C}_F(X)$ to its fiber $E(x)$ over x. This functor is exact and k–linear on Hom-sets. We show that it is also faithful. For this assume that E, F are object of $\mathcal{C}_F(X)$, and that $f, g \in \mathrm{Hom}(E, F)$, such that $\omega_x(f) = \omega_x(g)$, i.e. $f_x = g_x : E(x) \longrightarrow F(x)$. Then, $\mathrm{Ker}(f-g)$ is an object of $\mathcal{C}_F(X)$, since $\mathcal{C}_F(X)$ is an abelian category. In particular, $\mathrm{Ker}(f-g)$ is a vector bundle, hence all its fibres are of the same dimension. Because of the assumption that the fibre over x is zero-dimensional, we see that $\mathrm{Ker}(f-g) = 0$, hence $f = g$. Therefore, the quadruple $(\mathcal{C}_F(X), \otimes, \omega_x, \mathcal{O}_X)$ is a neutral Tannakian category over k. □

The quadruple $(\mathcal{C}_F(X), \otimes, \omega_x, \mathcal{O}_X)$ is a neutral Tannakian category over k. So it gives an affine group–scheme over k (Theorem 2.3.3).

Definition 3.4.6 The *F–fundamental group–scheme of X with the base point x* is the group–scheme associated to the neutral Tannakian category

$$(\mathcal{C}_F(X), \otimes, \omega_x, \mathcal{O}_X)\,.$$

This group–scheme will be denoted by $\pi_F(X,\, x)$.

Let $\mathbf{Rep}_k(\pi_F(X,\, x))$ be the category of finite dimensional k–representations of $\pi_F(X,\, x)$.

Lemma 3.4.7 *There exists a tautological principal $\pi_F(X,\, x)$–bundle over X.*

Proof. By the construction of $\pi_F(X,\,x)$, there is an equivalence of categories

$$\mathcal{C}_F(X) \longrightarrow \mathbf{Rep}_k(\pi_F(X,\,x)) \tag{3.5}$$

such that ω_x becomes a forgetful functor for $\mathbf{Rep}_k(\pi_F(X,\,x))$ (see [DM82, Theorem 2.11], [No76]). The resulting functor

$$T\,:\,\mathbf{Rep}_k(\pi_F(X,\,x)) \simeq (\mathcal{C}_F(X), \omega_x) \hookrightarrow \mathrm{Vect}(X) \tag{3.6}$$

satisfies the conditions of Theorem 3.1.18, and hence it defines a principal $\pi_F(X,\,x)$–bundle $\widetilde{X}$ over X. For any $V \in \mathbf{Rep}_k(\pi_F(X,\,x))$, the vector bundle $\widetilde{X} \times^{\pi_F(X,x)} V \longrightarrow X$ associated to the principal $\pi_F(X,\,x)$–bundle $\widetilde{X} \longrightarrow X$ for the $\pi_F(X,\,x)$–module V coincides with the vector bundle corresponding to V by the functor in (3.5). $\square$

Let $\varphi\,:\,X \longrightarrow Y$ be a morphism of smooth projective varieties defined over k and E an object of $\mathcal{C}_F(Y)$. Then φ^*E is an object of $\mathcal{C}_F(X)$. Consequently, φ induces a homomorphism of Tannakian categories from $(\mathcal{C}_F(Y), \omega_{f(x)})$ to $(\mathcal{C}_F(X), \omega_x)$. By [No76, Theorem 1.3], this determines a unique homomorphism of affine group schemes $\pi_F(X,\,x) \longrightarrow \pi_F(Y,\,f(x))$.

3.4.2 Some Properties of the F–fundamental Group Scheme

First we have

Proposition 3.4.8 *If $\varphi : X \longrightarrow Y$ is a flat surjective morphism of smooth projective varieties over a field k with $\varphi_*\mathcal{O}_X = \mathcal{O}_Y$. Then the induced homomorphism $\pi_F(X,\,x) \longrightarrow \pi_F(Y,\,\varphi(x))$ is faithfully flat.*

Proof. In view of Proposition 2.3.4(1), it suffices to show the following:

1. the functor $\omega^\varphi : \mathcal{C}_F(Y) \longrightarrow \mathcal{C}_F(X)$ is fully faithful, and
2. any Nori–semistable subbundle F of φ^*E, where $E \in \mathcal{C}_F(Y)$, is isomorphic to φ^*F', for some Nori–semistable subbundle $F' \subset E$.

Since φ is flat morphism, by [EGA1, 6.7.6.1], there is a canonical isomorphism

$$\mathscr{H}om_Y(E, F) \longrightarrow \varphi_*\mathscr{H}om_X(\varphi^*E, \varphi^*F)\,. \tag{3.7}$$

By taking global section, we get a bijection

$$\mathrm{Hom}_Y(E, F) \longrightarrow \mathrm{Hom}_X(\varphi^* E, \varphi^* F)\,. \tag{3.8}$$

This shows that ω^φ is fully faithful functor.

Let E be an object of $\mathcal{C}_F(Y)$, and let F be a Nori–semistable subbundle of $\varphi^* E$. Define $E' := \varphi^* E/F$, and denote the ranks of E, F and E' by r, r_1 and r_2 respectively. Let $X_y := \varphi^{-1}(y)$ be the fibre over a point $y \in Y$. Then the restriction $F_{|X_y}$ is degree zero subbundle of the trivial vector bundle $(\varphi^* E)_{|X_y} \cong \mathcal{O}^r_{X_y}$. Hence $F_{|X_y}$ is trivial. Similarly, $E'_{|X_y}$ is trivial. Since F is flat over Y and $\dim H^0(X_y, F_{|X_y}) = r_1$ for all $y \in Y$, by [Ha77, III, Corollary 12.9], it follows that $\varphi_* F$ is locally free of rank r_1. Similarly, $\varphi_* E'$ is locally free of rank r_2.

The quotient map $\varphi^* E \longrightarrow E'$ evidently factors through $\varphi^* \varphi_* E' \longrightarrow E'$. Consequently, $\varphi^* \varphi_* E' \longrightarrow E'$ is surjective homomorphism of vector bundles of the same rank r_2; hence it is an isomorphism. We have,

$$\begin{tikzcd}
0 \arrow[r] & \varphi_* F \arrow[r] & \varphi_* \varphi^* E \arrow[r, "f"] \arrow[dr, "g"'] & \varphi_* \varphi^* E/\varphi_* F \arrow[r] \arrow[d, "\pi"] & 0 \\
 & & & \varphi_*(\varphi^* E/F) &
\end{tikzcd}$$

Since rank of $\varphi_* \varphi^* E/\varphi_* F$ and $\varphi_*(\varphi^* E/F)$ are same, it follows that

$$\pi \,:\, \varphi_* \varphi^* E/\varphi_* F \longrightarrow \varphi_*(\varphi^* E/F)$$

is an isomorphism. Thus, we get an isomorphism $\varphi^* E/\varphi^* \varphi_* F \longrightarrow \varphi^* E/F$. Therefore, $\varphi^* \varphi_* F \longrightarrow F$ is also an isomorphism. Take $F' = \varphi_* F$. Since $\varphi^* F' \cong F$ is Nori–semistable, we conclude that F' is Nori–semistable. Indeed, let $f : C \longrightarrow Y$ be a non-constant morphism, where C a smooth projective curve. Then there exists a smooth projective curve C' with the map $g : C' \longrightarrow X$ and a non-constant map $h : C' \longrightarrow C$ such that the following diagram commutes:

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "\varphi"] \\
C \arrow[r, "f"'] & Y
\end{tikzcd}$$

This implies that $(\varphi \circ g)^* F' \cong (f \circ h)^* F'$. Since $\varphi^* F' \cong F$ and F is Nori–semistable, $h^*(f^* F')$ is semistable of degree zero. Let W be a subbundle of $f^* F'$. Then $h^* W$ is subbundle of $h^*(f^* F')$ and hence $\deg(h^* W) = \deg(h) \deg(W) \leq 0$. This implies that $f^* F'$ is semistable of degree zero. Therefore, we get a Nori–semistable subbundle F' of E such that $\varphi^* F'$ is isomorphic to F. □

Lemma 3.4.9 *Let $\psi : Y \longrightarrow X$ be a morphism of smooth projective varieties over k such that the following diagram*

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd} \tag{3.9}$$

is Cartesian, where F_X and F_Y are the absolute Frobenius morphisms of X and Y respectively. Let E be a vector bundle over Y. Then for any $n \geq 1$, there is a canonical isomorphism

$$(F_X^n)^*(R^i \psi_* E) \longrightarrow R^i \psi_*((F_Y^n)^* E)$$

for each $i \geq 0$.

Proof. Since the diagram in (3.9) is Cartesian, and F_X is flat, we have the following: For any vector bundle $E \longrightarrow Y$, the base change homomorphism

$$F_X^*(R^i \psi_*(E)) \longrightarrow R^i \psi_*(F_Y^* E) \tag{3.10}$$

is isomorphism for each $i \geq 0$ (see [Ha77, III, Proposition 9.3]). Fix any $i \geq 0$. We will prove the lemma using induction on n. For $n = 1$, this is the isomorphism in (3.10). Assume that we have an isomorphism

$$(F_X^{n-1})^*(R^i \psi_*(E)) \longrightarrow R^i \psi_*((F_Y^{n-1})^* E)\,. \tag{3.11}$$

Taking inverse image by F_X, the isomorphism in (3.11) gives an isomorphism

$$(F_X^n)^*(R^i \psi_*(E)) \longrightarrow F_X^* R^i \psi_*((F_Y^{n-1})^* E)\,. \tag{3.12}$$

Substituting E by $(F_Y^{n-1})^*E$ in (3.10), we get an isomorphism

$$F_X^*(R^i\psi_*((F_Y^{n-1})^*E)) \longrightarrow R^i\psi_*(F_Y^*((F_Y^{n-1})^*E)) \longrightarrow R^i\psi_*((F_X^n)^*E)\,. \tag{3.13}$$

By composing the isomorphisms in (3.12) and (3.13), we get an isomorphism

$$(F_X^n)^*(R^i\psi_*E) \longrightarrow R^i\psi_*((F_Y^n)^*E). \tag{3.14}$$

This completes the proof of the lemma. □

Base Change

Let k' be an algebraically closed extension of an algebraically closed field k of prime characteristic p. Let X be a smooth projective k–variety, and let $X_{k'}$ denote the base change $X \times_k \operatorname{Spec} k'$. We have the following commutative diagram

$$\begin{tikzcd} X_{k'} \arrow[r, "F_{X_{k'}}"] \arrow[d, "p_1"] & X_{k'} \arrow[d, "p_1"'] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

where $F_{X_{k'}}$ and F_X are the Frobenius morphisms on $X_{k'}$ and X respectively. Therefore,

$$F_X^*E \otimes_k k' = F_{X_{k'}}^*(E \otimes_k k')\,.$$

From this it follows that if E is an object of $\mathcal{C}_F(X)$, then $E \otimes_k k'$ is an object of $\mathcal{C}_F(X_{k'})$. Let $\mathcal{C}_{k'}$ be the Tannakian subcategory of

$$\mathcal{C}' = (\mathcal{C}_F(X_{k'}), \otimes, T_{x'}, \mathcal{O}_{X_{k'}})$$

whose objects are all vector bundles $E' \in \mathcal{C}_F(X_{k'})$ for which there exists $E \in \mathcal{C}_F(X)$ such that $E' \subset E \otimes k'$. Define $G := \pi_F(X_k, x)$, and let $\mathbf{Rep}_{k'}(G_{k'})$ denote the category of all finite dimensional k'–representations of $G_{k'} = G \times_k \operatorname{Spec} k'$.

Now following the argument as in the proof of [MS02, Proposition 3.1], we conclude that the category $\mathcal{C}'$ is equivalent to $\mathbf{Rep}_{k'}(G_{k'})$. By Proposition 2.3.4, the canonical homomorphism

$$h : \pi_F(X_{k'},\, x') \longrightarrow \pi_F(X,\, x) \times_k k' \tag{3.15}$$

is faithfully flat.

If the canonical homomorphism in (3.15) is an isomorphism then any stable vector bundle in $\mathcal{C}_F(X_{k'})$ is actually defined over k. For, let $E \in \mathcal{C}_F(X_{k'})$. Since h is closed immersion, by Proposition 2.3.4(2), there exists a vector bundle E_1 in $\mathcal{C}_F(X)$ such that $E \cong E_3/E_2$, where E_2 and E_3 are Nori–semistable subbundles of $E_1 \otimes k'$ with $E_2 \subseteq E_3 \subseteq E_1 \otimes k'$. If E is stable then it is a component of $\mathrm{gr}(E_1 \otimes k')$ (the associated graded object for the Jordan–Hölder filtration of $E_1 \otimes k'$). Since $\mathrm{gr}(E_1 \otimes k') = \mathrm{gr}(E_1) \otimes k'$, it follows that E is a bundle of the form $E' \otimes k'$, where E' is stable vector bundle over X.

But there are stable vector bundles in $\mathcal{C}_F(X_{k'})$ which are not defined over k; see [Pa07] for examples. Therefore, the homomorphism in (3.15) is not an isomorphism in general.

Remark 3.4.10 A vector bundle E over X is called F–trivial if $(F_X^i)^*E$ is trivial for some i. Note that the category of all F–trivial vector bundles form a Tannakian category; the corresponding affine group scheme is called *the local fundamental group scheme* which is denoted by $\pi^{\mathrm{loc}}(X,x)$ (see [MS08]). By applying the criterion of Proposition 2.3.4, it can be easily seen that there is a canonical flat surjective homomorphism

$$\alpha \,:\, \pi_F(X,\, x) \,\longrightarrow\, \pi^{\mathrm{loc}}(X,x)\,.$$

Relation with the S–fundamental Group Scheme

For a smooth projective curve defined over k, the S–fundamental group–scheme was introduced in [BPS06]. In [La09], this was generalized to smooth projective varieties defined over an algebraically closed field.

Let X be a smooth projective variety of dimension n over an algebraically closed field k. Fix a very ample hypersurface H on X.

Recall (see Section 3.1.2) that a torsion-free coherent sheaf $E \longrightarrow X$, the number

$$\mu(E) \,:=\, \frac{c_1(E)\cdot H^{n-1}}{\mathrm{rank} E} \,\in\, \mathbb{Q}$$

is called the *slope* of E. A vector bundle E is called *H–semistable* (respectively, *H–stable*) if for all subsheaves F of E with $\mathrm{rank}(F) < \mathrm{rank}(E)$, we have $\mu(F) \leq \mu(E)$ (respectively, $\mu(F) < \mu(E)$).

Let $\mathrm{Vect}_0^S(X)$ denote the full subcategory of the category of coherent sheaves on X whose objects are all strongly H–semistable reflexive sheaves with $c_1(E) \cdot H^{n-1} = 0$ and $c_2(E) \cdot H^{n-2} = 0$. By [La09, Theorem 4.1], a strongly H–semistable reflexive sheaf with $c_1(E) \cdot H^{n-1} = 0$ and $c_2(E) \cdot H^{n-2} = 0$ is locally free. The category $\mathrm{Vect}_0^S(X)$ does not depend on the choice of H ([La09, Proposition 4.5]). Fix a k–rational point $x \in X$ and define the fiber functor $\omega_x : \mathrm{Vect}_0^S(X) \longrightarrow \mathbf{Vect}(k)$ by sending E to its fiber $E(x)$. Then $(\mathrm{Vect}_0^S(X), \otimes, \omega_x, \mathcal{O}_X)$ is a neutral Tannakian category.

Definition 3.4.11 The affine k–group scheme Tannaka dual to the neutral Tannakian category $(\mathrm{Vect}_0^S(X), \otimes, \omega_x, \mathcal{O}_X)$ is denoted by $\pi^S(X, x)$ and it is called the *S–fundamental group scheme* of X with base point x.

Recall that a line bundle L over X is said to be *numerically effective* if the degree of the restriction of L to any irreducible curve C in X is non-negative. A vector bundle E over X is called numerically effective if and only if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is numerically effective. A vector bundle E is called *numerically flat* if both E and its dual $E^\vee$ are numerically effective.

If E is a vector bundle over a smooth projective curve C, we denote by $\delta(E)$ the minimum of the degrees of quotient line bundles of E. For a line bundle L, define $\delta(L) = \mathrm{degree}(L)$.

A vector bundle E is numerically effective if and only if for any finite morphism $f : C \longrightarrow X$ from a smooth projective curve C, we have $\delta(f^*E) \geq 0$ (see, [Ba71, p. 437]).

Remark 3.4.12 Let C be a smooth projective curve and V a vector bundle on C. Denote by $\delta^*(V)$ the minimum of the set $\{\mu(Q) | V \longrightarrow Q \longrightarrow 0\}$ and by $\delta(V)$ the minimum of the set $\{\mu(L) | V \longrightarrow L \longrightarrow 0,\ L \text{ is line bundle}\}$. Consider the Harder-Narasimhan filtration

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V$$

of the vector bundle V. Let $\mu_{\min}(V) := \mu(V/V_{n-1})$ and $\mu_{\max}(V) := \mu(V_1)$. The subbundle V_{n-1} is called the maximal destabilizing vector subbundle of V. Since

$\mu(V/V_{n-1}) \in \{\mu(Q)|V \longrightarrow Q \longrightarrow 0\}$, we have $\mu_{\min}(V) \geq \delta^*(V)$. If $V \longrightarrow Q \longrightarrow 0$ is a quotient of V, then Q is quotient of a semi-stable vector bundle V/V_{n-1}. From which it follows that $\mu_{\min}(V) \leq \mu(Q)$ for all quotients $V \longrightarrow Q \longrightarrow 0$. This proves that $\mu_{\min}(V) \leq \delta^*(V)$. Therefore, $\mu_{\min}(V) = \delta^*(V)$. It is obvious that $\delta(V) \geq \delta^*(V)$.

Proposition 3.4.13 *There exists a natural faithfully flat homomorphism*

$$\pi^S(X, x) \longrightarrow \pi_F(X, x).$$

Proof. Take any vector bundle E belonging to $\mathcal{C}_F(X)$. For any finite morphism $f : C \longrightarrow X$ from a smooth projective curve C, the pull–back f^*E is semistable. Consequently, $\mu_{\min}(f^*E) = 0$. Note that $\delta(V) \geq \mu_{\min}(V)$ for any vector bundle V over a smooth projective curve C. Consequently, $\delta(f^*E) \geq 0$, and hence E is numerically effective. Since E^* is also an object of $\mathcal{C}_F(X)$, it follows that E is numerically flat. By [La09, Proposition 5.1], we have a natural functor $\mathcal{C}_F(X) \longrightarrow \mathrm{Vect}_0^S(X)$ which is fully faithful and obviously satisfies the condition in Proposition 2.3.4(1). □

Product Formula

Suppose that k is an algebraically closed field. Let X and Y be two smooth projective varieties over k. Fix k–rational points x_0 and y_0 of X and Y, respectively.

For any closed point $y \in Y$, we have a canonical morphism

$$i_y : X \longrightarrow X \times_k Y$$

defined by

$$X \longrightarrow X \times_k \{y\} \longrightarrow X \times_k Y\,.$$

Similarly, for any closed point $x \in X$, we define

$$j_x : Y \longrightarrow X \times_k Y\,.$$

Let $p : X \times_k Y \longrightarrow X$ and $q : X \times_k Y \longrightarrow Y$ be the natural projections. The morphism $i_{y_0} : X \longrightarrow X \times_k Y$ induces a homomorphism

$$\pi_F(X, x_0) \longrightarrow \pi_F(X \times_k Y, x_0 \times y_0)$$

of affine group schemes. Since $p \circ i_{y_0} = \mathrm{Id}_X$, and $q \circ i_{y_0}$ is a constant map, it follows that the morphism

$$\psi : \pi_F(X \times_k Y,\, x_0 \times y_0) \longrightarrow \pi_F(X,\, x_0) \times \pi_F(Y,\, y_0)$$

is surjective.

Definition 3.4.14 A coherent sheaf E on X is called *bounded* if there is a scheme T of finite type over k and a coherent sheaf $\mathcal{E} \longrightarrow X \times T$ flat over T such that there are closed points $\{t_i\}_{i\geq 1}$ of T with the property that the restriction $\mathcal{E}_{|X\times\{t_i\}}$ is isomorphic to the pull-back $(F_X^i)^*E$.

Proposition 3.4.15 [BH09] *Let E be a bounded sheaf over X. Then*

(1) *For any morphism $g : Y \longrightarrow X$, where Y is a smooth projective variety, the pullback g^*E is bounded sheaf over Y.*

(2) *The coherent sheaf E is locally free.*

Proof. Since E is bounded, there is a family $\mathcal{E} \longrightarrow X \times T$ as in the Definition 3.4.14. Consider the family $(g \times \mathbf{1}_T)^*\mathcal{E} \longrightarrow Y \times T$, where $(g \times \mathbf{1}_T) : Y \times T \longrightarrow X \times T$. Since $F_Y^* g * E = g^* F_X^* E$, and $\mathcal{E}_{|X\times\{t_i\}} = (F_X^i)^*E$, we have

$$\big((g \times \mathbf{1}_T)^*\mathcal{E}\big)|_{Y\times\{t_i\}} = g^*\big(\mathcal{E}_{|X\times\{t_i\}}\big) = g^*(F_X^i)^*E = (F_Y^i)^*g^*E.$$

Therefore, g^*E is bounded sheaf.

To prove that E is locally free, it is enough to show that the fibre dimension $E \otimes k(x)$ is constant, where x runs over all closed points of X. For any g as in the first statement, and any closed point y of Y, we have

$$\dim(g^*E \otimes k(y)) = \dim(E \otimes k(g(y)).$$

Therefore, in view of the first statement, it suffices to prove that any bounded sheaf on a smooth projective curve is a locally free sheaf.

Let E be a bounded sheaf over a smooth projective curve C defined over k. Let E_{tor} be the torsion subsheaf of E, and let $E_f := E/E_{\mathrm{tor}}$ be the locally free quotient. We have

$$E = E_f \oplus E_{\mathrm{tor}}.$$

Hence for any $n \geq 1$,

$$(F_C^n)^* E_{\text{tor}} = \big((F_C^n)^* E\big)_{\text{tor}}, \tag{3.16}$$

where $\big((F_C^n)^*E\big)_{\text{tor}}$ is the torsion part of $\big((F_C^n)^*E\big)$. Let d be the dimension of $H^0(C, E_{\text{tor}})$. Then we have

$$\dim H^0\big(C, (F_C^n)^* E_{\text{tor}}\big) = p^n d, \tag{3.17}$$

where p is the characteristic of the field k. Since E is bounded, using semi-continuity of the dimension of global sections, we conclude that the sequence $\{\dim H^0\big(C, (F_C^n)^*E\big)\}_{n\geq 1}$ is bounded by a constant independent of n. Since $\dim H^0(C, \big((F_C^n)^*E\big) \geq \dim H^0(C, \big((F_C^n)^*E\big)_{\text{tor}})$, from (3.16) and (3.17), it follows that $H^0(C, E_{\text{tor}}) = 0$. Consequently, $E_{\text{tor}} = 0$.

Lemma 3.4.16 [BH09, Lemma 3.3] *Let X and Y be smooth projective varieties over k. Let $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ be the natural projections. Let E be a bounded sheaf on $X \times Y$. Then for each $i \geq 0$, the direct image $R^i p_* E$ and $R^i q_* E$ are bounded sheaves over X and Y respectively.*

Using Lemma 3.4.16 and Proposition 3.4.15, we conclude that for any $E \in \mathcal{C}_F(X \times_k Y)$, the direct images $R^i p_* E$ and $R^i q_* E$ are vector bundles over X and Y respectively.

Lemma 3.4.17 *Let E be a Frobenius–finite vector bundle over $X \times_k Y$. Then for each $i \geq 0$, the direct images $R^i p_* E \longrightarrow X$ and $R^i q_* E \longrightarrow Y$ are also Frobenius–finite vector bundles.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} X \times_k Y \arrow[r, "F_{X\times_k Y}"] \arrow[d, "p"'] & X \times_k Y \arrow[d, "p"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd} \tag{3.18}$$

and the Cartesian diagram

$$\begin{tikzcd} (X \times_k Y) \times_X X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \times_k Y \arrow[d, "p"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

By the properties of a fiber product, we have

$$(X \times_k Y) \times_X X \simeq (Y \times_k X) \times_X X \simeq X \times_k Y\,.$$

From this, it follows that the commutative diagram in (3.18) is Cartesian. Since F_X is flat, by [Ha77, III, Proposition 9.3], there is a canonical isomorphism

$$F_X^*(R^i p_* E) \longrightarrow R^i p_*(F_{X\times_k Y}^* E) \tag{3.19}$$

for each $i \geq 0$. Similarly, we have a canonical isomorphism

$$F_Y^*(R^i q_* E) \longrightarrow R^i q_*(F_{X\times_k Y}^* E)$$

for each $i \geq 0$. By Lemma 3.4.9, for any integer $n \geq 1$, there are canonical isomorphisms

$$(F_X^n)^*(R^i p_* E) \longrightarrow R^i p_*((F_{X\times_k Y}^n)^* E) \tag{3.20}$$

and

$$(F_Y^n)^*(R^i q_* E) \longrightarrow R^i q_*((F_{X\times_k Y}^n)^* E) \tag{3.21}$$

for each $i \geq 0$.

Suppose that E is a Frobenius–finite vector bundle over $X \times_k Y$. Therefore, there exist two distinct polynomials $f,\ g \in \mathbb{Z}[t]$ with non-negative coefficients such that $\widetilde{f}(E)$ is isomorphic to $\widetilde{g}(E)$. Let $\varphi : \widetilde{f}(E) \longrightarrow \widetilde{g}(E)$ be an isomorphism. Then

$$R^i p_*(\varphi) : R^i p_*(\widetilde{f}(E)) \longrightarrow R^i p_*(\widetilde{g}(E))$$

is an isomorphism for each $i \geq 0$. Using the isomorphism in (3.20), we conclude that the isomorphism $R^i p_*(\varphi)$ induces an isomorphism between $\widetilde{f}(R^i p_* E)$ and $\widetilde{g}(R^i p_* E)$, since $R^i p_*$ is additive functor. This completes the proof of the lemma. □

Theorem 3.4.18 *The canonical homomorphism*

$$\psi : \pi_F(X \times_k Y,\, x_0 \times y_0) \longrightarrow \pi_F(X,\, x_0) \times \pi_F(Y,\, y_0)$$

is an isomorphism.

Proof. It is enough to show that ψ is closed immersion. By Proposition 2.3.4(2), we only need to show that any vector bundle $E \in \mathcal{C}_F(X \times_k Y)$ is isomorphic to a quotient of $p^*E_1 \otimes q^*E_2$, for some $E_1 \in \mathcal{C}_F(X)$ and $E_2 \in \mathcal{C}_F(Y)$. Let

$$E_1 := p_*(E \otimes (q^* j_{x_0}^* E)^\vee) \text{ and } E_2 := j_{x_0}^* E\,,$$

where $(q^* j_{x_0}^* E)^\vee$ is the dual of $(q^* j_{x_0}^* E)$.

Since $E \otimes (q^* j_{x_0}^* E)^\vee$ is an object of $\mathcal{C}_F(X \times Y)$, there are Nori–semistable vector bundles V_1 and V_2 with

$$V_1 \subset V_2 \subset \bigoplus_{i=1}^{m} F_i\,,$$

where $F_i \in \mathrm{TFF}(X)$ such that

$$E \otimes (q^* j_{x_0}^* E)^\vee \text{ is isomorphic to } V_2/V_1\,.$$

Using Lemma 3.4.17 it follows that $p_*F_i \in \mathrm{TFF}$, and hence p_*V_1 and p_*V_2 are subbundles of Nori–semistable vector bundles. Since p_*V_1 and p_*V_2 are degree zero subbundles of a Nori–semistable vector bundle, it follows that p_*V_1 and p_*V_2 are themselves Nori–semistable. Note that $p_*(V_2/V_1)$ is a degree zero subbundle of $\left(\bigoplus p_*F_i\right)/p_*V_1$. This implies that E_1 is an object of $\mathcal{C}_F(X)$.

There is a natural morphism $p^*(p_*(E \otimes (q^* j^* E)^\vee)) \longrightarrow E \otimes (q^* j^* E)^\vee$ that gives the following homomorphism:

$$\eta \,:\, p^* p_*(E \otimes (q^* j_{x_0}^* E)^\vee) \otimes q^* j_{x_0}^* E \longrightarrow E\,. \tag{3.22}$$

Let (x, y) be a closed point of $X \times Y$. There is natural surjective homomorphism

$$\alpha \,:\, p^* p_*(E \otimes (q^* j_{x_0}^* E)^\vee)_{(x,y)} \longrightarrow H^0(Y, j_x^* E \otimes (j_{x_0}^* E)^\vee)\,.$$

By [BH09, Lemma 3.5], the vector bundles $j_x^* E$ and $j_{x_0}^* E$ over Y are isomorphic and hence the natural homomorphism

$$\beta \,:\, H^0(Y, j_x^* E \otimes (j_{x_0}^* E)^\vee) \otimes (q^* j_{x_0}^* E)_{(x,y)} \longrightarrow E_{(x,y)}$$

is surjective. The homomorphism in (3.22) induces a homomorphism

$$\eta_{(x,y)} \,:\, p^* p_*(E \otimes (q^* j_{x_0}^* E)^\vee)_{(x,y)} \otimes (q^* j_{x_0}^* E)_{(x,y)} \longrightarrow E_{(x,y)}\,. \tag{3.23}$$

Since $\beta \circ (\alpha \otimes 1) = \eta_{(x,y)}$, it follows that $\eta_{(x,y)}$ is surjective. This proves that the homomorphism η in (3.22) is surjective. □

Behavior under Étale Morphism

Let $\psi : Y \longrightarrow X$ be an étale morphism of smooth projective varieties over k. Then the following diagram

$$\begin{tikzcd} Y \arrow[r, "F_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

is Cartesian (see Proposition 3.1.13). By Lemma 3.4.9, for any $n \geq 1$, there is a canonical isomorphism

$$(F_X^n)^*(\psi_* E) \longrightarrow \psi_*((F_Y^n)^* E)\,. \tag{3.24}$$

Lemma 3.4.19 *Let $\psi : Y \longrightarrow X$ be an étale morphism of smooth projective varieties over k. Let E be a Frobenius–finite vector bundle over Y. Then the direct image $\psi_* E$ is also a Frobenius–finite vector bundle over X.*

Proof. Note that for any vector bundle V over Y, the direct image $\psi_* V$ is a vector bundle over X. Now, the lemma follows using the isomorphism in (3.24) and the argument in Lemma 3.4.17. □

Lemma 3.4.20 *Let $\psi : Y \longrightarrow X$ be an étale morphism of smooth projective varieties over k. Let E be a vector bundle over X. Then E is an object of $\mathcal{C}_F(X)$ if and only if $\psi^* E$ is an object of $\mathcal{C}_F(Y)$.*

Proof. If E is an object of $\mathcal{C}_F(X)$, then $\psi^* E$ is clearly an object of $\mathcal{C}_F(Y)$. Suppose that $\psi^* E$ is an object of $\mathcal{C}_F(Y)$. Note that for any vector bundle V over Y, the direct image $\psi_* V$ is a vector bundle over X. Since $\psi^* E \in \mathcal{C}_F(Y)$, there exist $V_1, V_2 \in \mathbf{Ns}(X)$ and $F_i \in \mathrm{TFF}(X)$ $(i = 1, \ldots, k)$ such that $V_1 \subseteq V_2 \subseteq \oplus_i F_i$ and $\psi^* E \cong V_2/V_1$. Using Lemma 3.4.19, it follows that $\psi_* F_i \in \mathrm{TFF}(X)$. This implies that $\psi_* \psi^* E \in \mathcal{C}_F(X)$. Since E is a Nori–semistable subbundle of $\psi_* \psi^* E$, it follows that E is an object of $\mathcal{C}_F(X)$. □

Lemma 3.4.21 *Let $\psi : Y \longrightarrow X$ be a finite étale morphism and E an object of $\mathcal{C}_F(Y)$. Then $\psi_* E$ is an object of $\mathcal{C}_F(X)$.*

Proof. Without loss of generality we may assume that the morphism ψ is a Galois covering. For, there always exists a morphism $h : Z \longrightarrow Y$ such that

$$\varphi := \psi \circ h : Z \longrightarrow X$$

is finite étale Galois cover. Since E is an object of $\mathcal{C}_F(Y)$, the pull-back h^*E is an object of $\mathcal{C}_F(Z)$. As φ is Galois, it follows that the vector bundle

$$\varphi_* h^* E \;\cong\; \psi_*(h_* h^* E)$$

is an object of $\mathcal{C}_F(X)$. Note that the direct image $\psi_* E$ is a degree zero subbundle of $\psi_*(h_* h^* E)$. Therefore, the vector bundle $\psi_* E$ is an object of $\mathcal{C}_F(X)$.

Now consider the following diagram

$$\begin{tikzcd}
Y \times G \arrow[r, "\simeq"] \arrow[rr, bend left, "\mu"] \arrow[dr, "p"'] & Y \times_X Y \arrow[r, "p"'] \arrow[d, "p"] & Y \arrow[d, "\psi"] \\
 & Y \arrow[r, "\psi"'] & X
\end{tikzcd}$$

where μ is the action of $G = \mathrm{Gal}(Y/X)$ on Y. Since ψ is flat, we have

$$\psi^* \psi_* E \;\cong\; p_* p^* E \;\cong\; p_* \mu^* E\,.$$

Note that $p_* \mu^* E \;=\; \bigoplus_{g \in G} g^* E$. This implies that $\psi^* \psi_* E$ is an object of $\mathcal{C}_F(Y)$. By Lemma 3.4.20, it follows that $\psi_* E$ is an object of $\mathcal{C}_F(X)$. □

Proposition 3.4.22 *Let $\psi : Y \longrightarrow X$ be a surjective finite étale morphism of smooth projective varieties over k. Then the induced homomorphism*

$$\pi_F(Y, y) \longrightarrow \pi_F(X, \psi(y))$$

is closed immersion.

Proof. By Proposition 2.3.4(2), we need to check that if E is an object of $\mathcal{C}_F(Y)$, then E is isomorphic to a Nori–semistable subquotient of an object of the form

$\psi^* E'$ with E' an object of $\mathcal{C}_F(X)$. Consider the following Cartesian diagram

$$\begin{tikzcd} Y \times_X Y \arrow[r, "p"] \arrow[d, "p"'] & Y \arrow[d, "\psi"] \\ Y \arrow[r, "\psi"'] & X \end{tikzcd}$$

where p denotes the natural projection. Since ψ is flat, we have $\psi^*(\psi_* E) \cong p_*(p^* E)$. Take $E' = \psi_* E$. Then, from Lemma 3.4.21 it follows that E' is an object of $\mathcal{C}_F(X)$. Since E is a sub–bundle of $p_*(p^* E)$, it follows that E is a degree zero subbundle of $\psi^* E'$, where E' is an object of $\mathcal{C}_F(X)$. $\square$

Corollary 3.4.23 *Let $\psi : Y \longrightarrow X$ be a finite étale morphism of smooth projective varieties over k. If $\psi_* \mathcal{O}_Y = \mathcal{O}_X$, then the induced homomorphism*

$$\pi_F(Y, y) \longrightarrow \pi_F(X, \psi(y))$$

is an isomorphism.

Proof. By Proposition 3.4.8, the induced homomorphism

$$\pi_F(Y, y) \longrightarrow \pi_F(X, \psi(y))$$

is faithfully flat. From Proposition 3.4.22, the homomorphism $\pi_F(Y, y) \longrightarrow \pi_F(X, \psi(y))$ is a closed immersion. $\square$

The case of Finite Field

In this subsection, we assume that our base field k is a finite field. The following theorem is due to Lange and Stuhler (see [LS77]):

Theorem 3.4.24 [LS77, Theorem 1.4] *Let E be a vector bundle over a smooth projective variety X defined over k. Then the following conditions are equivalent:*

1. *$(F_X^n)^* E$ is isomorphic to E for some $n > 0$.*

2. *E is étale trivializable.*

The implication (1) $\Rightarrow$ (2) also holds for smooth projective varieties defined over an arbitrary field of positive characteristic. If E is stable vector bundle on a smooth projective variety defined over an algebraically closed field, then the implication (2) ⇒ (1) is proved in [BD07, Theorem 1.1].

Let X be a smooth projective variety over k. We will assume that X admits a k–rational point. Fix a k–rational point x_0 of X.

Proposition 3.4.25 *There is a natural faithfully flat homomorphism*

$$\pi_F(X,\, x_0) \longrightarrow \pi^{\text{ét}}(X,\, x_0)\,.$$

Proof. By Theorem 3.4.24, we get a natural functor

$$\beta\,:\,\mathcal{C}^{\text{ét}}(X)\,\longrightarrow\,\mathcal{C}_F(X)$$

which is evidently fully faithful. The functor β gives a homomorphism $\widetilde{\beta}$: $\pi_F(X,\, x_0) \;\longrightarrow\; \pi_1(X, x_0)$. In view of Proposition 2.3.4(1), to prove that $\widetilde{\beta}$ is faithfully flat, it suffices to show that if F is a degree zero subbundle of an étale trivializable vector bundle E, then F is also étale trivializable.

Since E is étale trivializable, there is a finite étale Galois covering

$$\alpha\,:\,Y\,\longrightarrow\,X$$

such that the pull-back α^*E is trivializable. Note that α^*F is a degree zero subbundle of α^*E. Since any subbundle of degree zero of trivializable vector bundle is trivializable, we conclude that F is étale trivializable. □

In general, the above homomorphism need not be an isomorphism. To see this, we consider the following example (cf. [Bi09]). Recall that an elliptic curve C over a field of prime characteristic p is called *super–singular* if the induced homomorphism

$$F_C^*: H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_C)$$

vanishes. Let C be a super-singular elliptic curve defined over a finite field k. Note that such a curve exists over a finite field. Fix a non-zero element $\xi \in H^1(C, \mathcal{O}_C)$. Let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C \longrightarrow 0$$

be the extension given by ξ.

The cohomology class $F_C^*\xi = 0$ as C is super- singular. Hence, the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F_C^* E \longrightarrow \mathcal{O}_C \longrightarrow 0$$

splits. Therefore, $F_C^* E = \mathcal{O}_C \oplus \mathcal{O}_C$. This implies that E is an object of $\mathcal{C}_F(C)$.

Suppose that E is étale trivializable. Then by Theorem 3.4.24, there exists a positive integer n such that $(F_C^n)^* E$ is isomorphic to E. Since the cohomology class $(F_C^n)^*\xi = 0$, the short exact sequence

$$0 \longrightarrow (F_C^n)^*\mathcal{O}_C = \mathcal{O}_C \longrightarrow (F_C^n)^* E \cong E \longrightarrow \mathcal{O}_C = (F_C^n)^*\mathcal{O}_C \longrightarrow 0$$

splits, which is not possible. Hence, E is not étale trivializable.

Proposition 3.4.26 *There is a natural faithfully flat homomorphism*

$$\pi_F(X,\, x_0) \longrightarrow \pi^N(X,\, x_0)$$

Proof. Let E be an essentially finite vector bundle over X. Then there is a connected étale Galois covering $\beta : Y \longrightarrow X$ such that the vector bundle $\beta^* E \longrightarrow Y$ is F–trivial (see [BH07, p. 557]). Therefore, there exists a non-negative integer m such that $(F_Y^m)^*(\beta^* E)$ is isomorphic to a trivial vector bundle over Y. Since $F_X \circ \beta = \beta \circ F_Y$, the vector bundle $\beta^*((F_X^m)^* E)$ is isomorphic to a trivial vector bundle over Y. This implies that $(F_X^m)^* E$ is étale trivializable. By Theorem 3.4.24, there is an integer $n \geq 1$ such that $(F_X^{n+m})^* E$ is isomorphic to $(F_X^m)^* E$. This proves that any essentially finite vector bundle E over X is in $\mathcal{C}_F(X)$. By Proposition 2.3.4(1), we have a natural faithfully flat homomorphism

$$\pi_F(X,\, x_0) \longrightarrow \pi^N(X,\, x_0)\,.$$

□

Chapter 4

Real Parabolic Vector Bundles over a Real Curve

The notion of parabolic vector bundles over a compact Riemann surface was introduced by C. S. Seshadri [Se77] and their moduli studied in [MS80]. In this chapter, we consider real vector bundles with real parabolic structure over a real curve. For a suitable ramified covering $p : Y \longrightarrow X$ in the category of real curves, we give the correspondence between real equivariant vector bundles over Y and real parabolic vector bundles over X with some condition on weights.

4.1 Preliminaries

By a real curve we will mean a pair (X, σ), where X is a Riemann surface, and σ is an anti-holomorphic involution on X. Let $\sigma_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ be the conjugate map $z \mapsto \bar{z}$.

Proposition 4.1.1 *A continuous involution $\sigma : X \longrightarrow X$ on a Riemann surface X is an anti-holomorphic involution if and only if for every open subset U of X, the map $\tilde{\sigma} = \tilde{\sigma}_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(\sigma(U))$ defined by $f \mapsto \sigma_{\mathbb{C}} \circ f \circ \sigma$ is an isomorphism of rings.*

Proof. If $\sigma : X \longrightarrow X$ is an anti-holomorphic involution, then the map $\tilde{\sigma}_U$ defined as above is an isomorphism. Conversely, suppose that the map

$$\tilde{\sigma} = \tilde{\sigma}_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(\sigma(U))$$

defined by $f \mapsto \sigma_{\mathbb{C}} \circ f \circ \sigma$ is an isomorphism. For every pair of holomorphic charts $\psi_1 : U_1 \longrightarrow V_1 \subset \mathbb{C}$ and $\psi_2 : U_2 \longrightarrow V_2 \subset \mathbb{C}$ on X with $\sigma(U_1) \subset U_2$, the map

$$\psi_2 \circ \sigma \circ \psi_1^{-1} : V_1 \longrightarrow V_2$$

is anti-holomorphic, since $\sigma_{\mathbb{C}} \circ \psi_2 \circ \sigma$ is holomorphic. This proves that the map $\sigma : X \longrightarrow X$ is an anti-holomorphic involution. $\square$

Real Vector Bundles

Let (X, σ) be a real curve. A *real holomorphic vector bundle* $E \longrightarrow X$ is a holomorphic vector bundle, together with an anti-holomorphic involution σ^E of the total space E making the diagram

$$\begin{tikzcd} E \arrow[r, "\sigma^E"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

commutative, and such that, for all $x \in X$, the map $\sigma^E|_{E(x)} : E(x) \longrightarrow E(\sigma(x))$ is $\mathbb{C}$-antilinear:

$$\sigma^E(\lambda \cdot \eta) = \bar{\lambda} \cdot \sigma^E(\eta), \text{ for all } \lambda \in \mathbb{C} \text{ and all } \eta \in E(x).$$

A homomorphism between two real bundles (E, σ^E) and $(E', \sigma^{E'})$ is a homomorphism

$$f : E \longrightarrow E'$$

of holomorphic vector bundles over X such that $f \circ \sigma^E = \sigma^{E'} \circ f$.

A holomorphic subbundle F of a real holomorphic vector bundle E is said to be real subbundle of E if $\sigma^E(F) = F$.

Real $\mathcal{O}_X$–modules

Let (X, σ) be a real curve, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define an $\mathcal{O}_X$-module $\mathcal{F}^\sigma$ as follows. For any open subset U of X, $\mathcal{F}^\sigma(U) = \mathcal{F}(\sigma(U))$, and for every $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}^\sigma(U)$, $f \cdot s = \tilde{\sigma}_U(f)s$. Note that, $\sigma_U(f) \in \mathcal{O}_X(\sigma(U))$, and $s \in \mathcal{F}(\sigma(U))$, therefore, $f \cdot s \in \mathcal{F}^\sigma(U)$. It is easy to check that $\mathcal{F}^\sigma$ is an

$\mathcal{O}_X$-module.

Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Define $\varphi^\sigma : \mathcal{F}^\sigma \longrightarrow \mathcal{G}^\sigma$ as follows: For every open subset U of X,

$$\varphi_U^\sigma : \mathcal{F}^\sigma(U) \longrightarrow \mathcal{G}^\sigma(U), \quad \varphi_U^\sigma = \varphi_{\sigma(U)}$$

If $f \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}^\sigma(U)$ then

$$\varphi_U^\sigma(f \cdot s) = \varphi_{\sigma(U)}\big(\sigma_U(f)(s)\big) = \sigma_U(f)\varphi_{\sigma(U)}(s),$$

since $\varphi_{\sigma(U)}$ is an $\mathcal{O}_X(\sigma(U))$-linear. Therefore, $\varphi^\sigma(f \cdot s) = f \cdot \varphi_U^\sigma(s)$. It follows that φ^σ is a homomorphism of $\mathcal{O}_X$-modules.

Definition 4.1.2 A real structure on an $\mathcal{O}_X$–module $\mathcal{F}$ is an $\mathcal{O}_X$-module homomorphism $\sigma^{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}^\sigma$ such that $(\sigma^{\mathcal{F}})^\sigma \circ \sigma^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}}$. By a *real* $\mathcal{O}_X$*–module*, we mean a pair $(\mathcal{F}, \sigma^{\mathcal{F}})$, where $\mathcal{F}$ is an $\mathcal{O}_X$–module and $\sigma^{\mathcal{F}}$ is a real structure on an $\mathcal{O}_X$–module $\mathcal{F}$.

Let $(\mathcal{F}, \sigma^{\mathcal{F}})$ and $(\mathcal{G}, \sigma^{\mathcal{G}})$ be two real $\mathcal{O}_X$–modules. A morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of $\mathcal{O}_X$–modules is said to be a morphism of real $\mathcal{O}_X$–modules if $\sigma^{\mathcal{G}} \circ \varphi = \varphi^\sigma \circ \sigma^{\mathcal{F}}$.

We recall that a holomorphic vector bundle $\mathcal{F}$ on X is called *semistable* if for every non-zero subbundle $\mathcal{F}'$ of $\mathcal{F}$, the inequality

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F})$$

is valid. If strict inequality is valid for every non-zero proper subbundle $\mathcal{F}'$, then $\mathcal{F}$ is called *stable*. If $\mathcal{F}$ is a direct sum of stable vector bundles having the same slope, then $\mathcal{F}$ is called *polystable*.

Definition 4.1.3 A real holomorphic vector bundle (E, σ^E) over a real curve is said to be *real semistable* (respectively, *real stable*) if for every proper real subbundle $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \leq \mu(E) \qquad \text{(respectively,} \quad \mu(\mathcal{F}) < \mu(E)).$$

Let (E, σ^E) be a real holomorphic vector bundle over a real curve (X, σ). Then (E, σ^E) is said to be *real polystable* if $E = \oplus_{i=1}^n E_i$, where $(E_i, \sigma^E|_{E_i})$ is real stable subbundle of E satisfying $\mu(E_i) = \mu(E)$ for $i = 1, \ldots, n$.

Remark 4.1.4 Let E be a real holomorphic vector bundle over a real curve X. Then the corresponding locally free $\mathcal{O}_X$–module $\mathcal{E}$ is a real $\mathcal{O}_X$–module.

To see this, for every open subset U of X, we define

$$\sigma_U^{\mathcal{E}} : \mathcal{E}(U) \longrightarrow \mathcal{E}(\sigma(U))$$

by $s \mapsto \sigma_E \circ s \circ \sigma$. Then $\sigma_U^{\mathcal{E}}$ is an abelian group homomorphism. For every pair of open subsets U and V of X with $U \supset V$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{E}(U) \arrow[r, "\sigma_U^{\mathcal{E}}"] \arrow[d, "\rho_V^U"'] & \mathcal{E}(\sigma(U)) \arrow[d, "\rho_{\sigma(V)}^{\sigma(U)}"] \\ \mathcal{E}(V) \arrow[r, "\sigma_V^{\mathcal{E}}"'] & \mathcal{E}(\sigma(V)) \end{tikzcd}$$

For $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$, we have

$$\begin{aligned} \sigma_U^{\mathcal{E}}(fs)(\sigma(x)) &= \sigma_E \circ s \circ \sigma(\sigma(x)) \\ &= \sigma_E(f(x)s(x)) \\ &= \sigma_{\mathbb{C}}(f(x))\sigma_E(s(x)) \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}_U(f)\sigma_U^{\mathcal{E}}(s)(\sigma(x)) &= (\sigma_{\mathbb{C}} \circ f \circ \sigma)(\sigma(x))(\sigma_E \circ s \circ \sigma)(\sigma(x)) \\ &= \sigma_{\mathbb{C}}(f(x))\sigma_E(s(x)) \end{aligned}$$

Therefore, $\sigma_U^{\mathcal{E}}(fs) = \tilde{\sigma}_U(f)\sigma_U^{\mathcal{E}}(s)$, for every $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$. For $s \in \mathcal{E}(U)$, we have

$$\sigma_{\sigma(U)}^{\mathcal{E}} \circ \alpha_U(s) = \sigma_{\sigma(U)}^{\mathcal{E}}(\sigma_E \circ s \circ \sigma) = \sigma_E \circ (\sigma_E \circ s \circ \sigma) \circ \sigma = s.$$

Conversely, if $\mathcal{E}$ is a locally free real $\mathcal{O}_X$–module then the corresponding holomorphic vector bundle is a real holomorphic vector bundle.

For, let E denote the corresponding vector bundle over X. Then $\mathcal{E}$ can be identified with the sheaf of holomorphic sections of E. Define $\sigma_E : E \longrightarrow E$ as follows: Let $e \in E$. Then $e \in E(x)$ and there exists a section $s \in \mathcal{E}(U)$ such that the image of the stalk s_x in $E(x)$ is e. Let e' denote the image of the stalk $\sigma_U^{\mathcal{E}}(s)_{\sigma(x)}$ in $E(\sigma(x))$. We define $\sigma_E(e) = e'$. Then σ_E is independent of the choice of the section. The following diagram

$$\begin{tikzcd} E \arrow[r, "\sigma_E"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

commutes. Since $\sigma \circ \pi = \pi \circ \sigma_E$, the map $\pi \circ \sigma_E$ is an anti-holomorphic map. Since π is holomorphic, it follows that σ_E is an anti-holomorphic map. Note that $\sigma_E^2 = 1_E$. Therefore, (E, σ_E) is a real vector bundle over (X, σ).

4.2 Real Parabolic Vector Bundles

Definition 4.2.1 Let (E, σ^E) be a real vector bundle over a real curve (X, σ_X). Let S be a non-empty finite subset of X such that $\sigma(S) = S$.

By *real quasi-parabolic structure* on (E, σ^E) over S, we mean for each $x \in S$, there is a strictly decreasing flag

$$E(x) = F^1E(x) \supset F^2E(x) \supset \cdots \supset F^{k_x}E(x) \supset F^{k_x+1}E(x) = 0$$

of linear subspaces in $E(x)$ satisfying the following property:

(RP1) σ^E preserve the flags, i.e., $\sigma_x^E(F^iE(x)) = F^iE(\sigma_X(x))$.

We define

$$r_j = \dim(F^jE(x)) - \dim(F^{j+1}E(x)).$$

The integer k is called the length of the flag and the sequence $(r_1, \ldots r_{k_x})$ is called the type of the flag.

A *real parabolic structure* on (E, σ^E) over S is a real quasi-parabolic structure on (E, σ^E) over S as above, together with a sequence of real numbers $0 \leq \alpha_1^x < \cdots < \alpha_{k_x}^x < 1$, which are called weights corresponding to the subspaces $(F^1E(x), F^2E(x), \ldots, F^{k_x}E(x))$, with the following property:

(RP2) the weights over x and $\sigma_X(x)$ are same.

We set

$$d_xE = \sum_{j=1}^{k} r_j\alpha_j,$$

where $r_j = \dim(F^jE(x)) - \dim(F^{j+1}E(x))$.

The parabolic degree, denoted by pdeg(E), is defined by

$$\mathrm{pdeg}(E) = \deg(E) + \sum_{x \in S} d_x E, \tag{4.1}$$

where $\deg(E)$ denotes the topological degree of E, and we define the parabolic slope by

$$\mathrm{p}\mu(E) = \frac{\mathrm{pdeg}(E)}{\mathrm{rank}(E)}. \tag{4.2}$$

The points in S are called the real parabolic points.

Definition 4.2.2 Given two real parabolic bundles (E_1, σ^{E_1}) and (E_2, σ^{E_2}) over (X, σ_X), a *real parabolic morphism* is a homomorphism $\psi : (E_1, \sigma^{E_1}) \longrightarrow (E_2, \sigma^{E_2})$ of real vector bundles which respects the real parabolic structures, i.e., for each real parabolic point x with the real parabolic structures on E_l at x for $l = 1, 2$ given by

$$E_l(x) = F^1 E_l(x) \supset F^2 E_l(x) \supset \ldots F^{k_x} E_l(x) \supset 0,$$

$$0 \leq \alpha_1^l < \alpha_2^l < \ldots \alpha_{k_x}^l < 1,$$

we require that $\psi(x)$ satisfies

$$\alpha_i^1 > \alpha_j^2 \Longrightarrow \psi(x)(F^i E_1(x)) \subseteq F^{j+1} E_2(x). \tag{4.3}$$

An isomorphism $\psi : (E_1, \sigma^{E_1}) \longrightarrow (E_2, \sigma^{E_2})$ is said to be an isomorphism of real parabolic bundles if ψ and ψ^{-1} are real parabolic morphisms.

Remark 4.2.3 We can replace condition (4.3) by the following equivalent condition on $\psi(x)$. Given the weight α_i^1, let α_j^2 be the smallest weight such that $\alpha_i^1 \leq \alpha_j^2$, then we require

$$\psi(x)(F^i E(x)) \subseteq F^j E_2(x). \tag{4.4}$$

Lemma 4.2.4 *If $\psi : E_1 \longrightarrow E_2$ and $\varphi : E_2 \longrightarrow E_3$ are morphism of real parabolic bundles, then $\varphi \circ \psi$ is a real parabolic morphism.*

Proof. First note that $\varphi \circ \psi$ is a morphism of real holomorphic vector bundles. Suppose $x \in X$ is a real parabolic point. We use the notation $\{F_j^n, \alpha_j^n\}$ for the weighted flag in E_n at x for $n = 1, 2, 3$. Given the weight α_i^1, let α_j^2 be the smallest weight with $\alpha_i^1 \leq \alpha_j^2$. Then by condition (4.4), $\psi(x)(F_i^1) \subseteq F_j^2$. Also, if α_k^3 is the smallest weight with $\alpha_j^2 \leq \alpha_k^3$, then $\varphi(x)(F_j^2) \subset F_k^3$. Thus we see that $(\varphi \circ \psi)(x)(F_i^1) \subseteq F_k^3$. On the other hand, let $\alpha_{k'}^3$ be the smallest weight with $\alpha_i^1 \leq \alpha_{k'}^3$. Since $\alpha_i^1 \leq \alpha_{k'}^3$ we see that $\alpha_{k'}^3 \leq \alpha_k^3$. Thus $F_k^3 \subseteq F_{k'}^3$ and hence $(\varphi \circ \psi)(x)(F_i^1) \subseteq F_{k'}^3$. □

We denote by $\mathbf{RP}(X)$ the category whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S, and morphisms are real parabolic morphisms.

Remark 4.2.5 Given a short exact sequence of real holomorphic bundles over (X, σ_X)

$$0 \longrightarrow E_1 \xrightarrow{\iota} E_2 \xrightarrow{\pi} E_3 \longrightarrow 0,$$

it is easy to see that a real parabolic structure on E_2 determines a unique real parabolic structure on E_1 and E_3. Conversely, real parabolic structures on E_1 and E_3 determine a real parabolic structure on E_2. We call E_1 with this canonical real parabolic structure, a *real parabolic subbundle* of E_2 and E_3 a *real parabolic quotient* (cf. [Se82], [MS80]).

Indeed, assume that we have a real parabolic structure on E_2. Then at each real parabolic point $x \in X$, we have the flag

$$E_2(x) = F_2^1(x) \supset F_2^2(x) \supset \cdots \supset F_2^{r_2}(x) \supset 0$$

with the following weights

$$0 \leq a_{2_x}^1 < a_{2_x}^2 < \cdots < a_{2_x}^{r_2} < 1.$$

We define the real parabolic structure on E_1 as follows:

Let $H_i = \iota^{-1}(F_2^i(x))$. Then we obtain a sequence of subspaces

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{r_2}.$$

To obtain a strictly increasing sequence of subspaces, we can choose a subset $\{i_1, \ldots, i_{r_1}\} \subseteq \{1, \ldots, r_2\}$ such that

$$H_1 = \cdots = H_{i_1} \supset H_{i_1+1} = \cdots = H_{i_2} \supset \cdots \supset H_{i_{r_1-1}+1} = \cdots = H_{i_{r_1}}.$$

We set $F_1^j(x) = H_{i_j}$ and $a^j_{1_x} = a^{i_j}_{2_x}$ for $j = 1, \ldots, r_1$. This gives at each real parabolic point $x \in X$ the following flag for $E_1(x)$

$$E_1(x) = F_1^1(x) \supset F_1^2(x) \supset \cdots \supset F_1^{r_1}(x) \supset 0$$

with the following weights

$$0 \leq a^1_{1_x} < a^2_{1_x} < \cdots < a^{r_1}_{1_x} < 1.$$

Since E_1 is a real subbundle of E_2, we have the following commutative diagram:

$$\begin{tikzcd} E_1(x) \arrow[r, "\iota"] \arrow[d, "\sigma_x^{E_1}"'] & E_2(x) \arrow[d, "\sigma_x^{E_2}"] \\ E_1(\sigma(x)) \arrow[r, "\iota"'] & E_2(\sigma(x)) \end{tikzcd}$$

From the commutativity of above diagram, it easy to see that $\sigma_x^{E_1}(F_1^i(x)) = F_1^i(\sigma(x))$ and $a^i_{1_x} = a^i_{1_{\sigma(x)}}$ for $i = 1, \ldots, r_1$. This shows that E_1 with above weighted flag structure over real parabolic points is a real parabolic vector bundle on (X, σ) with real parabolic structure over S.

Definition 4.2.6 A real parabolic vector bundle (E, σ^E) is said to be *real parabolic semistable* (respectively, *real parabolic stable*) if for every real subbundle F of the real vector bundle E, we have

$$\mathrm{p}\mu(F) \leq \mathrm{p}\mu(E) \qquad \text{(respectively, } \quad \mathrm{p}\mu(F) < \mathrm{p}\mu(E)),$$

where F is a real parabolic subbundle of E.

We say that a real parabolic vector bundle (E, σ^E) is real parabolic polystable if E is a direct sum of real parabolic stable bundles with the same slope $\mathrm{p}\mu(E)$.

4.3 Construction of the Covering

Let (X, σ_X) be a real curve, and $S \subset X$ be a non-empty finite subset of X such that $\sigma(S) = S$. Consider the divisor $D = \sum_{x \in S} x$ on X. Let L be the line bundle corresponding to the divisor D on X.

We will show that the line bundle L is real holomorphic line bundle over

(X, σ_X). It is enough to give a real structure on the $\mathcal{O}_X$-module $\mathcal{L} = \mathcal{O}_X(D)$. Let U be an open subset of X, and let $s \in \Gamma(U, \mathcal{O}_X(D))$. Let $u \in U$ and (V, z) be a holomorphic chart centred at u, where $z = x + iy$. Then, $(\sigma(V), z_\sigma)$, where $z_\sigma = x \circ \sigma - iy \circ \sigma$, is a holomorphic chart centred at $\sigma(u)$. Let $\sum_{k=-n}^{\infty} b_k z^k$ be the power series expansion of s with respect to a holomorphic chart (V, z). Then, the power series expansion of $\overline{s \circ \sigma}$ with respect to a holomorphic chart $(\sigma(V), z_\sigma)$ is $\sum_{k=-n}^{\infty} \overline{b_k} z_\sigma^k$. Since $\sigma(S) = S$, it follows that $\overline{s \circ \sigma} \in \Gamma(\sigma_X(U), \mathcal{O}_X(D))$.

For $U \subset X$ open, we define a map

$$\sigma_U^{\mathcal{L}} : \Gamma(U, \mathcal{O}_X(D)) \longrightarrow \Gamma(\sigma_X(U), \mathcal{O}_X(D))$$

by $s \mapsto \overline{s \circ \sigma}$. Then, $\sigma_U^{\mathcal{L}}$ is well-defined, since $\sigma_X(S) = S$ and $\mathrm{ord}_x(s) = \mathrm{ord}_{\sigma(x)}(\overline{s \circ \sigma})$. Since σ is an involution, $\sigma_{\sigma(U)}^{\mathcal{L}} \circ \sigma_U^{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}(U)}$. Thus, we get an $\mathcal{O}_X$-module homomorphism

$$\sigma^{\mathcal{L}} : \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)^\sigma$$

such that $(\sigma^{\mathcal{L}})^\sigma \circ \sigma^{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$. We denote by σ^L the corresponding anti-holomorphic involution on L.

Let $N \geq 2$ be a positive integer. Consider the holomorphic map

$$\Psi : L \longrightarrow L^{\otimes N}$$

given by $v \mapsto v^{\otimes N}$

Let $s \in H^0(X, L^{\otimes N}) - \{0\}$ be a real holomorphic section such that $\mathrm{div}(s) = ND$. Let $Y = \Psi^{-1}(s(X))$ and let $p : Y \longrightarrow X$ be the restriction of the natural projection $\pi : L \longrightarrow X$. Then, Y gets a canonical structure of a complex manifold of dimension 1 such that p is a holomorphic map. For, notice that $p^{-1}(X \setminus S)$ has a canonical structure of a complex manifold of dimension 1. Let $y \in p^{-1}(S)$. Let $(U \cong \mathbb{D}, \psi)$ be a holomorphic chart centred at $p(y)$ on X. Then, there exists a homeomorphism $\varphi : V = p^{-1}(U) \longrightarrow \mathbb{D}$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "p"'] & \mathbb{D} \arrow[d, "\tau_N"] \\ & \mathbb{D} \end{tikzcd}$$

commutes. Let (V, φ) be a chart at y on Y. If (V_1, φ_1) and (V_2, φ_2) are two such charts, then by using removable singularity theorem, it follows that (V_1, φ_1) and (V_2, φ_2) are holomorphically compatible. Consequently, we get a canonical structure of a complex manifold of dimension 1 on Y. Note that $p : Y \longrightarrow X$ is a finite holomorphic map of complex manifolds of dimension 1. Hence, p is an open map. For $x \in S$, the preimage $p^{-1}(x)$ is singleton, and hence Y is connected. Therefore, Y is a compact Riemann surface.

Note that

$$Y = \{y \in L \mid y^{\otimes N} \in \mathrm{Image}(s)\}.$$

For $y \in Y$, we have $y^{\otimes N} = s(x)$ for some $x \in X$. From this, we have

$$\sigma^{L^N}(y^{\otimes N}) = (\sigma^L(y))^{\otimes N} = \sigma^{L^N}(s(x)).$$

Since s is a real holomorphic section, we have $\sigma^{L^N}(s(x)) = s(\sigma(x))$. It follows that $\sigma^L(y) \in Y$. Let

$$\sigma_Y : Y \longrightarrow Y$$

be the restriction of the anti-holomorphic involution $\sigma^L : L \longrightarrow L$. Therefore, Y is a real curve and we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\sigma_X"'] & X \end{tikzcd} \tag{4.5}$$

where $\sigma_Y : Y \longrightarrow Y$ is an anti-holomorphic involution.

Consider the action of the multiplicative group $\mathbb{C}^*$ on the total space of L. The action of the subgroup

$$\mu_N := \{c \in \mathbb{C} \mid c^N = 1\}$$

preserves the real curve Y. Clearly, the action of μ_N satisfies the following: for $y \in Y$ and $c \in \mu_N$, we have $\sigma_Y(c \cdot y) = c^{-1}\sigma_Y(y)$. Also, note that for each $x \in X \setminus S$, the cardinality of $p^{-1}(x)$ is N. It follows that $p : Y \longrightarrow X$ is a Galois covering with Galois group $\Gamma = \mu_N$.

We summarize the above discussion in the following:

Lemma 4.3.1 *Let (X, σ_X) be a real curve, and $S \subset X$ be a finite subset of X with $\sigma_X(S) = S$. For any positive integer $N \geq 2$, there exists an N-fold cyclic cover $p : Y \longrightarrow X$ which is ramified precisely over each point of S, such that the following diagram*

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ X \arrow[r, "\sigma_X"'] & X \end{tikzcd}$$

commutes, where σ_Y is an anti-holomorphic involution on Y. Moreover, the action of Γ on Y has the following property:

$$\sigma_Y(gy) = g^{-1}\sigma_Y(y) \text{ for all } g \in \Gamma \text{ and } y \in Y\,.$$

Equivariant real vector bundles

Let (Y, σ_Y) be a real curve. Let G be a finite group acting holomorphically and effectively on Y. with the property that $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$ for all $g \in G$.

Definition 4.3.2 Let Y be a Riemann surface, and let G be a subgroup of $\mathrm{Aut}(Y)$. An *admissible chart* for G at a point $y \in Y$ is a chart $\varphi : U \longrightarrow \mathbb{D}$ on Y centred at y such that:

- U is G_y–invariant, where G_y denotes the isotropy subgroup of G at y; and
- $g(U) \cap U = \emptyset$ for all $g \in G \setminus G_y$.

We call U an *admissible neighbourhood* of y.

Remark 4.3.3 Let Y be a Riemann surface, and let G be a properly discontinuous subgroup of $\mathrm{Aut}(Y)$. Then the isotropy group G_y is finite and cyclic for each $y \in Y$, and has unique generator g_y such that for every admissible chart $\varphi : U \longrightarrow \mathbb{D}$ at y, we have

$$\varphi(g_y(x)) = \exp(2\pi\sqrt{-1}/m_y)\varphi(x) \text{ for all } x \in U,$$

where m_y is the cardinality of G_y. We call g_y the *isotropy generator* at y.

Definition 4.3.4 A *real G–equivariant vector bundle* on (Y, σ_Y) consists of the following data: a real holomorphic vector bundle (W, σ^W) on (Y, σ_Y), and a lift of the natural action of G on Y to W such that

(a) the bundle projection $\pi : W \longrightarrow Y$ is G-equivariant;

(b) if $y \in Y$ and $g \in G$, the map $W_y \longrightarrow W_{g \cdot y}$, given by $v \mapsto g \cdot v$ is linear isomorphism.

(c) the following diagram

$$\begin{tikzcd} G \times W \arrow[r] \arrow[d, "(\mathrm{inv}, \sigma^W)"'] & W \arrow[d, "\sigma^W"] \\ G \times W \arrow[r] & W \end{tikzcd}$$

commutes, where $\mathrm{inv} : G \longrightarrow G$ is an inverse map $g \mapsto g^{-1}$.

Let W_1 and W_2 are two G–equivariant real vector bundles on (Y, σ_Y). A real homomorphism $\varphi : W_1 \longrightarrow W_2$ of real vector bundles is called a morphism of real G–equivariant vector bundles if φ is G–equivariant map.

Definition 4.3.5 A G–equivariant holomorphic vector bundle E over a Riemann surface Y is said to be *G–semistable* (respectively, *G–stable*) if for every proper G–invariant coherent subsheaf $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \le \mu(E) \qquad (\text{respectively,} \quad \mu(\mathcal{F}) < \mu(E))\,.$$

Definition 4.3.6 A G–equivariant real holomorphic vector bundle (E, σ^E) over a real curve is said to be *G–real semistable* (respectively, *G–real stable*) if for every proper G–invariant real coherent subsheaf $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \le \mu(E) \qquad (\text{respectively,} \quad \mu(\mathcal{F}) < \mu(E)).$$

A G–equivariant real holomorphic vector bundle E over a real curve Y is said to be *G–real polystable* if $E = \oplus_{i=1}^{n} E_i$, where E_i is G–invariant real subbundle of E which is G–real stable satisfying $\mu(E_i) = \mu(E)$ for $i = 1, \ldots, n$.

4.4 Correspondence

Let (X, σ_X) be a real curve, and let $S \subset X$ be a finite subset of X such that $\sigma_X(S) = S$. let $N \ge 2$ be a positive integer. Let $p : Y \longrightarrow X$ be an N-fold cyclic ramified covering as in the Lemma 4.3.1.

Lemma 4.4.1 *Let W be a Γ–equivariant real vector bundle on (Y, σ_Y). Then $p_*^\Gamma W$ is a real vector bundle on (X, σ_X).*

Proof. For $U \subset X$ open, we define $\sigma_U^{p_*^\Gamma W} : \Gamma\big(p^{-1}(U), W\big)^\Gamma \longrightarrow \Gamma\big(\sigma_Y\big(p^{-1}(U)\big), W\big)^\Gamma$ by $s \mapsto \sigma_{p^{-1}(U)}^W(s)$. We first show that $\sigma^{p_*^\Gamma W}$ is well defined: Let $y \in \sigma_Y\big(p^{-1}(U)\big)$ and $g \in \Gamma$. Then

$$g \cdot \sigma_{p^{-1}(U)}^W(s)(y) = g \cdot \sigma^W\big(s(\sigma_Y(y))\big).$$

On the other hand, we have

$$\begin{aligned}
\sigma_{p^{-1}(U)}^W(s)(g \cdot y) &= \sigma^W\big(s(\sigma_Y(g \cdot y))\big) \\
&= \sigma^W\big(s(g^{-1} \cdot \sigma_Y(y))\big) \\
&= \sigma^W\big(g^{-1} \cdot s(\sigma_Y(y))\big) \\
&= g \cdot \sigma^W\big(s(\sigma_Y(y))\big).
\end{aligned}$$

Hence, $\sigma_{p^{-1}(U)}^W(s) \in \Gamma\big(\sigma_Y\big(p^{-1}(U)\big), W\big)^\Gamma$. For $f \in \Gamma(U, \mathcal{O}_X)$ and $s \in \Gamma\big(p^{-1}(U), W\big)^\Gamma$, we have $\sigma_U^{p_*^\Gamma W}(f \cdot s) = \sigma_{p^{-1}(U)}^W((f \circ p) \cdot s) = (f \circ p) \cdot \sigma_{p^{-1}(U)}^W(s) = f \cdot \sigma_U^{p_*^\Gamma W}(s)$. Hence, $\sigma^{p_*^\Gamma W} : p_*^\Gamma W \longrightarrow \big(p_*^\Gamma W\big)^\sigma$ is an $\mathcal{O}_X$–module homomorphism, which is a real structure on $p_*^\Gamma W$. $\square$

Lemma 4.4.2 *Let E be a real vector bundle over Y. Then there is natural isomorphism $\psi : E \longrightarrow p_*^G p^* E$ of real vector bundles over X.*

Proof. Let U be an open subset of Y and a section $s \in \Gamma(U, E)$. Then a section s defines naturally a section $t \in \Gamma(p^{-1}(U), p^*E)^G$ as follows:

$$t(y) = s(p(y)) \text{ for all } y \in p^{-1}(U).$$

Since $p(g \cdot y) = p(y)$ for all $g \in G$, a section t is G–invariant. Define

$$\psi_U : \Gamma(U, E) \longrightarrow \Gamma(p^{-1}(U), p^*E)^G = \Gamma(U, p_*^G p^* E)$$

by $\psi_U(s) = t$. Let $s \in \Gamma(U, E)$. Then $\sigma^{p^*E} \circ \psi_U(s)(y) = \sigma^{p^*E} \circ \psi_U(s)(\sigma_Y(y)) = \sigma^{p^*E}(s(p(\sigma_Y(y)))) = \sigma^E(s(\sigma_X(p(y))) = \sigma^E(s)(p(y)) = \psi_U(\sigma^E(s))$. From this, it follows that ψ is a morphism of real vector bundles. It is enough to check that the stalk map

$$\psi_x : E_x \longrightarrow (p_*^G p^* E)_x$$

is isomorphism for all $x \in X$.

Injectivity: Let $\theta \in E_x$ such that $\psi_x(\theta) = 0$. Then there exists a section $s \in \Gamma(U, E)$ such that $\psi_U(s)_x = 0$, i.e., there exists an open subset $V \subset U$ such that $\psi_U(s)|_V = 0$. This means that $s(p(y)) = 0$ for all $y \in p^{-1}(V)$, i.e., $s|_V = 0$.

Surjectivity: Let $\eta \in (p_*^G p^* E)_x$. Then there exists a section $t \in \Gamma(U, p_*^G p^* E) = \Gamma(p^{-1}(U), p^* E)^G$ such that $t_x = \eta$. Define a section $s \in \Gamma(U, E)$ by $s(x) = t(y)$, where $y \in p^{-1}(x)$. Since t is G–invariant section, s is well-defined. Clearly, $\psi_U(s) = t$ and hence $\psi_U(s)_x = t_x = \eta$. □

Remark 4.4.3 Let E be a Γ–equivariant vector bundle. Consider the following canonical map $\varphi : (p_*^\Gamma E)_{p(y)} \longrightarrow E_y$ defined as follows:

Let $\theta \in (p_*^\Gamma E)_{p(y)}$. Then there exists an open neighbourhood U of $p(y)$ in X and a section $s \in \Gamma(p^{-1}(U), E)^\Gamma$ such that $s_{p(y)} = \theta$. Since $y \in p^{-1}(U)$, we define $\varphi(\theta) = s_y$. Note that the map φ is well-defined. This canonical bijection induces a canonical isomorphism $\varphi(y) : (p_*^\Gamma E)(p(y)) \longrightarrow E(y)$.

Proposition 4.4.4 *Let W be a Γ–equivariant real vector bundle over Y. Then the vector bundle $p_*^\Gamma W$ over X is a real parabolic vector bundle over X with real parabolic structure over S.*

Proof. Let W be a Γ–equivariant real vector bundle over Y. Then the invariant direct image $p_*^\Gamma E$ defines a holomorphic vector bundle over X. The real structure on W induces a real structure on $p_*^\Gamma W$. Therefore, $p_*^\Gamma W$ is a real vector bundle over X.

Let $y \in Y$ be a ramified point of p over $x \in S$, and let ξ be the isotropy generator of $\Gamma = \Gamma_y$ at y (see Remark 4.3.3). By Remark 4.4.3, the fibre $W(y)$ is canonically identified with the fibre $p_*^\Gamma W(x)$. Note that $W(y)$ is a Γ–module. For the generator ξ of Γ, the distinct eigen-values of the operator $\xi : W(y) \longrightarrow W(y)$ will be $\omega^{k_1}, \ldots, \omega^{k_{r_x}}$ with their multiplicities $n_1, \ldots, n_{r_x}$ respectively, where $0 \leq k_1 < k_2 < \cdots < k_{r_x} < N$. Let V_i be the ω^{k_i}–eigenspace of ξ and define

$$F_y^i = V_i \oplus \cdots \oplus V_{r_x}$$

with associated weight $a_i = k_i/N$ for $i = 1, \ldots, r_x$. Then

$$p_*^\Gamma W(x) \simeq W(y) = F_y^1 \supset F_y^2 \supset \cdots \supset F_y^{r_x} \supset F_y^{r_x+1} = 0$$

is a flag with weights $0 \le a_1 < a_2 < \cdots < a_n < 1$. The multiplicity of the weight a_i is $n_i = \dim(F_y^i/F_y^{i+1})$. Since the following diagram

$$\begin{tikzcd} W(y) \arrow[r, "\xi_y"] \arrow[d, "\sigma_y^W"'] & W(y) \arrow[d, "\sigma_y^W"] \\ W(\sigma(y)) \arrow[r, "\xi_{\sigma(y)}^{-1}"'] & W(\sigma(y)) \end{tikzcd}$$

commutes, we have

$$\sigma_y^W(V_i) = \omega^{-k_i}\text{-eigenspace for } \xi_{\sigma(y)}^{-1} \text{ in } W(\sigma(y)).$$

Let v' be a eigenvector of $\xi_{\sigma(y)}^{-1}$ for the eigenvalue ω^{-k_i}. Then we have $\xi_{\sigma(y)}(v') = \omega^{k_i}v'$. This shows that

$$\sigma_y^W(V_i) = \omega^{k_i}\text{-eigenspace for } \xi_{\sigma(y)} \text{ in } W(\sigma(y)).$$

Therefore, we obtain a real parabolic structure on $p_*^\Gamma W$ over S. □

Proposition 4.4.5 *Let (E, σ^E) be a real parabolic vector bundle on X with parabolic structure over S, all of whose weights are integral multiples of $1/N$. Then there exists a Γ–equivariant real vector bundle W on Y such that $(p_*^\Gamma W)$ is isomorphic to E as a real parabolic vector bundle.*

Proof. Let E be a real parabolic vector bundle over X, i.e., for each $x \in S$

$$E(x) = F_x^1 \supset F_x^2 \supset \cdots \supset F_x^{r_x} \supset 0$$

$$0 \le a_x^1 < a_x^2 < \cdots a_x^{r_x} < 1,$$

where $a_x^i = k_i^x/N, 0 \le k_i^x < N$, satisfying the following:

$$\sigma_x^E(F_x^i) = F_{\sigma(x)}^i \text{ and } a_x^i = a_{\sigma(x)}^i \text{ for all } i = 1, \ldots, r_x. \tag{4.6}$$

Consider weights $0 \leq \alpha_x^1 \leq \cdots \leq \alpha_x^n < N$ according to their multiplicities, where $\alpha_x^i = k_i^x/N$ for all i. Define a map $\Delta_x : \mathbb{C}^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ by

$$\Delta_x(z) = \begin{pmatrix} z^{k_1^x} & & 0 \\ & \ddots & \\ 0 & & z^{k_n^x} \end{pmatrix}$$

Let y be a ramified point over x. Choose admissible neighbourhoods U_y of y and $U_{\sigma(y)}$ of $\sigma(y)$ in Y such that $\sigma(U_y) = U_{\sigma(y)}$ (in case if $x = \sigma(x)$ then we can choose U_y so that $\sigma(U_y) = U_y$). We may assume that U_y's are pairwise disjoint and p^*E is trivial over U_y for all $y \in p^{-1}(S)$.

Let $U = Y \setminus p^{-1}(S)$ and $V = \cup_{y \in p^{-1}(S)} U_y$. Consider a vector bundle E_1 over U defined by $p^*E|_U$. Then E_1 is naturally a Γ–equivariant real vector bundle over U. For $y \in p^{-1}(S)$, let $\varphi_y : p^*E|_{U_y} \xrightarrow{\simeq} U_y \times \mathbb{C}^n$ be an isomorphism. let $\{e_1, \ldots, e_n\}$ be a flag basis for $E(x) \overset{\text{can}}{\simeq} p^*E(y)$, where $x = p(y)$. By an isomorphism φ_y, we get a basis of $\mathbb{C}^n$ which we also denote by $\{e_1, \ldots, e_n\}$. Now, consider a vector bundle E_2 over V to be a trivial vector bundle $V \times \mathbb{C}^n$ with the following Γ–action: $\tau_1 : V \times \mathbb{C}^n \longrightarrow V \times \mathbb{C}^n$ define by $\tau_1(z, v) = (\omega z, \Delta_x(\omega) v), \quad z \in U_y$. Let $\sigma^{V \times \mathbb{C}^n} : V \times \mathbb{C}^n \longrightarrow V \times \mathbb{C}^n$ be the anti-holomorphic involution induced from p^*E. The action of Γ on $E_1|_{U \cap V} = (U \cap V) \times \mathbb{C}^n$ is given by $\tau_{12} : (U \cap V) \times \mathbb{C}^n \longrightarrow (U \cap V) \times \mathbb{C}^n, \ (z, v) \mapsto (\omega z, v)$ and a vector bundle $E_2|_{U \cap V} = (U \cap V) \times \mathbb{C}^n$ have the following Γ–action $\tau_{21} : (U \cap V) \times \mathbb{C}^n \longrightarrow (U \cap V) \times \mathbb{C}^n$ defined by $\tau_{21}(z, v) = (\omega z, \Delta_x(\omega) v), \ z \in U_y \cap U$. Define a map $\delta : (U \cap V) \times \mathbb{C}^n \longrightarrow (U \cap V) \times \mathbb{C}^n$ by $\delta(z, v) = (z, \Delta_x(z) v), \ z \in U_y \cap U$. Then the following diagram commutes:

$$\begin{tikzcd} (U \cap V) \times \mathbb{C}^n \arrow[r, "\tau_{12}"] \arrow[d, "\delta"'] & (U \cap V) \times \mathbb{C}^n \arrow[d, "\delta"] \\ (U \cap V) \times \mathbb{C}^n \arrow[r, "\tau_{21}"'] & (U \cap V) \times \mathbb{C}^n \end{tikzcd}$$

For, let $(z, v) \in (U \cap V) \times \mathbb{C}^n$. If $z \in U_y \cap U$, then we have

$$\begin{aligned} \delta \circ \tau_{12}(z, v) &= \delta(\omega z, v) \\ &= (\omega z, \Delta_x(\omega z) v) \end{aligned}$$

and

$$\begin{aligned}\tau_{21}\circ\delta(z,v) &= \tau_{21}(z,\Delta_x(z)v)\\ &= (\omega z,\Delta_x(\omega)\Delta_x(z)v)\end{aligned}$$

This implies that $\delta\circ\tau_{12}(z,v)=\tau_{21}\circ\delta(z,v)$. Therefore, $\delta\circ\tau_{12}=\tau_{21}\circ\delta$.

Similarly, the following diagram commutes:

$$\begin{tikzcd}(U\cap V)\times\mathbb{C}^n \arrow[r, "\sigma_{12}"] \arrow[d, "\delta"'] & (U\cap V)\times\mathbb{C}^n \arrow[d, "\delta"] \\ (U\cap V)\times\mathbb{C}^n \arrow[r, "\sigma_{21}"'] & (U\cap V)\times\mathbb{C}^n\end{tikzcd}$$

where $\sigma_{12}=\sigma_{21}=\sigma^{V\times\mathbb{C}^n}|_{(U\cap V)\times\mathbb{C}^n}$. For, let $(z,v)\in(U\cap V)\times\mathbb{C}^n$. If $z\in U_y\cap U$, we have

$$\begin{aligned}\delta\circ\sigma_{12}(z,v) &= \delta(\bar z,\sigma^{V\times\mathbb{C}^n}(v))\\ &= (\bar z,\Delta_{\sigma(x)}(\bar z)\sigma^{V\times\mathbb{C}^n}(v))\end{aligned}$$

and

$$\begin{aligned}\sigma_{21}\circ\delta(z,v) &= \sigma_{21}(z,\Delta_x(z)v)\\ &= (\bar z,\sigma^{V\times\mathbb{C}^n}(\Delta_x(z)v))\end{aligned}$$

From (4.6), it follows that $\delta\circ\sigma_{12}=\sigma_{21}\circ\delta$.

By glueing, we obtain a Γ–equivariant real vector bundle W over Y. From Lemma 4.4.2, we have an isomorphism $\psi':E\longrightarrow p_*^\Gamma p^*E$ of real vector bundles. We define $\psi:E\longrightarrow p_*^\Gamma W$ as follows: For $U\subset X$ open and a section $s\in\Gamma(U,E)$, we define $\psi_U(s):p^{-1}(U)\longrightarrow W$ by $\psi_U(s)(y)=[\psi'_U(s)(y)]$. By the construction of W and the fact that $\psi'_U(s)\in\Gamma(p^{-1}(U),p^*E)^\Gamma$, it follows that $\psi_U(s)\in\Gamma(p^{-1}(U),W)^\Gamma$. From the proof of the Lemma 4.4.2, it follows that ψ is an isomorphism of real vector bundles. From the construction of W, we conclude that ψ is an isomorphism of real parabolic vector bundles. □

Let $\mathbf{RP}(X,N)$ denote the full sub-category of $\mathbf{RP}(X)$ whose objects are real parabolic vector bundles on (X,σ_X) with parabolic structure over S having the property that all the weights are integral multiples of $1/N$. Let $\mathbf{RE}_\mathbf{\Gamma}(Y)$ denote the category whose objects are all Γ–equivariant real vector bundles on $(Y,\ \sigma_Y)$ and morphisms are morphisms of Γ–equivariant real vector bundles.

Theorem 4.4.6 *There is a canonical functor $\Psi:\mathbf{RP}(X,N)\longrightarrow\mathbf{RE}_\mathbf{\Gamma}(Y)$ which is an equivalence of categories.*

Proof. We first define a functor $\Psi : \mathbf{RE}_{\mathbf{\Gamma}}(Y) \longrightarrow \mathbf{RP}(X,N)$ as follows: Let W be a Γ–equivariant real vector bundle on Y of rank n. Then $p_*^{\Gamma}W$ is a locally free sheaf of rank n on X. The real structure on W induces a real structure on $p_*^{\Gamma}W$ (see Lemma 4.4.1). We define $\Psi(W)$ to be the corresponding real vector bundle on X. By Proposition 4.4.4, we obtain a real parabolic structure on $\Psi(W)$ with real parabolic structure over S having the property that all the weights are integral multiple of $1/N$. This implies that $\Psi(W)$ is an object of $\mathbf{RP}(X,N)$. Let $\varphi : W_1 \longrightarrow W_2$ be a morphism in $\mathbf{RE}_{\mathbf{\Gamma}}(Y)$. Then $\Psi(\varphi) : \Psi(W_1) \longrightarrow \Psi(W_2)$ is defined as follows: For $U \subseteq X$ open and a section $s \in \Gamma(U, \Psi(W_1))$, we define

$$\Psi(\varphi)_U(s) := \varphi_{p^{-1}(U)}(s).$$

Since φ is Γ–equivariant, $\varphi_{p^{-1}(U)}(s) \in \Gamma(p^{-1}(U), W_2)^{\Gamma}$. Therefore, $\Psi(\varphi)$ is well-defined. For $U \subset X$ open and $s \in \Gamma(p^{-1}(U), W_1)^{\Gamma}$, we have

$$\Psi(\varphi)^{\sigma}_{\sigma_Y(p^{-1}(U))}(\sigma^{p_*^{\Gamma}W_1}(s)) = \varphi \circ \sigma^{W_1} \circ s \circ \sigma_Y$$

and

$$\sigma^{p_*^{\Gamma}W_2}_{p^{-1}(U)}(\Psi(\varphi)_U(s)) = \sigma^{W_2} \circ \varphi \circ s \circ \sigma_Y.$$

Since $\sigma^{W_2} \circ \varphi = \varphi \circ \sigma^{W_1}$, we conclude that $\Psi(\varphi)$ is a morphism of real vector bundles on X.

Let $x \in S$ be a real parabolic point. Then we have

$$\Psi(W_1)(x) = F_1^1 \supset F_2^1 \supset \cdots \supset F_{r_1}^1 \supset 0$$

with weights $0 \leq a_1 < a_2 < \cdots < a_{r_1} < 1$, where $a_i^1 = k_i/N$ and

$$\Psi(W_2)(x) = F_1^2 \supset F_2^2 \supset \cdots \supset F_{r_2}^2 \supset 0$$

with weights $0 \leq b_1 < b_2 < \cdots < b_{r_2} < 1$, where $b_j = h_j/N$.

To show that $\Psi(\varphi)$ is a morphism of parabolic vector bundle, we must show that $\Psi(\varphi)(x)$ satisfy condition

$$\Psi(\varphi)(x)(F_i^1) \subset F_{j+1}^2 \text{ whenever } a_i > b_j.$$

Equivalently, writing $\Psi(\varphi)(x) = (\Psi(\varphi)_{ij}(x))$ in terms of flag bases of $\Psi(W_1)(x)$ and of $\Psi(W_1)(x)$, it requires that $\Psi(\varphi)_{ij}(x) = 0$ whenever $\alpha_i > \beta_j$, where $0 \le \alpha_1 \le \cdots \le \alpha_m < 1$ and $0 \le \beta_1 \le \cdots \le \beta_n < 1$ are the weights repeated according to their multiplicities.

Choose an admissible neighbourhood U_y of y in Y. We may assume that $U_y \simeq \mathbb{D}$ with $W_1 \simeq \mathbb{D} \times \mathbb{C}^m$ and $W_2 \simeq \mathbb{D} \times \mathbb{C}^n$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\mathbb{C}^m$ and $\{d_1, \ldots, d_n\}$ for $\mathbb{C}^n$ so that the action of ξ on $\mathbb{C}^m$ is given by

$$\begin{pmatrix} \omega^{k_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{k_m} \end{pmatrix}$$

where $0 \le k_1 \le \cdots \le k_m < N$ and the action of ξ on $\mathbb{C}^n$ is given by

$$\begin{pmatrix} \omega^{h_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{h_n} \end{pmatrix}$$

where $0 \le h_1 \le \cdots \le h_n < N$. It follows that $\{e_1, \ldots, e_m\}$ and $\{d_1, \ldots, d_n\}$ gives flag basis for $\Psi(W_1)(x)$ and $\Psi(W_2)(x)$ respectively. Now a Γ–invariant section of W_1 on U_y can be given by a Γ–equivariant holomorphic map $f : \mathbb{D} \longrightarrow \mathbb{C}^m$ in these coordinates. We can write

$$f(z) = \sum_i f_i(z) e_i$$

in terms of the basis $\{e_1, \ldots, e_m\}$. Since $\xi \cdot f \cdot \xi^{-1}(z) = f(z)$, we have

$$f_i(z) = \omega^{k_i} f_i(\overline{\omega} z).$$

Write

$$f_i(z) = \sum_{l=0}^{\infty} a_l z^l.$$

Then we have

$$f_i^l(z) = \sum_{k=l}^{\infty} a_k z^{k-l}$$

and

$$\omega^{k_i} f_i^l(\overline{\omega} z) = \omega^{k_i} \sum_{k=l}^{\infty} a_l(\overline{\omega})^{2k-l} z^{k-l}.$$

Evaluating at $z = 0$, we have $a_l = a_l(\overline{\omega})^l \omega^{k_i}$ which implies that $a_l = 0$ unless $l \equiv k_i (\mathrm{mod} N)$. Therefore, each $f_i(z)$ satisfies

$$f_i(z) = z^{k_i} \sum_{l=0}^{\infty} b_l (z^N)^l = z^{k_i} \hat{f}_i(z^N)$$

where $\hat{f}_i(z^N)$ can be consider as a section of $\Psi(W_1)$ near x. Since φ is Γ–equivariant, $\varphi_U(s) = t$ is Γ–invariant section of W_2 on U_y. Applying the above consideration to t, we get

$$g_j(z) = z^{h_j} \hat{g}_j(z^N), \text{ where } \hat{g}_j(z^N) = \sum_{k=0}^{\infty} c_k (z^N)^k.$$

We write φ_U as the matrix (φ_{ij}). Then we have

$$\sum_i \varphi_{ij}(z) f_i(z) = g_j(z) = z^{h_j} \hat{g}_j(z^N)$$

Write $\Psi(\varphi)$ also as a matrix $(\Psi(\varphi)_{ij})$ near x then we have

$$\sum_i \varphi_{ij}(z) z^{k_i} \hat{f}_i(z^N) = \sum_i z^{h_j} \Psi(\varphi)_{ij}(z^N) \hat{f}_i(z^N).$$

From this, it follows that

$$\varphi_{ij}(z) = z^{h_j - k_i} \Psi(\varphi)_{ij}(z^N). \tag{4.7}$$

Since $\varphi_{ij}(z)$ is bounded as $z \longrightarrow 0$, we get

$$\Psi(\varphi)_{ij}(x) = 0 \text{ whenever } h_j < k_i. \tag{4.8}$$

Therefore, we obtain a canonical functor $\Psi : \mathbf{RE}_{\mathbf{\Gamma}}(Y) \longrightarrow \mathbf{RP}(X, N)$. Let φ_1 and φ_2 are morphism from W_1 to W_2 in $\mathbf{RE}_{\mathbf{\Gamma}}(Y)$. Suppose that $\Psi(\varphi_1) = \psi(\varphi_2)$. Let $y \in Y$ and $\theta \in (W_1)_y$. Since $(W_1)_y$ is canonically isomorphic to $(p_*^{\Gamma} W_1)_x$, there exists a section $s \in \Gamma(p^{-1}(U), W_1)^{\Gamma}$, where $y \in p^{-1}(U)$ and $x \in U \subset X$, such that $s_y = \theta$. Since $\Psi(\varphi_1) = \psi(\varphi_2)$, it follows that $(\varphi_1)_y(\theta) = (\varphi_2)_y(\theta)$.

This implies that $\varphi_1 = \varphi_2$. Conversely, given a real parabolic homomorphism $\psi : \Psi(W_1) \longrightarrow \Psi(W_2)$, we define $\varphi : W_1 \longrightarrow W_2$ near y using the equation (4.7). Since ψ is real parabolic morphism, φ is well defined. This proves that Ψ is fully faithful functor. To show that Ψ is an equivalence of categories, we only need to check that Ψ is essentially surjective, which follows from the Proposition 4.4.5. □

Proposition 4.4.7 *A real parabolic vector bundle (E, σ^E) in $\mathbf{RP}(X, N)$ is real parabolic semistable if and only if the corresponding Γ–equivariant real vector bundle W on Y is semistable in the usual sense.*

Proof. Let E be real parabolic semistable vector bundle over X. Let W be the corresponding Γ–equivariant real vector bundle over Y. Note that

$$\mathrm{pdeg}(E) = \frac{\deg(W)}{N}, \tag{4.9}$$

where N is the order of the Galois group of the covering $p : Y \longrightarrow X$ (see [Se70, p. 165]). Let

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{l-1} \subset W_l = W$$

be the Harder–Narasimhan filtration of W. Since $\sigma^W : W \longrightarrow W^\sigma$ is an isomorphism, we have the following filtration

$$0 = \sigma^W(W_0) \subset \sigma^W(W_1) \subset \sigma^W(W_2) \subset \cdots \subset \sigma^W(W_{l-1}) \subset \sigma^W(W_l) = W^\sigma. \tag{4.10}$$

We also have another filtration

$$0 = {W_0}^\sigma \subset {W_1}^\sigma \subset {W_2}^\sigma \subset \cdots \subset {W_{l-1}}^\sigma \subset {W_l}^\sigma = W^\sigma. \tag{4.11}$$

Since filtrations in (4.10) and (4.11) of W^σ satisfy the conditions of the Harder–Narasimhan filtration, by the uniqueness of the Harder–Narasimhan filtration we conclude that

$$\sigma^W(W_i) = {W_i}^\sigma$$

for all $i = 1, \ldots l$. Thus, W_1 is a real semistable subbundle of W. Similarly, by the uniqueness of W_1 it follows that W_1 is left invariant under the action of Γ on W. Therefore, W_1 is G–equivariant real semistable subbundle of W. For every subsheaf $\mathcal{F}$ of W, we have $\mu(\mathcal{F}) \leq \mu(W_1)$. By the correspondence, $p_*^\Gamma W_1$ is real parabolic subbundle of E. Since E is real parabolic semistable, using (4.9), we have

$$\mu(W_1) \leq \mu(W).$$

Since W_1 is maximal semistable subbundle of W, we have $\mu(W_1) = \mu(W)$. This proves that W is semistable.

Conversely, assume that W is semistable. Let F be a proper real parabolic subbundle of E. Let V be a corresponding Γ–equivariant real vector bundle over Y. By the construction, it follows that V is a Γ–equivariant real subbundle of W. Since W is semistable, we have $\mu(V) \leq \mu(W)$. Using (4.9), we have $\mathrm{p}\mu(F) \leq \mathrm{p}\mu(E)$. This shows that E is real parabolic semistable. □

Remark 4.4.8 Using (4.9), it can be easily seen that a real parabolic vector bundle E is real parabolic stable if and only if the corresponding Γ–equivariant real vector bundle W is Γ–real stable. Consequently, a real parabolic vector bundle E is real parabolic polystable if and only if the corresponding Γ–equivariant real vector bundle W is Γ–real polystable.

Appendix A

Category Theory

In this appendix, we will recall some facts in category theory, which are used in this thesis.

A:1 A category $\mathcal{C}$ is said to be an *additive category* if

(a) there exists a zero object in $\mathcal{C}$,

(b) there exist finite coproducts in $\mathcal{C}$, and

(c) each of the morphisms sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ carries the structures of an abelian group such that the composition of morphisms is bilinear with respect to the addition of these groups.

A:2 Let $\mathcal{C}$ be an additive category. Let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. Then we have the following commutative diagram

$$\begin{tikzcd} Ker(f) \arrow[r, "q"] & A \arrow[r, "p"] \arrow[d, "f"] & CokKer(f) \arrow[d, dotted, "h"] \\ Cok(f) & B \arrow[l, "p'"] & KerCok(f) \arrow[l, "q'"] \end{tikzcd}$$

where h is uniquely determined by f.

An additive category with kernels and cokernels, where for each morphism f the uniquely determined morphism $h : \mathrm{CokKer}(f) \longrightarrow \mathrm{KerCok}(f)$ is an isomorphism, is called an *abelian category*.

A:3 Recall that a *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{C}'$ which satisfies the following conditions:

(a) $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$.

(b) For all $X, Y \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(c) For all $X \in \mathrm{Ob}(\mathcal{C})$, the identity morphism of X in $\mathcal{C}'$ equals the identity morphism of X in $\mathcal{C}$.

(d) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C}')$, the composition function

$$\mathrm{Hom}_{\mathcal{C}'}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}'}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(X, Z)$$

is the restriction of the composition function

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

A:4 We say that a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a *full subcategory* of $\mathcal{C}$ if for all $X, Y \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

A full subcategory $\mathcal{C}'$ is called *strict full subcategory* if for any object X, all objects isomorphic to X belong to $\mathrm{Ob}(\mathcal{C}')$.

A:5 Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and $\varphi : F \to G$ be a morphism of functors. Then, φ is an isomorphism of functors if and only if for all $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\varphi(X) : F(X) \to G(X)$ is an isomorphism of objects in $\mathcal{D}$.

A:6 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then we get a new functor $\mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet)) : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ defined by the following assignments:

(a) For all $(Y, X) \in \mathrm{Ob}(\mathcal{D}^{\mathrm{op}} \times \mathcal{C})$

$$\mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet))(X, Y) = \mathrm{Hom}_{\mathcal{D}}(Y, F(X)).$$

(b) If (Y, X) and (Y', X') are objects in $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$, the function

$$\begin{aligned} \mathrm{Hom}((Y, X), (Y', X')) &\to \mathrm{Hom}(\mathrm{Hom}_{\mathcal{D}}(Y, F(X)), \mathrm{Hom}_{\mathcal{D}}(Y', F(X'))) \\ (g^{\mathrm{op}}, f) &\mapsto \mathrm{Hom}(g, F(f)) \end{aligned}$$

is given by

$$\mathrm{Hom}(g, F(f))(u) = F(f) \circ u \circ g$$

for all morphisms $g : Y' \to Y$ in $\mathcal{D}$, $f : X \to X'$ in $\mathcal{C}$, and $u : Y \to F(X)$ in $\mathcal{D}$.

If $G : \mathcal{D} \to \mathcal{C}$ is a functor then, $\mathrm{Hom}_{\mathcal{C}}(G(\bullet), \bullet) : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ can be similarly defined.

A:7 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. We say that G *is a left adjoint of* F, or that F *is a right adjoint of* G, if there exists an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(\bullet, F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(G(\bullet), \bullet)$$

of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$; in that case, φ is called an *adjunction between* F *and* G.

A:8 By **A:5** the adjunction φ attaches to each object (Y, X) in $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$, a bijection of sets

$$\varphi(Y, X) : \mathrm{Hom}_{\mathcal{D}}(Y, F(X)) \to \mathrm{Hom}_{\mathcal{C}}(G(Y), X) \tag{4.12}$$

which is functorial in Y and X, i.e., if $g : Y' \to Y$ is a morphism in $\mathcal{D}$, and if $f : X \to X'$ is a morphism in $\mathcal{C}$, then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(Y, F(X)) \arrow[r, "\varphi(Y,X)"] \arrow[d, "\mathrm{Hom}(g,F(f))"'] & \mathrm{Hom}_{\mathcal{C}}(G(Y), X) \arrow[d, "\mathrm{Hom}(G(g),f)"] \\
\mathrm{Hom}_{\mathcal{D}}(Y', F(X')) \arrow[r, "\varphi(Y',X')"'] & \mathrm{Hom}_{\mathcal{C}}(G(Y'), X')
\end{tikzcd}$$

commutes.

A:9 By substituting $Y = F(X)$ in (4.12) we get,

$$\varphi(F(X), X) : \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \to \mathrm{Hom}_{\mathcal{C}}(G(F(X)), X).$$

Recall that $\varphi(X, F(X))(\mathbf{1}_{F(X)})$ is the *counit morphism of* X with respect to the adjunction φ. We will denote it by σ_X. For every object Y in $\mathcal{D}$, the *unit morphism of* Y with respect to the adjunction φ, is defined to be the morphism $\rho_Y : Y \to F(G(Y))$ in $\mathcal{D}$ such that

$$\varphi_{Y,G(Y)}(\rho_Y) = \mathbf{1}_{G(Y)}.$$

A:10 A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* (respectively, *full*, *fully faithful*) if for all $X, X' \in \mathrm{Ob}(\mathcal{C})$, the function

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(X'))$$

is injective (respectively, surjective, bijective). A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object Y in $\mathcal{D}$, there exists an object X in $\mathcal{C}$ such that $F(X)$ is isomorphic to Y. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathbf{1}_{\mathcal{C}}$ and $F \circ G \cong \mathbf{1}_{\mathcal{D}}$.

A:11 A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

A:12 Let $\mathcal{C}$ be a category. Then for every object $Y \in \mathcal{C}$, we get the *functor* Hom,

$$\mathrm{Hom}(\bullet, Y) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set},$$

which is defined by the following assignments:

(a) $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Ob}(\mathbf{Set}), \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(b) If X and X' are objects in $\mathcal{C}^{\mathrm{op}}$, then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}(X, X') &\to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(X, Y), \mathrm{Hom}_{\mathcal{C}}(X', Y)) \\ u^{\mathrm{op}} &\mapsto \mathrm{Hom}(u, Y) := \mathrm{Hom}(u, \mathbf{1}_Y), \end{aligned}$$

where $\mathrm{Hom}(u, \mathbf{1}_Y)(w) = w \circ u$ for $u \in \mathrm{Hom}_{\mathcal{C}}(X', X)$ and $w \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

It is easy to verify that $\mathrm{Hom}(\bullet, Y) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is indeed a functor.

A:13 Let $\mathcal{C}$ be a category. We say that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if there exists an object $Y \in \mathrm{Ob}(\mathcal{C})$ such that $F \cong \mathrm{Hom}(\bullet, Y)$. An object $Y \in \mathrm{Ob}(\mathcal{C})$ is called a *representing object for* F; we also say that F *is representable by* Y. By Yoneda embedding, a representing object is unique up to a canonical isomorphism.

A:14 Let $\mathcal{C}$ be a category, and let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. Let $Y \in \mathrm{Ob}(\mathcal{C})$. An element $\xi \in F(Y)$ is called a *universal element* for F if it satisfies the following condition: for every object $X \in \mathrm{Ob}(\mathcal{C})$ and for every element

$\mu \in F(X)$, there exists a unique morphism $w : X \to Y$ in $\mathcal{C}$ such that $\mu = F(w)(\xi)$.

A:15 If a functor $F : \mathcal{C} \to \mathbf{Set}$ is representable by X, then there exist a universal element $\xi \in F(X)$ for F. A *representation of* F is a pair (X, ξ), where $X \in \mathrm{Ob}(\mathcal{C})$, and $\xi \in F(X)$ is a universal element for F.

A:16 Let $\mathcal{C}$ be a category, and let S be an object in $\mathcal{C}$. Then, an S-object, or an object over S, is a pair (X, u), where X is an object in $\mathcal{C}$, and $u : X \longrightarrow S$ is a morphism in $\mathcal{C}$, called the structure morphism.

An S-morphism, or a morphism over S, from an S-object (X', u') to another (X, u) is a morphism $f : X' \longrightarrow X$ in $\mathcal{C}$, such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[dr, "u'"'] & X \arrow[d, "u"] \\ & S \end{tikzcd}$$

commutes. We thus get a category $\mathcal{C}/S$ of S-objects and S-morphism. This category has a final object $(S, 1_S)$. If X and Y are S-objects, $\mathrm{Hom}_{\mathcal{C}/S}(X, Y)$ is also denoted by $\mathrm{Hom}_S(X, Y)$.

A:17 If (X, u) and (Y, v) are S-objects, then a fibred product of X and Y over S is a product of X and Y in $\mathcal{C}/S$. It is therefore, a triple $(X \times_S Y, p_1, p_2)$, where $X \times_S Y$ is an S-object, and $p_1 : X \times_S Y \longrightarrow X$ and $p_2 : X \times_S Y \longrightarrow Y$ are S-morphisms, called the canonical projections with the following universal property: For every S-object T, the function

$$\mathrm{Hom}_S(T, X \times_S Y) \longrightarrow \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)$$
$$h \mapsto (p_1 \circ h, p_2 \circ h)$$

is a bijection. Equivalently, for every object T in $\mathcal{C}$, and for every pair of morphisms $f : T \longrightarrow X$ and $g : T \longrightarrow Y$ in $\mathcal{C}$ such that $u \circ f = v \circ g$, there exists a unique morphism $h : T \longrightarrow X \times_S Y$ such that the diagram

$$\begin{tikzcd}
 & & X \arrow[dr, "u"] & \\
T \arrow[r, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & X \times_S Y \arrow[ur] \arrow[dr] & & S \\
 & & Y \arrow[ur, "v"'] &
\end{tikzcd}$$

commutes. We denote h by $(f,g)_s$ or just (f,g).

The fibred product of two S-objects, if it exists, is unique up to a canonical isomorphism. If S is a final object in $\mathcal{C}$, then a fibred product over S is a product in $\mathcal{C}$. If $\mathcal{C} = \mathbf{Set}$, the all fibred product exists in $\mathcal{C}$.

A:18 We say that a diagram in $\mathcal{C}$

$$\begin{tikzcd}
Z \arrow[r, "\varphi"] \arrow[d, "\psi"'] & X \arrow[d, "u"] \\
Y \arrow[r, "v"'] & S
\end{tikzcd}$$

is Cartesian if

(a) it is commutative, and

(b) the triple (Z, φ, ψ) is a fibred product of (X,u) and (Y,v) over S, or equivalently, the function

$$\begin{gathered}
\mathrm{Hom}_S(T,Z) \longrightarrow \mathrm{Hom}_S(T,X) \times \mathrm{Hom}_S(T,Y) \\
h \mapsto (\varphi \circ h, \psi \circ h)
\end{gathered}$$

is a bijection for every S-object T.

A:19 Let (X,u) and (Y,v) be two S-objects. Then we have a functor $F : \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Set** which maps

$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T,X) \times_{\mathrm{Hom}_{\mathcal{C}}(T,S)} \mathrm{Hom}_{\mathcal{C}}(T,Y)\,,$$

where the fibred product on the right side is with respect to the functions

$$\mathrm{Hom}(T,u) : \mathrm{Hom}(T,X) \longrightarrow \mathrm{Hom}(T,S)$$

$$\mathrm{Hom}(T, v) : \mathrm{Hom}(T, Y) \longrightarrow \mathrm{Hom}(T, S)\,.$$

A triple $(X \times_S Y, p_1, p_2)$ is a fibred product of X and Y over S iff the object $X \times_S Y$ in $\mathcal{C}$ represents the functor F, and $(p_1, p_2) \in F(X \times_S Y)$ is a universal element, that is, for every object T in $\mathcal{C}$, the function

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(T, X \times_S Y) &\longrightarrow F(T) \\ h &\mapsto (F(h^{op})(p_1, p_2)) \end{aligned}$$

is a bijection. Equivalently, for every object T in $\mathcal{C}$, $\big(\mathrm{Hom}_{\mathcal{C}}(T, X \times_S Y), \mathrm{Hom}(T, p_1), \mathrm{Hom}(T, p_2)$ is a fibred product of $\mathrm{Hom}(T, X)$ and $\mathrm{Hom}(T, Y)$ over $\mathrm{Hom}(T, S)$ in **Set**.

A:20 Let $\mathcal{C}$ be a category with a final object S. We assume that finite products exist in $\mathcal{C}$. A *group object* of $\mathcal{C}$ is an object G of $\mathcal{C}$, together with a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow$ **Grp** into the category of groups, whose composite with forgetful functor **Grp** $\longrightarrow$ **Set** is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(\bullet, G)$. Equivalently, a group object is an object G, together with a group structure on $\mathrm{Hom}_{\mathcal{C}}(X, G)$ for each object X of $\mathcal{C}$, so that the function $f^* : \mathrm{Hom}_{\mathcal{C}}(Y, G) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, G)$ associated with an arrow $f : X \longrightarrow Y$ in $\mathcal{C}$ is always a homomorphism of groups.

A group object in the category of topological spaces is called a *topological group*. A group object in the category of schemes over a scheme S is called a *group scheme over S*.

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