Some Problems on Modular Forms

By

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Certificate

This is to certify that the Ph.D. thesis titled "**Some Problems on Modular Forms**" by **Jaban Meher** is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent and original work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Thesis Supervisor: ____________________ B. Ramakrishnan

Place:____________________

Date: ____________________

Declaration

The author hereby declares that the work in the thesis titled "**Some Problems on Modular Forms**", submitted for Ph.D. degree to the **Homi Bhabha National Institute** has been carried out under the supervision of Professor B. Ramakrishnan. Whenever contributions of others are involved, every effort is made to indicate that clearly, with due reference to the literature. The author attests that the work is original and has not been submitted in part or full by the author for any degree or diploma to any other institute or university.

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Jaban Meher
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Synopsis

Title of thesis: Some Problems on Modular Forms.

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This thesis contains some of my work on modular forms during my stay at Harish-Chandra Research Institute as a research scholar.

1. Generalized Modular Functions

In [15], Knopp and Mason introduced the notion of a generalized modular function. A generalized modular function f on $\Gamma_0(N)$ is a holomorphic function on the upper half-plane $\mathcal{H}$, meromorphic at cusps, that transforms under $\Gamma_0(N)$ like a usual modular function, namely $f(\frac{az+b}{cz+d}) = f(\gamma \circ z) = \chi(\gamma)f(z)$, for all $\gamma \in \Gamma_0(N)$, where $\chi : \Gamma_0(N) \longrightarrow \mathbb{C}^\times$ is a homomorphism with $\chi(\gamma) = 1$ for γ parabolic with trace $= 2$. We abbreviate GMF for a generalized modular function. Let f be a non-zero GMF on $\Gamma_0(N)$. Then by a theorem of Eholzer and Skoruppa [5, 9], each GMF f has a product expansion

$$f(z) = q^h \prod_{n\geq 1}(1-q^n)^{c(n)} \qquad (0 < |q| < \epsilon), \quad q = e^{2\pi i z}, \quad z \in \mathcal{H},$$

where $h \in \mathbb{Z}$ and $c(n)(n \geq 1)$ are uniquely determined complex numbers.

If $f : \mathcal{H} \to \mathbb{C}$ is a function and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, we define the slash operator by,

$$(f|\gamma)(z) := f(\gamma \circ z) = f\left(\frac{az+b}{cz+d}\right).$$

For $M|N$, let $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ be a set of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$. We define a "norm" of f w.r.t. $\mathcal{R}$ by,

$$\mathcal{N}^M_{\mathcal{R},N}(f) := \prod_{\nu=1}^{r} f|\gamma_\nu.$$

Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ be the Fricke involution. In [19], W. Kohnen proved that for a non-zero GMF f on $\Gamma_0(N)$ with empty divisor, the exponents $c(n)(n \in \mathbb{N})$ take infinitely many different values. In Chapter 1, which is a joint work with W. Kohnen [KM], we show that $c(p)(p\ prime)$ take infinitely many different values, by assuming certain extra conditions on the GMF. More precisely, we prove the following theorem. We prove this theorem by using a positive proportion result regarding values not taken by the Fourier coefficients of cusp forms lying in the subspace of newforms, due to Ono and Skinner [39].

Theorem 0.0.1. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and assume that f has algebraic Fourier coefficients. Suppose further that $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R},N}(f|W_N)$ are constant for every divisor M of N, $M \neq N$. Then the exponents $c(p)$ (p prime) take infinitely many different values.*

Next, we prove the following result using Landau's theorem on Dirichlet series with non-negative coefficients, coupled with the fact that the abscissa of absolute convergence of the Hecke L-function of a non-zero cusp form of weight 2 is exactly $3/2$.

Theorem 0.0.2. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and suppose that the $c(n)\,(n \geq 1)$ are real. Then the $c(n)\,(n \geq 1)$ change signs infinitely often.*

2. Newforms of Half-integral Weight on $\Gamma_0(8N)$ and $\Gamma_0(16N)$

In 1973, G. Shimura introduced the theory of modular forms of half-integral weight and obtained a correspondence between modular forms of half-integral weight and integral weight. After the works of Shimura and Niwa, Shintani gave a correspondence in the reverse direction. In [49, 50], J. -L. Waldspurger proved a remarkable result in showing that the square of the Fourier coefficients of a half-integral Hecke eigenform f is proportional to the special value (at the center) of the L- function of the corresponding Hecke eigenform F of integral weight twisted with a quadratic character, where f maps to F by Shimura correspondence. In 1980's, W. Kohnen characterized certain subspace, called the Kohnen plus space w.r.t. certain operators and further extended the Shimura correspondence to the plus space. Kohnen also studied the analogous Atkin-Lehner theory of newforms in the plus space on $\Gamma_0(4N)$, where N is odd and square-free and obtained the Shintani lifting which is adjoint to the Shimura-Kohnen map. As a consequence, he obtained the explicit form of the Waldspurger theorem for the newforms in the plus space. Analogous theory of newforms for the full space on $\Gamma_0(4N)$, N odd and square-free was obtained by Manickam, Ramakrishnan and Vasudevan in [31]. Manickam in his thesis obtained the theory of newform for the full space on $\Gamma_0(8N)$, N odd square-free [27, 28]. In 1990's, M. Ueda initiated the work extending the result of Kohnen for general N. Recently, M. Ueda and S. Yamana [48] established Kohnen's theory on $\Gamma_0(8N)$, N square-free odd integer, but without considering the Shimura-Kohnen lifts and Waldspurger's formula for the newforms. In Chapter 2 of the thesis, which is a joint work with M. Manickam and B. Ramakrishnan [MMR], we establish the theory of newforms for both the plus spaces on $\Gamma_0(8N)$ and on $\Gamma_0(16N)$ for N odd and square-free. We carry out this theory by using the isomorphism proved by Ueda by means of trace formulas [45, 46]. We define the Shimura-Kohnen maps on these plus spaces and prove that the spaces of newforms of half-integral weight (in the respective plus spaces) on $\Gamma_0(8N)$ and $\Gamma_0(16N)$ are isomorphic to the respective spaces of newforms of integral weight on $\Gamma_0(2N)$ and $\Gamma_0(4N)$. Further, by using the results of Ueda [45, 46], we develop the theory of newforms for the full space on $\Gamma_0(16N)$ with trivial character and also for the primitive quadratic character modulo 8. We give below the precise statements of results proved in Chapter 2.

Before stating the theorems, we give the necessary notations. For a positive integer M with $4|M$, let $S_{k+1/2}(M, \chi)$ denote the space of modular forms of weight $k + 1/2$ on $\Gamma_0(M)$ with χ, a Dirichlet character modulo M.

If χ is trivial, it is denoted by $S_{k+1/2}(M)$. For any positive integer n, the operators $U(n)$ and $B(n)$ are defined on formal sums as follows:

$$U(n): \sum_{m\geq 1} a(m)q^m \mapsto \sum_{m\geq 1} a(mn)q^m,$$

$$B(n): \sum_{m\geq 1} a(m)q^m \mapsto \sum_{m\geq 1} a(m)q^{nm}.$$

For a prime $p|M$, $U(p^2)$ is the Hecke operator on the space $S_{k+1/2}(M,\chi)$ and $B(n)$ maps $S_{k+1/2}(M,\chi)$ into $S_{k+1/2}(Mn,\chi\chi_n)$, where χ_n is the quadratic character $\left(\frac{n}{\cdot}\right)$. For $Q|M$ with $(Q,M/Q)=1$, let $W(Q)$ denote the analogous Atkin-Lehner W-operator on the space $S_{k+1/2}(M,\chi)$. The Kohnen plus space on the space $S_{k+1/2}(4M,\chi)$ for $8\nmid M$ is defined by,

$$S^+_{k+1/2}(4M,\chi) = \left\{ f \in S_{k+1/2}(4M,\chi) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2,3 \pmod 4 \right\}. \tag{0.0.1}$$

From now onwards, N denotes an odd positive square-free integer. Our first result gives the decomposition of $S^+_{k+1/2}(8N)$ into old and newforms.

Theorem 0.0.3. *We have the following decompositions:*

$$S^+_{k+1/2}(8N) = S^{+,new}_{k+1/2}(8N) \bigoplus S^{+,old}_{k+1/2}(8N), \tag{0.0.2}$$

$$S^{+,old}_{k+1/2}(8N) = \bigoplus_{\substack{rd|2N \\ d<2N, 2\nmid r}} S^{+,new}_{k+1/2}(4d)|U(r^2) \bigoplus_{\substack{rd|2N \\ d<2N, 2|r}} S^{+,new}_{k+1/2}(4d)|U(r^2)L_+, \tag{0.0.3}$$

where the operator L_+ is defined by,

$$L_+: \sum_{m\geq 1} a(m)q^m \mapsto \sum_{\substack{m\geq 1 \\ (-1)^k m \equiv 0,1 \pmod 4}} a(m)q^m.$$

Next, we define the Shimura-Kohnen map on $S^+_{k+1/2}(8N)$. Let t be a square-free integer with $(-1)^k t > 0$ and let D $(= t$ or $4t$ according as t is 1 or $2,3$ modulo 4) be the corresponding fundamental discriminant. The t-th Shimura map on $S_{k+1/2}(8N)$ is defined as

$$f|\mathscr{S}_{t,8N}(z) = \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ (d,2N)=1}} \left(\frac{4t}{d}\right) d^{k-1} a_f(|t|n^2/d^2) \right) q^n. \tag{0.0.4}$$

We now define the D-th Shimura-Kohnen map $\mathscr{S}^+_{D,8N}$ on $S^+_{k+1/2}(8N)$ by

$$f|\mathscr{S}^+_{D,8N}(z) = f|\mathscr{S}_{t,8N}|U(2)(z), \quad f \in S^+_{k+1/2}(8N). \tag{0.0.5}$$

Since $U(2) : S_{2k}(4N) \to S_{2k}(2N)$ and $f|\mathscr{S}_{t,8N} \in S_{2k}(4N)$, it follows that $\mathscr{S}^+_{D,8N}$ maps $S^+_{k+1/2}(8N)$ into $S_{2k}(2N)$. Next, we show that the Hecke operator $U(4)$ maps the space $S^{+,new}_{k+1/2}(8N)$ isomorphically onto the space $S^{new}_{k+1/2}(4N)$. As a consequence of this fact, we obtain the Waldspurger theorem for the newforms belonging to $S^{+,new}_{k+1/2}(8N)$. Let F be a normalized newform in $S^{new}_{2k}(2N)$ and let g be the corresponding newform in $S^{new}_{k+1/2}(4N)$ such that F is the Shimura image of g. For a square-free integer $t \equiv 1 \pmod 4$ with $(-1)^k t > 0$, $(t, N) = 1$, let $L(F, \chi_t, k)$ be the special value of the L-function of F twisted with the character $\chi_t = \left(\frac{t}{\cdot}\right)$. Let $f = \sum_{n\geq 1} a_f(n)q^n \in S^{+,new}_{k+1/2}(8N)$ be the newform such that $f|U(4) = g$. Then F is the newform corresponding to f under the Shimura-Kohnen map. With f, F as above, we have the following (Waldspurger theorem):

$$\frac{|a_f(4|t|)|^2}{\langle f|U(4), f|U(4)\rangle} = 2^{\nu(N)-1}\frac{(k-1)!}{\pi^k}|t|^{k-1/2}\frac{L(F,\chi_t,k)}{\langle F, F\rangle}, \tag{0.0.6}$$

where $\nu(N)$ is the number of prime factors of N and $\langle , \rangle$ denotes the Petersson inner product.

Next, we obtain the theory of newforms on the Kohnen plus space $S^+_{k+1/2}(16N)$.

Theorem 0.0.4. *We have the following decompositions:*

$$S^+_{k+1/2}(16N) = S^{+,new}_{k+1/2}(16N) \oplus S^{+,old}_{k+1/2}(16N), \tag{0.0.7}$$

$$\begin{aligned} S^{+,old}_{k+1/2}(16N) = \left(\oplus_{\substack{rd|N\\ d<N}} S^{+,new}_{k+1/2}(16d)|U(r^2)\right) \oplus \left(\oplus_{\substack{d|N\\ rd|2N}} S^{+,new}_{k+1/2}(8d)|U(r^2)^*\right) \\ \oplus \left(\oplus_{\substack{d|N\\ rd|4N}} S^{+,new}_{k+1/2}(4d)|U(r^2)^*\right), \end{aligned} \tag{0.0.8}$$

where $U(r^2)^*$ is $U(r^2)$ if $2 \nmid r$, is equal to $U(r^2)B(4)$ if $2\|r$ and is equal to $U(r^2/16)B(4)$ if $4|r$.

Using the above decompositions of the spaces, we show that the dimensions of $S^{+,new}_{k+1/2}(16N)$ and $S^{new}_{k+1/2}(8N)$ are the same. Moreover, we prove that $S^{+,new}_{k+1/2}(16N)$ and $S^{new}_{2k}(4N)$ are isomorphic under a linear combination of Shimura-Kohnen maps.

For the full space $S_{k+1/2}(16N)$, we prove the following theorem.

Theorem 0.0.5. *The space $S_{k+1/2}(16N)$ contains no newforms in it and the whole space is obtained upon duplicating forms in $S_{k+1/2}(8N)$. The decomposition* *is given by,*

$$
\begin{aligned}
S_{k+1/2}(16N) = {} & \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2) \\
& \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4)L_+U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8)W(8)U(r^2) \\
& \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|B(4)U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4)B(4)U(r^2) \\
& \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|U(r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|B(4)U(r^2) \\
& \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|U(2)B(2)U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \\
& \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|U(r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|U(2)B(2)U(r^2).
\end{aligned}
\tag{0.0.9}
$$

Finally, we study the theory of newforms on $S_{k+1/2}(16N, \chi_2)$, where χ_2 is the even quadratic character modulo 8 given by $\left(\frac{2}{\cdot}\right)$. We prove the following result.

Theorem 0.0.6. *The space $S_{k+1/2}(16N, \chi_2)$ is decomposed as*

$$S_{k+1/2}(16N, \chi_2) = S^{new}_{k+1/2}(16N, \chi_2) \oplus S^{old}_{k+1/2}(16N, \chi_2),$$

where the above directsum is orthogonal with respect to the Petersson inner product and the decomposition of $S^{old}_{k+1/2}(16N, \chi_2)$ is given by

$$
\begin{aligned}
S^{old}_{k+1/2}(16N, \chi_2) = {} & \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(2r^2) \\
& \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8)W(8)U(r^2) \\
& \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|U(2r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|B(2)U(r^2) \\
& \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(8d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|B(2)U(r^2) \\
& \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|W(8)U(r^2) \oplus \oplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(16d, \chi_2)|U(r^2).
\end{aligned}
\tag{0.0.10}
$$

Moreover, the spaces $S^{new}_{k+1/2}(16N, \chi_2)$ and $S^{new}_{2k}(8N)$ are isomorphic as Hecke modules.

3. Products of Eigenforms

The space of modular forms of a fixed weight on the full modular group has a basis of simultaneous eigenvectors for all the Hecke operators. A modular form is called an eigenform if it is a simultaneous eigenvector for all the Hecke operators. A natural question that arises is that whether the product of two eigenforms (may be of different weights) is an eigenform. This question was taken up by W. Duke [8] and E. Ghate [10]. They proved that there are only finitely many cases where this phenomenon occurs. A more general question (i.e., the Rankin-Cohen bracket of two eigenforms) was studied by D. Lanphier and R. Takloo-Bighash [25]. They showed that in this case also the phenomenon occurs in only finitely many cases. Recently, Beyerl et al. [4] have extended this result to a certain class of nearly holomorphic modular forms. In Chapter 3 of the thesis, we extend these results to some cases of quasimodular and nearly holomorphic eigenforms [M]. We consider the product of two quasimodular eigenforms and the product of nearly holomorphic eigenforms. Finally, we generalize the results of Ghate [11] to the case of Rankin-Cohen brackets.

(i) Quasimodular forms:

Let M_k and $\widetilde{M}_k$ denote respectively the space of modular forms and the space of quasimodular forms of weight k on $SL_2(\mathbb{Z})$. It is known that the differential operator $D = \frac{1}{2\pi i}\frac{d}{dz}$ takes $\widetilde{M}_k$ to $\widetilde{M}_{k+2}$. Let T_n be the n^{th} Hecke operator. The following result follows by straightforward verification.

Proposition 0.0.7. *If $f \in \widetilde{M}_k$, then $(D^m(T_n f))(z) = \frac{1}{n^m}(T_n(D^m f))(z)$, for $m \geq 0$. Moreover, we have $D^m f$ is an eigenform for T_n iff f is. In this case, if λ_n is the eigenvalue of T_n associated to f, then $n^m\lambda_n$ is the eigenvalue of T_n associated to $D^m f$.*

For $k \geq 2$, let E_k be the Eisenstein series of weight k, which is an eigenform. For $k \geq 4$, E_k is a modular form and E_2 is a quasimodular form. For $k \in \{12, 16, 18, 20, 22, 26\}$, let Δ_k denote the unique normalized cusp form of weight k on $SL_2(\mathbb{Z})$. Following the method of proof of Theorem 3.1 of [4], we prove the following result.

Theorem 0.0.8. *Let $f \in M_k$, $g \in M_l$. For $r, s \geq 0$, assume that $D^r f \in \widetilde{M}_{k+2r}$, $D^s g \in \widetilde{M}_{l+2s}$ are eigenforms. Then $(D^r f)(D^s g)$ is an eigenform only in the following cases.*

1. $E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14}, \quad E_4\Delta_{12} = \Delta_{16},$
   $E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$
   $E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$
   $E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}.$

(*These are the modular cases given in [8] and [10]*)

2. $(DE_4)E_4 = \frac{1}{2}DE_8$

Using the properties of Bernoulli numbers, we prove the next result.

Theorem 0.0.9. *For $k \geq 2$ and $r, s \geq 0$, $(D^r E_2)(D^s E_k)$ is not an eigenform.*

As a consequence of the above theorem, we get the following corollary.

Corollary 0.0.10. *Let $f \in M_k$ be an eigenform. Then $(D^r E_2)f$ is an eigenform iff $r = 0$ and $f \in \mathbb{C}\Delta_{12}$.*

(ii) Nearly holomorphic modular forms:

Let $E_2^*(z) = E_2(z) - \frac{3}{\pi Im(z)}$. Then E_2^* is a nearly holomorphic modular form of weight 2 which is an eigenform. Using the same methods of proof as in the case of quasimodular forms, we prove the following theorem.

Theorem 0.0.11. *Let f be a normalized eigenform in M_k. Then $E_2^* f$ is an eigenform iff $f = \Delta_{12}$.*

(iii) Rankin-Cohen brackets of eigenforms:

Let $M_k(\Gamma_1(N))$ (respectively $M_k(N, \chi)$), $S_k(\Gamma_1(N))$, (respectively $S_k(N, \chi)$) and $\mathcal{E}_k(\Gamma_1(N))$ (respectively $\mathcal{E}_k(N, \chi)$) be the spaces of modular forms, cusps forms and Eisenstein series of weight $k \geq 1$ on $\Gamma_1(N)$ (respectively on $\Gamma_0(N)$ with character χ) respectively. We have an explicit basis $\mathcal{B}$ for $M_k(\Gamma_1(N))$ consisting of common eigenforms for all Hecke operators T_n with $(n, N) = 1$ as described in the work of Ghate [11]. An element of $M_k(\Gamma_1(N))$ is called an almost everywhere eigenform or a.e. eigenform in short, if it is a constant multiple of an element of $\mathcal{B}$.

We have an explicit basis of eigenforms of $\mathcal{E}_k(\Gamma_1(N))$ described in the following theorem (see for example [35], Theorems 4.7.1 and 4.7.2).

Theorem 0.0.12. *Let ψ_1 and ψ_2 be Dirichlet characters mod M_1 and mod M_2 respectively, such that $\psi_1\psi_2(-1) = (-1)^k$, where $k \geq 1$. Also assume that:*

- *if $k = 2$ and ψ_1 and ψ_2 both are trivial, then $M_1 = 1$ and M_2 is a prime number.*
- *otherwise, ψ_1 and ψ_2 are primitive characters.*

Put $M = M_1M_2$ *and* $\psi = \psi_1\psi_2$. *Then there is an element* $f = f_k(z, \psi_1, \psi_2) = \sum_{n=0}^{\infty} a_n q^n$ *such that* $L(s; f) = L(s, \psi_1)L(s-k+1, \psi_2)$, *with explicit description of* a_n, $n \geq 0$. *The modular form* $f_k(z, \psi_1, \psi_2) \in \mathcal{E}_k(M, \psi)$ *is an eigenvector for the Hecke operators of level* M. *Modulo the relation* $f_1(z, \psi_1, \psi_2) = f_1(z, \psi_2, \psi_1)$ *when* $k = 1$, *the set of elements* $f_k(Qz, \psi_1, \psi_2)$, *where* $QM_1M_2|N$ *form a basis of* $\mathcal{E}_k(N, \psi)$ *consisting of common eigenforms of all the Hecke operators* T_n *of level* N, *with* $(n, N) = 1$.

For $f \in M_{k_1}(N, \chi)$ and $g \in M_{k_2}(N, \psi)$, the m^{th} Rankin-Cohen bracket of f and g is defined by,

$$[f, g]_m(z) = \sum_{r+s=m} (-1)^r \binom{m+k_1-1}{s}\binom{m+k_2-1}{r} f^{(r)}(z)g^{(s)}(z),$$

where $f^{(r)}(z) = D^r f(z)$ and $g^{(s)}(z) = D^s g(z)$.
Following the methods of [11], we prove the following theorems.

Theorem 0.0.13. *Let* $k_1, k_2 \geq 1$ *be integers and* $N \geq 1$ *be a square-free integer and* $m \geq 1$. *If* $g \in S_{k_1}(\Gamma_1(N))$ *and* $h \in S_{k_2}(\Gamma_1(N))$ *are a.e. eigenforms, then* $[g, h]_m$ *is not an a.e. eigenform.*

Theorem 0.0.14. *Let* k_1, k_2, k, m *be positive integers such that* $k = k_1 + k_2 + 2m$ *and* N *be square-free.*

1. *Let* $k_1 \geq 3$ *and* $k_2 \geq k_1 + 2 > 2$. *Suppose that* $g \in S_{k_1}(N, \chi)$ *is an a.e. eigenform which is a newform and* $h \in \mathcal{E}_{k_2}(N, \psi)$ *is an a.e. eigenform. If dim* $(S_k^{new}(N, \chi\psi)) \geq 2$, *then* $[g, h]_m$ *is not an a.e. eigenform.*
2. *Let* $k_1, k_2 \geq 3$, $|k_1 - k_2| \geq 2$. *Let* $g = f_{k_1}(z, \chi_1, \chi_2) \in \mathcal{E}_{k_1}(N, \chi)$ *and* $h = f_{k_2}(z, \psi_1, \psi_2) \in \mathcal{E}_{k_2}(N, \psi)$ *be a.e. eigenforms as mentioned in Theorem 0.0.12 with* χ *and* ψ *primitive characters. If dim* $(S_k^{new}(N, \chi\psi)) \geq 2$, *then* $[g, h]_m$ *is not an a.e. eigenform.*

List of publications and preprints:

1. [KM] W. Kohnen and J. Meher, *Some remarks on the q-exponent of generalized modular functions*, Ramanujan J. **25**, (2011), no.1, 115-119.
2. [MMR] M. Manickam, J. Meher and B. Ramakrishnan, *Newforms of Half-integral weight on* $\Gamma_0(8N)$ *and* $\Gamma_0(16N)$, Preprint 2011.
3. [M] J. Meher, *Some Remarks on Rankin-Cohen Brackets of Eigenforms*, (Revised version submitted to Int. J. Number Theory).

Table of Contents

Chapter 1

Notations and Preliminaries

In this chapter we give some basic definitions and results that will be used in the thesis.

1.1 Notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For $a, b \in \mathbb{Z}$ we write $a|b$ when b is divisible by a. For $z \in \mathbb{C}$, Re z denotes the real part of z and Im z denotes the imaginary part of z, and we define $q = e^{2\pi i z}$ with $i = \sqrt{-1}$. Let $\mathcal{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ be the complex upper half-plane. For $a, b \in \mathbb{Z}$, the symbol $a \pmod b$ means that a varies over a complete set of congruent classes modulo b and (a, b) means the greatest common divisor of a and b. For $k \in \mathbb{N}$, let $\sigma_k(n)$ denote the k-th divisor function defined by $\sigma_k(n) = \sum_{d|n} d^k$. The k-th Bernoulli number B_k is defined as the coefficients of the series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$

For a commutative ring R, we denote the set of all $n \times n$ matrices with entries in R by $M_n(R)$.

1.2 Modular Forms of integral weight

The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

For a positive integer N, we define the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows.

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, c \equiv 0 \pmod N \right\},$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

The group

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad - bc > 0 \right\}$$

acts on $\mathcal{H}$ by

$$\gamma z := \frac{az+b}{cz+d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } z \in \mathcal{H}.$$

Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and a positive integer k, we define

$$f|_k\gamma(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function such that

$$f|_k\gamma(z) = f(z), \text{ for all } \gamma \in \Gamma_0(N).$$

Then f is said to be holomorphic at all cusps of $\Gamma_0(N)$ if

$$f|_k\gamma(z) = \sum_{n=0}^{\infty} a_\gamma(n) q^{n/N}, \text{ for all } \gamma \in SL_2(\mathbb{Z}).$$

If in addition, the constant terms $a_\gamma(0)$ are zero for all $\gamma \in \Gamma$, then we say that f vanishes at all the cusps.

Definition 1.2.1. (Modular form of weight k, level N and character χ) Let χ be a Dirichlet character modulo N. A modular form of weight k, level N and character χ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

(i) $f|_k\gamma(z) = \chi(d) f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

(ii) f is holomorphic at all the cusps of $\Gamma_0(N)$.

If in addition, f vanishes at all the cusps, then we say that f is a cusp form.

The spaces of modular forms and cusp forms of weight k, level N and character χ are $\mathbb{C}$-vector spaces denoted by $M_k(N,\chi)$ and $S_k(N,\chi)$ respectively. If χ is a trivial character, then we write these spaces as $M_k(N)$ and $S_k(N)$ respectively.
Let $k \geq 4$ be an even integer. The Eisenstein series of weight k is defined by

$$E_k(z) := \frac{1}{2} \sum_{\substack{(0,0)\neq(c,d)\in\mathbb{Z}^2,\\ (c,d)=1}} (cz+d)^{-k}. \tag{1.2.1}$$

This is a modular form of weight k for the full modular group $SL_2(\mathbb{Z})$. It has the following Fourier series expansion:

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.2.2}$$

The first few Eisenstein series are as follows.

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \\ E_6 &= 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n, \\ E_8 &= 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n, \\ E_{10} &= 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n, \\ E_{12} &= 1 + \frac{65520}{691} \sum_{n\geq 1} \sigma_{11}(n) q^n, \\ E_{14} &= 1 - 24 \sum_{n\geq 1} \sigma_{13}(n) q^n. \end{aligned} \tag{1.2.3}$$

Consider the following function defined by

$$\triangle(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n. \tag{1.2.4}$$

Then $\triangle(z)$ is a cusp form of weight 12. The function $\tau(n)$ is called the Ramanujan tau function.

Definition 1.2.2. (Petersson inner product) Let $f, g \in M_k(N,\chi)$ such that at least one of them is a cusp form. Then the Petersson inner product of f and g is defined by

$$\langle f, g \rangle = \frac{1}{i_N} \int_{\mathcal{F}_N} f(z) \overline{g(z)} y^{k-2} dx dy,$$

where $\mathcal{F}_N$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$, i_N is the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$ and $z = x + iy$.

Definition 1.2.3. (Hecke operators) For any function $f \in M_k(N,\chi)$, the n^{th} Hecke operator T_n on f is defined by

$$(T_n f)(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

The Hecke operator T_n maps the space $M_k(N,\chi)$ into itself. It also preserves the space $S_k(N,\chi)$. Moreover, if $(n,N)=1$, then T_n is a hermitian operator with respect to the Petersson inner product.

1.3 Newforms in $S_k(N,\chi)$

Let χ be a Dirichlet character modulo N and $c(\chi)$ be its conductor. For any integers M and d satisfying $c(\chi)|M$ and $Md|N$, let χ' be the Dirichlet character modulo M induced by the character χ. Then, we have

$$f(z) \in S_k(M,\chi') \Rightarrow f(dz) \in S_k(N,\chi).$$

We denote by $S_k^{old}(N,\chi)$, the linear subspace of $S_k(N,\chi)$ spanned by all forms of the type $f(dz)$ where $d|N$ and $f \in S_k(M,\chi')$ for some $M < N$ such that $dM|N$ and χ' is the character modulo M which is induced by the character χ modulo N. The subspace $S_k^{old}(N,\chi)$ of $S_k(N,\chi)$ is called the space of oldforms in $S_k(N,\chi)$. Then the space of newforms $S_k^{new}(N,\chi)$ is defined to be the orthogonal complement of the space $S_k^{old}(N,\chi)$ in $S_k(N,\chi)$ with respect to the Petersson inner product. For details we refer to [2] and [26].

1.4 Modular forms of half-integral weight

We begin by recalling some basic facts regarding modular forms of half-integral weight. For complex numbers $0 \neq z, x$, we let $z^x = e^{x \log z}$, $\log z = \log|z| + i \arg z$, $-\pi < \arg z \le \pi$. Let ζ be a fourth root of unity. Let G denote the four-sheeted covering of $GL_2^+(\mathbb{Q})$ defined as the set of all ordered pairs $(\alpha, \phi(z))$, where $\phi(z)$ is a holomorphic function on $\mathcal{H}$ such that $\phi^2(z) = \zeta^2 \frac{cz+d}{\sqrt{\det \alpha}}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. Then G is a group with the multiplicative rule: $(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z))$. Let $k \ge 2$ be a natural number. For a complex valued function f defined on the upper half-plane $\mathcal{H}$ and an element $(\alpha, \phi(z)) \in G$, define the stroke operator by

$$f|_{k+1/2}(\alpha, \phi(z))(z) = \phi(z)^{-2k-1} f(\alpha z).$$

We omit the subscript $k+1/2$ wherever there is no ambiguity. For $\Gamma_0(4)$ and its subgroups, we take the lifting $\Gamma_0(4) \to G$ as the collection $\{(\alpha, j(\alpha, z))\}$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, and $j(\alpha, z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz+d)^{1/2}$. Here $\left(\frac{c}{d}\right)$ denotes the generalized quadratic residue symbol and $\left(\frac{-4}{d}\right)^{1/2}$ is equal to 1 or i according as d is 1 or 3 modulo 4.

Definition 1.4.1. (Modular form of weight $k+1/2$ for $\Gamma_0(4M)$) Let M be a natural number. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k+1/2$ for $\Gamma_0(4M)$ with character χ (modulo $4M$) if

$$f|_{k+1/2}(\gamma, j(\gamma, z))(z) = \chi(d) f(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M)$$

and f is holomorphic at all the cusps of $\Gamma_0(4M)$. If further, f vanishes at all cusps, then it is called a cusp form.

The set of modular forms (resp. cusp forms) defined as above forms a complex vector space denoted by $M_{k+1/2}(4M, \chi)$ (resp. $S_{k+1/2}(4M, \chi)$). If χ is a trivial character, then the space is denoted by $M_{k+1/2}(4M)$ (resp. $S_{k+1/2}(4M)$). The Fourier expansion of a modular form f at the cusp infinity is usually written as $f(z) = \sum_{n\geq 0} a_f(n) q^n$. For any positive integer n, the operators $U(n)$ and $B(n)$ are defined on formal sums as follows:

$$U(n): \sum_{m\geq 0} a(m) q^m \mapsto \sum_{m\geq 0} a(mn) q^m,$$

$$B(n): \sum_{m\geq 0} a(m) q^m \mapsto \sum_{m\geq 0} a(m) q^{nm}.$$

For any $f \in M_{k+1/2}(4M, \chi)$ and a prime $p \nmid 2M$, we define the Hecke operator $T(p^2)$ by

$$f|T(p^2) = \sum_{n\geq 0} \{a_f(np^2) + \chi(p)\left(\frac{(-1)^k n}{p}\right) p^{k-1} a_f(n) + \chi(p^2) p^{2k-1} a_f(n/p^2)\} q^n$$

and using

$$T(p^{2(n+1)}) = T(p^2) T(p^{2n}) - p^{2k-1} T(p^{2(n-1)}), \quad (n \geq 1)$$

and

$$T(n^2 m^2) = T(n^2) T(m^2), \quad (n, m) = (n, 2M) = (m, 2M) = 1,$$

one can extend the definition of $T(n^2)$ for $n \in \mathbb{N}$, $(n, 2M) = 1$. The operators $T(n^2)$ for $n \in \mathbb{N}$, $(n, 2M) = 1$ and $U(n^2)$ for $n \in \mathbb{N}$, $n|2M$ are Hecke operators on $M_{k+1/2}(4M, \chi)$.

Further, $B(n)$ maps $M_{k+1/2}(4M, \chi)$ into $M_{k+1/2}(4Mn, \chi\chi_n)$, where χ_n is the quadratic character $\left(\frac{n}{\cdot}\right)$. Finally for forms $f, g \in M_{k+1/2}(N, \chi)$ with f or g is a cusp form, the Petersson inner product is defined by

$$\langle f, g \rangle = \frac{1}{i_{4M}} \int_{\mathcal{F}_{4M}} f(z)\overline{g(z)}v^{k-3/2}dudv,$$

where $\mathcal{F}_{4M}$ is a fundamental domain for the action of $\Gamma_0(4M)$ on $\mathcal{H}$, i_{4M} is the index of $\Gamma_0(4M)$ in $SL_2(\mathbb{Z})$ and $z = u + iv$.

Chapter 2

Generalized Modular Functions

2.1 Introduction

In this chapter we discuss some properties of exponents of q-product expansions of certain class of generalized modular functions (GMF) on the Hecke congruence subgroup $\Gamma_0(N)$. The results of this chapter are contained in a joint work with W. Kohnen [22].

Let f be a non-zero GMF. Then by a theorem of Eholzer and Skoruppa [9], each GMF f has a product expansion

$$f(z) = cq^h \prod_{n \geq 1} (1 - q^n)^{c(n)} \qquad (0 < |q| < \epsilon), \quad q = e^{2\pi i z}, \quad z \in \mathcal{H},$$

where $h \in \mathbb{Z}$ and c, $c(n) (n \geq 1)$ are uniquely determined complex numbers.

It was proved in [21] that for each square-free integer $N \geq 11$, one can find a GMF f on $\Gamma_0(N)$ such that f has no zeros on $\mathcal{H}$ and the q-exponents $c(n)$ $(n \geq 1)$ take infinitely many different values. This result and the proof given are actually easily seen to be valid for arbitrary integer $N \geq 11$. Kohnen proved in [19] that for any non-constant GMF f with empty divisor, the q-exponents $c(n)$ $(n \geq 1)$ take infinitely many different values. The first result of the chapter sharpens the above statement under certain conditions on f. Let $\mathrm{div}(f)$ denote the divisor of f, i.e., the set of zeros and poles of f in $\mathcal{H}$ and at all cusps. Our second result shows that under the hypothesis that the divisor of f empty, $c(n)$ $(n \geq 1)$ change signs infinitely often, provided that $c(n)$ are real numbers.

2.2 Preliminaries

Definition 2.2.1. (Generalized modular function) A generalized modular function f on $\Gamma_0(N)$ is a holomorphic function on $\mathcal{H}$, meromorphic at cusps such that

$$f\left(\frac{az+b}{cz+d}\right) = \chi(\gamma)f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $\chi : \Gamma_0(N) \to \mathbb{C}^\times$ is a homomorphism with $\chi(\gamma) = 1$ for γ parabolic with trace $= 2$.

M. I. Knopp and G. Mason [15] introduced the notion of a generalized modular function. We abbreviate GMF for a generalized modular function. For further details on GMF, we refer to [15].

Let $g = \sum_{n\geq 1} b(n)q^n$ be a cusp form on $\Gamma_0(N)$. Then the L-series associated to this cusp form is given by

$$L(g,s) = \sum_{n\geq 1} \frac{b(n)}{n^s}.$$

It is known that $L(g,s)$ can be analytically continued over the whole s-plane to an entire function.

2.3 Overview of Earlier Works

Theorem 2.3.1. (*Knopp-Mason, [15]*) *If f is a GMF on $\Gamma_0(N)$, then $g = \frac{1}{2\pi i}\frac{f'}{f}$ is a modular function of weight 2 with trivial character. If the GMF f doesn't have any zero or pole in $\mathcal{H}$ as well as at cusps, then g is a cusp form. Conversely, if g is a cusp form of weight 2 on $\Gamma_0(N)$, then there exists a GMF f on $\Gamma_0(N)$ such that $g = \frac{1}{2\pi i}\frac{f'}{f}$ and f is uniquely determined up to multiplication with non-zero scalars.*

Remark 2.3.1. Since $g = \frac{1}{2\pi i}\frac{f'}{f}$, the exponents of q-expansion of f and the Fourier coefficients of g are related. In fact, if $g(z) = \sum_{n\geq 1} b(n)q^n$ is a cusp form, then

$$b(n) = -\sum_{d|n} dc(d), \qquad (n \geq 1). \tag{2.3.1}$$

Regarding the properties of $c(n)$, we have a theorem of Kohnen and Martin, which is the following.

Theorem 2.3.2. (*Kohnen-Martin, [21]*) *For each square-free integer $N \geq 11$, there exists a GMF f on $\Gamma_0(N)$ with div(f) $= \emptyset$ such that the exponents $c(n)$ $(n \geq 1)$ take infinitely many different values.*

Remark 2.3.2. The result and the proof given in [21] are actually easily seen to be valid for any integer $N \geq 11$ and to hold for any non-constant f, if one exploits the fact proved in [15] that GMFs f with $\mathrm{div}(f) = \emptyset$ correspond to cusp forms of weight 2 by taking logarithmic derivatives.

More generally, Kohnen [17] has proved the following result for any GMF with empty divisor.

Theorem 2.3.3. (*Kohnen, [17]*) *For any non-constant GMF f on $\Gamma_0(N)$ with div(f) $=$ $\emptyset$, $c(n)$ take infinitely many different values.*

2.4 Main Results

To state our results, we define certain operators on a GMF. Let f be a GMF on $\Gamma_0(N)$ and $M|N$. Assume that $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ is a set of representatives for $\Gamma_0(N)$ modulo $\Gamma_0(M)$, then we define a "norm" of f w.r.t. $\mathcal{R}$ by

$$\mathcal{N}^M_{\mathcal{R},N}(f) := \prod_{\nu=1}^{r} f|_0\gamma_\nu. \tag{2.4.1}$$

Lemma 2.4.1. *Let f be a GMF on $\Gamma_0(N)$ and $M|N$. If we have two sets of representatives $\mathcal{R}$ and $\mathcal{R}'$ for $\Gamma_0(N)\backslash\Gamma_0(M)$, then $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R}',N}(f)$ differ by a non-zero scalar. Further, for any set of representatives $\mathcal{R}$, $\mathcal{N}^M_{\mathcal{R},N}(f)$ is a GMF.*

Proof. Let $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ and $\mathcal{R}' = \{\alpha_1, \ldots, \alpha_r\}$ be two sets of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$. Then for each γ_i there exists a unique α_j such that $\gamma_i{\alpha_j}^{-1} \in \Gamma_0(N)$. Since f is a GMF on $\Gamma_0(N)$, we get

$$f|_0(\gamma_i{\alpha_j}^{-1}) = \chi(\gamma_i{\alpha_j}^{-1})f,$$

where χ is the character associated to the GMF f. Thus,

$$f|_0\gamma_i = \chi(\gamma_i{\alpha_j}^{-1})f|_0\alpha_j,$$

this implies

$$\prod_{i=1}^{r} f|_0\gamma_i = \prod_{j=1}^{r} \chi(\gamma_i{\alpha_j}^{-1})f|_0\alpha_j$$

$$\Rightarrow \mathcal{N}^M_{\mathcal{R},N}(f) = (\prod_{j=1}^{r} \chi(\gamma_i {\alpha_j}^{-1}))\mathcal{N}^M_{\mathcal{R}',N}(f).$$

This proves the first assertion. The second assertion follows from the first, since if $\{\gamma_1, \ldots, \gamma_r\}$ is a set of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$, then for any $\gamma \in \Gamma_0(M)$, $\{\gamma_1\gamma, \ldots, \gamma_r\gamma\}$ is another set of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$. □

The "trace" operator on the space of cusp forms $S_k(N)$ is defined as

$$F|Tr^M_N = \sum_{\nu=1}^{r} F|_k\gamma_\nu. \tag{2.4.2}$$

The trace operator maps $S_k(N)$ to $S_k(M)$.

Remark 2.4.1. The norm and trace operators have a relation. Let $g \in S_2(N)$ be the cusp form corresponding to a GMF f, i.e., $g = \frac{1}{2\pi i}\frac{f'}{f}$. Then, we have

$$\frac{1}{2\pi i}\frac{\mathcal{N}^M_{\mathcal{R},N}(f)'}{\mathcal{N}^M_{\mathcal{R},N}(f)} = \frac{1}{2\pi i}\sum_{\nu=1}^{r}\frac{(f|_0\gamma_\nu)'}{(f|_0\gamma_\nu)} = \sum_{\nu=1}^{r}\frac{1}{2\pi i}\left(\frac{f'}{f}\right)|_2\gamma_\nu = \sum_{\nu=1}^{r} g|_2\gamma_\nu = g|Tr^M_N. \tag{2.4.3}$$

With respect to the trace operator, one has a nice characterization of the space of newforms $S^{new}_k(N)$ obtained by A. P. Ogg. Below we give a version from [26].

Theorem 2.4.2. (*Li, [26]*) *If $F \in S_k(N)$, then $F \in S^{new}_k(N)$ iff for all primes $p|N$, we have $F|Tr^{N/p}_N = 0 = (F|_kH_N)|Tr^{N/p}_N$, where H_N is the Fricke involution defined by, $z \mapsto \frac{-1}{Nz}$.*

Remark 2.4.2. Let f be a GMF on $\Gamma_0(N)$. Since $H_N\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} d & -c/N \\ bN & a\end{pmatrix} H_N$, for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$, it follows that $f|_0H_N$ is also a GMF on $\Gamma_0(N)$.

We now state the main results of this chapter.

Theorem 2.4.3. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and assume that f has algebraic Fourier coefficients. Suppose further that $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R},N}(f|_0H_N)$ are constants for every divisor $M|N$, $M \neq N$. Then the exponents $c(p)$ (p prime) take infinitely many different values.*

Theorem 2.4.4. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and suppose that the $c(n)\,(n \geq 1)$ are real. Then the $c(n)\,(n \geq 1)$ change signs infinitely often.*

2.5 Proofs

2.5.1 Proof of Theorem 2.4.3

By assumption f has algebraic Fourier coefficients. Therefore, the same is true for $g = \frac{1}{2\pi i}\frac{f'}{f}$. Then, by bounded denominator argument, there exists an integer $A \in \mathbb{N}$ such that the Fourier coefficients of Ag are algebraic integers. Hence replacing f by f^A we may assume without loss of generality that g has integral algebraic Fourier coefficients. By hypothesis, $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R},N}(f|_0 H_N)$ are constants for every divisor $M|N$, $M \neq N$. This implies that $g|Tr^M_N$ and $(g|_2 H_N)|Tr^M_N$ are zero for each $M|N$, $M \neq N$. Hence, by Theorem 2.4.2, we conclude that $g \in S_2^{new}(N)$. Let us write the Fourier expansion of g as

$$g(z) = \sum_{n\geq 1} b(n)q^n.$$

Since $g = \frac{1}{2\pi i}\frac{f'}{f}$, we get the following relation.

$$b(n) = -\sum_{d|n} dc(d), \qquad (n \geq 1).$$

In particular, for each prime p, we have the relation

$$b(p) = -c(1) - pc(p).$$

Now assume on contrary that $c(p)$ take finitely many different values. Then using the Deligne's bound

$$b(p) \ll_g \sqrt{p}$$

for the Fourier coefficients $b(p)$ of g, where the constant depends only on g, we get

$$-c(1) - pc(p) \ll_g \sqrt{p}.$$

Thus, for sufficiently large prime p, the above relation implies that $c(p) = 0$. Hence, for sufficiently large prime p, we get the following relation.

$$b(p) = -c(1) = b(1). \tag{2.5.1}$$

On the other hand, since $0 \neq g \in S_2^{new}(N)$ has integral algebraic Fourier coefficients, by a result of Ono and Skinner (Lemma on p. 459, [39]), there exists a positive proportion of primes p with $b(p) \neq \alpha$, for any fixed algebraic integer α. This gives a contradiction to (2.5.1). This proves our theorem.

2.5.2 Proof of Theorem 2.4.4

To prove this theorem, we first recall Landau's theorem on Dirichlet series with non-negative coefficients. See (Theorem 11.13, [1]) for more details.

Theorem 2.5.1. *(Landau) Let $h(s)$ be represented in the half-plane $\sigma = Re(s) > c$ by the Dirichlet series*

$$h(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where c is finite, and assume that $a(n) \geq 0$ for all $n \geq n_0$, for some n_0. If the Dirichlet series has a finite abscissa of convergence σ_c, then $h(s)$ has a singularity on the real axis at the point $s = \sigma_c$.

Now we recall a Lemma of Kohnen.

Lemma 2.5.2. (*Kohnen, [20]*) *Suppose that $0 \neq F \in S_k(N)$. Then the abscissa of absolute convergence of the L-series $L(F,s)$ associated to F is $\frac{k+1}{2}$.*

We are now ready to prove Theorem 2.4.4. Since f is not constant, $g \neq 0$. The identity between the exponents $c(n)$ of the product expansion of f and the Fourier coefficients $b(n)$ of g can be rewritten as an identity between Dirichlet series

$$L(g,s) = -\zeta(s) \sum_{n \geq 1} \frac{c(n)}{n^{s-1}}, \qquad \sigma = Re(s) > \frac{3}{2}. \tag{2.5.2}$$

Let us assume on the contrary that $c(n) \geq 0$ for almost all n. Since $L(g,s)$ extends to an entire function and $\zeta(s)$ is holomorphic for $\sigma > 1$, by Theorem 2.5.1, the series

$$\sum_{n \geq 1} \frac{c(n)}{n^{s-1}}$$

converges in the range $\sigma > 1$. Further, almost all of its coefficients are non-negative by hypothesis. Therefore the convergence in this range must be absolute. From (2.5.2) we therefore conclude that $L(g,s)$ is absolutely convergent for $\sigma > 1$. This is a contradiction, since the abscissa of absolute convergence of $L(g,s)$ is exactly $3/2$ by Lemma 2.5.2. This completes the proof.

Chapter 3

Newforms of Half-integral Weight

3.1 Introduction

In 1973, G. Shimura introduced the theory of modular forms of half-integral weight and obtained a correspondence between modular forms of half-integral weight and integral weight. After the works of G. Shimura and S. Niwa, T. Shintani gave a correspondence in the reverse direction. In [49, 50], J. -L. Waldspurger proved a remarkable result in showing that the square of the Fourier coefficients of a half-integral Hecke eigenform f is proportional to the special value (at the center) of the L- function of the corresponding Hecke eigenform F of integral weight twisted with a quadratic character, where f maps to F by the Shimura correspondence. In 1980's, W. Kohnen characterized certain subspace, called the Kohnen plus space w.r.t. certain operators and further extended the Shimura correspondence to the plus space. Kohnen also studied the analogous Atkin-Lehner theory of newforms in the plus space on $\Gamma_0(4N)$, where N is odd and square-free and obtained the Shintani lifting which is adjoint to the Shimura-Kohnen map. As a consequence, he obtained the explicit form of the Waldspurger theorem for the newforms in the plus space. Analogous theory of newforms for the full space on $\Gamma_0(4N)$, N odd and square-free was obtained by M. Manickam, B. Ramakrishnan and T. C. Vasudevan in [31]. Manickam in his thesis obtained the theory of newforms for the full space on $\Gamma_0(8N)$, N odd square-free [27, 28].

Let $M = 2^a N$, N is an odd square-free positive integer, a is an integer, $0 \leq a \leq 2$. Let χ be a real Dirichlet character modulo M and χ is primitive modulo 8 if $a = 2$. By explicit computation of traces, in [45], M. Ueda proved that there exists a Hecke equivariant isomorphism between the spaces $S_{k+1/2}(4M, \chi)$ and $S_{2k}(2M)$. In this chapter, which is a joint work with M. Manickam and B. Ramakrishnan, we set up the theory of newforms for both the Kohnen plus space and the full space on these spaces $S_{k+1/2}(2^{a+2}N,\chi)$, for $0 \leq a \leq 2$ and χ primitive modulo 8 if $a = 2$. As mentioned above, the theory of newforms is known for the spaces when $a = 0$ [17, 18, 31]. If $a = 1$, the theory is known for the full space by the work of Manickam [27, 28] and for the Kohnen plus space by a recent work of M. Ueda and S. Yamana [48].

Using Ueda's isomorphism as a main tool, we establish the theory of newforms for the remaining cases for both the Kohnen plus space and the full space. Using the explicit dimension formulas, we observe that the Shimura maps $\mathscr{S}_{t,16N}$ (see §3.2 for the definition) map the space $S_{k+1/2}(16N)$ into $S_{2k}(4N)$ and thereby we deduce that there is no newform space on $S_{k+1/2}(16N)$. However, we establish the theory of newforms for $S_{k+1/2}(16N,\chi)$, which is compatible with the integral weight theory when χ is a primitive character modulo 8.

3.2 Newforms on the plus space $S^+_{k+1/2}(8N)$

From now onwards, let us assume that N is an odd positive integer. In [17, 18], Kohnen introduced a subspace of $S_{k+1/2}(4N)$ (referred to as the Kohnen plus space) as follows:

$$S^+_{k+1/2}(4N) = \left\{ f \in S_{k+1/2}(4N) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2, 3 \pmod 4 \right\}. \tag{3.2.1}$$

In [48], Ueda and Yamana considered the same for $S_{k+1/2}(8N)$ and they defined the plus space by

$$S^+_{k+1/2}(8N) = \left\{ f \in S_{k+1/2}(8N) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2, 3 \pmod 4 \right\}. \tag{3.2.2}$$

In this section, we develop the theory of newforms in the Kohnen plus space $S^+_{k+1/2}(8N)$, where N is an (odd) square-free natural number.

3.2.1 W-operators

For $p|2N$, let W_p denote the Atkin-Lehner W-operator on $S_{2k}(2N)$. For $p = 2$, we define the analogous Atkin-Lehner W-operators $W(4)$ on $S_{k+1/2}(4N)$ and $W(8)$ on $S_{k+1/2}(8N,\chi)$ as follows:

$$W(4) = \left(\begin{pmatrix} 4a & b \\ 4Nc & 4 \end{pmatrix},\ 2^{1/2}\, e^{\frac{i\pi}{4}} (Nc\tau + 1)^{1/2} \right),$$

where a, b, c are integers satisfying $4a - Nbc = 1$ and $b \equiv 1 \pmod 4$.

$$W(8) = \left(\begin{pmatrix} 8x & y \\ 8Nw & 8 \end{pmatrix},\ 8^{1/4}\, e^{i\pi/4} (Nw\tau + 1)^{1/2} \right), \tag{3.2.3}$$

where x, y, w are integers such that $y \equiv 1 \pmod 8$, $8x - Nwy = 1$. We also let

$$W_*(4) = \left(\begin{pmatrix} 4u & v \\ 4Nr & 8 \end{pmatrix},\ 2^{1/2}\ e^{\frac{i\pi}{4}}(Nr\tau + 2)^{1/2} \right),$$

where r, u, v are integers satisfying $8u - Nrv = 1$ and $v \equiv 1 \pmod 8$.

Remark 3.2.1. The W-operators defined above are independent of the choice of the integers $a, b, c, x, y, w, r, u, v$. Since $W_*(4)B(2) = W(8)$ on $S_{k+1/2}(8N, \chi)$, by applying the operator $U(2)$ we see that $W_*(4) = W(8)U(2)$ on $S_{k+1/2}(8N, \chi)$. Also, note that $W_*(4) = W(4)$ on $S_{k+1/2}(4N)$ (see [27, 28] for details). The operator $W(8)$ maps $S_{k+1/2}(8N, \chi)$ into $S_{k+1/2}(8N, \chi\chi_8)$ and $W(8)^2 = I$ on $S_{k+1/2}(8N, \chi)$, where χ is the principal character or $\chi = \chi_8$ (we denote by χ_8, the real quadratic character modulo 8 defined by $\chi_8(n) = \left(\frac{2}{n}\right)$, the extended Jacobi symbol) and I denotes the identity operator.

We now obtain a characterization of $S^+_{k+1/2}(4N)$ and $S^{new}_{k+1/2}(4N)$ (N odd square-free) in terms of the operator $U(2)W(8)$.

Proposition 3.2.1. *Let N be an odd square-free integer.*
(i) If $f \in S_{k+1/2}(4N)$, then

$$f|U(2)W(8) = \chi_8(2k+1)\ 2^{k/2-1/4} f \tag{3.2.4}$$

if and only if $f \in S^+_{k+1/2}(4N)$ and
(ii)

$$f|U(2)W(8) = -\chi_8(2k+1)\ 2^{k-1} f \tag{3.2.5}$$

if and only if $f \in S^{new}_{k+1/2}(4N)$.

Proof. For (i) we refer to [27, 28]. Let us now prove (ii). Let $f \in S^{new}_{k+1/2}(4N)$. Then

$$f|U(4)W(4) = -\chi_8(2k+1)2^{k-1} f.$$

Therefore, using the fact that $f|W(8)^2 = f$, $f|W_*(4) = f|W(8)U(2)$, we have

$$\begin{aligned} f|U(4)W(4) &= -\chi_8(2k+1)2^{k-1} f, \\ f|U(2)W(8)W(8)U(2) &= -\chi_8(2k+1)2^{k-1} f|W(4), \\ f|U(2)W(8)W_*(4) &= -\chi_8(2k+1)2^{k-1} f|W(4). \end{aligned}$$

Now applying $W_*(4)^{-1}$ both sides and using the fact that $W(4)W_*(4)^{-1} = I$ on $S_{k+1/2}(8N)$, we get

$$f|U(2)W(8) = -\chi_8(2k+1)2^{k-1} f.$$

Retracing the above steps, we get the converse. □

3.2.2 Certain operators

In this section we give some general facts and so we assume that $M \geq 1$ is any integer. Let $\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}\right)$ and $\xi' = \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, e^{-\pi i/4}\right)$. Then

$$\xi + \xi' : S_{k+1/2}(4M) \longrightarrow S_{k+1/2}(8M), \tag{3.2.6}$$

and on formal Fourier series $\sum a_n q^n$, it transforms as

$$\sum a_n q^n | (\xi + \xi') = \chi_8(2k+1)\sqrt{2} \left(\sum_{(-1)^k n \equiv 0,1 \pmod 4} a_n q^n - \sum_{(-1)^k n \equiv 2,3 \pmod 4} a_n q^n \right). \tag{3.2.7}$$

This observation proves the following lemma.

Lemma 3.2.2. *Let $L_+ := \frac{1}{2}\left(\frac{\chi_8(2k+1)}{\sqrt{2}}(\xi + \xi') + I\right)$, where I is the identity operator. Then L_+ maps the space $S_{k+1/2}(4M)$ into the space $S_{k+1/2}(8M)$. It acts as identity on the plus space $S^+_{k+1/2}(4M)$:*

$$S^+_{k+1/2}(4M)|L_+ = S^+_{k+1/2}(4M).$$

Moreover, if $f = \sum_{n \geq 1} a_f(n) q^n \in S_{k+1/2}(4M)$, then $f|L_+$ has the following Fourier expansion.

$$f|L_+ = \sum_{(-1)^k n \equiv 0,1 \pmod 4} a_f(n) q^n.$$

Proposition 3.2.3. *We have the following assertions.*

(i) The operator L_+ is injective on $S^{new}_{k+1/2}(4M)$, where M is square-free. Moreover, if M is odd and square-free, then L_+ is injective on $S_{k+1/2}(4M)$.

(ii) If $f \in S_{k+1/2}(4M)$, then $f|L_+T(p^2) = f|T(p^2)L_+$, for any prime p satisfying $(p, 2M) = 1$.

(iii) If $f, g \in S_{k+1/2}(4M)$, then $\langle f|L_+, g\rangle = \langle f, g|L_+\rangle$.

Proof. Let $F \in S^{new}_{2k}(2M)$ be a normalized newform. Then the theory of newforms on $S_{k+1/2}(4M)$ gives a unique (upto a constant) non-zero newform $f \in S^{new}_{k+1/2}(4M)$, which corresponds to F under the Shimura correspondence. Hence, using Waldspurger's formula together with the fact that $L(F, \chi_D, k) \neq 0$ for some fundamental discriminant

D, $(-1)^k D > 0$ and $(D, 2M) = 1$, we get $a_f(|D|) \neq 0$. This proves the injectivity of L_+ on $S^{new}_{k+1/2}(4M)$. Let M be odd and square-free. Using the facts that

$$L_+ U(4) = U(4) \quad \text{on} \quad S_{k+1/2}(4M)$$

and $U(4)$ is an automorphism of $S_{k+1/2}(4M)$, we get the injectivity of L_+ on $S_{k+1/2}(4M)$. This proves (i). A direct computation gives (ii). Using the definition of L_+, we have

$$\langle f|\xi, g\rangle = \langle f, g|\xi'\rangle \quad \text{and} \quad \langle f|\xi', g\rangle = \langle f, g|\xi\rangle, \quad \text{where} \quad f, g \in S_{k+1/2}(4M),$$

from which it follows that $\langle f|L_+, g\rangle = \langle f, g|L_+\rangle$. This completes the proof. □

Shimura-Kohnen map on $S^+_{k+1/2}(8M)$.

Let t be a square-free integer with $(-1)^k t > 0$ and let D ($= t$ or $4t$ according as t is 1 or $2, 3$ modulo 4) be the corresponding fundamental discriminant. The t-th Shimura map on $S_{k+1/2}(4M)$ is defined as

$$f|\mathscr{S}_{t,4M}(\tau) = \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ (d,2M)=1}} \left(\frac{4t}{d}\right) d^{k-1} a_f(|t|n^2/d^2) \right) q^n. \tag{3.2.8}$$

When M is odd, we define the D-th Shimura-Kohnen map $\mathscr{S}^+_{D,M}$ on $S^+_{k+1/2}(8M)$ by

$$f|\mathscr{S}^+_{D,8M}(\tau) = f|\mathscr{S}_{t,8M}|U(2)(\tau), \quad f \in S^+_{k+1/2}(8M). \tag{3.2.9}$$

Since $U(2) : S_{2k}(4M) \to S_{2k}(2M)$, it follows immediately (using the mapping property of the Shimura map $\mathscr{S}_{t,8M}$) that $\mathscr{S}^+_{D,8M}$ maps $S^+_{k+1/2}(8M)$ into $S_{2k}(2M)$.

3.2.3 The plus space $S^+_{k+1/2}(8N)$

In this section, we assume that $N \geq 1$ is an odd square-free integer. Define the space of oldforms in $S^+_{k+1/2}(8N)$ as follows.

$$S^{+,old}_{k+1/2}(8N) = S^{old}_{k+1/2}(4N)|L_+, \tag{3.2.10}$$

and define the space of newforms in $S^+_{k+1/2}(8N)$ as

$$S^{+,new}_{k+1/2}(8N) = S^{new}_{k+1/2}(4N)|L_+.$$

Using Proposition 3.2.3, we get $S^{+,old}_{k+1/2}(8N)$ and $S^{+,new}_{k+1/2}(8N)$ are orthogonal with respect to the Petersson inner product. Hence, we define the Kohnen plus space on $S_{k+1/2}(8N)$ by

$$S^+_{k+1/2}(8N) := S_{k+1/2}(4N)|L_+. \tag{3.2.11}$$

Theorem 3.2.4. *(i) We have the orthogonal direct sum decomposition:*

$$S^{+}_{k+1/2}(8N) = S^{+,new}_{k+1/2}(8N) \oplus S^{+,old}_{k+1/2}(8N),$$

where

$$S^{+,old}_{k+1/2}(8N) = \bigoplus_{\substack{rd|2N \\ d<2N}} S^{+,new}_{k+1/2}(4d)|U(r^2)_{+}, \tag{3.2.12}$$

with the sums over various d's are orthogonal.

(ii) "Multiplicity 1" theorem holds on $S^{+,new}_{k+1/2}(8N)$.

(iii)

$$U(4) : S^{+,new}_{k+1/2}(8N) \longrightarrow S^{new}_{k+1/2}(4N)$$

is an isomorphism.

(iv) (Waldspurger formula): We have

$$\frac{|a_f(4|D|)|^2}{\langle f|U(4), f|U(4)\rangle} = 2^{\nu(N)-1}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(F,\chi_D,k)}{\langle F,F\rangle}, \tag{3.2.13}$$

where D is a fundamental discriminant with $(D,2N)=1$, $\nu(N)$ is the number of prime factors of N and $L(F,\chi_D,k)$ is the special value of the L-function of F twisted with the quadratic character χ_D at $s=k$.

Proof. As in [31], we have

$$S^{old}_{k+1/2}(4N) = \bigoplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(4d)|U(r^2) \bigoplus_{\substack{d|N \\ rd|2N}} S^{+,new}_{k+1/2}(4d)|U(r^2). \tag{3.2.14}$$

Applying L_+ on both the sides and using Proposition 3.2.3, we have

$$S^{+,old}_{k+1/2}(8N) = \bigoplus_{\substack{rd|N \\ d<N}} S^{+,new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_{+}.$$

Further, the nature of the operator $L+$ and the multiplicity one theorem on $S^{new}_{k+1/2}(4N)$ give the validity of the multiplicity one result on $S^{+,new}_{k+1/2}(8N)$.
If $f \in S^{+,new}_{k+1/2}(8N)$, then $f|U(4) \in S_{k+1/2}(4N)$. In addition, if f is a Hecke eigenform, then $f|U(4)$ is also a Hecke eigenform which is not in the plus space. Therefore, by the theory of newforms on $S_{k+1/2}(4N)$, $f|U(4) \in S^{new}_{k+1/2}(4N)$. Thus, $U(4) : S^{+,new}_{k+1/2}(8N) \to S^{new}_{k+1/2}(4N)$. Also for any $g \in S_{k+1/2}(4N)$, we know that $g|L_+U(4) = g|U(4)$. From this we get (iii).

We now derive the Waldspurger formula. Let F be a normalized newform in $S_{2k}^{new}(2N)$. Then by the work of [31], there exits a newform $g \in S_{k+1/2}^{new}(4N)$ which corresponds to F under the Shimura correspondence. Let $f = g|L_+$. Then $f|U(4) = g$. Therefore, the Waldspurger formula obtained in [32, Corollary 1], and the fact that $g = f|U(4)$ give the required formula (iv). □

3.3 Remark about a result of Ueda and Yamana

In this section, we make an observation about Proposition 4 of [48]. It says that the operator $\tilde{Y}(8)$ (defined on page 4 of [48]) maps the space $S_{k+1/2}(8N)$ into $S_{k+1/2}^{+}(8N)$ and further, $4^{-1}\chi_8(2k+1)\tilde{Y}(8)$ is an involution on $S_{k+1/2}^{+}(8N)$. In the following, we first show that $\tilde{Y}(8) = \epsilon\ 2^{-3k/2+9/4}U(8)W(8)$ on $S_{k+1/2}(8N)$, where ϵ is a fourth root of unity. Next, we show that $U(8)W(8)$ does not preserve the space $S_{k+1/2}^{+,new}(8N)$. As a consequence, Proposition 4 of [48] can not be true.

We begin with the following lemma, which follows from a straightforward computation.

Lemma 3.3.1.

$$\tilde{Y}(8) = \epsilon 2^{-3k/2+9/4}U(8)W(8) \quad on \quad S_{k+1/2}(8N).$$

We now prove that $U(8)W(8)$ is not a constant times the identity function on $S_{k+1/2}^{+,new}(8N)$. Let $f \in S_{k+1/2}^{+,new}(8N)$ and assume that $f|U(8)W(8) = \lambda f$ for some constant λ. Let $g = f|U(4)$. Then $g \in S_{k+1/2}^{new}(4N)$. Now by Proposition 3.2.1, we have

$$-\chi_8(2k+1)2^{k-1}g = g|U(2)W(8) = f|U(8)W(8) = \lambda f \in S_{k+1/2}^{+,new}(8N), \tag{3.3.1}$$

which is a contradiction.

3.4 Newforms on the plus space $S_{k+1/2}^{+}(16N)$

In this section we extend the results of the previous section to forms on $\Gamma_0(16N)$ (N is odd and square-free.) We need the following orthogonal decomposition of $S_{k+1/2}(8N)$ (in [27] and [28] a slightly different decomposition was given).

$$S_{k+1/2}(8N) = S_{k+1/2}^{new}(8N) \oplus S_{k+1/2}^{old}(8N), \tag{3.4.1}$$

where the space of oldforms $S^{old}_{k+1/2}(8N)$ has the following decomposition.

$$\begin{aligned} S^{old}_{k+1/2}(8N) = &\bigoplus_{\substack{d<N \\ rd|N}} S^{new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{new}_{k+1/2}(4d)|U(r^2) \\ &\bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+. \end{aligned} \tag{3.4.2}$$

We need to show only that for a fixed divisor $d|N$, the sum $S^{+,new}_{k+1/2}(4d)+S^{+,new}_{k+1/2}(4d)|U(4)+S^{+,new}_{k+1/2}(4d)|U(4)L_+$ is direct. For some constants α, β, γ and a newform $f \in S^{+,new}_{k+1/2}(4d)$, if we have

$$\alpha f + \beta f|U(4) + \gamma f|U(4)L_+ = 0,$$

then applying the operator $U(4)$, we get

$$\alpha f|U(4) = -(\beta+\gamma) f|U(16),$$

from which we conclude that $\alpha = 0$. Since the sum $S^{+,new}_{k+1/2}(4d)|U(4) \oplus S^{+,new}_{k+1/2}(4d)|U(4)L_+$ is a direct sum, it follows that $\beta = \gamma = 0$. This proves the required direct sum.

We define the space $S^{+,new}_{k+1/2}(16N)$ as follows.

$$S^{+,new}_{k+1/2}(16N) = S^{new}_{k+1/2}(8N)|L_+. \tag{3.4.3}$$

We also define $S^{+,old}_{k+1/2}(16N)$ as follows.

$$\begin{aligned} S^{+,old}_{k+1/2}(16N) = &\sum_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(8d)|U(r^2)L_+ + \sum_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \\ &+ \sum_{rd|N} S^{+,new}_{k+1/2}(8d)|U(4r^2)B(4) + \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \\ &+ \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)B(4) + \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+ \end{aligned} \tag{3.4.4}$$

Since L_+ and $B(4)$ commute with the Hecke operators $T(p^2)$ for $(p, 2N) = 1$, each of the distinct eigenspaces of the decomposition of $S^{+,old}_{k+1/2}(16N)$ is orthogonal to $S^{+,new}_{k+1/2}(16N)$. Thus, the following orthogonal direct sum defines the plus space:

$$S^{+}_{k+1/2}(16N) = S^{+,new}_{k+1/2}(16N) \oplus S^{+,old}_{k+1/2}(16N). \tag{3.4.5}$$

Theorem 3.4.1. *Let t be a square-free integer with $(-1)^k t > 0$. Then the t-th Shimura map $\mathscr{S}_{t,16N}$ maps the space $S^{+}_{k+1/2}(16N)$ into the space $S_{2k}(4N)$.*

Proof. Let us first show that $\mathscr{S}_{t,16N}$ maps the newforms space $S^{+,new}_{k+1/2}(16N)$ into the newforms space $S^{new}_{2k}(4N)$. Since $U(4)$ is zero on $S^{+,new}_{k+1/2}(16N)$, the plus space property shows that $\mathscr{S}_{t,16N}$ is non-zero on $S^{+,new}_{k+1/2}(16N)$ only when $t \equiv 1 \pmod 4$. It is known that $\mathscr{S}_{t,16N}$ maps $S^{+,new}_{k+1/2}(16N)$ into the space $S_{2k}(8N)$. By the theory of newforms, and the definition of $S^{+,new}_{k+1/2}(16N)$, the image is contained in the old class generated by $S^{new}_{2k}(4N)$. So, the image belongs to $S^{new}_{2k}(4N) \oplus S^{new}_{2k}(4N)|B(2)$. But $f|U(4) = 0$ if $f \in S^{+,new}_{k+1/2}(16N)$ implies that $f|\mathscr{S}_{t,16N}U(2) = 0$. Therefore, for a newform $F \in S^{new}_{2k}(4N)$, we have $(\alpha F + \beta F|B(2))|U(2) = 0$, which implies that $\beta = 0$. Hence, $f|\mathscr{S}_{t,16N} = F \in S^{new}_{2k}(4N)$. This proves that $\mathscr{S}_{t,16N}$ maps $S^{+,new}_{k+1/2}(16N)$ into the space $S^{new}_{2k}(4N)$. Next we consider the case of oldforms. We have

$$\begin{aligned} f|U(r^2)|\mathscr{S}_{t,16N} &= f|\mathscr{S}_{t,16N}|U(r),\ r|N \\ f|U(4)B(4)|\mathscr{S}_{t,16N} &= f|\mathscr{S}_{t,16N}|U(2)B(2), \quad f \in S^{+,old}_{k+1/2}(16N). \end{aligned} \tag{3.4.6}$$

Let $rd|N$, $d < N$. Since $U(4) = 0$ on $S^{new}_{k+1/2}(8d)$, $S^{new}_{k+1/2}(8d)|L_+$ contains forms whose n-th Fourier coefficient is non-zero only when $(-1)^k n \equiv 1 \pmod 4$. This implies that

$$S^{new}_{k+1/2}(8d)|U(r^2)L_+|\mathscr{S}_{t,16N} = S^{new}_{k+1/2}(8d)|U(r^2)|\mathscr{S}_{t,16N} \in S_{2k}(4N),$$

when $t \equiv 1 \pmod 4$, and when $t \equiv 2, 3 \pmod 4$, the image is zero. Next, when $rd|N$ and $f \in S^{+,new}_{k+1/2}(8d)$, using (3.4.6), it follows that both $f|U(r^2)|\mathscr{S}_{t,16N}$ and $f|U(4r^2)B(4)|\mathscr{S}_{t,16N}$ belong to the space $S_{2k}(4N)$. By a similar reasoning, we see that $f|U(r^2)|\mathscr{S}_{t,16N}$, $f|U(4r^2)B(4)|\mathscr{S}_{t,16N}$ and $f|U(4r^2)L_+|\mathscr{S}_{t,16N}$ belong to the space $S_{2k}(4N)$, where $rd|N$ and $f \in S^{+,new}_{k+1/2}(4d)$. This completes the proof. □

Remark 3.4.1. If $t \equiv 1 \pmod 4$ with $(-1)^k t > 0$, from the above theorem, we observe that the Shimura-Kohnen map $\mathscr{S}^{+}_{t,16N}$ on the plus space $S^{+}_{k+1/2}(16N)$ is the same as the Shimura map $\mathscr{S}_{t,16N}$.

The main results of this section is presented in the following theorem.

Theorem 3.4.2. (*i*) *The spaces* $S^{+,new}_{k+1/2}(16N)$ *and* $S^{new}_{2k}(4N)$ *are isomorphic under a linear combination of Shimura maps. In particular, 'multiplicity 1' theorem holds good in the space of newforms* $S^{+,new}_{k+1/2}(16N)$.

(ii) One has

$$S^{+,old}_{k+1/2}(16N) = \bigoplus_{\substack{rd|N\\ d<N}} S^{new}_{k+1/2}(8d)|U(r^2)L_+ \bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2)$$
$$\bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(4r^2)B(4) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \tag{3.4.7}$$
$$\bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)B(4) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+,$$

where the sums over various d's are orthogonal.

(iii) There is a linear combination of Shimura maps which is an isomorphism between the spaces $S^{+}_{k+1/2}(16N)$ and $S_{2k}(4N)$, mapping the space of newforms to newforms and the space of oldforms to oldforms.

Proof. We have already shown that for a square-free $t \equiv 1 \pmod 4$, $(-1)^k t > 0$, $\mathscr{S}_{t,16N}$ maps the newforms space $S^{+,new}_{k+1/2}(16N)$ into the newforms space $S^{new}_{2k}(4N)$. Moreover, as L_+ is injective on $S^{new}_{k+1/2}(8N)$, by the definition of $S^{+,new}_{k+1/2}(16N)$ and from the theory of newforms in $S_{k+1/2}(8N)$, it follows that the spaces $S^{+,new}_{k+1/2}(16N)$ and $S^{new}_{k+1/2}(4N)$ have equal dimensions. Therefore, it follows that there is a linear combination of the Shimura maps $\mathscr{S}_{t,16N}$ which is an isomorphism between these spaces. This proves (i).
We now prove that the decomposition is direct. We use inductive method to obtain this decomposition. Note that the direct sums over various d's are orthogonal as they belong to different eigenspaces. We observe that there exists a square-free t, $(t, 2N) = 1$ with $t \equiv 1 \pmod 4$ and $(-1)^k t > 0$ such that $a_f(|t|) \neq 0$, for a newform $f \in S^{+,new}_{k+1/2}(4d)$, where $d|N$. Such property is also true in $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$, for $d|N$. Since the operator L_+ lifts the spaces $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ into $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ respectively, we have the validity of the said property for newforms in $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$, $d|N$. Using this, we obtain the following facts.

$$S^{+,new}_{k+1/2}(4d) \cap S^{+,new}_{k+1/2}(4d)|U(4)B(4) = \{0\}, \qquad d|N, \tag{3.4.8}$$

$$S^{+,new}_{k+1/2}(8d) \cap S^{+,new}_{k+1/2}(8d)|U(4)B(4) = \{0\}, \qquad d|N. \tag{3.4.9}$$

Now we prove that

$$S^{+,new}_{k+1/2}(4d)|U(4)L_+ \cap S^{+,new}_{k+1/2}(4d)|U(4)B(4) = \{0\}, \qquad d|N \tag{3.4.10}$$

Let $f \in S^{+,new}_{k+1/2}(4d)$ be a newform such that $f|U(4)L_+ = \mathbb{C}f|U(4)B(4)$. Applying $U(4)$ both sides, we get $f|U(16) = f|U(4)$. Then by applying the Shimura map, it will lead to

$\mathbb{C}F|U(4) = F|U(2)$, a contradiction, where $F \in S_{2k}(4d)$ is the corresponding newform under the himura map. Now using the Shimura map $S_{t,16N}$ and the theory of newforms of integral weight, we obtain the following fact.

$$S^{+,new}_{k+1/2}(4d)|U(r_1^2) \cap S^{+,new}_{k+1/2}(4d)|U(r_2^2) = \{0\}, \qquad d|4N, r_1, r_2|N, r_1 \neq r_2. \tag{3.4.11}$$

Then the direct sums follows easily using induction on the number of prime factors of N. The fact that the sum

$$S^{+,new}_{k+1/2}(4d) + S^{+,new}_{k+1/2}(4d)|U(4)B(4) + S^{+,new}_{k+1/2}(4d)|U(4)L_+,$$

for each $d|N$ is a direct sum follows in a similar way as was done at the beginning of this section. This completes the proof. □

3.5 Newform theory on $S_{k+1/2}(16N)$

In this section, we extend the theory of newforms to the space $S_{k+1/2}(16N)$, where N is an odd square-free positive integer. Note that the space $S_{k+1/2}(8N)$ is isomorphic to $S_{k+1/2}(8N, (\frac{2}{\cdot}))$, isomorphism given by the operator $W(8)$. We now define the space of oldforms in $S_{k+1/2}(16N)$.

$$\begin{aligned} S^{old}_{k+1/2}(16N) = &\sum_{rd|N} \left(S^{+,new}_{k+1/2}(4d) + S^{+,new}_{k+1/2}(4d)|U(4) + S^{+,new}_{k+1/2}(4d)|U(4)L_+\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{+,new}_{k+1/2}(4d)|U(8)B(2) + S^{+,new}_{k+1/2}(4d)|B(4) + S^{+,new}_{k+1/2}(4d)|U(4)B(4)\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{new}_{k+1/2}(4d) + S^{new}_{k+1/2}(4d)|B(4) + S^{new}_{k+1/2}(4d)|U(2)B(2)\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{+,new}_{k+1/2}(8d) + S^{new}_{k+1/2}(8d) + S^{+,new}_{k+1/2}(16d)\right)|U(r^2). \end{aligned} \tag{3.5.1}$$

We shall show that the above is a direct sum. Clearly, by using the theory of newforms on $S^{+}_{k+1/2}(4d)$, $S_{k+1/2}(4d)$, $S^{+}_{k+1/2}(8d)$, $S_{k+1/2}(8d)$ and $S^{+}_{k+1/2}(16d)$, where $d|N$, the eigenclasses generated by the three newform spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ form a direct sum. Note that the eigenclasses of $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ belong to the eigenclasses of $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ respectively. So, it is enough to show the direct sum within each of the eigenclasses.

In the following cases, let d be a positive divisor of N such that $rd|N$.

Case (i): We first consider the eigenclass $S^{+,new}_{k+1/2}(4d)$. Let $f \in S^{+,new}_{k+1/2}(4d)$ be an eigenform such that

$$\alpha_1 f + \alpha_2 f|U(4) + \alpha_3 f|U(4)L_+ + \alpha_4 f|U(8)B(2) + \alpha_5 f|B(4) + \alpha_6 f|U(4)B(4) = 0, \quad (3.5.2)$$

where $\alpha_i \in \mathbb{C}$. Applying $U(4)$ to the above equation, we get $\alpha_5 = 0$. Then comparing the n^{th} Fourier coefficients, where $(-1)^k n \equiv 3 \pmod 4$, we get $\alpha_2 = 0$. Now comparing the n^{th} coefficients, where $n \equiv 2 \pmod 4$, we get $\alpha_4 = 0$. Thus the above equation reduces to the following equation

$$\alpha_1 f + \alpha_3 f|U(4)L_+ + \alpha_6 f|U(4)B(4) = 0. \quad (3.5.3)$$

Applying $U(4)$ to the above equation, we get $(\alpha_1 f - \alpha_6 f)|U(4) = -\alpha_3 f|U(16)$. This implies that $\alpha_3 = 0$. Thus, we get $\alpha_1 f + \alpha_6 f|U(4)B(4) = 0$. Now comparing the n^{th} coefficients, where $(-1)^k n \equiv 1 \pmod 4$, we get $\alpha_1 = 0 = \alpha_6$. This proves the direct sum within the eigenclass $S^{+,new}_{k+1/2}(4d)$.

Case (ii): We consider the eigenclass $S^{new}_{k+1/2}(4d)$. By comparing the Fourier coefficients and using the fact that there always exists n such that $a_f(n) \neq 0$ with $(-1)^k n \equiv 3 \pmod 4$, where $f = \sum_{n\geq 1} a_f(n) q^n \in S^{new}_{k+1/2}(4d)$, we get the following direct sum.

$$S^{new}_{k+1/2}(4d)|U(r^2) \oplus S^{new}_{k+1/2}(4d)|B(4)U(r^2) \oplus S^{new}_{k+1/2}(4d)|U(2)B(2)U(r^2) \oplus S^{+,new}_{k+1/2}(8d)|U(r^2).$$

Case (iii): Since $S^{new}_{k+1/2}(8d)|L_+ = S^{+,new}_{k+1/2}(16d)$, we see that

$$S^{new}_{k+1/2}(8d)|U(r^2) \cap S^{+,new}_{k+1/2}(16d)|U(r^2) = \{0\}.$$

This completes the proof of the direct sum decomposition of $S^{old}_{k+1/2}(16N)$.

Since the spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$, $S^{+,new}_{k+1/2}(8d)$, $S^{new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ are isomorphic (under the Shimura correspondence) to the spaces $S^{new}_{2k}(d)$, $S^{new}_{2k}(2d)$, $S^{new}_{k+1/2}(2d)$, $S^{new}_{2k}(4d)$ and $S^{new}_{2k}(4d)$ respectively, we see that

$$\begin{aligned}\dim S^{old}_{k+1/2}(16N) &= \sum_{rd|N} \left(6 \dim S^{new}_{2k}(d) + 4 \dim S^{new}_{2k}(2d) + 2 \dim S^{new}_{2k}(4d)\right) \\ &= 2 \sum_{rd|N} \left(3 \dim S^{new}_{2k}(d) + 2 \dim S^{new}_{2k}(2d) + \dim S^{new}_{2k}(4d)\right) \quad (3.5.4) \\ &= 2 \dim S_{2k}(4N).\end{aligned}$$

Let us now compute the dimensions of the spaces $S_{2k}(4N)$ and $S_{k+1/2}(16N)$. Using [33], we have

$$\dim S_{2k}(4N) = \frac{2k-1}{12} 4N \prod_{p|2N} \left(1 + \frac{1}{p}\right) - \frac{3}{2} 2^{\nu(N)} = \frac{(2k-1)}{2} \prod_{p|N} (p+1) - 3 \cdot 2^{\nu(N)-1}, \quad (3.5.5)$$

where $\nu(N)$ is the number of prime factors of N. Now, using [7], we get

$$\begin{aligned}\dim S_{k+1/2}(16N) &= \frac{2k-1}{24}16N\prod_{p|2N}\left(1+\frac{1}{p}\right) - \frac{\zeta(k,16N,1)}{2}\prod_{p|N}\lambda(r_p,s_p,p)\\ &= (2k-1)\prod_{p|N}(p+1) - 3\cdot 2^{\nu(N)}.\end{aligned} \tag{3.5.6}$$

(In the above we have used the dimension formula as given in [38][Theorem 1.56, p.16]. Note that $\zeta(k,16N,1) = \lambda(r_2,s_2,2) = 6$ and $\lambda(r_p,s_p,p) = 2$, for $p|N$.) Equations (3.5.5) and (3.5.6) imply that $\dim S_{k+1/2}(16N) = 2\dim S_{2k}(4N)$. However, from (3.5.4), $\dim S^{old}_{k+1/2}(16N) = 2\dim S_{2k}(4N)$. Therefore, it follows that $S^{new}_{k+1/2}(16N) = \{0\}$.

Summarizing, we have proved the following theorem.

Theorem 3.5.1. *(i) For N odd and square-free, the space $S_{k+1/2}(16N)$ contains no newforms in it and the whole space is obtained upon duplicating forms in $S_{k+1/2}(8N)$.*

(ii) The space $S_{k+1/2}(16N)$ has the following decomposition:

$$\begin{aligned}&S_{k+1/2}(16N)\\ &= \oplus_{rd|N}\Big(S^{+,new}_{k+1/2}(4d)\oplus S^{+,new}_{k+1/2}(4d)|U(4)\oplus S^{+,new}_{k+1/2}(4d)|U(4)L_+\oplus S^{+,new}_{k+1/2}(4d)|U(8)B(2)\\ &\qquad \oplus S^{+,new}_{k+1/2}(4d)|B(4)\oplus S^{+,new}_{k+1/2}(4d)|U(4)B(4)\Big)|U(r^2)\\ &\oplus\oplus_{rd|N}\Big(S^{new}_{k+1/2}(4d)\oplus S^{new}_{k+1/2}(4d)|B(4)\oplus S^{new}_{k+1/2}(4d)|U(2)B(2)\oplus S^{+,new}_{k+1/2}(8d)\Big)|U(r^2)\\ &\qquad \oplus\oplus_{rd|N}\Big(S^{new}_{k+1/2}(8d)\oplus S^{+,new}_{k+1/2}(16d)\Big)|U(r^2).\end{aligned} \tag{3.5.7}$$

(iii) For square-free integer t with $(-1)^k t > 0$, the Shimura map $\mathscr{S}_{t,16N}$ has the following mapping property:

$$\mathscr{S}_{t,16N} : S_{k+1/2}(16N) \longrightarrow S_{2k}(4N).$$

There is a linear combination of Shimura maps which is an isomorphism between the spaces $S_{k+1/2}(16N)$ and $S_{2k}(4N)$

3.6 Newform theory on $S_{k+1/2}(16N,\chi_8)$

In this section, we study the theory of newforms on $S_{k+1/2}(16N,\chi_8)$, where χ_8 is the even quadratic character modulo 8, and N is an odd square-free positive integer. Define the space of oldforms in $S_{k+1/2}(16N, \chi_8)$ as follows.

$$\begin{aligned} S^{old}_{k+1/2}(16N,\chi_8) = &\sum_{rd|N}\left(S^{+,new}_{k+1/2}(4d)|B(2) + S^{+,new}_{k+1/2}(4d)|U(2)\right)U(r^2) \\ &+\sum_{rd|N}\left(S^{+,new}_{k+1/2}(4d)|U(8) + S^{+,new}_{k+1/2}(4d)|U(8)W(8)B(2)\right)U(r^2) \\ &+\sum_{rd|N}\left(S^{new}_{k+1/2}(4d)|U(2) + S^{new}_{k+1/2}(4d)|B(2)\right)U(r^2) \\ &+\sum_{rd|N} S^{+,new}_{k+1/2}(8d)|B(2)U(r^2) + \sum_{rd|N} S^{new}_{k+1/2}(8d)|B(2)U(r^2) \\ &+\sum_{rd|N} S^{new}_{k+1/2}(8d)|W(8)U(r^2) + \sum_{rd|N, d<N} S^{new}_{k+1/2}(16d,\chi_8)|U(r^2). \end{aligned} \tag{3.6.1}$$

We shall show that the above is a direct sum. As before, we only have to show the direct sum within each of the eigenclasses.

Case (i): We consider the eigenclass of $S^{+,new}_{k+1/2}(4d)$. Let $f \in S^{+,new}_{k+1/2}(4d)$ be an eigenform and suppose that

$$\alpha_1 f|B(2) + \alpha_2 f|U(2) + \alpha_3 f|U(8) + \alpha_4 f|U(8)W(8)B(2) = 0. \tag{3.6.2}$$

Comparing the n^{th} Fourier coefficients when $(-1)^k n \equiv 3 \pmod 4$, we see that $\alpha_3 = 0$. Now applying the operator $U(2)W_*(4)$ to the above equation and using the facts that $W(8)W_*(4) = U(2)$ on $S_{k+1/2}(8d)$, $W_*(4) = W(4)$ on $S_{k+1/2}(4d)$, $U(4) = 2^k W(4)$ on $S^+_{k+1/2}(4d)$ and $W(4)^2$ is the identity operator on $S_{k+1/2}(4d)$, we conclude that $\alpha_2 = 0$. Thus, the above equation reduces to

$$2^{-k}\alpha_1 f|U(4) + \alpha_4 f|U(16) = 0.$$

The above equality is true only when $\alpha_1 = 0 = \alpha_4$.

Case (ii): We consider the eigenclass of $S^{new}_{k+1/2}(4d)$. Let $f = \sum_{n\geq 1} a_f(n)q^n \in S^{new}_{k+1/2}(4d)$ be an eigenform such that

$$\alpha_1 f|U(2) + \alpha_2 f|B(2) + \alpha_3 f|L_+B(2) = 0. \tag{3.6.3}$$

Applying the operator $U(2)$ on both the sides, we get $\alpha_3 = 0$. Now comparing the odd Fourier coefficients, we see that $a_f(n) = 0$ for $n \equiv 2 \pmod 4$. This implies that f is in the plus space, giving a contradiction. Hence $\alpha_1 = 0$, which implies that $\alpha_2 = 0$.

Case (iii): We consider the eigenclass of $S^{new}_{k+1/2}(8d)$. By applying the operator $U(2)$ to both the spaces $S^{new}_{k+1/2}(8d)|B(2)$ and $S^{new}_{k+1/2}(8d)|W(8)$ and using the fact that $W(8)U(2) =$

$W_*(4)$ maps $S_{k+1/2}(8d)$ into $S_{k+1/2}(4d)$, we get $S^{new}_{k+1/2}(8d)|B(2) \cap S^{new}_{k+1/2}(8d)|W(8) = \{0\}$. We have thus shown that the following decomposition of the space of oldforms in $S_{k+1/2}(16N, \chi_8)$ is direct.

$$\begin{aligned} S^{old}_{k+1/2}(16N, \chi_8) = {} & \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(2r^2) \\ & \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8)W(8)B(2)U(r^2) \\ & \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|U(2r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|B(2)U(r^2) \\ & \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(8d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|B(2)U(r^2) \\ & \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|W(8)U(r^2) \oplus \oplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(16d, \chi_8)|U(r^2). \end{aligned} \tag{3.6.4}$$

Define the space of newforms in $S_{k+1/2}(16N, \chi_8)$ to be the orthogonal complement (with respect to the Petersson scalar product) of $S^{old}_{k+1/2}(16N, \chi_8)$ in $S_{k+1/2}(16N, \chi_8)$. It is already known that the spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(8d)$ are isomorphic (respectively) to $S^{new}_{2k}(d)$, $S^{new}_{2k}(2d)$, $S^{new}_{2k}(4d)$. Using induction on the number of prime factors of N, it follows that the space $S^{new}_{k+1/2}(16d, \chi_8)$ is isomorphic to $S^{new}_{2k}(8d)$, $d|N$ and $d < N$. Now, comparing the dimension of the space $S^{old}_{2k}(8N)$, we see that the spaces $S^{old}_{k+1/2}(16N, \chi_8)$ and $S^{old}_{2k}(8N)$ have equal dimension. In [45], Ueda has shown that the spaces $S_{k+1/2}(16N, \chi_8)$ and $S_{2k}(8N)$ are isomorphic when N is odd and square-free. Therefore, combining all these facts, it follows that the space $S^{new}_{k+1/2}(16N, \chi_8)$ is isomorphic to $S^{new}_{2k}(8N)$.

Theorem 3.6.1. *Let N be an odd square-free natural number and let χ_8 denote the even quadratic character modulo 8. Then the space $S_{k+1/2}(16N, \chi_8)$ can be decomposed as*

$$S_{k+1/2}(16N, \chi_8) = S^{new}_{k+1/2}(16N, \chi_8) \oplus S^{old}_{k+1/2}(16N, \chi_8),$$

where the above directsum is orthogonal with respect to the Petersson product and the decomposition of $S^{old}_{k+1/2}(16N, \chi_8)$ is given by (3.6.4). Moreover, the spaces $S^{new}_{k+1/2}(16N, \chi_8)$ and $S^{new}_{2k}(8N)$ are isomorphic under a linear combination of Shimura maps.

Chapter 4

Products of Eigenforms

4.1 Introduction

The space of modular forms of a fixed weight on the full modular group $SL_2(\mathbb{Z})$ has a basis of simultaneous eigenvectors for all Hecke operators. A modular form is called an eigenform if it is a simultaneous eigenvector for all Hecke operators. A natural question to ask is that whether the product of two eigenforms (may be of different weights) is an eigenform. This question was considered by W. Duke [8] and E. Ghate [10]. They proved that there are only finitely many cases where this phenomenon happens. Later, a more general question in the case of Rankin-Cohen bracket of two eigenforms was studied by D. Lanphier and R. Takloo-Bighash [25]. In this case also, they showed that except for a finitely many cases the Rankin-Cohen brackets of two eigenforms is not an eigenform. Recently, J. Beyerl, K. James, C. Trentacoste and H. Xue [4] have proved that this phenomenon extends to a certain class of nearly holomorphic modular forms. More explicitly, they have proved that there is only one additional case apart from the cases listed in [8] and [10] for which the product of two nearly holomorphic eigenforms of certain type is a nearly holomorphic eigenform.

In this chapter, we extend this result for a few more types of modular forms. First, we consider the case of quasimodular eigenforms and then the case of Rankin-Cohen brackets of almost everywhere (in short a.e.) eigenforms, the latter case generalizes the work of Ghate [11] on products of a.e. eigenforms. Finally, we consider the case of products of nearly holomorphic eigenforms. The results of this chapter are contained in [34].

4.2 Preliminaries

(a) Quasimodular forms:

Definition 4.2.1. (Quasimodular form of weight k and depth s) Let $k \geq 2$, $s \geq 0$ be integers. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is defined to be a quasimodular form of weight k, depth s on $SL_2(\mathbb{Z})$, if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j,$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and f_s is holomorphic at infinity and not identically vanishing. By convention, the zero function is a quasimodular form of depth 0 for each weight.

The concept of quasimodular form was first introduced by M. Kaneko and D. Zagier in [14].

Remark 4.2.1. It is a fact that if f is a quasimodular form of weight k and depth s on $SL_2(\mathbb{Z})$, not identically zero, then k is even and $s \leq k/2$.

A fundamental quasimodular form is the Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \quad q = e^{2\pi i z}.$$

It is a quasimodular form of weight 2 and depth 1 on $SL_2(\mathbb{Z})$. It is also easy to see that any modular form of weight k is a quasimodular form of weight k and depth 0.

The vector space of all quasimodular forms of weight k and depth $\leq s$ on $SL_2(\mathbb{Z})$ is denoted by $\widetilde{M}_k^{\leq s}$. Let $\widetilde{M}_k = \cup_s \widetilde{M}_k^{\leq s}$ and $\widetilde{M}_* = \oplus_k \widetilde{M}_k$ denote the space of quasimodular forms of weight k and the graded ring of all quasimodular forms on $SL_2(\mathbb{Z})$ respectively. Then it is known that $\widetilde{M}_* = \mathbb{C}[E_2, E_4, E_6]$ and the differential operator $D = \frac{1}{2\pi i}\frac{d}{dz}$ takes $\widetilde{M}_k$ to $\widetilde{M}_{k+2}$. It can also be seen that if f is a modular form, then Df is not a modular form. But using the Eisenstein series E_2 and Df, one can construct a new modular form. In fact, if $f \in M_k$ then we see that

$$Df - \frac{k}{12} E_2 f \in M_{k+2}.$$

It is a cusp form iff f is so. For details on quasimodular forms we refer to [6] and [14].

For an integer $n \geq 1$, The Hecke operator T_n on $\widetilde{M}_k$ is defined in a similar way as we define in the case of modular forms. For any $f \in \widetilde{M}_k$, T_n acts on f by

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right). \tag{4.2.1}$$

It is a fact that T_n preserves $\widetilde{M_k}$. Let $f \in \widetilde{M_k}$. We say that f is an eigenform if it is an eigenvector for all Hecke operators T_n, $n \geq 1$.

(b) Almost everywhere (a.e.) eigenforms:

Let $M_k(\Gamma_1(N))$ (respectively $M_k(N,\chi)$), $S_k(\Gamma_1(N))$, (respectively $S_k(N,\chi)$) and $\mathcal{E}_k(\Gamma_1(N))$ (respectively $\mathcal{E}_k(N,\chi)$) be the spaces of modular forms, cusps forms and Eisenstein series of weight $k \geq 1$ on $\Gamma_1(N)$ (respectively on $\Gamma_0(N)$ with character χ) respectively. The space $\mathcal{E}_k(\Gamma_1(N))$ is orthogonal to $S_k(\Gamma_1(N))$ in the space $M_k(\Gamma_1(N))$ with respect to the Petersson inner product. In other words,

$$M_k(\Gamma_1(N)) = S_k(\Gamma_1(N)) \oplus \mathcal{E}_k(\Gamma_1(N)).$$

Each of the above spaces can also be decomposed as follows.

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N,\chi), \tag{4.2.2}$$

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(N,\chi), \tag{4.2.3}$$

$$\mathcal{E}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{E}_k(N,\chi), \tag{4.2.4}$$

where the direct sum varies over all Dirichlet characters χ modulo N.

Each of the spaces in the above decomposition has an explicit basis consisting of modular forms that are eigenvectors of all the Hecke operators T_n with n coprime to N. For $S_k(N,\chi)$, this basis is constructed from newforms of level M, $M|N$. Let M be a positive integer divisible by the conductor $c(\chi)$ of χ. We say that $f \in S_k(M,\chi)$ is primitive if it is a normalized newform, i.e., it is an eigenvector of all the Hecke operators of level M, it is in the newform space $S_k^{new}(M,\chi)$ and the first coefficient of the Fourier expansion of f is 1. Then the set of cusp forms

$$\bigcup_{M}\bigcup_{Q} \{f(Qz)|\ f \text{ is a primitive form of } S_k(M,\chi)\},$$

where M varies through all positive integers satisfying $M|N$ and $c(\chi)|M$, and Q varies through all positive integers dividing N/M, forms a basis of $S_k(N,\chi)$ consisting of common eigenforms of all Hecke operators T_n with n coprime to N. A basis for $\mathcal{E}_k(N,\chi)$ consisting of modular forms that are eigenvectors of all Hecke operators T_n with n coprime to N is given in the following theorem. We refer to Theorems 4.7.1 and 4.7.2 of [35] for more details.

Theorem 4.2.1. *Let χ_1 and χ_2 be Dirichlet characters modulo M_1 and M_2 respectively, such that $\chi_1\chi_2(-1) = (-1)^k$ with $k \geq 1$. Also assume that*

- *if $k = 2$ and both χ_1 and χ_2 are principal characters, then $M_1 = 1$ and M_2 is a prime number,*
- *otherwise, χ_1 and χ_2 are primitive characters.*

Put $M = M_1M_2$ and $\chi = \chi_1\chi_2$. Then there is an element $f = f_k(z, \chi_1, \chi_2) = \sum_{n=0}^{\infty} a_n q^n$ such that $L(s, f) = L(s, \chi_1)L(s - k + 1, \chi_2)$,

$$a_o = \begin{cases} 0 & \text{if } k = 1, \text{ and } \chi_1 \text{ is non-principal,} \\ & \text{or if both } \chi_1 \text{ and } \chi_2 \text{ are non-principal,} \\ (M-1)/24 & \text{if } k = 2, \text{ and both } \chi_1 \text{and } \chi_2 \text{ are non-principal,} \\ -B_{k,\chi}/(2k) & \text{otherwise,} \end{cases}$$

where $B_{k,\chi}$ is the generalized Bernoulli number associated with a primitive Dirichlet character χ of conductor $c(\chi)$, defined by

$$\sum_{a=1}^{c(\chi)} \frac{\chi(a)te^{at}}{e^{c(\chi)t} - 1} = \sum_{k=0}^{\infty} \frac{B_{k,\chi}}{k!} t^k,$$

and $a_n = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}$, for $n \geq 1$.

The modular form $f_k(z, \chi_1, \chi_2) \in \mathcal{E}_k(M, \chi)$ is an eigenvector for the Hecke operators of level M. Modulo the relation $f_1(z, \chi_1, \chi_2) = f_1(z, \chi_2, \chi_1)$ when $k = 1$, the set of elements $f_k(Qz, \chi_1, \chi_2)$, where $QM_1M_2|N$ form a basis of $\mathcal{E}_k(N, \chi)$ consisting of common eigenforms of all the Hecke operators T_n of level N, with $(n, N) = 1$.

Let $\mathcal{B}$ denote the explicit basis of $M_k(\Gamma_1(N))$ obtained as above. An element of $M_k(\Gamma_1(N))$ is called an almost everywhere eigenform (or a.e. eigenform in short) if, upto a scalar multiple, it is an element of $\mathcal{B}$.

For $k > 2$ and ψ, a Dirichlet character modulo N, we define the Eisenstein series

$$E_k^{(N,\psi)}(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \overline{\psi}(d)(cz + d)^{-k}, \tag{4.2.5}$$

where $z \in \mathcal{H}$ and the sum varies over all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ modulo $\Gamma_\infty = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} |n \in \mathbb{Z}\}$. It is known that $E_k^{(N,\psi)} \in \mathcal{E}_k(N, \psi)$ ([35]).

Atkin-Lehner Involutions

Let $N \geq 1$ be an arbitrary integer and let $Q|N$ be such that $(Q, N/Q) = 1$. Then the Atkin-Lehner involution W_Q is defined by

$$W_Q = \begin{pmatrix} Qx & y \\ Nv & Qw \end{pmatrix},$$

where $x, y, v, w \in \mathbb{Z}$ satisfying $x \equiv 1 \pmod{N/Q}$, $y \equiv 1 \pmod{Q}$ and the determinant of W_Q is Q. W_Q acts on the space $M_k(N, \chi)$ in the usual way, i.e., if $f \in M_k(N, \chi)$, then

$$f|W_Q(z) = \det(W_Q)^{k/2}(Nvz + Qw)^{-k} f\left(\frac{Qxz + y}{Nvz + Qw}\right).$$

Now, let us decompose $\chi = \chi_Q \chi_{N/Q}$ into its Q and N/Q parts. It is well known (see Proposition 1.1 of [3]) that W_Q maps

$$W_Q : M_k(N, \chi_Q \chi_{N/Q}) \to M_k(N, \overline{\chi}_Q \chi_{N/Q})$$

and that it takes cusp forms to cusp forms. Moreover, since W_Q commutes, up to a constant, with T_n for all n with $(n, N) = 1$, it takes a.e. eigenforms to a.e. eigenforms (see Proposition 1.2 of [3]). In fact W_Q takes primitive cusp forms to primitive cusp forms (up to multiplication by a constant).

(c) Nearly holomorphic modular forms:

Definition 4.2.2. (Nearly holomorphic modular form) A nearly holomorphic modular form $F(z)$ of weight k on $SL_2(\mathbb{Z})$ is a polynomial in $\frac{1}{Im(z)}$ whose coefficients are holomorphic functions on $\mathcal{H}$ such that

$$(cz + d)^{-k} F\left(\frac{az + b}{cz + d}\right) = F(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The function E_2^* defined by

$$E_2^*(z) = E_2(z) - \frac{3}{\pi Im(z)}$$

is a nearly holomorphic modular form of weight 2. The space of nearly holomorphic modular forms of weight k on $SL_2(\mathbb{Z})$ is denoted by $\widehat{M_k}$. Any modular form of weight k is trivially a nearly holomorphic modular form of weight k.

Definition 4.2.3. (Maass-Shimura operator) The Maass-Shimura operator δ_k on $f \in \widehat{M_k}$ is defined by

$$\delta_k(f) = \left(\frac{1}{2\pi i} \left(\frac{k}{2iIm(z)} + \frac{\partial}{\partial z} \right) f \right)(z).$$

It is known that δ_k maps $\widehat{M_k}$ to $\widehat{M}_{k+2}$. In particular δ_k maps M_k to $\widehat{M}_{k+2}$. For details on nearly holomorphic modular forms we refer to [43]. The Hecke operators T_n on $\widehat{M_k}$ is defined in the same way as in (4.2.1). A nearly holomorphic modular form is called an eigenform if it is an eigenform for all Hecke operators T_n, where $n \in \mathbb{N}$.

4.3 Overview of Earlier Works

For $k \in \{12, 16, 18, 20, 22, 26\}$, let Δ_k denote the unique normalized cusp form of weight k on $SL_2(\mathbb{Z})$.

Theorem 4.3.1. (*Duke [8], Ghate [10]*) *The product of two eigenforms on $SL_2(\mathbb{Z})$ is not an eigenform except for the following cases.*

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14},$$
$$E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$$
$$E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$$
$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}.$$

Definition 4.3.1. (Rankin-Cohen bracket) Let $f \in M_{k_1}(N, \chi)$ and $g \in M_{k_2}(N, \psi)$. Then the m^{th} Rankin-Cohen bracket of f and g is given by,

$$[f, g]_m(z) = \sum_{r+s=m} (-1)^r \binom{m+k_1-1}{s} \binom{m+k_2-1}{r} f^{(r)}(z) g^{(s)}(z),$$

where $f^{(r)}(z) = D^r f(z)$ and $g^{(s)}(z) = D^s g(z)$.

It is known that $[f, g]_m \in M_{k_1+k_2+2m}(N, \chi\psi)$ and $[f, g]_m$ is a cusp form if $m \geq 1$.

Theorem 4.3.1 was generalized to Rankin-Cohen brackets of eigenforms by Lanphier and Takloo-Bighash [25]. They proved the following theorem.

Theorem 4.3.2. (*Lanphier–Takloo-Bighash, [25]*) *There are only a finite number of triples (f, g, m) with the property that f and g are normalized eigenforms on $SL_2(\mathbb{Z})$ and $[f, g]_m$ is again an eigenform. The following describes all the possibilities.*

1. $[E_4, E_6]_0 = E_{10}$ *and* $[E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}$.

2. *If* $k, l \in \{4, 6, 8, 10, 14\}$ *and* $m \geq 1$ *with* $k + l + 2m \in \{12, 16, 18, 20, 22, 26\}$, *then*
$$[E_k, E_l]_m = c_m(k, l)\Delta_{k+l+2m},$$
where
$$c_m(k, l) = -\frac{2l}{B_l}\binom{m + l - 1}{m} + (-1)^{m+1}\frac{2k}{B_k}\binom{m + k - 1}{m}.$$

3. *If* $k \in \{4, 6, 8, 10, 14\}$ *and* $m \geq 0$ *with* $l, k + l + 2m \in \{12, 16, 18, 20, 22, 26\}$, *then*
$$[E_k, \Delta_l]_m = c_m(l)\Delta_{k+l+2m}, \quad \textit{where} \quad c_m(l) = \binom{m + l - 1}{m}.$$

In [11], Ghate considered the product of two a.e. eigenforms of square-free level. More precisely, he proved the following theorems.

Theorem 4.3.3. (*Ghate, [11]*) *Let* $k_1, k_2 \geq 1$ *be integers and* $N \geq 1$ *be a square-free integer. If* $g \in S_{k_1}(\Gamma_1(N))$ *and* $h \in S_{k_2}(\Gamma_1(N))$ *are a.e. eigenforms, then* gh *is not an a.e. eigenform.*

Theorem 4.3.4. (*Ghate, [11]*) *Let* $k_1, k_2 \geq 3$ *be integers,* $k = k_1 + k_2$*,* $g \in S_{k_2}(N, \psi)$ *and* $h \in \mathcal{E}_{k_1}(N, \chi)$ *be a.e. eigenforms. Assume that* N *is square-free and* g *is a newform. If* $\dim(S_k^{new}(N, \psi\chi)) \geq 2$ *then* gh *is not an a.e. eigenform.*

Theorem 4.3.5. (*Ghate, [11]*) *Let* $k_1, k_2 \geq 3$ *be integers and* $k = k_1 + k_2$*. Let* $g = f_{k_2}(z, \psi_1, \psi_2) \in \mathcal{E}_{k_2}(N, \psi)$ *and* $h = f_{k_1}(z, \chi_1, \chi_2) \in \mathcal{E}_{k_1}(N, \chi)$ *be eigenforms as in Theorem 4.2.1 with* ψ *and* χ *primitive. Assume also that* N *is square-free and* $k_2 \neq k/2$*. If*
$$\dim(S_k^{new}(N, \psi\chi)) \geq \begin{cases} 1 & \textit{when } gh \in \mathcal{E}_k(N, \psi\chi), \\ 2 & \textit{when } gh \in S_k(N, \psi\chi), \end{cases}$$
then gh *is not an a.e. eigenform.*

Recently, Beyerl et al.[4] have proved that in the case of a certain class of nearly holomorphic modular forms, the product of any two nearly holomorphic eigenforms is not an eigenform, except for a finitely many explicitly given cases.
For $r \geq 1$, let $\delta_k^{(r)} := \delta_{k+2r-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k$ and assume that $\delta_k^{(0)}$ is the identity map.

Theorem 4.3.6. (*Beyerl et al., [4]*) *Let* $f \in M_k$ *and* $g \in M_l$ *be such that* $\delta_k^{(r)}(f)$ *and* $\delta_l^{(s)}(g)$ *are eigenforms. Then* $\delta_k^{(r)}(f)\delta_l^{(s)}(g)$ *is an eigenform only in the following cases:*

(1) $E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14}, \quad E_4\Delta_{12} = \Delta_{16},$
$E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20}, \quad E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$
$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}.$

(2) $\delta_4(E_4)E_4 = \frac{1}{2}\delta_8(E_8)$

Note: We remark that the cases mentioned in case (1) above are the same modular cases given in [8] and [10].

4.4 Main Results

(a) Quasimodular forms:

Theorem 4.4.1. *Let $f \in M_k$, $g \in M_l$. For $r, s \geq 0$, assume that $D^r f \in \widetilde{M}_{k+2r}$, $D^s g \in \widetilde{M}_{l+2s}$ are eigenforms. Then $(D^r f)(D^s g)$ is an eigenform only in the following cases.*

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14},$$
$$E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$$
$$E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$$
$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26},$$
$$(DE_4)E_4 = \frac{1}{2}DE_8.$$

Theorem 4.4.2. *Let $f \in \widetilde{M}_k^{\leq p}$ and $g \in \widetilde{M}_l^{\leq q}$ be eigenforms. Assume that $p < k/2$ and $q < l/2$, then except for the cases mentioned in Theorem 4.4.1, fg is not an eigenform.*

Since $\widetilde{M}_2$ is generated by E_2, E_2 is an eigenform.

Remark 4.4.1. Following the same proof as in the case of M_k, one can prove that a quasimodular form in $\widetilde{M}_k$ with non-zero constant Fourier coefficient is an eigenform iff $f \in \mathbb{C}E_k$. Also f is an eigenform iff $D^r f$ is an eigenform (See Proposition 4.5.3).

In the following we consider product of derivatives of E_k.

Theorem 4.4.3. *For $k \geq 2$ and $r, s \geq 0$, $(D^r E_2)(D^s E_k)$ is not an eigenform.*

As a consequence of the above theorem, we get the following corollary.

Corollary 4.4.4. *Let $f \in M_k$ be an eigenform. Then $(D^r E_2)f$ is an eigenform iff $r = 0$ and $f \in \mathbb{C}\Delta_{12}$.*

(b) Almost everywhere (a.e.) eigenforms:
We now state the results which generalize the results of Ghate (Theorem 4.3.3, 4.3.4, 4.3.5) for the Rankin-Cohen brackets.

Theorem 4.4.5. *Let $k_1, k_2 \geq 1$ be integers and $N \geq 1$ be a square-free integer and $m \geq 1$. If $g \in S_{k_1}(\Gamma_1(N))$ and $h \in S_{k_2}(\Gamma_1(N))$ are a.e. eigenforms, then $[g,h]_m$ is not an a.e. eigenform.*

Theorem 4.4.6. *Let k_1, k_2, k, m be positive integers such that $k = k_1 + k_2 + 2m$ and N be a square-free integer. Then we have the following.*

1. *For $k_1 \geq 3$ and $k_2 \geq k_1 + 2 > 2$, suppose that $g \in S_{k_1}(N,\chi)$ is an a.e. eigenform which is a newform and $h \in \mathcal{E}_{k_2}(N,\psi)$. If $\dim(S_k^{new}(N,\chi\psi)) \geq 2$, then $[g,h]_m$ is not an a.e. eigenform.*

2. *For $k_1, k_2 \geq 3$, $|k_1 - k_2| \geq 2$, let $g = f_{k_1}(z,\chi_1,\chi_2) \in \mathcal{E}_{k_1}(N,\chi)$ and $h = f_{k_2}(z,\psi_1,\psi_2) \in \mathcal{E}_{k_2}(N,\psi)$ be a.e. eigenforms as described earlier with χ and ψ being primitive. If $\dim(S_k^{new}(N,\chi\psi)) \geq 2$, then $[g,h]_m$ is not an a.e. eigenform.*

(c) Nearly holomorphic modular forms:
In the case of nearly holomorphic modular forms, we prove the following theorem.

Theorem 4.4.7. *Let $f \in M_k$ be an eigenform. Then $E_2^* f$ is an eigenform iff $f = \mathbb{C}\Delta_{12}$.*

4.5 Proofs

4.5.1 Proof of Theorem 4.4.1

Before we proceed to the proof of the theorem, we present some preliminary results which will be needed in the proof. Recall the following proposition from [4].

Proposition 4.5.1. *(Beyerl et al., [4]) Let $f \in M_k$ and $g \in M_l$. Then*

$$\delta_k^{(r)}(f)\delta_l^{(s)}(g) = \sum_{j=0}^{r+s} \frac{1}{\binom{k+l+2j-2}{j}} \left(\sum_{m=max(j-r,0)}^{s} (-1)^{j+m} \frac{\binom{s}{m}\binom{r+m}{j}\binom{k+r+m-1}{r+m-j}}{\binom{k+l+r+m+j-1}{r+m-j}} \right) \delta_{k+l+2j}^{(r+s-j)}([f,g]_j(z)).$$

Both sides of the equality in the above proposition are polynomials in $\frac{1}{Im(z)}$, where the coefficients are holomorphic functions on the upper half plane $\mathcal{H}$. Comparing the constant coefficients on both the sides of the equality, we get the following corollary.

Corollary 4.5.2. *Let $f \in M_k$ and $g \in M_l$. Then*

$$D^r(f)D^s(g) = \sum_{j=0}^{r+s} \frac{1}{\binom{k+l+2j-2}{j}} \left(\sum_{m=max(j-r,0)}^{s} (-1)^{j+m} \frac{\binom{s}{m}\binom{r+m}{j}\binom{k+r+m-1}{r+m-j}}{\binom{k+l+r+m+j-1}{r+m-j}} \right) D^{r+s-j}([f,g]_j(z)).$$

Next, we prove the following proposition which shows the commuting relation between the operators D and T_n.

Proposition 4.5.3. *If $f \in \widetilde{M}_k$, then $(D^r(T_n f))(z) = \frac{1}{n^r}(T_n(D^r f))(z)$, for $r \geq 0$. Moreover, we have $D^r f$ is an eigenform for T_n iff f is. In this case, if λ_n is the eigenvalue of T_n associated to f, then $n^r\lambda_n$ is the eigenvalue of T_n associated to $D^r f$.*

Proof. We have

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

So

$$D(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \frac{1}{2\pi i} \frac{n}{d^2} \frac{df}{dz} \left(\frac{nz+bd}{d^2}\right).$$

Again one computes that

$$T_n(Df)(z) = n \left[n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \frac{1}{2\pi i} \frac{n}{d^2} \frac{df}{dz} \left(\frac{nz+bd}{d^2}\right) \right],$$

from which we obtain

$$D(T_n f)(z) = \frac{1}{n}(T_n(Df)).$$

By using induction on r, we get the first assertion of the proposition.
Now assume that f is an eigenform. So, $(T_n f)(z) = \lambda_n f(z)$. Then applying D^r on both the sides and using the first assertion, we obtain the following:

$$T_n(D^r f)(z) = n^r \lambda_n (D^r f)(z).$$

Hence, $D^r f$ is an eigenform. Conversely, let us assume that $D^r f$ is an eigenform for T_n with eigenvalue β_n. Then $T_n(D^r f)(z) = \beta_n (D^r f)(z)$. Again using the first assertion, we obtain

$$D^r(T_n f)(z) = \frac{1}{n^r} T_n(D^r f)(z) = \frac{\beta_n}{n^r} D^r f(z),$$

$$\Rightarrow D^r(T_n f - \tfrac{\beta_n}{n^r} f) = 0.$$

$$\Rightarrow T_n f - \tfrac{\beta_n}{n^r} f \text{ is a constant.}$$

Since $T_n f - \frac{\beta_n}{n^r} f$ is a quasimodular form and any non-zero constant function is not a quasimodular form of positive weight, we get

$$T_n f(z) = \frac{\beta_n}{n^r} f(z).$$

Hence f is an eigenform, proving the proposition. □

Now we prove a result on a sum of eigenforms with distinct weights.

Proposition 4.5.4. *Suppose that $\{f_i\}_i$ is a collection of modular forms of distinct weights k_i. Then $\sum_{i=1}^{t} a_i D^{(r-\frac{k_i}{2})}(f_i)$, $a_i \in \mathbb{C}^*$ is an eigenform iff each $D^{(r-\frac{k_i}{2})}(f_i)$ is an eigenform and each function has the same set of eigenvalues.*

Proof. We prove the proposition for $t = 2$ and the proof follows by induction. Let $f \in M_k$ and $g \in M_l$ be such that $D^r f$ and $D^{(r+\frac{k}{2}-\frac{l}{2})}(g)$ are eigenforms with same eigenvalues. If $T_n(D^r(f)) = \mu_n D^r(f)$ and $T_n(D^{(r+\frac{k}{2}-\frac{l}{2})}(g)) = \mu_n D^{(r+\frac{k}{2}-\frac{l}{2})}(g)$, then by linearity of T_n,

$$T_n(D^r(f) + D^{(r+\frac{k}{2}-\frac{l}{2})}(g)) = \mu_n(D^r(f) + D^{(r+\frac{k}{2}-\frac{l}{2})}(g)).$$

Hence, $D^r(f) + D^{(r+\frac{k}{2}-\frac{l}{2})}(g)$ is also an eigenform.
Conversely, suppose that $D^r(f)+D^{(r+\frac{k}{2}-\frac{l}{2})}(g)$ is an eigenform. Then by Proposition 4.5.3 and linearity of D^r, $f + D^{(\frac{k-l}{2})}(g)$ is also an eigenform. Write

$$T_n(f + D^{(\frac{k-l}{2})}(g)) = \lambda_n(f + D^{(\frac{k-l}{2})}(g)).$$

Using the linearity of T_n and Proposition 4.5.3, we get

$$T_n(f) + n^{\frac{k-l}{2}} D^{(\frac{k-l}{2})}(T_n(g)) = \lambda_n f + \lambda_n D^{(\frac{k-l}{2})}(g).$$

Rearranging this, we get

$$T_n(f) - \lambda_n f = D^{(\frac{k-l}{2})}(\lambda_n g - n^{\frac{k-l}{2}} T_n(g)).$$

Now, we see that the left hand side is a modular form and the right hand side is not a modular form since $k \neq l$. Hence both sides must be equal to zero. Thus we have

$$T_n(f) = \lambda_n f \quad \text{and} \quad T_n(g) = \lambda_n n^{\frac{-(k-l)}{2}} g.$$

Therefore, f is an eigenvector for T_n with eigenvalue λ_n and g is an eigenvector for T_n with eigenvalue $\lambda_n n^{\frac{-(k-l)}{2}}$. Then by Proposition 4.5.3, we see that $D^{(\frac{k-l}{2})}(g)$ is an eigenvector for T_n with eigenvalue λ_n. Therefore, f and $D^{(\frac{k-l}{2})}(g)$ are eigenvectors for T_n with eigenvalues λ_n. So $D^r(f)$ and $D^{(\frac{k-l}{2}+r)}(g)$ must have the same eigenvalue with respect to T_n. Hence, for all $n \in \mathbb{N}$, $D^r(f)$ and $D^{(\frac{k-l}{2}+r)}(g)$ must be eigenforms with the same eigenvalues. □

Using the above proposition, we prove the following lemma.

Lemma 4.5.5. *Let $l < k$ and $f \in M_k$, $g \in M_l$ be eigenforms. Then $D^{(\frac{k-l}{2})}(g)$ and f do not have the same eigenvalues.*

Proof. Suppose on the contrary that they have the same eigenvalues. Let g have eigenvalues $\lambda_n(g)$, then by Proposition 4.5.3, f has eigenvalues $n^{\frac{k-l}{2}}\lambda_n(g)$. Thus the Fourier expansion of f is of the following form

$$f(z) = \sum_{n\geq 1} cn^{\frac{k-l}{2}}\lambda_n(g)q^n + c_0,$$

for some constants c and c_0. Thus, we have

$$f(z) = \frac{1}{(2\pi i)^{(k-l)/2}}\frac{d^{(k-l)/2}}{dz^{(k-l)/2}}g(z) + c_0.$$

This says that f equals some derivative of g plus some constant. However, from direct computation, we see that this function is not modular, which contradicts that f is modular. □

We shall need a special case of the above lemma.

Corollary 4.5.6. *Let $k > l$ and $f \in M_k$, $g \in M_l$ be eigenforms. Then $D^{(\frac{k-l}{2}+r)}(g)$ and $D^r(f)$ do not have the same eigenvalues.*

Next, we need the following lemma.

Lemma 4.5.7. *Let $D^r(f) \in \widetilde{M}_{k+2r}$, $D^s(g) \in \widetilde{M}_{l+2s}$. In the following cases $[f,g]_m \neq 0$.*
Case 1 : *f a cusp form, g not a cusp form.*
Case 2 : *$f = g = E_k$, m even.*
Case 3 : *$f = E_k$, $g = E_l$, $k \neq l$.*

Proof. Case 1: Let $f = \sum_{n\geq 1} a_nq^n$, $g = \sum_{n\geq 0} b_nq^n$ and let n_0 be the smallest positive integer for which $a_{n_0} \neq 0$. Then a direct computation of the coefficient of q^{n_0} in $[f,g]_m$ yields

$$a_{n_0}b_0(-1)^m\binom{m+k-1}{m} \neq 0.$$

Case 2: In this case, the q-coefficient of $[E_k, E_k]_m$ is

$$2 \cdot \frac{-2k}{B_k}\binom{m+k-1}{m},$$

which is non-zero.

Case 3: Without loss of generality, let us assume that $k > l$. Then the coefficient of q in the Fourier expansion of $[E_k, E_l]$ is

$$\frac{-2l}{B_l}\binom{m+k-1}{m} + (-1)^m \frac{-2k}{B_k}\binom{m+l-1}{m}.$$

If m is even, then both of these terms are non-zero and of same sign, and so the sum is non-zero. If m is odd, then using the fact that $|B_k| > |B_l|$ for $l > 4$ and l even, we get

$$\begin{aligned}\left|\frac{B_k}{k}\binom{m+k-1}{m}\right| &= \left|\frac{(m+k-1)\dots(k+1)B_k}{m!}\right| \\ &> \left|\frac{(m+l-1)\dots(l+1)B_l}{m!}\right| = \left|\frac{B_l}{l}\binom{m+l-1}{m}\right|.\end{aligned} \tag{4.5.1}$$

Thus, for $l > 4$ and m odd, the coefficient of q is non-zero. If $l = 4$, then also (4.5.1) holds true for $m > 1$. For $m = 1$, (4.5.1) simplifies to $|B_k| > |B_l|$, which is true for $(k, l) \neq (8, 4)$. Now if $m = 1$, $k = 8$ and $l = 4$, then $B_k = B_l$ and hence $\frac{-2l}{B_l}\binom{m+k-1}{m} + (-1)^m \frac{-2k}{B_k}\binom{m+l-1}{m} \neq 0$. This proves the lemma. □

We shall need a criterion for an eigenform, given in the following lemma.

Lemma 4.5.8. *If* $f(z) = \sum_{n\geq 0} a_n q^n \in \widetilde{M}_k$ *is a non-zero eigenform, then* $a_1 \neq 0$.

Proof. If T_m is the mth Hecke operator, then the Fourier expansion of $T_m f(z)$ is given by

$$T_m f(z) = \sum_{n\geq 0}\left(\sum_{d|(m,n),d>0} d^{k-1} a_{mn/d^2}\right) q^n. \tag{4.5.2}$$

If f is an eigenform, then for all $m \geq 1$, there exist $\lambda_m \in \mathbb{C}$ such that $T_m f = \lambda_m f$. Substituting in (4.5.2), we get

$$\sum_{n\geq 0}\left(\sum_{d|(m,n),d>0} d^{k-1} a_{mn/d^2}\right) q^n = \lambda_m \sum_{n\geq 0} a_n q^n. \tag{4.5.3}$$

Now, comparing the coefficients of q both the sides in the above equation, we get

$$\lambda_m a_1 = a_m,$$

for all $m \geq 1$.
If $a_1 = 0$, then $a_m = 0$ for all $m \geq 1$, and so $f = a_0$. Since any non-zero constant is not a quasimodular form of positive weight, this implies that $a_0 = 0 = f$. This proves the lemma. □

Finally, we prove a lemma, which is a driving force to prove the theorem.

Lemma 4.5.9. *Let $D^r(f) \in \widetilde{M}_{k+2r}$ and $D^s(g) \in \widetilde{M}_{l+2s}$ be eigenforms. Let us assume that not both f and g are cusp forms. Then in the expansion given in Corollary 4.5.2, either the term involving $[f,g]_{r+s}$ is non-zero, or the term involving $[f,g]_{r+s-1}$ is non-zero.*

Proof. There are three cases.
Case 1: $f = g = E_k$. If $r+s$ is even, then by Lemma 4.5.7, $[f,g]_{r+s} \neq 0$ and from Corollary 4.5.2, we see that the coefficient of $[f,g]_{r+s}$ is non-zero. If $r+s$ is odd, then by Lemma 4.5.7, $[f,g]_{r+s-1}$is non-zero. Now, the coefficient of $[f,g]_{r+s-1}$ is also non-zero, because if it were zero, after simplification we would have $k = -(r+s)+1 \leq 0$, a contradiction.
Case 2: If f is a cusp form and g is not a cusp form, then by Lemma 4.5.7, $[f,g]_{r+s}$ is non-zero and by computing the coefficient, we see that the term involving $[f,g]_{r+s}$ is non-zero.
Case 3: If $f = E_k$, $g = E_l$, $k \neq l$, then again by Lemma 4.5.7, $[f,g]_{r+s} \neq 0$ and thus the term involving $[f,g]_{r+s}$ is non-zero. □

We are now ready to prove Theorem 4.4.1. By Corollary 4.5.2, we have

$$D^r(f)D^s(g) = \sum_{j=0}^{r+s} \alpha_j D^{(r+s-j)}([f,g]_j), \quad \text{for} \quad \alpha_j \in \mathbb{C}.$$

By Proposition 4.5.4, the above sum is an eigenform iff every summand is either an eigenform with a single common eigenvalue or is zero. Again by Corollary 4.5.6, $\alpha_j D^{(r+s-j)}([f,g]_j)$ are always of different eigenvalues for different j. Hence for $D^r(f)D^s(g)$ to be an eigenform, all but one term in the summation must be zero and the remaining non-zero term must be an eigenform.

If both f and g are cusp forms, then by Lemma 4.5.8, $D^r(f)D^s(g)$ is not an eigenform. Otherwise, from Lemma 4.5.9 either the term involving $[f,g]_{r+s}$ or the term involving $[f,g]_{r+s-1}$ is non-zero. By Theorem 4.3.2, this is an eigenform for finitely many cases only. Hence, there are only finitely many f, g, r, s such that $D^r(f)D^s(g)$ becomes an eigenform. So, we have to rule out the remaining cases not mentioned in the theorem.

In particular, consider the cases for which $[f,g]_{r+s}$ is an eigenform. From Theorem 4.3.2, we find that there are 29 cases with g a cusp form and 81 cases with f and g both Eisenstein series. We also must consider the infinite class with $f = g = E_k$ and $r+s$ odd, where $[f,g]_{r+s} = 0$.

For the last case when $f = g = E_k$ and $r+s$ is odd, we have $[f,g]_{r+s} = 0$. Then by Lemma 4.5.9, the term involving $[f,g]_{r+s-1}$ is non-zero. If $r+s-1 = 0$, then this is covered in the modular cases. Otherwise, $r+s-1 \geq 2$. In each of these cases, one computes that the term involving $[f,g]_0$ is not an eigenform. Thus there are two non-zero terms. Then applying Proposition 4.5.4 and Corollary 4.5.6, we conclude that $D^r(f)D^s(g)$ is not an eigenform.

Now, consider the rest. In the last finitely many cases, we find computationally that there are two non-zero terms involving $[f,g]_0$ and $[f,g]_{r+s}$. Therefore, in these cases also the fact that $D^r(f)D^s(g)$ is not an eigenform follows from Proposition 4.5.4 and Corollary 4.5.6. Below we consider a typical case

$$D(E_4)\cdot D(E_4) = \frac{-1}{45}[E_4,E_4]_2 + 0\cdot D([E_4,E_4]_1) + \frac{10}{45}D^2([E_4,E_4]_0),$$

$$D(E_6)\cdot E_8 = \frac{-1}{14}[E_6,E_8]_1 + \frac{3}{7}D([E_6,E_8]_0),$$

which can not be eigenforms because of the fact that there are multiple terms of different weights.

4.5.2 Proof of Theorem 4.4.2

To prove the theorem, we recall a result given in ([6], Proposition 20 (iii)).

Lemma 4.5.10. *If $p < k/2$, then $\widetilde{M}_k^{\leq p} = \oplus_{r=0}^{p} D^r(M_{k-2r})$.*

Now, if $f \in \widetilde{M}_k^{\leq p}$ and $p < k/2$, then by the above lemma, we have

$$f = \sum_{r=0}^{p} D^r(f_r), \quad \text{for} \quad f_r \in M_{k-2r}.$$

By hypothesis, f is an eigenform. Therefore, applying Proposition 4.5.4 and Corollary 4.5.6, we conclude that $f = D^r(f_r)$ for some r. Similarly, we get $g = D^s(g_s)$ for some $g_s \in M_{l-2s}$. Now applying Theorem 4.4.1, we get the desired result.

4.5.3 Proof of Theorem 4.4.3

We first prove the following proposition.

Proposition 4.5.11. *Let $f \in M_k$ be an eigenform. Then $E_2 f$ is an eigenform if and only if $f \in \mathbb{C}\Delta_{12}$.*

Proof. Since $D\Delta_{12} = E_2\Delta_{12}$, by Proposition 4.5.3, $E_2\Delta_{12}$ is an eigenform.
Conversely, suppose that $E_2 f$ is an eigenform with eigenvalues β_n and

$$f = \sum_{m \geq 0} a_m q^m \in M_k$$

is an eigenform with eigenvalues λ_n. Since $g = Df - \frac{k}{12}E_2 f \in M_{k+2}$, we have

$$T_n(g) = T_n(Df) - \frac{k}{12}T_n(E_2 f) = n\lambda_n Df - \frac{k}{12}n\lambda_n E_2 f + \frac{k}{12}(n\lambda_n - \beta_n)E_2 f \in M_{k+2}. \quad (4.5.4)$$

As $E_2 f$ is not a modular form and $n\lambda_n g = n\lambda_n(Df - \frac{k}{12}E_2 f)$ is a modular form, we have $n\lambda_n = \beta_n$ for all $n \geq 1$. Thus $g = Df - \frac{k}{12}E_2 f \in M_{k+2}$ is an eigenform with eigenvalues $n\lambda_n$. Now suppose that f is a non cusp eigenform, so we may assume that $f = E_k$. This implies that g has to be a non cusp eigenform of weight $k+2$ and hence $g = \alpha E_{k+2}$ for some $\alpha \in \mathbb{C}$. Then applying T_n to $\alpha E_{k+2} = DE_k - \frac{k}{12}E_2 E_k$, we get

$$T_n(\alpha E_{k+2}) = T_n(DE_k - \frac{k}{12}E_2 E_k)$$

$$\Rightarrow \sigma_{k+1}(n) = n\sigma_{k-1}(n).$$

Thus, we get for all $n \geq 1$, $\sigma_{k+1}(n) = n\sigma_{k-1}(n)$, which is not true. Hence f can not be a non cusp eigenform.

If f is a cusp form, without loss of generality we may assume that f is normalized. Let $g = \sum_{m \geq 1} b_m q^m$. Since $b_1 = 1 - \frac{k}{12}$, we have for all $n \geq 1$,

$$b_n = na_n\left(1 - \frac{k}{12}\right). \quad (4.5.5)$$

Now computing the values of b_n from $Df - \frac{k}{12}E_2 f$ in terms of a_n and then substituting in the previous equation, we see that $a_2 = -24$, $a_3 = 252$ and $a_4 = -1472$. These are nothing but the second, third and fourth Fourier coefficients of Δ_{12} respectively. But Theorem 1 of [12] says that if $f_1 = \sum_{n \geq 1} a_n(f_1)q^n$ and $f_2 = \sum_{n \geq 1} a_n(f_2)q^n$ are two cuspidal eigenforms on $\Gamma_0(N)$ of different weights, then there exists $n \leq 4(\log(N) + 1)^2$ such that $a_n(f_1) \neq a_n(f_2)$. Applying this theorem to $f_1 = f$, $f_2 = \Delta_{12}$ and $N = 1$, we conclude that $k = 12$. Thus we have $f = \Delta_{12}$. □

Remark 4.5.1. Since $DE_2 = \frac{E_2^2 - E_4}{12}$ and DE_2, E_4 are eigenforms with different eigenvalues, E_2^2 is not an eigenform.

Now, we are ready to prove Theorem 4.4.3. By Lemma 4.5.8, Proposition 4.5.11 and Remark 4.5.1, we only have to prove in the following cases that it is not an eigenform.

1. $r = 0$ and $s \geq 1$,
2. $r \geq 1$ and $s = 0$.

Let us assume on contrary that $(D^s E_k)E_2$ is an eigenform for $s \geq 1$. Let

$$\frac{-B_k}{2k}(D^s E_k)E_2 = \sum_{n\geq 1} a_n q^n$$

be the normalized form. The first few coefficients in the Fourier expansion of the normalized form are as follows:

$$a_1 = 1,\ a_2 = 2^s\sigma_{k-1}(2) - 24,$$

$$a_3 = 3^s\sigma_{k-1}(3) - 24\{2^s\sigma_{k-1}(2) + 3\},$$
$$a_4 = 4^s\sigma_{k-1}(4) - 24\{3^s\sigma_{k-1}(3) + 3\cdot 2^s\sigma_{k-1}(2) + 4\}.$$

Since $\frac{-B_k}{2k}(D^s E_k)E_2$ is a normalized eigenform, we have

$$a_4 = a_2^2 - 2^{k+2s+1}. \tag{4.5.6}$$

Substituting the values of a_2 and a_4 in the above equation, we get

$$4^s\sigma_{k-1}(4) - 24\{3^s\sigma_{k-1}(3) + 3\cdot 2^s\sigma_{k-1}(2) + 4\} = 2^{2s}\sigma_{k-1}(2)^2 - 48\cdot 2^s\sigma_{k-1}(2) + 576 - 2^{k+2s+1}$$

$$\Rightarrow 24\{3^s(1+3^{k-1}) + 3\cdot 2^s(1+2^{k-1}) + 4\} - 48\cdot 2^s(1+2^{k-1}) + 576 - 3\cdot 2^{k+2s-1} = 0$$
$$\Rightarrow 3^s(1+3^{k-1}) + 3\cdot 2^s(1+2^{k-1}) + 4 - 2^{s+1}(1+2^{k-1}) + 24 - 2^{k+2s-4} = 0$$

$$\Rightarrow 3^s(1+3^{k-1}) + 2^s(1+2^{k-1}) - 2^{k+2s-4} + 28 = 0 \tag{4.5.7}$$

$$\Rightarrow 3^s(1+3^{k-1}) + 2^s + 28 = 2^{k+s-4}(2^s - 2^3). \tag{4.5.8}$$

We also have $a_6 = a_2 a_3$.

$$\Rightarrow 6^s\sigma_{k-1}(6) - 24\{5^s\sigma_{k-1}(5) + 3\cdot 4^s\sigma_{k-1}(4) + 4\cdot 3^s\sigma_{k-1}(3) + 7\cdot 2^s\sigma_{k-1}(2) + 6\}$$
$$= (2^s\sigma_{k-1}(2) - 24)\{3^s\sigma_{k-1}(3) - 24(2^s\sigma_{k-1}(2) + 3)\}$$
$$\Rightarrow 5^s\sigma_{k-1}(5) + 3\cdot 4^s\sigma_{k-1}(4) + 4\cdot 3^s\sigma_{k-1}(3) + 7\cdot 2^s\sigma_{k-1}(2) + 6 - 3^s\sigma_{k-1}(3)$$
$$-2^s\sigma_{k-1}(2)(2^s\sigma_{k-1}(2) + 3) + 24(2^s\sigma_{k-1}(2) + 3) = 0$$

$$\Rightarrow 5^s\sigma_{k-1}(5)+3^{s+1}\sigma_{k-1}(3)+2^{2s+1}\sigma_{k-1}(2)^2+7\cdot 2^{s+2}\sigma_{k-1}(2)-3\cdot 2^{k+2s-1}+78=0. \quad (4.5.9)$$

In the above equalities, we have used the fact that $\sigma_{k-1}(6) = \sigma_{k-1}(2)\sigma_{k-1}(3)$ and $\sigma_{k-1}(4) = \sigma_{k-1}(2)^2 - 2^{k-1}$.

Now if $s \leq 3$, then the left hand side of (4.5.8) is positive, but the right hand side of the equation is non-positive. Thus $s \geq 4$. Now from (4.5.7) , we have

$$3^s\left(\frac{1+3^{k-1}}{4}\right)+2^{s-2}(1+2^{k-1})+7=2^{k+2s-6}. \quad (4.5.10)$$

If $k \equiv 2 \pmod 4$ and s is odd, then $7+3^s\left(\frac{1+3^{k-1}}{4}\right) \equiv 2 \pmod 4$, but the remaining terms of (4.5.10) are divisible by 4, giving a contradiction. If $k \equiv 2 \pmod 4$ and $s \equiv 0 \pmod 4$, then $3^s\left(\frac{1+3^{k-1}}{4}\right)+2^{s-2}(1+2^{k-1})+7 \equiv 0 \pmod 5$, but 5 does not divide 2^{k+2s-6}. This gives a contradiction. If $k \equiv 2 \pmod 4$ and $s \equiv 2 \pmod 4$, then $3^{s+1}\sigma_{k-1}(3)+2^{2s+1}\sigma_{k-1}(2)^2+7\cdot 2^{s+2}\sigma_{k-1}(2)-3\cdot 2^{k+2s-1}+78 \equiv 4 \pmod 5$, but the remaining term of left hand side of (4.5.9) is divisible by 5, giving a contradiction. If $k \equiv 0 \pmod 4$ and s is even or $s \equiv 1 \pmod 4$, then we get a contradiction from (4.5.10) and if $k \equiv 0 \pmod 4$ and $s \equiv 3 \pmod 4$, we get a contradiction from (4.5.9). This proves the theorem in the first case.

Now, we prove the theorem in the second case. Let us assume on contrary that $(D^rE_2)E_k$ is an eigenform for $r \geq 1$. Let $\frac{-1}{24}(D^rE_2)E_k = \sum_{n\geq 1} b_nq^n$ be the normalized eigenform. We have the first few coefficients of the expansion as follows:

$$b_1 = 1,\ b_2 = 3\cdot 2^r - \frac{2k}{B_k},$$

$$b_3 = 4\cdot 3^r - \frac{2k}{B_k}(3\cdot 2^r + \sigma_{k-1}(2)),$$

$$b_4 = 7\cdot 4^r - \frac{2k}{B_k}(4\cdot 3^r + 3\cdot 2^r\sigma_{k-1}(2) + \sigma_{k-1}(3)).$$

Since $\frac{-1}{24}(D^rE_2)E_k$ is a normalized eigenform, we have

$$b_4 = b_2^2 - 2^{k+2r+1}. \quad (4.5.11)$$

Substituting the values of b_2 and b_4 in the above equation, we get

$$7\cdot 4^r - \frac{2k}{B_k}\left(4\cdot 3^r + 3\cdot 2^r\sigma_{k-1}(2) + \sigma_{k-1}(3)\right) = \left(3\cdot 2^r - \frac{2k}{B_k}\right)^2 - 2^{k+2r+1}$$

$$\Rightarrow \left(\frac{2k}{B_k}\right)^2 + \frac{2k}{B_k}\left(4\cdot 3^r + 3\cdot 2^r\sigma_{k-1}(2) + \sigma_{k-1}(3) - 3\cdot 2^{r+1}\right) + 3^2\cdot 2^{2r} - 7\cdot 2^{2r} - 2^{k+2r+1} = 0$$

$$\Rightarrow \left(\frac{2k}{B_k}\right)^2 + \frac{2k}{B_k}\left(4\cdot 3^r + 3\cdot 2^r(1+2^{k-1}) + 1 + 3^{k-1} - 3\cdot 2^{r+1}\right) + 2^{2r+1}(1-2^k) = 0$$

$$\Rightarrow \left(\frac{2k}{B_k}\right)^2 + \frac{2k}{B_k}\left(4\cdot 3^r + 3\cdot 2^r(2^{k-1}-1) + 1 + 3^{k-1}\right) + 2^{2r+1}(1-2^k) = 0$$

$$\Rightarrow \frac{2k}{B_k} = \frac{-b \pm \sqrt{b^2 + 2^{2r+3}(2^k-1)}}{2}, \tag{4.5.12}$$

where

$$b = 4\cdot 3^r + 3\cdot 2^r(2^{k-1}-1) + 1 + 3^{k-1}. \tag{4.5.13}$$

Since $\frac{2k}{B_k}$ is a rational number, $b^2 + 2^{2r+3}(2^k-1)$ is a perfect square. Again, since 2 divides b, $\frac{2k}{B_k}$ is an integer. This implies that $k \in \{2, 4, 6, 8, 10, 14\}$. Since the case when $k = 2$ is considered in the earlier case, we have to consider the cases $k \in \{4, 6, 8, 10, 14\}$.

$\underline{k = 4}$

In this case, $\frac{2k}{B_k} = -240$. Since $\frac{2k}{B_k}$ is negative, from (4.5.12), we get

$-b - \sqrt{b^2 + 2^{2r+3}(2^4-1)} = -480$

$\Rightarrow b^2 + 15\cdot 2^{2r+3} = (b-480)^2$

$\Rightarrow b = 240 - 2^{2r-3}$.

Substituting this value of b in (4.5.13), we get

$$2^{2r-3} + 4\cdot 3^r + 21\cdot 2^r - 212 = 0. \tag{4.5.14}$$

Now, we see that (4.5.14) is not satisfied for any positive integer r, giving a contradiction.

$\underline{k = 6}$

Here $\frac{2k}{B_k} = 504$. Since $\frac{2k}{B_k}$ is positive in this case, from (4.5.12), we get

$-b + \sqrt{b^2 + 2^{2r+3}(2^6-1)} = 1008$

$\Rightarrow b = 2^{2r-2} - 504$.

Substituting this value of b in (4.5.13), we get

$$4^{r-1} - 4\cdot 3^r - 93\cdot 2^r - 748 = 0. \tag{4.5.15}$$

Since the left hand side of the above equation is positive for $r \geq 11$ and is non-zero for any positive integer $r < 11$, we get a contradiction.

$\underline{k = 8}$

Since $\frac{2k}{B_k} = -480$ is negative in this case, we have from (4.5.12)

$-b - \sqrt{b^2 + 2^{2r+3}(2^8-1)} = -960$

$\Rightarrow b = 480 - 17\cdot 2^{2r-4}$.

Then from (4.5.13), we get

$$17\cdot 2^{2r-4} + 4\cdot 3^r + 3\cdot 2^r(2^7-1) + 3^7 + 1 = 480. \tag{4.5.16}$$

Now, the left hand side of the above equation is greater than 480 for any integer $r \geq 1$. Thus, this case is excluded.

$\underline{k = 10}$

Here we have $\frac{2k}{B_k} = 264$ is positive. Then from (4.5.12), we have

$-b + \sqrt{b^2 + 2^{2r+3}(2^{10} - 1)} = 528$

$\Rightarrow b = 31 \cdot 2^{2r-2} - 264.$

Then from (4.5.13), we get

$$31 \cdot 4^{r-1} - 4 \cdot 3^r - 3 \cdot 2^r(2^9 - 1) - (3^9 + 265) = 0. \tag{4.5.17}$$

Here the left hand side of the above equation is positive for any integer $r \geq 8$ and is non-zero for any positive integer $r < 8$. Thus, this case is also excluded.

$\underline{k = 14}$

Here $\frac{2k}{B_k} = 24$ is positive. Then from (4.5.12), we have

$-b + \sqrt{b^2 + 2^{2r+3}(2^{14} - 1)} = 48$

$\Rightarrow b = 5161 \cdot 2^{2r-2} - 24.$

Substituting this value of b in (4.5.13), we get

$$5161 \cdot 4^{r-1} - 4 \cdot 3^r - 3 \cdot 2^r(2^{14} - 1) - (3^{13} + 25) = 0. \tag{4.5.18}$$

Here the left hand side of the above equation is positive for any integer $r \geq 6$ and is non-zero for any positive integer $r < 6$. Thus, this case is also excluded. This concludes the proof of the theorem.

4.5.4 Proof of Corollary 4.4.4

Applying Theorem 4.4.3, we get $D^r(E_2)E_k$ is not an eigenform. If f is a cusp form, then for $r \geq 1$, $D^r(E_2)f$ has Fourier coefficients starting from q^2. So in this case also it is not an eigenform by Lemma 4.5.8. Now applying Proposition 4.5.11, we get the required result.

4.5.5 Proof of Theorem 4.4.5

We recall the following result given in [24] (Proposition 1(ii)).

Lemma 4.5.12. *(Lanphier, [24]) If $g \in M_{k_1}(N, \chi)$ and $h \in M_{k_2}(N, \psi)$ and $k = k_1 + k_2 + 2m$, then $[g, h]_m|_k W_Q = [g|_{k_1} W_Q, h|_{k_2} W_Q]$.*

Now, assume on the contrary that

$$[g, h]_m = f \tag{4.5.19}$$

is an a.e. eigenform. Then $f(z) = \alpha f_0(Qz)$, where $M|N$, $Q|(N/M)$, $\alpha \in \mathbb{C}^*$ and $f_0 \in S_k(M, \chi)$ is a normalized newform. Since N is square-free, for any divisor Q of N, $(Q, N/Q) = 1$. We also compute that

$$\begin{aligned} f|_k W_Q &= Q^{k/2}(Nvz + Qw)^{-k} f\left(\frac{Qxz + y}{Nvz + Qw}\right) \\ &= \alpha Q^{-k/2}(Nvz/Q + Qw)^{-k} f_0\left(\frac{Qxz + y}{Nvz/Q + w}\right) \\ &= \alpha\chi(w) Q^{-k/2} f_0(z), \end{aligned}$$

since $\begin{pmatrix} Qx & y \\ Nv/Q & w \end{pmatrix} \in \Gamma_0(M)$.

Applying the operator W_Q to (4.5.19) and by Lemma 4.5.12, we get $[g|_{k_1} W_Q, h|_{k_2} W_Q]_m = f|_k W_Q = \text{const.}\ f_0$. This gives a contradiction since the Fourier expansion of f_0 (being primitive) starts with q, whereas the Fourier expansion of $[g|_{k_1} W_Q, h|_{k_2} W_Q]_m$ starts with at least q^2, a contradiction. This proves the theorem.

4.5.6 Proof of Theorem 4.4.6

We recall Proposition 6 of [51].

Proposition 4.5.13. (*Zagier, [51]*) *Let k_1, k_2, m be integers satisfying $k_2 \geq k_1 + 2 > 2$ and let $k = k_1 + k_2 + 2m$. If $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \chi\psi)$ and $g(z) = \sum_{n=0}^{\infty} b_n q^n \in M_{k_1}(N, \chi)$, then*

$$\langle f, [g, E_{k_2}^{(N,\psi)}]_m \rangle = \frac{\Gamma(k-1)\Gamma(k_2 + m)}{(4\pi)^{k-1} m! \Gamma(k_2)} \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k_1+k_2+m-1}},$$

where $\langle , \rangle$ is the Petersson inner product and $E_{k_2}^{N,\psi)}$ is the Eisenstein series defined by (4.2.5).

Next, we recall Proposition 1 of [11], which gives the action of the W_Q operator on a.e. eigenforms belonging to $\mathcal{E}_k(N, \psi)$.

Proposition 4.5.14. (*Ghate, [11]*) *Let $k \geq 3$. Let $f_k(z, \psi_1, \psi_2) \in \mathcal{E}_k(N, \psi)$ be an a.e. eigenform as described in Theorem 4.2.1. Let $Q_1 = (Q, M_1)$, $Q_2 = (Q, M_2)$. Let $\psi_1 = \psi_{Q_1}\psi_{M_1/Q_1}$ and $\psi_2 = \psi_{Q_2}\psi_{M_2/Q_2}$ be the decompositions of ψ_1 and ψ_2 into their Q and prime to Q parts respectively. Set*

$$\alpha = \frac{\psi_{Q_2}(-M_2/Q_2)\psi_{M_2/Q_2}(Q_2)}{\psi_{Q_1}(M_2/Q_2)\psi_{M_1/Q_1}(Q_2)},$$

then

$$f_k(z, \psi_1, \psi_2)|_k W_Q = \alpha \; Q^{k/2} {Q_2}^{-k} f_k(\frac{Qz}{Q_1 Q_2}, \overline{\psi}_{Q_2}\psi_{M_1/Q_1}, \overline{\psi}_{Q_1}\psi_{M_2/Q_2}).$$

The Eisenstein series $E_k^{(N,\psi)}$ are related to the a.e. eigenforms given in Theorem 4.2.1 in the following way (See Lemma 1 of [11]).

Lemma 4.5.15. (*Ghate, [11]*) *For $k \geq 3$, we have*

$$E_k^{(N,\psi)}(z) = \frac{-2k}{B_{k,\psi}} f_k(Qz, \psi_0, \psi_2),$$

where ψ_0 is the principal character of level $M_1 = 1$, ψ_2 is the primitive character of conductor M_2 associated to ψ and $Q = N/M_2$.

We now prove the following propositions.

Proposition 4.5.16. *Let k, k_1, k_2, m be positive integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2m$. Let $g \in S_{k_1}(N, \chi)$ be an a.e. eigenform which is a newform and $h = E_{k_2}^{(N,\psi)} \in \mathcal{E}_{k_2}(N, \psi)$. If $\dim(S_k^{new}(N, \chi\psi)) \geq 2$, then $[g, h]_m$ is not an a.e. eigenform.*

Proof. Assume on the contrary that $[g, h]_m \in S_k(N, \chi\psi)$ is an a.e. eigenform. Since $\dim(S_k^{new}(N, \chi\psi)) \geq 2$, there exists a newform $f \in S_k^{new}(N, \chi\psi)$ which is orthogonal to $[g, h]_m$ with respect to Petersson inner product. By Theorem 4.5.13, we also have

$$\langle f, [g, E_{k_2}^{(N,\psi)}]_m \rangle = \frac{\Gamma(k-1)\Gamma(k_2+m)}{(4\pi)^{k-1} m! \Gamma(k_2)} \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k_1+k_2+m-1}}.$$

Since f is orthogonal to $[g, h]_m = [g, E_{k_2}^{(N,\psi)}]_m$, the left hand side of the above identity is zero. But the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^s}$ has an Euler product which is absolutely convergent for $\mathrm{Re}(s) > k_1 + \frac{k_2}{2} + m$. Hence, for $\mathrm{Re}(s) > k_1 + \frac{k_2}{2} + m$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^s}$ does not vanish. Since $k_1 > 2$, we have $k_1 + k_2 + m - 1 > k_1 + \frac{k_2}{2} + m$. Therefore, $\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k_1+k_2+m-1}} \neq 0$. This gives a contradiction, since the left hand side is zero, whereas the right hand side is not zero. □

Proposition 4.5.17. *Let k, k_1, k_2, m be positive integers satisfying the same conditions as in Proposition 4.5.16. Suppose that $g = f_{k_1}(z, \chi_1, \chi_2)$ is an a.e. eigenform as described in Theorem 4.2.1 with χ primitive and $h = E_{k_2}^{(N,\psi)} \in \mathcal{E}_{k_2}(N, \psi)$. If $\dim(S_k^{new}(N, \chi\psi)) \geq 2$, then $[g, h]_m$ is not an a.e. eigenform.*

Proof. Assume that $[g, h]_m$ is an a.e. eigenform. Since $\dim(S_k^{new}(N, \chi\psi)) \geq 2$, there exists a newform $f \in S_k^{new}(N, \chi\psi)$ which is orthogonal to $[g, h]_m$ with respect to Petersson inner product. Applying Theorem 4.5.13, we get

$$\langle f, [g, E_{k_2}^{(N,\psi)}]_m \rangle = \frac{\Gamma(k-1)\Gamma(k_2+m)}{(4\pi)^{k-1} m! \Gamma(k_2)} \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k_1+k_2+m-1}}.$$

According to the choice of f, the left hand side is zero. We shall show that the right hand side is not zero, i.e., the series on the right hand side does not vanish. Let

$$D(k-m-1, f, g) = \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k-m-1}}.$$

We have from Lemma 1 of [42],

$$D(k-m-1, f, g) = L(k-m-1, f, \chi_1) L(k-m-k_1, f, \chi_2) / L(k-k_1, \psi), \quad (4.5.20)$$

where $L(k-m-1, f, \chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n) a_n}{n^{k-m-1}}$, $L(k-m-k_1, f, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_2(n) a_n}{n^{k-m-k_1}}$ and $L(k-k_1, \psi) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^{k-k_1}}$. The denominator of (4.5.20) is a finite non-zero quantity and each term of the numerator is not zero since $k_2 \geq k_1 + 2$. Thus the right hand side is not zero. This completes the proof. □

We are now ready to prove the theorem. In the notation of Theorem 4.2.1, we may write $h = f_{k_2}(Qz, \psi_1, \psi_2)$, where $Q | (N/M_1M_2)$. Since N is square-free, for any divisor Q of N, we have $(Q, N/Q) = 1$. Now applying the W-operator W_{N/QM_2} on h and using Proposition 4.5.14 and Lemma 4.5.15, we get

$$h|_{k_2} W_{N/QM_2} = c_1 \cdot f_{k_2}\left(\frac{Nz}{M_1M_2}, \psi_0, \overline{\psi}_1 \psi_2\right) = c_2 \cdot E_{k_2}^{(N, \overline{\psi}_1 \psi_2)},$$

where ψ_0 is the principal character and c_1, c_2 are some constants. Now assume on the contrary that $[g, h]_m$ is an a.e. eigenform. Applying W_{N/QM_2} to $[g, h]_m$ and using Lemma 4.5.12, we see that

$$[g|_{k_1} W_{N/QM_2}, h|_{k_2} W_{N/QM_2}]_m \in S_k(N, \chi_{QM_2} \overline{\chi}_{N/QM_2} \overline{\psi}_1 \psi_2)$$

is an a.e. eigenform. Since the W-operator is an isomorphism and it takes newform space to newform space, we have $\dim(S_k^{new}(N, \chi_{QM_2}\overline{\chi}_{N/QM_2}\overline{\psi}_1\psi_2)) \geq 2$. Now applying Proposition 4.5.16 to $g|_{k_1}W_{N/QM_2} \in S_{k_1}(N, \chi_{QM_2}\overline{\chi}_{N/QM_2})$ and $h|_{k_2}W_{N/QM_2} = c_2 \cdot E_{k_2}^{(N,\overline{\psi}_1\psi_2)}$, we conclude that $[g|_{k_1}W_{N/QM_2}, h|_{k_2}W_{N/QM_2}]_m$ is not an a.e. eigenform. This gives a contradiction, proving the first assertion.

If $k_2 - k_1 \geq 2$, then as in the proof of the previous assertion, applying the operator W_{N/M_2} to h and g, we get $h|_{k_2}W_{N/M_2} = c_3 \cdot E_{k_2}^{(N,\overline{\psi}_1\psi_2)}$ and $g|_{k_1}W_{N/M_2}$ continues to be a form with primitive character, where c_3 is a constant. Then applying Proposition 4.5.17, we get the result. If $k_1 - k_2 \geq 2$, then interchanging the role of g and h gives the required result. This proves the theorem.

4.5.7 Proof of Theorem 4.4.7

To prove the theorem, we first recall Proposition 2.5 from [4].

Proposition 4.5.18. (*Beyerl et al., [4]*) *Let $f \in M_k$. Then $\delta_k^{(r)}(f)$ is an eigenform for T_n iff f is. In this case, if λ_n denotes the eigenvalue of T_n associated to f, then the eigenvalue of T_n associated to $\delta_k^{(r)}(f)$ is $n^r\lambda_n$.*

For any modular form $f \in M_k$, we have

$$\delta_k(f) - \frac{k}{12}E_2^* f = Df - \frac{k}{12}E_2 f \in M_{k+2}. \tag{4.5.21}$$

Moreover, $\delta_k(f) - \frac{k}{12}E_2^* f$ is a cusp form if f is a cusp form. Then $\delta_{12}(\Delta_{12}) - E_2^*\Delta_{12}$ is a cusp form of weight 14, and since there is no non-zero cusp form of weight 14 on $SL_2(\mathbb{Z})$, we get

$$\delta_{12}(\Delta_{12}) = E_2^*\Delta_{12}.$$

Now applying Proposition 4.5.18, we see that $E_2^*\Delta_{12}$ is an eigenform.
Conversely, assume that $f \in M_k$ is an eigenform such that $E_2^* f$ is an eigenform. Then applying (4.5.21) and proceeding as in the proof of Proposition 4.5.11, we conclude that $f \in \mathbb{C}\Delta_{12}$. This completes the proof of the theorem.

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