INTRODUCTION TO COXETER GROUPS

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This is a set of notes of a course given by me at HRI, during May-June 2018, intended for circulation among participants. As it is a work in progress, any corrections/ suggestions for improvements are welcome, in fact solicited. I thank the participants Ms. Sumana Hatui, Mr. Rahul Kaushik, Mr. Tushar Kanta Naik and Mr. Nishant Rathee for kindly preparing these notes for circulation.

1. Some historical remarks

Coxeter groups appear crucially in many situations in mathematics. It has a long and rich history (see [1], p. 249-264). Coxeter groups and Coxeter-Dynkin diagrams appear in many classification contexts: regular polyhedra in Euclidean Spaces (classified by Schläfli in 1852); semi-simple Lie Algebras over complex numbers (classified by Killing and Cartan; Van der Warden by 1933); quivers with finitely many indecomposables; cluster algebras with finitely many seeds; singularities of hypersurfaces (see [9]); theory of hypergeometric functions; crystallographic, etc.,

The modern theory of Coxeter groups developed around the fundamental contributions by : H.S.M. Coxeter, E.Witt, J.Tits; M. Davis; M. Salvetti; and E. B. Vinberg.

In [3], Coxeter determined all finite, irreducible groups of Euclidean displacements which are generated by reflections, completing the work of Cartan [5], [6], who had determined all "crystallographic" reflection groups (that is, the ones appearing as automorphism groups of finite root systems; equivalently, those having an embedding in an infinite discrete group of displacements). In 1941, Stifel remarked that the Weyl groups appearing in the classification, due to W. Killing and E. Cartan, of finite dimensional simple Lie algebras over the field $\mathbb{C}$ of complex numbers, are exactly the groups generated by reflections that leave invariant lattices in finite dimensional Euclidean space (i.e., free abelian groups of maximal rank). Chevalley and Harish-Chandra gave a uniform proof for the bijective correspondence between the crystallographic groups and complex semi-simple Lie algebras. In [2], Coxeter further showed that a finite group generated by reflections of an Euclidean or of an affine space if, and only if, it admits a presentation of the form

$$\langle s_1, s_2, \ldots, s_n \mid s_i^2, (s_i s_j)^{m_{i,j}} \rangle,$$

where $M = (m_{i,j})$ is a $n \times n$ matrix with $m_{i,i} = 1$ for each i and $m_{i,j} = m_{j,i} \in \mathbb{Z}^{\geq 2}$ for $i \neq j$. Following Tits [11], groups given by such presentations are now called Coxeter groups whether the group, or the generating set in the presentation, is finite or not.

Coxeter [4] and Witt [13] observed that irreducible, infinite groups of Euclidean displacements generated by reflections correspond bijectively, up to isomorphism, to simple Lie algebras over $\mathbb{C}$. Witt gave a new characterization of infinite discrete groups of Euclidean displacements. Those groups and the groups generated by reflections in hyperbolic spaces are all given by Coxeter presentations. This motivated the study of the groups defined by Coxeter presentations: first, in terms of geometric realization by Witt, and Coxeter and Moser; and then, combinatorially and algebraically, by Tits [11]. Good sources for the basic theory, with particular emphasis on realisation of Coxeter groups as groups generated by reflections in Euclidean and affine spaces are, [11], [10]. For the theory of Coxeter groups generated by reflections in hyperbolic spaces, [12] provides a good beginning.

The general theory of Coxeter groups was pioneered by Tits in 1960. This historically important, unpublished work finally appeared in print in ([11], Vol 1, chap 43, p. 803-818).

In fifties, Tits set for himself the task of geometrically understanding simple Lie groups over $\mathbb{C}$, especially the groups of exceptional type. His idea was to construct a geometric/combinatorial object for each simple group G over $\mathbb{C}$ of Lie type, on which G acts, revealing much of the structure of G. Guiding examples were the projective spaces of dimension n for the groups $PSL_{n+1}(k)$ and the polar spaces for classical matrix groups, constructed by Fredenthal and Tits (see [25]). The structure he constructed are now called Tits buildings or spherical buildings. Given the Bruhat decomposition for each of these groups and the central role of the corresponding Weyl groups in the structure theory of these groups (so called theory of groups with (B, N)-pairs), it was natural to study the structure of Coxeter groups and look for a natural geometric structure for these groups. The buildings Tits constructed for G generalizes the Cayley graph for the Weyl group of G.

Tits showed that the Coxeter group $W(M)$ defined by the Coxeter matrix $M = (m_{i,j})$ has a faithful representation ρ on the real vector space V with a basis $\{e_s : s \in S\}$

indexed by the set S of defining generators of $W(M)$. Further, $W(M)$ leaves invariant a symmetric (possibly degenerate) bilinear form B on V defined in terms of M (in fact, by setting $B(e_s, e_{s'}) = -\cos(\pi/m_{s,s'})$ or -1 according as $m_{s,s'}$ is finite or not). Thus, $\rho(W)$ is a subgroup of the orthogonal group $O_B(V)$. We note that the set of generators in the presentation of $W(M)$, or the group $W(M)$, could be infinite.

Thus, Coxeter groups are linear groups and share all the properties of linear groups: solvability of word problem, residual finiteness, virtual torsion-freeness, etc., They are automatic groups and also Cat(0) groups. In particular, each Coxeter group acts geometrically on a simply connected, non-positively curved, piecewise Euclidean, cell complex, known as Davis complex, with Moussong metric (see [8]). Thus, they fit into many powerful theories of geometric group theory and are geometrically very nice.

The action ρ of W on V is regular on a cone $\Im$ (now called a Tits cone) in V with vertex at the origin. The cone $\Im$ is the union of the images $\{w(\sigma) : w \in W\}$ of a simplicial cone σ in V appearing as the intersection of certain half-spaces in V, each defined by the hyperplanes orthogonal (relative to B) to a basis element e_s of V.

The Tits cone $\Im$ is V if B is positive definite and a half-space of V with the vertex of $\Im$ on the boundary if B is positive semi-definite. Otherwise, it is a cone properly contained in a half-space of V. For more on Tits cone, see [12]. Further, B is positive definite if, and only if, $W(M)$ is finite; and B is positive semi-definite if, and only if, $W(M)$ is an affine reflection group; i.e., the semi-direct product of free abelian group on n generators by a finite Coxeter group on n generators.

References

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2. Coxeter groups

'Coxeter's theory of refection groups is the only case known to me in which an interesting class of presentations characterise an interesting class of groups'- John Conway

Objective: To present an introduction to the theory of Coxeter groups with emphasis on structural aspects of the theory.

2.1. **Free monoid on I:** Let I be a set (of any cardinality) and let I^* denote the free monoid consisting of words (i.e. finite sequences) in I, with concatenation as the monoid operation. The empty word is the identity element of I^*. We write $f = (i_1, i_2, \dots, i_t) \in I^*$ as $i_1 i_2 \dots i_t$ and say that *length* of f is t and denote it by $l(f)$. The words $P_m(i,j) = iji\dots$ of length m, $i, j \in I$ and $m \geqslant 2$ appear prominently in what follows. Note that $P_m(i,j) = (ij)^{m/2}$ if m is even and $(ij)^{(m-1)/2}i$ if m is odd.

2.2. **The length function l:** Let G be a group and S be a set of generators of G such that $1_G \notin S$ and, for each $s \in S, s^{-1} \in S$. Throughout, we assume that there is a bijection $i \to s_i, i \in I$ and $s_i \in S$, from I to S. This bijection extends to a monoid homomorphism from I^* to G. The image $s_{i_1} s_{i_2} \dots s_{i_t}$ of $f = i_1, \cdots, i_t \in I^*$ in G is written as r_f. For $g \in G$, the *length* $l_S(g)$ of g is the minimum of the lengths of $f \in I^*$ such that $r_f = g$. Thus, $l_S = 0$ or 1 accordingly as $g = I_G$ or $g \in S$. An expression of g as r_f with $l(f) = l_S(g)$ is said to be a *reduced expression* of g. We write $l_S(g)$ as $l(g)$ if there is no ambiguity about

S. Some times, it is of interest to consider the length function on G relative to the set $S^G = \{gsg^{-1} : g \in G, s \in S\}$.

Proposition 2.1. *Let $g, g' \in G$. Then, the following hold:*

(i) $l(gg') \leq l(g) + l(g')$,

(ii) $l(g^{-1}) = l(g)$,

(iii) $|l(g) - l(g'^{-1})| \leq l(gg'^{-1})$.

Proof. (i) and (ii) are clear. Replacing g by gg'^{-1} in (i) and using (ii), we get

$$l(g) - l(g') \leq l(gg'^{-1}), \tag{2.1}$$

$$l(gg'^{-1}) = l(g'g^{-1}) \tag{2.2}$$

Interchanging g and g' in (2.1) and using (2.2), we get

$$l(g') - l(g) \leq l(g'g^{-1}) = l(gg'^{-1}). \tag{2.3}$$

Now, (iii) follows from (2.1) and (2.3). □

G can be considered as a metric space relative to the matrix d_S defined by $d_S(g, g') = l(gg'^{-1})$. This metric is invariant under the left translations of G.

2.3. **The Cayley graph $\Gamma_G(S)$:** The Cayley graph $\Gamma = \Gamma_G(S)$ associated with (G, S) as above is the labelled, directed graph with vertex set G and a directed edge from g to gs (respectively, gs to g), labelled s (respectively, s^{-1}), for all $g \in G, s \in S$. The direction of the edge is ignored if s is an involution. Since S generates G, Γ is connected. For each $g \in G$, the permutation τ_g of G taking $x \in G$ to gx is a label preserving automorphism of Γ. For $g, g' \in G$, $l(gg'^{-1})$ is the length of the shortest path in Γ from g to g'. A shortest path in Γ from g to g' is called a *geodesic* from g to g'. A subset A of G is said to be *convex* if A contains each vertex of each geodesic in Γ connecting any two elements of A.

2.4. **The simplicial complex $\Sigma_G(S)$:** The simplicial complex $\Sigma = \Sigma_G(S)$ associated with (G, S) as above is the partially ordered set (or *poset*, for short) of left cosets $\{g\langle S'\rangle : g \in G, S' \subsetneq S\}$, with the partial order '$A \preccurlyeq B$ if, and only if, $B \subseteq A$'. If $A \preccurlyeq B$, we say that A is a *face* of B. Maximal elements of Σ are called *chambers*. They are the cosets of the trivial subgroup of G and can be - and, are - identified with the elements of G. Maximal elements of $\Sigma \setminus \{\{g\} : g \in G\}$ are called *panels* of Σ. If $s \in S$ is an involution, then $g\langle s\rangle$ is a panel for each $g \in G$; and $\{g\}$ and $\{gs\}$ are the only chambers in G with

$g\langle s\rangle$ as a face. The permutation τ_g of G defined in (2.3) for $g \in G$ is an automorphism of Σ. The map $\tau \mapsto \tau_g$ is an injective homomorphism from G to the group of automorphisms of Σ. If $s \in S$ is an involution, then, for each $g \in G$, $\rho_{gsg^{-1}}$ fixes the panel $g_1\langle s_1\rangle$ if, and only if, $gsg^{-1} = g_1 s_1 g_1^{-1}$ and in this case, it interchanges the chambers g_1 and $g_1 s_1$. The set of all panels fixed by ρ_r for a conjugate r of an element of S is called a *wall* in Σ.

We can also consider Σ as a metric space with the distance d_Σ defined as $d_\Sigma\big(g_1\langle S_1\rangle, g_2\langle S_2\rangle\big) = min\{l(x_1^{-1}x_2) : x_i \in g_i(S_i)\}$ for $g_i \in G$ and $S_i \subseteq S$, $i = 1, 2$.

2.5. **Coxeter matrix M, Coxeter graph Γ_M and Coxeter group $W(M)$ and Artin group $A(M)$:** A function from $I \times I$ to $\mathbb{N} \cup \{\infty\}$, taking $(i, j) \in I \times I$ to $m_{i,j}$, is called a *Coxeter matrix* over I if $m_{i,i} = 1$ and $m_{i,j} = m_{j,i} \geq 2$ or ∞ for all $i, j \in I$, $i \neq j$. We write $M = (m_{i,j})$.

The *Coxeter diagram* Γ_M of type M is a graph with I as the set of vertices and a 2-subset $\{i, j\}$ of I is called an *edge* if $m_{i,j} \geq 3$ and the edge is labelled $m_{i,j}$ if $m_{i,j} \geq 4$ or ∞. Note that this diagram encapsulates all the information in M.

Different conventions of labelling is convenient and used sometimes. Here is one: for $i, j \in I, i \neq j, \{i, j\}$ is not an edge if $m_{i,j} = \infty$, is an unlabelled edge if $m_{i,j} = 2$ and an edge with label $m_{i,j}$ if $2 < m_{i,j} \neq \infty$. However, in this notes, unless specified, we always use the previous labelling.

Definition 2.2. *The Coxeter diagram Γ_M is said to be:*

(i) indecomposable if Γ_M is a connected graph;
(ii) right-angled if $m_{i,j} \in \{2, \infty\}$ for all $i \neq j$;
(iii) of finite type if $|I| < +\infty$;
(iv) of spherical type if $m_{i,j} \neq \infty$ for all i, j;
(v) of even type if $m_{i,j}$ is even or ∞ for all $i \neq j$;
(vi) universal right-angled if $m_{i,j} = \infty$ for all $i \neq j$; and
(vii) of large type if $m_{i,j} = 3$ for all i, j.

Definition 2.3. *A pair (G, S) as in section (2.2) is said to be:*

(i) a pre-Coxeter system of type M if $s_i s_j$ has order $m_{i,j}$ for all $i, j \in I$;
(ii) a Coxeter system of type M if G has the presentation

$$\langle S \mid (s_i s_j)^{m_{i,j}}, i, j \in I,\ m_{i,j} \neq \infty\rangle;$$

(iii) an Artin-Tits system of type M if G has the presentation

$$\langle S \mid r_{P_m(i,j)} = r_{P_m(j,i)}, i \neq j,\ m = m_{i,j} \neq \infty \rangle;$$

The group G is said to be a *Coxeter group of type M*, and is written as $W(M)$ in case (ii); and G is said to be an *Artin group*, and sometimes also called an *Artin-Tits group* of type M, and is written as $A(M)$ in case (iii). A Coxeter group of type M is said to be *indecomposable*, *right-angled*, etc., if so is its defining Coxter diagram.

Though much is known about the structure and representation theory of the Coxeter groups, very little is known about the Artin groups. The following problems about Artin groups seem to be basic:

(1) Is every Artin group torsion-free?
(2) Is the centre of each nonspherical Artin group trivial?
(3) Does each Artin group has a solvable word problem?
(4) Does every Artin group satisfies the $K(\pi, 1)$-conjecture?

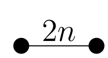
In contrast, the solutions to (1), (2), (3) and (4) for Coxeter groups is known.

Note: (1): $s \in S$ is an involution in (i) and in (ii) (see Theorem 2.4) and is of infinite order in (iii). The presentation in (ii) can also be written as in (iii), allowing $i = j$ also. The relation $r_{P_m(i,j)} = r_{P_m(j,i)}$ is called a *braid relation* of length m.
(2) Many fundamental results about Coxeter groups give criteria for a pre-Coxeter system of type M to be a Coxeter system of type M.
(3) The Coxeter diagram Γ_M clearly defines the group $W(M)$. A fundamental question in the subject is to decide whether the group structure of a Coxeter group determines M.

More precisely,

Question (1): if (G, S) and (G, S') are Coxeter systems, does there exist an automorphism of G mapping S onto S'?

This problem, and some variants of it, are called *rigidity problems*. The answer to this question is negative as Coxeter groups for the following two diagrams



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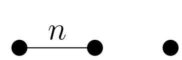
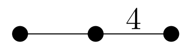


are isomorphic, and in fact to the dihedral group of order $4n$.

Exercise: The diagrams



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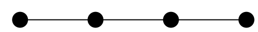

give the same group.

However, the answer to the Question (1) is conjectured to be positive except for specific list of counter examples. For the current status on this conjecture, see

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2.6. **Alternating Coxeter and Artin groups:** Let $F(S)$ denote the free group on S. The map from S to the multiplicative group $\{-1, 1\}$, taking each element s of S to -1, extends to a group homomorphism η from $F(S)$ to $\{-1, 1\}$. Since the relations in the presentation of $A(M)$, as well as of $W(M)$, are all of even length, η defines a surjective homomorphism from $A(M)$ to $\{-1, 1\}$. This further factors through $W(M)$. By abuse of notation, we denote each of three homomorphisms with domains $F(S)$, $A(M)$ and $W(M)$ with the common range $\{-1, 1\}$ by η; and also the image of $s \in S \subseteq F(S)$ in $A(M)$ as well as in $W(M)$ also by s. Thus, s is non-trivial and, if $|S| > 1$, $W(M)$ is not simple. The kernels of η in $W(M)$ and in $A(M)$ are respectively called the *alternating Coxeter group of type M* and the *alternating Artin group of type M*.

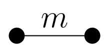
Since length of each relation in the presentation of G, $G \in \{A(M), W(M)\}$ is even, the relation $\eta(xy) = \eta(x)\eta(y)$ for $x, y \in G$ translates into $l(xy) = l(x) + l(y) \pmod 2$. Consequently, for each $s \in S$, $l(s) = 1 \pmod 2$ and $s \neq 1_G$. If r is a conjugate of s, $s \in S$ in W, then : $l(r)$ is odd, $l(rx) \neq l(x) \neq l(xr)$. Further by Proposition 2.1, $|l(sx) - l(x)|$ and $|l(xs) - l(x)|$ are at most $l(s) = 1$, $\{l(xs),\ l(sx)\} \subseteq \{l(x) + 1, l(x) - 1\}$ for each

$x \in G$. We show that, for all $i, j \in I$, order of $s_i s_j$ in $W(M)$ is $m_{i,j}$ (Theorem 2.4 (i)). This would imply that ρ is injective on S. The map ρ from $F(S)$ to $A(M)$ is also known to be injective on S.

2.7. **Some examples and remarks:**

(1) If $|S| = 1$, then $W(M) \simeq C_2$ and $A(M) \simeq (\mathbb{Z}, +)$.

(2) If $|S| = 2$ and Γ is

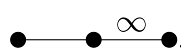


$m \geq 2$ or ∞,

then $W(\Gamma)$ is a dihedral group of order $2m$.

(3) If the Coxeter diagram is disconnected and the labelling considered is the former defined (respectively, the latter suggested) in section (2.5), then the Coxeter group defined by the diagram is the direct product (respectively, the free product) of the Coxeter groups defined by its connected components. This statement is true if the word 'Coxeter group' is replaced by 'Artin group' in the statement.

(4) If

M: 

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then $W(M) \simeq PGL_2(\mathbb{Z})$. We can take

$$s_1 = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

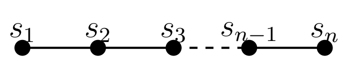
$$s_2 = \pm \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$s_3 = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(see [10], Section 5.1 for a hint)

(5) If

Γ_n: 

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then $W(\Gamma_n) \simeq S_{n+1}$.

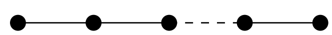
Notation: For integers $i, j, i < j$, we denote the set of integers k such that $i \leq k \leq j$ by $[i, j]$.

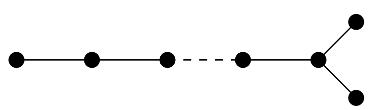
Proof of (5). The map taking $k \in [1, n]$ to the transposition $t_k = (k, k+1) \in S_{n+1}$ extends to a group homomorphism θ from $W(\Gamma_n)$ onto S_{n+1}, because $\{t_k\}$ satisfies the defining relations of $W(\Gamma_n)$. That this homomorphism is surjective, follows because: $W_1 = \langle s_1, s_2, \ldots, s_{n-1} \rangle \simeq S_n$ by induction hypothesis; and using the relations for $W(\Gamma_n)$, we see that $\{W_1,\ W_1 s_n \ldots s_i : i \in [1, n]\}$ are all the mutually distinct cosets of W_1 in W. So, $|W(\Gamma_n)| = (n+1)W_1 = (n+1)!$ and θ is also injective. □

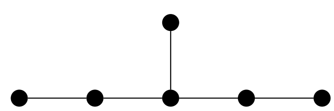
($\mathcal{S}$) **Coxeter diagrams of irreducible, finite Coxeter groups:** Coxeter groups defined by the diagrams in ($\mathcal{S}.i$), ($\mathcal{S}.ii$) below, is the complete list of finite reflection groups, upto isomorphism, acting irreducibly on Euclidean spaces.

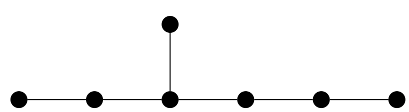
($\mathcal{S}.i$) **Crystallographic root systems:**

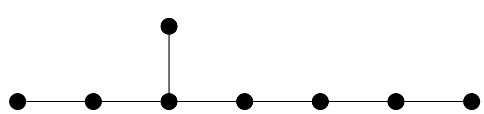
A_n:  ($n \geq 1$ vertices)

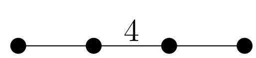
$B_n = C_n$:  ($n \geq 2$ vertices)

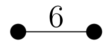
D_n:  ($n \geq 4$ vertices)

E_6 : 

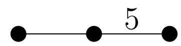
E_7 : 

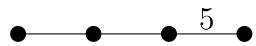
E_8 : 

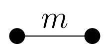
F_4: 

G_2: 

($\mathcal{S}.ii$) **Non-crystallographic root systems:**

H_3: 

H_4: 

$I_2(m)$: 

($m = 5$ or $m \geq 7$)

FIGURE 1

Remark 2.4. (1) The subscript n in the diagram $\mathfrak{X}_n$ above denotes the number of nodes and the dimension of the Euclidean spce on which the corresponding Coxeter groups acts faithfully and irreducibly. No two of these groups are isomorphic. This is also the complete list of irreducible Coxeter systems, for which the bilinear form associated in section 2.8 is positive definite.
(2) Except for the types H_3, H_4 and $I_2(m), m = 5$ or $m \geq 7$, $W(\mathfrak{X}_n)$ is the Weyl group of a finite dimensional simple Lie algebra of type $\mathfrak{X}_n$ over $\mathbb{C}$ and is the stabilizer, in the orthogonal group of rank n, of the corresponding root system. These are called *crystallographic groups*. They stabilize a lattice (i.e. a free abelian subgroup of rank n) in $\mathbb{R}^n$.
(3) The following groups $W(\mathfrak{X}_n)$ are the groups of symmetries of the regular polytopes in the Euclidean spaces $\mathbb{R}^n$. For groups, we use the notation of: J. Conway, et. al, Atlas of Finite groups, Camb. Univ. Press.

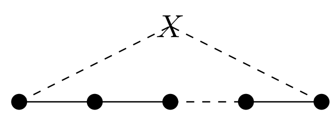
- $W(A_n) \simeq S_{n+1}$ and is the group of symmetries of a regular n-simplex in $\mathbb{R}^{n+1}$ and, of its dual.
- $W(B_n) \simeq 2^n S_n, n \geq 2$ and is the group of symmetries of a n-dimensional hypercube in $\mathbb{R}^n$ and, of its dual.
- $W(D_n) \simeq 2^{n-1} S_n, n \geq 4$. This is **not** the symmetry group of a regular polytope.
- $W(F_4) \simeq 2^3 : S_4 : S_3$ and is the group of symmetries of a regular $(3, 4, 3)$-polytope in $\mathbb{R}^4$. This has 24 octahedral faces and is self dual. The set of vertices of this polytope is the root system of type F_4.

- $W(I_2(m)) \simeq D_{2m}$ and is the group of symmetries of a regular m-gon in $\mathbb{R}^2$.
- $W(H_3) \simeq 2 \times A_5$ and is the group of symmetries of a icosahedron (with twenty triangular faces) in $\mathbb{R}^3$ and of its dual, a regular dodecahedron (with twelve pentagonal faces) in $\mathbb{R}^3$.
- $W(H_4) \simeq 2A_5 \times (2 \times A_5)$ and is the group of symmetries of the regular cell in $\mathbb{R}^4$ (this has 600 tetrahedrol faces) and of its dual, the 600-solid in $\mathbb{R}^4$.

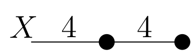
($\mathcal{A}$) **Coxeter diagrams for irreducible affine Coxeter groups:** The Coxeter groups for the diagrams $\tilde{\mathfrak{X}}_n$ in Figure 2 below act on the Euclidean space $\mathbb{R}^n$ irreducibly as affine transformations. The number of nodes in $\tilde{\mathfrak{X}}_n$ is $n+1$. If we delete the node of $\tilde{\mathfrak{X}}_n$ marked "X" and the dotted edge(s) incident with it, we get a crystallographic spherical diagram $\mathfrak{X}_n$ and $W(\tilde{\mathfrak{X}}_n)$ is the semi-direct product of a free abelian group of rank n extended by the finite Coxeter group $W(\mathfrak{X}_n)$.

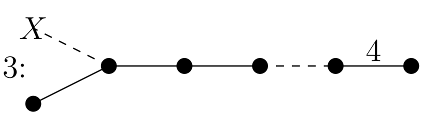
Following are the irreducible affine Coxeter diagrams:

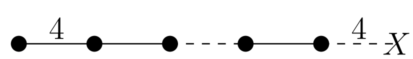
$\tilde{A}_1$: 

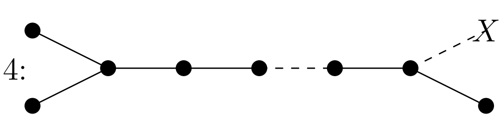
$\tilde{A}_n$: 

$(n+1$ nodes $n \geq 2)$

$\tilde{B}_2$: 

$\tilde{B}_n, n \geq 3$: 

$\tilde{C}_n, n \geq 3$: 

$\tilde{D}_n, n \geq 4$: 

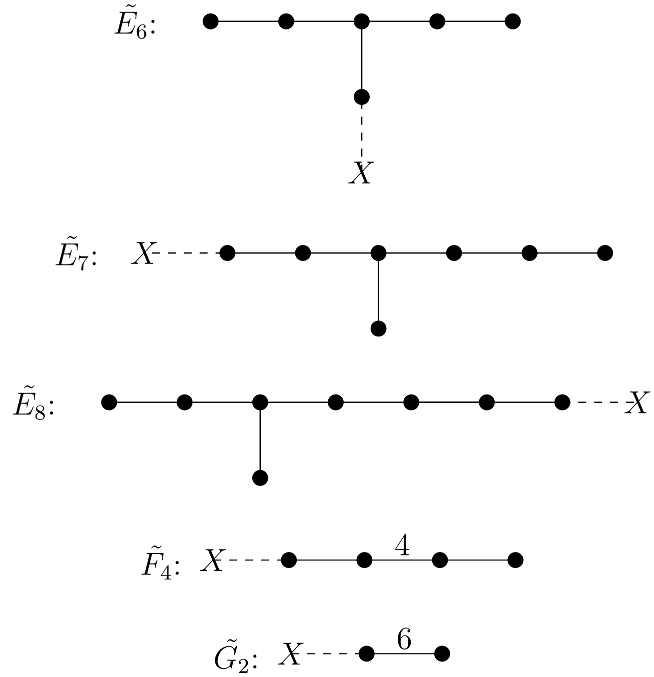


FIGURE 2

No two of these diagrams are isomorphic. This is the complete list of irreducible Coxeter systems with positive indefinite quadratic form. Each diagram $\tilde{\mathfrak{X}}_n$ encodes a simplex σ_n in $\mathbb{R}^n$, unique up to scaling, with the following properties: the vertices of $\tilde{\mathfrak{X}}_n$ are in bijection with the facets (faces of codimension one) of σ_n, and the faces labelled s, s' intersect in the dihedral angle $\pi/m_{s,s'}$. These properties are enough to describe the simplicies associated with $\tilde{\mathfrak{X}}_n$, $n \geq 2$. The simplices associated with $\tilde{\mathfrak{X}}_n, n \geq 2$, are the only simplices in $\mathbb{R}^n$ where the dihedral angles are π/m for an integer $m > 1$. The diagram $\tilde{A}_1$ corresponds to a 1-simplex in $\mathbb{R}^1$ whose faces are its end points. These faces do not intersect and this is indicated by the label ∞ on the unique edge in $\tilde{A}_1$.

2.8. **Linear representation of a Coxeter group $W(S)$ of the type M:** Let (W, S) be a Coxeter system over I of type M and $i \to s_i$ be a bijection from I to S. In this section, we present a real linear representaion ρ of $W(S)$, due to Tits, and deduce that the cannonical map from the free group on S to $W(S)$ is injective on S (Theorem 2.4). Tits also showed that ρ is faithful, thus proving that Coxeter groups are linear groups (see [1], chapter V, section 4, corollary 2).

Let V be the vector space over $\mathbb{R}$ with basis $\{e_i : i \in I\}$. Let B be the (possibly degenrate) symmetric bilinear form on V defined by setting $B(e_i, e_j) = -cos(\pi/m_{i,j})$ or -1 according as $m_{i,j}$ is finite or not. Thus, $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$. The form B is called a *Tits form*. For a subset A of V, we define $A^\perp = \{v \in V : B(v, a) = 0$ for each $a \in A\}$. We denote by $O_B(V)$ the orthogonal group defined by B.

For each $v \in V$ with $B(v, v) = 1$, we denote by r_v the reflection of V in the hyperplane $V^\perp$; explicitly, $r_v(x) = x - 2B(v, x)v$ for $x \in V$. Since $r_v(x) = -v$ and ${r_v}^2(x) = x$ for each $x \in V$, r_v is an involution. Also, $r_v = r_{-v}$ for each $v \in V$ with $B(v, v) = 1$. Further, for $x, y \in V$, $B(r_v(x), r_v(y)) = B(x, y)$. So, $r_v \in O_B(V)$. We write r_{e_i} as r_i for $i \in I$.

Theorem 2.4 (Tits)**.** *The following hold:*

(i) $|r_i| = 2$ *and* $|r_ir_j| = m_{i,j}$ *for* $i, j \in I$. *Consequently, the map* $s_i \to r_i$ *from* $S \subseteq F(S)$ *to* $O_B(V)$ *is injective and extends to a unique group homomorphism* ρ *from* W *to* $O_B(V)$. *Further, order of* s_is_j *in* W *is* $m_{i,j}$.

(ii) *If* $r_i \in \langle r_j : j \in J\rangle \leq O_B(V)$ *for some* $i \in I$ *and* $J \subseteq I$, *then* $i \in J$.

Proof. By definition, $r_i \neq r_i^2 = Id$ on V for each $i \in I$. Now choose $i, j \in I$ and let V_{ij} denote the subspace of V spanned by e_i, e_j. Then, both r_i and r_j map V_{ij} to itself. Further

$$\begin{aligned} r_ir_j(e_i) &= r_i\big(e_i - 2B(e_i, e_j)e_j\big) = -e_i - 2B(e_i, e_j)\big(e_j - 2B(e_i, e_j)e_i\big) \\ &= \big(4B(e_i, e_j)^2 - 1)\big)e_i - 2B(e_i, e_j)e_j, \text{ and} \\ r_ir_j(e_j) &= r_i(-e_j) = -\big(e_j - 2B(e_i, e_j)e_i\big) = 2B(e_i, e_j)e_i - e_j. \end{aligned}$$

If $B(e_i, e_j) = 1$ (i.e., if $m_{i,j} = \infty$), then

$$(r_ir_j)(e_i + e_j) = 3e_i + 2e_j - 2e_i - e_j = e_i + e_j,$$

and

$$(r_ir_j)(e_i) = 3e_i + 2e_j = 2(e_i + e_j) + e_i.$$

So

$$(r_ir_j)^m(e_i) = 2m(e_i + e_j) + e_i$$

and $r_ir_j \leq O_B(V)$ is of infinite order.

If $B(e_i, e_j) \neq 1$ (i.e., if $m_{i,j} < +\infty$), then V_{ij} admits an orthonormal basis: in fact, if $f = -B(e_i, e_j)e_i + e_j$, then $B(f, f) = 1 - B(e_i, e_j)^2 > 0$ and $\{e_i, f/\sqrt{B(f, f)}\}$ is an orthogonal basis for V_{ij}. Consequently, $V = V_{ij} \oplus V_{ij}^{\perp}$ and V_{ij} is isomorphic to the Euclidean plane.

Let $(-,-)$ be the standard inner product on $\mathbb{R}^2$. Choose the elements $f_1 = (1,0)$ and $f_2 = (-\cos(\pi/m_{i,j}), \sin(\pi/m_{i,j}))$ of $\mathbb{R}^2$. The map taking e_k to $f_k, k = 1,2$, extends to a unique isometric isomorphism ϕ from V_{ij} to $\mathbb{R}^2$; since $\sin(\pi/m_{i,j}) \neq 0$. Let $T_k = \phi \circ r_k \circ \phi^{-1} \in GL(\mathbb{R}^2)$, $k = 1,2$. Then, for $x \in \mathbb{R}^2$, $T_k(x) = x - 2(x, f_k)f_k$, i.e., T_1 and T_2 are two reflections of the Euclidean plane whose axes make an angle of $\pi/m_{i,j}$. Thus, their product is a rotation of $\mathbb{R}^2$ about the origin by an angle $2\pi/m_{i,j}$. So, T_1T_2 is of order $m_{i,j}$ and r_1r_2 induces an element of order $m_{i,j}$ on V_{ij}. Since both r_i and r_j act as identity on $V_{ij}^{\perp}$, order of r_ir_j indeed is $m_{i,j}$. Now, from the presentation of W, the remaining statements of (i) follows.

(ii) Let $V_J = \langle e_j : j \in J \rangle \leq V$ and $G_J = \langle r_j : j \in J \rangle \leq \rho(W)$. For each $g \in G_J$ and $v \in V, g(v) \in v + V_J$. Since $r_i \in G_J$, $-e_i = r_i(e_i) \in e_i + V_J$. Thus, $e_i \in V_J$ and $i \in J$. □

From now on, we write $\rho(w)(x)$ as wx for $w \in W$ and $x \in V$.

Remark 2.5.

(1) If $x = \lambda_s e_s + \lambda_{s'} e_{s'} \in V_{ij}, \lambda_s, \lambda_{s'} \in \mathbb{R}$, then

$$B(x,y) = \lambda_s^2 + \lambda_{s'}^2 - 2\lambda_s\lambda_{s'}\cos(\pi/m) = (\lambda_s - \lambda_{s'}\cos(\pi/m))^2 + \lambda_s^2 \sin^2(\pi/m).$$

Thus, the restriction of B to $V_{s,s'}$ is positive. It is positive definite if $m_{s,s'} \neq \infty$ and, in this case, $\langle s, s' \rangle$ is finite. If $m_{s,s'} = \infty$, then B is positive indefinite, $\langle s, s' \rangle$ is infinite and $Rad(B)$ is one-dimensional. In general, W is finite if B is positive definite. If B is positive indefinite, W is infinite and radical of B is one dimensional. (see [1](section 4, chapter V) and [10](section 6.5)).

(2) Let $w \in W, s_i \in S$ with $i \in I$, $r = ws_iw^{-1} \in W$ and $\alpha = \rho(w)(e_i) \in V$. Since $\rho(W) \subseteq O_B(V)$, B is $\rho(W)$-invariant. So, for all $x \in V$,

$$\begin{aligned} \rho(r)(x) &= \rho(ws_iw^{-1})(x) = \rho(w)\big(r_i(\rho(w)^{-1}(x))\big) \\ &= \rho(w)\big(\rho(w)^{-1}(x) - 2B(\rho(w)^{-1}(x), e_i)e_i\big) \\ &= x - 2B(\rho(w^{-1})(x), e_i)\rho(w)(e_i) \\ &= x - 2B(x, \rho(w)(e_i))\rho(w)(e_i) = r_\alpha(x). \end{aligned}$$

So $\rho(r)$ is the reflection of V in the plane $\alpha^{\perp}$. Note that $B(\alpha, \alpha) = B(e_i, e_i) = 1$.

Proposition 2.5. *Let (W, S) be an indecomposable Coxeter system of type M, and let V, B and ρ be as above. Then the following hold:*

(i) Radical of B contains every W-invariant proper subspace of V.

(ii) An endomorphism T of V commutes with $\rho(W)$ for each $w \in W$ if, and only if, $T = \lambda I_V$ for some $\lambda \in \mathbb{R}$.

(iii) If B is non-degenerate, then ρ is an irreducible representation of W.

(iv) If B is degenerate, then ρ is not completely reducible and W is infinite.

Proof. (i) Let U be a proper subspace of V such that $w(U) \subseteq U$ for each $w \in W$. Suppose that $e_i \in U$ for some $i \in I$. Since (W, S) is indecomposable, there exist $j \in I$ such that $e_j \notin U$ and $B(e_i, e_j) \neq 0$. But then, $B(e_i, e_j)e_j = e_i - r_{e_j}(e_i) \in U$ and so, $e_j \in U$, a contradiction. So $e_i \notin U$ for each $i \in I$.

Now for $x \in U$ and $i \in I$, $B(x, e_i)e_i = x - r_{e_i}(x) \in U$. So, $B(x, e_i) = 0$ for each $i \in I$ and $U \subseteq Rad(B)$, completing the proof of (i).

(ii) For each $i \in I$,

$$-T(e_i) = T(r_i(e_i)) = r_i(T(e_i)) = T(e_i) - B(T(e_i), e_i)e_i.$$

So $T(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{R}$. If T is non-trivial, then $\lambda_i \neq 0$ for some i. Now the kernel of $(T - \lambda_i I)$ is W-invariant (because, so is T), non-trivial (because it contains i) and is not in the $Rad(B)$ (because $B(e_i, e_i) = 1$). By (i), it is equal to V and so $T = \lambda_i Id$ on V.

(iii) If B is non-degenerate, then, by (i), any W-invariant subspace is trivial.

(iv) If B is degenerate, then $Rad(B)$ is non-trivial and, by (i), has no W-invariant complement in V. So (iv) follows from Maschke's theorem. □

2.9. Here, we introdice Coxeter cpmplexes associated with a Coxeter system of type M and indicate their prominent appearence in Tits buildings. Throughout (2.9), let G be a group and S denote a set of its involutory generators, indexed by I. Vertices of the Cayley graph $\Gamma = \Gamma_G(S)$ (see (2.3)) are called *chambers* of Γ and paths in Γ are called *galleries* in Γ.

(a) Type of a gallery in Γ: The *type* of a gallery $\nu = (g{=}g_0, g_1, \ldots, g_k)$ in Γ is the word $f = (i_1, i_2, \ldots, i_k)$ in I, where $g_i = g_{i-1}s_i$ for $i \in [1, k]$. We write ν as ν_f if $g_0 = I_G$ and as $g\nu_f$ in general. We say that ν_f is of *type* $J, J \subseteq I$, (or ν_f is a *J-gallery*) if

$\{i_1, i_2, \ldots, i_k\} \subseteq J$. Since $|s_i| = 2$ for each $i \in I$, for each $g \in G$ and $i \in I$, there is a unique edge labelled s_i incident with g. Consequently, there is a unique gallery $g\nu_f$ in Γ of a given type $f = (i_1, i_2, \ldots, i_k)$ from g to gr_f; namely, $g\nu_f = (gg_0, gg_1, \ldots, gg_k)$, where $g_0 = I_G$, $g_t = s_{i_1} s_{i_2} \ldots s_{i_t}$ for all $t \in [1, k]$. In particular, $g_k = r_f$. For $g, g' \in G$, $f \mapsto g\nu_f$ is a bijection from the set of words f in I with $r_f = g^{-1}g'$ to the set of galleries $g\nu_f$ in Γ from g to g'. Note that $g\nu_f$ is of type f.

(b) Residues and coresidues in Γ: For $J \subseteq I$, define an equivalence relation '$\sim_J$' on G by setting $g \sim_J g'$ if either $g = g'$ or Γ has a gallery of type J from g to g'. Equivalence classes in G relative to $\sim_J$ (respectively, relative to $\sim_{I \setminus J}$) are called *J-residues* (respectively, *J-coresidues*) in Γ. The J-residue in Γ containing an element g of G is the coset gG_J of the subgroup $G_J = \langle s_j : j \in J \rangle$ in G. Thus, if $J = \emptyset$, then the J-residues in Γ are the elements of G, and the set G is the only J-coresidue in Γ. Further, the terms '$\{i\}$-residue' and 'i-panel' refer to the same object, viz., $g\langle s_i \rangle$.

(c) $\Gamma_W(S)$ as a thin building of type M: From Theorem 2.4 (i), the Cayley graph $\Gamma_W(S)$ associated with a Coxeter system (W, S) of type M has the following properties: each i-panel is incident with two chambers of Γ (since $|s_i| = 2$); for each $w \in W$, the map taking $i \in I$ to the i-panel with w as a face is a bijection from I to the set of all panels in Γ with w as a face (because $s_i \neq s_j$ for $i \neq j$); and for all $i, j \in S$, $i \neq j$, each $\{i, j\}$-residue has $2m_{ij}$ chambers (since s_i, s_j generate a dihedral subgroup of W of order $2m_{ij}$). For $x, y \in W$, there is a gallery of type f, $f \in I^*$, from x to y if, and only if, $r_f = x^{-1}y$; and, when it exists, it is unique. Thus, $\Gamma_W(S)$ is a thin building of type M (see Example 2.6 below). In view of the remarks above, we identify $i \in I$ with $s_i \in S$ for each $i \in I$ and talk of panels, types of galleries, etc., in terms of S only.

(d) Coxeter complexes as apartments in buildings: Coxeter complexes are crucial substructures of a building Tits constructed as a natural geometric structure for each simple group of Lie type. Here we contend ourself by giving the four distinct, but equivalent, definitions of a building Tits gave to indicate the role of Coxeter complexes in buildings.

(d.i) The earliest definition Tits gave in [25] is in terms of simplicial complexes. Before we give the definition of a building, we recall a few concepts related to simplicial complexes. A *simplicial complex with vertex set $\mathcal{V}$* is a non-empty collection $\triangle$ of finite (possibly empty) subsets of $\mathcal{V}$ (called *simplicies*) such that every singleton subset of a non-empty set $\mathcal{V}$ is in $\triangle$ (called a *vertex*); and every subset B of a simplex A is a simplex (called a *face* of A). The cardinality r of A is called the *rank* of A and $r-1$ is called the *dimension* of A.

Equivalently, a *simplicial complex* can be thought of as a non-empty poset $\triangle$ possessing the following properties:

(α) any two elements of $\triangle$ have a greatest lower bound; and

(β) for each $a \in \triangle$, the poset $\triangle_{\leq a} = \{b \in \triangle : b \leq a\}$ is isomorphic to the power set of a set of cardinality r (considered as a poset with inclusion as the partial order). We say that *rank* of a is r and *dimension* of a is $r-1$.

Maximal simplicies in $\triangle$ are called *chambers* in $\triangle$. A simplicial complex is said to be *regular of rank* k if all maximal simplicies have rank k.

A regular simplicial complex $\triangle$ of rank r is called a *chamber complex* if all chambers have rank r and any two chambers can be connected by a gallery in $\triangle$; that is, given any two chambers c, d in $\triangle$, there is a finite sequence $(c = c_0, \ldots, c_t = d)$ of chambers in $\triangle$ such that consecutive chambers c_{i-1}, c_i share a simplex of rank $r-1$. A chamber complex $\triangle$ is said to be *thin, rigid or thick* according as each simplex of rank $r-1$, is on two, at least two or more than two chambers respectively. With these definitions, we are ready to give the first definition of a building.

First definition of a building (Tits): A *building* is a pair $\mathcal{B} = (\triangle, \mathcal{A})$, where $\triangle$ is a simplicial complex and $\mathcal{A}$ is a family of subcomplexes of $\triangle$ (called *apartments*), such that the following holds:

(B-1) $\triangle$ is the union of its apartments;

(B-2) each apartment is a thin simplicial complex;

(B-3) any two given simplicies of $\triangle$ are contained in an apartment;

(B-4) if Σ and Σ' are apartments containing two simplicies A and B, then there is a simplicial isomorphism from Σ to Σ' fixing A and B;

Remark 2.6.

• Taking A and B to be empty in (B-4), we see that any two apartments are isomorphic. Further thinnes of an apartment can be shown to imply that every apartment is a Coxeter complex. So, we can talk of a building of type M, where M is the Coxeter matrix which is the common type of the apartments in $\triangle$.

• $\triangle$ is also a chamber complex, because if A and B are maximal simplicies, by (B-3), they are contained in an apartment $\mathcal{A}$. So, they have the same rank and are connected by a gallery in $\mathcal{A}$ and so also in $\triangle$. Thus, $\triangle$ is connected.

Example 2.6. *The Coxeter system $\Sigma = \Sigma_W(S)$ defined in (2.4) is a thin building (with only one apartment, namely Σ).*

Example 2.7. *Let V be a right vector space of dimension $n+1$ over a field or a skew-field K. Let $\triangle$ be the poset of all proper subspaces U of V; that is, $\{0\} < U < V$, partially ordered by inclusion. An apartment is the subposet A_φ of $\triangle$ consisting of all proper subspaces of V generated by subsets of a given basis φ of V. Then, $(\triangle, \{A_\varphi : \varphi$ a basis of $V\})$ is a building of type A_n (see [22]).*

(d.ii) Second definition of a building (Tits): Let (W, S) be a Coxeter system of type M. A *building $\mathcal{B}$ of type M* is a pair $(\triangle, \delta)$, where $\triangle$ is a non empty set (whose elements are called *chambers*) and δ is a map from $\triangle \times \triangle$ to the Coxeter group W of type M(called the *Weyl distance*) such that, for all $c, d \in \triangle$, the following statements hold:

(Wd-1) $\delta(c, d) = I_W$ if, and only if, $c = d$.

(Wd-2) If $\delta(c, d) = w$ and $c' \in \triangle$ is such that $\delta(c', c) = s \in S$, then $\delta(c', d) = sw$ or w. If, in addition, $l(sw) = l(w) + 1$, then $\delta(c', d) = sw$.

(Wd-3) If $\delta(c, d) = w$, then, for each $s \in S$, there exists $c' \in \triangle$ such that $\delta(c', c) = s$ and $\delta(c', d) = sw$.

Example 2.8. *Take $\triangle = W$ and define δ by setting $\delta(x, y) = x^{-1}y$ for $x, y \in W$.*

Example 2.9. *Let V be as in Example 2.7 and let $\triangle$ now denote the set of all maximal flags of proper subspaces of V; that is, sequences $F = (U_1, \dots, U_n)$ of subspaces of V, where dimension of U_i is i and $U_i < U_{i+1}$ for each i. Let $W \simeq S_{n+1}$, the Coxeter group corresponding to the diagram A_n in Figure 1. For $F = (U_1, \dots, U_n)$ and $F' = (U_1', \dots, U_n')$ in $\triangle$, let $\delta(F, F') \in S_{n+1}$ be the map taking $i \in [1, n+1]$ to $\sigma(i) = min\{j \in [1, n+1] : U_i \le U_{i-1} + U_j'\}$.*

We take $U_0 = \{0\}$ and $U_{n+1} = V$. For $F, F' \in \triangle$, the map $\delta(F, F')$ is a permutation of the set $[1, n+1]$ (see [27], p.48) and $(\triangle, \delta)$ is a building of type A_n.

(d.iii) To present the third definition of a building, we introduce the concept of a chamber system. For each $s \in S$, define the equivalence relation '$\sim_s$' on W by setting $x \sim_s y, x, y \in$

W, if, and only if, $\delta(x,y) \in \langle s \rangle$. Then, the pair $(\triangle, \{\sim_s : s \in S\})$ is a chamber system in the sense of the following definition:

Definition 2.10. *A chamber system $\mathcal{C}$ over a set I is a pair $(\triangle, \{\sim_i : i \in I\})$, where $\triangle$ is a non empty set and, for each $i \in I$, '$\sim_i$' is an equivalence relation on $\triangle$ such that for each $i \in I$, each equivalence class relative to $\sim_i$ has atleast two elements; and, for $c, d \in \triangle$ and $i, j \in I$, $c \sim_i d$, $c \sim_j d$ imply that either $c = d$ or $i = j$.*

Note that a chamber system $\mathcal{C}$ can also be seen as a graph with vertex set $\triangle$ and each edge assigned a unique label (more colourfully, a 'colour') which is an element of I.

Elements of $\triangle$ are called *chambers* and an equivalence class relative to $\sim_i$ is called an *i-panel*. The chamber system $\mathcal{C}$ is said to be *thin, rigid, thick* according as each i-panel of $\mathcal{C}$ has two, atleast two, atleast three chambers, respectively.

A *gallery* in $\mathcal{C}$ from a chamber c to a chamber d is a finite sequence $\nu = (c{=}c_0, \ldots, c_k{=}d)$ of elements of $\triangle$ such that $c_{t-1} \sim_{i_{t-1}} c_t$ for each $t \in [1,k]$; its *length* is t; it is of *type* $f = (i_1, \ldots, i_k) \in I^*$ and is written as ν_f ; it is *closed* if $c_k = c_0$ and *simple* if $c_{t-1} \neq c_t$ and $i_{t-1} \neq i_t$ for each $t \in [1,k]$ (taking the relation $i_0 \neq i_1$ to be vacuous); and ν is said to be a *geodesic* if its length is the least among the lengths of the galleries in $\mathcal{C}$ from c to d. We say that a subset A of $\triangle$ is *convex* if A contains each chamber of each geodesic in $\mathcal{C}$ connecting any two distinct chambers of A.

Definition 2.11. *Let (W,S) be a Coxeter system of type $M = (m_{i,j})_{i,j \in I}$. A chamber system $\mathcal{C}$ over I is said to be of* type M *if the following holds:*
***(Cox-**1)* *for $i, j \in I$, $i \neq j$, and an integer n, $1 \leq n < 2m_{i,j}$, $\mathcal{C}$ contains no simple, closed galleries of type $P_n(i,j)$;*
***(Cox-**2)* *for $i, j \in I$, $i \neq j$, and $m = m_{i,j} \neq \infty$, if $\mathcal{C}$ contains a simple gallery of type $P_m(i,j)$, then $\mathcal{C}$ also contains a simple gallery of type $P_m(j,i)$ with the same extremities.*

Example 2.12. *With W, S, M as above, $(W, \{\sim_s : s \in S\})$ is a chamber system of type M with the equivalence relation '$\sim_i$' defined on W by $w \sim_i w'$ if $w' = ws$.*

Example 2.13. *Let $V, \triangle, F, F' \in \triangle$ be as in Example 2.9 and $I = [1,n]$. For $i \in I$, define the equivalence relation '$\sim_i$' on $\triangle$ by saying that $F \sim_i F'$ if $U_t = U'_t$ for each $t \in I, t \neq i$. Then $(\triangle, \{\sim_i : i \in I\})$ is a thick chamber system of type M over I.*

The following example is quite general and fundamental.

Example 2.14. Coset chamber system: *Let G be a group; B be a subgroup of G; $\{P_i\}_{i\in I}$ be a collection of subgroups of G, each containing B; and $\triangle = \{gB : g \in G\}$. For $i \in I$, let '$\sim_i$' be the equivalence relation on $\triangle$ defined by setting $gB{\sim_i}g'B$ if, and only if, $gP_i = g'P_i$. The resulting chamber system $(\triangle, \{\sim_i : i \in [n]\})$ is written as $(G, B, \{P_i\}_{i\in I})$.*

Example 2.12 is the case with $G = W$, $B = \langle Id\rangle$ and $P_i = \langle s_i\rangle$, $s_i \in S$. For a simple algebraic group G, the chamber complex Tits used in constructing the building associated with G is $(G, B, \{P_i\}_{i\in I})$, where B is a Borel subgroup of G and $\{P_i\}$ to be the set of all minimal parabolic subgroups of G containing B.

We now present the:

Third definition of a building (Tits): Let (W, S) be a Coxeter system of type $M = (m_{i,j})_{i,j\in I}$. A *building $\mathcal{B}$ of type M* is a pair $(\mathcal{C}, \delta)$, where $\mathcal{C} = (\triangle, \{\sim_i : i \in I\})$ is a chamber system over I and $\delta : \triangle \times \triangle \longrightarrow W$ is a map such that, for each reduced word f in S (relative to the Coxeter system (W, S)) and for each $x, y \in \triangle \times \triangle$, $\delta(x, y) = r_f = x^{-1}y \in W$ if, and only if, there is a gallery in $\mathcal{C}$ of type f from x to y.

δ is called the *Weyl distance* and W is called the *Weyl group*.

Example 2.15. *$(\mathcal{C}, \delta)$, where $\mathcal{C}$ is as in Example 2.12 and $\delta : W \times W \to W$, taking $(x, y) \in W \times W$ to $x^{-1}y \in W$, is a thin building.*

Given two buildings $\mathcal{B} = (\mathcal{C}, \delta)$ and $\mathcal{B}' = (\mathcal{C}', \delta')$ of the same type M and the same Weyl group W, a map π from a subset X of $\mathcal{C}$ to $\mathcal{C}'$ is said to be an *isometry* if $\delta'(\pi(x), \pi(y)) = \delta(x, y)$ for all $x, y \in X$.

A fundamental result due to Tits says that an isometry from any subset of Σ_M to a building $\mathcal{B}$ of type M extends to an isometry from Σ_M to $\mathcal{B}$, ([26], 3.7.4). In view of this, a building of type M has plenty of apartments.

(d.iv) Fourth definition of a building (Tits):

This is based on the concept of a geometry of type M due to Buekenhout and the characterization of buildings among geometries of a given type M. Here, generalized polygons are the basic structures involved in the definition of a geometry. See [26] for this elegant theory.

2.10. Reflections, walls and half spaces in $\Gamma{=}\Gamma_G(S)$: Let (G, S) be as in the beginning of (2.9). Let

$$\mathcal{R} = S^G = \{gsg^{-1} | g \in G, s \in S\},$$

$\mathcal{H}$ be a set in bijection with $\mathcal{R}$ and $\mathfrak{h} = \mathcal{R} \times \{-1, 1\}$. We call elements of $\mathcal{R}$ *reflections* in Γ. Some times, elements of S are also called *fundamental reflections* in Γ. Elements of $\mathcal{H}$ are called *walls* in Γ and those of $\mathfrak{h}$ are called *half spaces* of Γ.

We denote by r_H and H_r the reflection and the wall, respectively, corresponding to $H \in \mathcal{H}$ and $r \in \mathcal{R}$ under the fixed bijection between $\mathcal{R}$ and $\mathcal{H}$ mentioned above. We identify H_r with the union P_r of all panels fixed by the automorphism τ_r of Γ defined in (2.3). The chambers incident with a panel fixed by τ_r are interchanged by τ_r. When G is a Coxeter group, $G \setminus P_r$ has two connected components which we indicate by the elements $(r,\ 1)$ and $(r,\ -1)$ of $\mathfrak{h}$, and are interchanged by τ_r.

We transport the action of G on $\mathcal{R}$ by conjugation (with the action of $g \in G$ on $r \in R$ written as grg^{-1}) to an action φ of G on $\mathcal{H}$, by defining

$$\varphi(g)(H_r) := H_{grg^{-1}},\ \ g \in G,\ r \in \mathcal{R}.$$

We abbreviate $\varphi(g)(H_r)$ as gH_r. Note that, for $g \in G$ and $H \in \mathcal{H},\ r_{g(H)} = gr_Hg^{-1}$.

For $r = xsx^{-1} \in \mathcal{R}$, $\tau_r \in Aut(\Gamma)$ fixes an s'-pannel $g\langle s' \rangle$ if, and only if, $r = gs'g^{-1}$, and, in this case, τ_r interchanges the chambers of the panel $g\langle s' \rangle$. We then say that H_r *separates* the chambers g and gs' of Γ.

Notation: For $f = (s_1,\ \ldots,\ s_k) \in S^*$, define $g_0 := r_0 = Id_G,\ g_i = s_1 \ldots s_i \in G$ and $r_i = g_{i-1}s_ig_{i-1}^{-1} \in \mathcal{R}$ for all $i \in [1,\ k]$. We write $\varphi(f) = \{r_1, \ldots, r_k\} \subseteq \mathcal{R}$, $r(f) = (r_1, \ldots, r_k) \in \mathcal{R}^*$, and for $r \in \mathcal{R}$, define $n(f,\ r) = \{i \in [1, k] : r_i = r\}$ and $\eta(f, r) = (-1)^{n(f,r)}$.

Note that $r_f = r_k \ldots r_1 \in G$ and $n(s, s) = 1$ for $s \in S$. If f is the empty word in S, then $r_f = 1_G$ and $n(f,\ r) = 0$ for each $r \in \mathcal{R}$.

For $g,\ g' \in G$ and $f = (s_1, \ldots, s_k) \in S^*$ such that $r_f = g^{-1}g'$, $(gg_0, \ldots, gg_k)$, where g_i is as above, is a gallery in Γ of type f from g to gr_f. We denote it by $g\gamma_f$; and when $g = 1_G$, as γ_f. The involutory automorphism $\tau_{gr_ig^{-1}}$ of Γ interchanges the chambers gg_{i-1} and gg_i of the s_i-panel $gg_{i-1}\langle s_i \rangle$. We say that the gallery $g\nu_f$ *crosses* the walls $h_{gr_1g^{-1}}, \ldots, h_{gr_kg^{-1}}$. Thus, for $r \in \mathcal{R}$, $n(f, r)$ is the number of times the gallery ν_f crosses the wall $\mathcal{H}_r$. If $f \in S^*$ is such that r_f is a reduced expression of the element $r_f \in G$ in S, then $n(f,\ r) = 0$ or 1 for each $r \in \mathcal{R}$ in view of the following lemma.

Lemma 2.16. *With the notation above, if $r_i = r_j$ for some $i, j \in [1,\ k]$, $i < j$, and $f' = (s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k) \in S^*$, then $r_f = r_{f'}$.*

Proof. Write $r_i = r_j$ as r, $r \in \mathcal{R}$, and consider the gallery $\nu_f = (1_G = g_0,\ g_1, \ldots, g_k = g)$ in Γ. Its type is f. The automorphism τ_r of Γ is involutory, type preserving and interchanges the chambers of each of the 2-sets $\{g_{i-1},\ g_i\}$ and $\{g_{j-1},\ g_j\}$. So, it maps the part γ_1 of γ_f from g_i to g_{i-1} to a gallery γ_2, namely $(g_{i+1}, \ldots, g_{j-1})$, of the same type from g_{i-1} to g_j. Replacing the part ν_1 of ν_f by ν_2, we get a gallery ν' of type f' such that $r_f = r_{f'}$. □

For a general pair (G, S) as above, the set of walls crossed by different geodesics in Γ with the same initial and the final chambers might be different. Remarkably, for a Coxeter system, all geodesic in Γ with the same initial and final chambers cross the same set of walls in Γ. (see Theorem 0.4(ii)).

2.11. Characterizations of Coxeter systems: We start with

Lemma 2.17. *Let G be a group and S be a set of involutions of G generating G. For each $s \in S$, define a map ρ_s from $\mathfrak{h}$ to itself by setting, for each $(r, \epsilon) \in \mathfrak{h}$,*

$$\rho_s(r, \epsilon) = \begin{cases} (srs, \epsilon); & \text{if } r \neq s \\ (s, -\epsilon); & \text{if } r = s. \end{cases}$$

then:

(i) $\rho_s \neq (\rho_s)^2 = id$ *on* $\mathfrak{h}$*; consequently,* ρ_s *is a permutation of* $\mathfrak{h}$*;*
(ii) for $f = (s_1, \ldots, s_k) \in S^*$,

$$\rho_{s_k}\rho_{s_{k-1}}...\rho_{s_1}((r, \epsilon)) = (g^{-1}rg,\ \epsilon(-1)^{n(f,r)}),$$

where $g = r_f = s_1 \ldots s_k \in G$.

Proof. (i) is clear. We prove (ii) by induction on k. The cases $k = 0$ and $k = 1$ are clear. Let $k > 1$, $f' = (s_1, \ldots, s_{k-1}) \in S^*$ and $g' = s_1 \ldots s_{k-1}$. Using induction hypothesis,

$$\begin{aligned} \rho_{s_k}(\rho_{s_{k-1}} \ldots \rho_{s_1})((r, \epsilon)) &= \rho_{s_k}(g'^{-1}rg',\ \epsilon(-1)^{n(f',\, r)}) \\ &= (g^{-1}rg,\ \epsilon(-1)^{n(f',\, r)+\delta_{s_k,\, r}}) \\ &= (g^{-1}rg,\ \epsilon(-1)^{n(f',\, r)})), \end{aligned}$$

because $r(f) = (r(f'), g'^{-1}s_k g')$. So, (ii) holds. □

Lemma 2.18. *let $(G,\ S)$ be a Coxeter system of type M; $s,\ s' \in S$ be such that $m = m_{s,s'} \neq \infty$, $f = (s, s', s, \ldots, s, s') \in S^*(2m$ terms$)$, and $r \in \mathcal{R}$. Then, $n(f, r) = 0$ or 2.*

Proof. For $1 \leq j \leq 2m$, $r_j = (ss')^{j-1}s$. Since the order of ss' in G is m, the elements $r_1, \ldots, r_m$ of the dihedral subgroup $\langle s,\ s'\rangle$ are mutually distinct. Further, $r_j = r_{j+m}$ for each $j \in [1, m]$. So, for each $r \in \mathcal{R}$, $n(f, r) = 2$ if, and only if, $r = (ss')^{j'-1}s$ with $j' \in \{j,\ j+m\}$ for some $j \in [1,\ m]$, and $n(f,\ r) = 0$ otherwise. □

We now show that each of the conditions $(A), (D), (E), (F)$ stated in what follows on the pair $(G,\ S)$ characterise the Coxeter systems $(W,\ S)$. We start with

(A) the action condition on $(G\ S)$**:** *There is a homomorphism ρ from G to the group sym$(\mathfrak{h})$ of permutations of $\mathfrak{h}$ such that, for each $s \in S$, $\rho(s)$ maps $(r,\ \epsilon) \in \mathfrak{h}$ to $(r, -\epsilon) \in \mathfrak{h}$ if $r = s$ and to $(srs,\ \epsilon) \in \mathfrak{h}$ if $r \neq s$.*

We write $\rho(g)$ as ρ_g throughout.

Proposition 2.19. *Assume that (A) holds for (G, S). Then, for $g \in G$, define $\mathcal{R}(g) = \{r \in \mathcal{R} : \rho_{g^{-1}}((r, 1)) = (g^{-1}rg, -1)$.*

- *(i) A reflection $r \in \mathcal{R}$ belongs to $\mathcal{R}(g)$ if, and only if, r occurs an odd number of times in $r(f)$ for each $f \in S^*$ such that $r_f = g$ (see Notation, page 22). In particular, $\mathcal{R}(g) \subseteq \varphi(f)$ for each $f \in S^*$ such that $r_f = g$ and for $r \in \mathcal{R}$, $\eta(f,\ r) = 1$ if $r \in \mathcal{R}(g)$ and zero otherwise.*
- *(ii) For any word $f = (s_1, \ldots, s_k)$ in S such that r_f is a reduced expression for g in S, we have $r_i \neq r_j$ for all $i,\ j \in [1,\ k], i \neq j$. Consequently, $\mathcal{R}(g) = \varphi(f)$.*
- *(iii) $|R(g)| = l(g)$.*

Proof. Let $f = (s_1, \ldots, s_k) \in S^*$ be such that $r_f = g$. First, note that, by Lemma 2.17(ii), for $r \in \mathcal{R}$,

$$\rho_{s_{i-1}} \ldots \rho_{s_1}((r,\ 1)) = (g_{i-1}^{-1} r g_{i-1},\ \epsilon)$$

for some $\epsilon \in \{-1,\ 1\}$. By the action of ρ_{s_i} on $\mathfrak{h}$ specified in (A), the second coordinate of its image under ρ_{s_i} will be $-\epsilon$ if, and only if, $g_{i-1}^{-1} r g_{i-1} = s_i$, that is, $r = g_{i-1} s_i g_{i-1}^{-1}$. So $\rho_{g^{-1}}((r, 1)) = (g^{-1}rg,\ (-1)^p)$, where p is the number of terms r_i in $r(f)$ such that $r = r_i$. So (i) follows.

(ii) follows from (i) and Lemma 2.16.

(iii) folllows from (i) and (ii). □

For $g \in G$ and $r \in \mathcal{R}$, we write $\eta(g, r) = 1$ if $r \in r(f)$ where f is the type of a geodesic in Γ from 1_G to g, and zero otherwise.

We now give another discription of $\mathcal{R}(g)$. Let $\mathcal{R}_g$ denote the set of all elements $g_1 s g_1^{-1}$ of G corresponding to the triples $(g_1,\ s,\ g_2) \in G \times S \times G$ such that $g = g_1 s g_2$ and $l(g) = l(g_1) + 1 + l(g_2)$.

Proposition 2.20. *If $(G,\ S)$ satisfies (A), then $\mathcal{R}(g) = \mathcal{R}_g$ for each $g \in G$.*

Proof. If $r \in \mathcal{R}(g)$ and $s_1 \dots s_d,\ s_i \in S$, is a reduced expression of g in S, then, by Proposition 2.19 (i), $r = (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1}$. Taking $g_1 = s_1 \dots s_{i-1}$, $s = s_i$ and $g_2 = s_{i+1} \dots s_d$, we see that $r \in \mathcal{R}_g$.

Now, let $r = g_1 s g_1^{-1} \in \mathcal{R}_g$ with $g_1,\ s,\ g_2$ as in the definition of $\mathcal{R}_g$. If $s'_1 \dots s'_m$ and $s''_1 \dots s''_n$ are reduced expression for g' and g'' in $\mathcal{S}$, respectively, then $(s'_1, \dots, s'_m,\ s,\ s''_1, \dots, s''_n)$ is a reduced expression in S for g and $r \in r(f)$. By Proposition 2.19 (ii) and (i), the elements of $r(f)$ are mutually distinct and $r \in \mathcal{R}(g)$. □

(D) Deletion condition on (G, S): *If $f = (s_1, \dots, s_k) \in S^*$ is such that $l(r_f) < k$, then there exist $i, j \in [1,\ k]$, $i \neq j$, such that $r_f = r_{f'}$ for $f' = (s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_k)$.*

Proposition 2.21. *If (A) holds for $G(S)$, then so does (D).*

Proof. If f is as in the statement of (D), then, $s_1 \dots s_k$ is not a reduced expression for $r_f \in G$ in S. By Proposition 2.19 (ii), $r_i = r_J$ for some $i\ j \in [1,\ m]$, $i \neq j$ and (D) now follows from Lemma 2.16. □

Proposition 2.22. *Let (W, S) be a Coxeter system of type M. Then,*

(i) $(W,\ S)$ satisfies (A) and so (D);

(ii) for $w \in W$ and $(r,\ \epsilon) \in \mathfrak{h}$, $\rho_w((r,\ \epsilon)) = (w^{-1} r w,\ \epsilon\eta(w^{-1},\ r))$.

Proof. (i) For each $s \in S$, the map ρ_s defined in (A) is an involution. So $\rho_s \in sym(\mathfrak{h})$. If $s_1,\ s_2 \in S$ and $m = m_{s_1, s_2} \neq \infty$, then, by Lemma 2.17 (ii) and Lemma 2.18, $\rho_s \rho_{s'}$ is of order m. So, by the presentation of $W(S)$, the map $s \to \rho_s$ from S to $sym(\mathfrak{h})$ extends to a group homomorphism from W to $sym(\mathfrak{h})$. This, together with proposition 2.21, imply (i).

(ii) follows from Lemma 2.17 (ii). □

(E) **The exchange condition on (G, S):** (Matsumoto) *For $g \in G$, $s \in S$, if $s_1 \dots s_k$ is a reduced expression for g in S, then, either $l(sg) = l(g) + 1$ or else there exists $i \in [1,\ k]$ such that*

$$s_1 \dots s_k = s s_1 \dots \hat{s}_i \dots s_k.$$

Equivalently,

$$s_1 \dots s_i = ss_1 \dots s_{i-1}.$$

The condition (E) says that, if $l(sg) < l(g)$, then any geodesic γ in $\Gamma_G(S)$ from 1_G to g, $g \in S$, can be modified to a geodesic from 1_G to g with $\{1,\ s\}$ as the initial edge and dropping an appropriate edge of γ.

The condition (E) rules out the possibility $l(sg) = l(g)$ for $g \in G$ and $s \in S$. Consequently, $\{l(sg),\ l(gs)\} \subseteq \{l(g)+1, l(g)-1\}$ for $g \in G$, $s \in S$. Further, lengths of different expressions of g in S have the same parity.

Proposition 2.23. *The condition (E) holds for a Coxeter system (W, S).*

Proof. For $w \in W$ and $s \in S$, $l(sw) = l(w) + 1$ or $l(w) - 1$ by (2.6). If $l(sw) = l(w) - 1$ and $f = (s_1, \dots, s_k) \in S^*$ is such that r_f is a reduced expression for w, then, by the condition (D) which holds in a Coxeter system (see Proposition 2.22 (ii)), $sw = f'$, where $f' = (s = s_0, \dots, \hat{s_i}, \dots . \hat{s_j} \dots, s_k)$ for some $i,\ j \in [0, k]$, $i \neq j$. Since r_f is a reduced expression for w in S, it follows that $i = 0$ and $sw = s_1 \dots \hat{s_j} \dots s_k$. Thus (E) holds for (W, S). □

Applying the condition (E) to g^{-1}, we get the (right) exchange condition (E') stated below on $(G,\ S)$ which is clearly equivalent to (E).

(E') *For $g \in G$, $s \in S$, if $s_1 \dots s_k$ be a reduced expression of g in S, then either $l(gs) = l(g) + 1$ or there exists $i \in [1,\ k]$ such that*

$$s_1 \dots s_k = s_1 \dots \hat{s_i} \dots s_k s.$$

Equivqlently,

$$s_i \dots s_k = s_{i+1} \dots s_k s.$$

Proposition 2.24. *Asuume that the condition (E) holds for $(G,\ S)$. If $s_{i_1} \dots s_{i_n}$ and $s_{j_1} \dots s_{j_n}$ are reduced expressions of g, $g \in G$ and $s_{i_t},\ s_{j_t} \in S$, then $\{s_{i_1}, \dots, s_{i_n}\} = \{s_{j_1}, \dots, s_{j_n}\}$.*

Proof. Deny. Let n be the smallest integer for which there is a counter example g, $g \in G$, with $l(g) = n$. Then, $n \neq 1$ by Theorem 2.4(i). Since $(s_{i_1} \dots s_{i_n})s_{j_n} = s_{j_1} ... s_{j_{n-1}}$ is of length less then n, by (E'),

$$s_{j_1} \dots s_{j_{n-1}} = (s_{i_1} \dots s_{i_n})s_{j_n} = s_{i_1} \dots \hat{s}_{i_t} \dots s_{i_n}.$$

for some $t \in [1,\ n]$. By the choice of n, $\{s_{j_1}, \dots, s_{j_{n-1}}\} \subseteq \{s_{i_1}, \dots, s_{i_n}\}$. Considering

$$s_{j_1}(s_{i_1} \dots s_{i_n}) = s_{j_2} \dots s_{j_n}$$

instead and using (E), we see that $\{s_{j_2}, \dots, s_{j_n}\} \subseteq \{s_{i_1}, \dots, s_{i_n}\}$. So, $\{s_{j_1}, \dots, s_{j_n}\} \subseteq \{s_{i_1}, \dots, s_{i_n}\}$ and, by symmetry, the proposition follows. ☐

Definition: Let $g \in G$ and $g = s_1 \dots s_k$, $s_i \in S$, be a reduced expression. Define

$$S_g = \{s_1, \dots, s_k\}.$$

In view of Proposition 2.24, if $(G,\ S)$ satisfies (E), then S_g is independent of the reduced expression of $g \in S$. This, in particular, implies that each element of G, admits only a finite number of reduced expressions in S. (see Remark 2.12 (1))

Corollary 2.25. *If $(G,\ S)$ satisfies (E), then S is a minimal generating set for G. Consequently, if G is finitely generated, then it admits a finite subset S of G such that $(G,\ S)$ satisfies (E).*

Corollary 2.26. *If $(G\ S)$ satisfies (E), then, for each $g \in G$, $\Delta(g) = \{s \in S : l(sg) \leq l(g)\}$ is a finite subset of S.*

Proof. For each $s \in \Delta(g)$, by (E), there is a reduced expression for g in S starting with s. Since the number of reduced expressions of g in S is finite, the corollary follows. ☐

(F) Folding condition on $(G,\ S)$**:** *If $g \in G$ and $s,\ s' \in S$ are such that $l(sg) = l(g) + 1 = l(gs')$, then either $l(sgs') = l(g) + 2$ or $sgs' = g$.*

Proposition 2.27. *If (E) holds for $(G,\ S)$, then so does (F).*

Proof. Let $g = s_1 \dots s_k$, $s_i \in S$, be a reduced expression for g in S. Then, $s_1 \dots s_k s'$ is a reduced expression for gs' in S. By (E), either $l(sgs') = l(g)+2$ or there exists $i \in [1,\ k+1]$ such that $sgs' = s_1 \dots \hat{s}_i \dots s_k s_{k+1}$ (writing s' as s_{k+1}). Since $l(sg) = l(g) + 1$, $i = k + 1$ and $sgs' = g$. ☐

Proposition 2.28. *If (F) holds for $(G,\ S)$, then so does (D).*

Proof. Let $g = s_1 \dots s_d$, $s_i \in S$, and $l(g) < d$. We show, by induction on d, that there exists $i,\ j \in [1,\ d]$, $i < j$, such that $g = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_d$. By induction hypothesis, we may assume that $s_1 \dots s_{d-1}$ and $s_2 \dots s_d$ are both reduced expressions in S. Note that $d \geq 2$ and consider $g' = s_2 \dots s_{d-1}$. Then, $l(s_1 g') = l(g') + 1 = l(g' s_d) < l(g') + 2$. So, by (F), $s_1 g' s_d = g'$; that is $g = \hat{s}_1 s_2 \dots s_{d-1} \hat{s}_d$. □

Proposition 2.29. *If (D) holds for $(G,\ S)$, then so does (E).*

Proof. Let $g = s_1 \dots s_k$, $s_i \in S$, be a reduced expression for g in S and $s_0 \in S$ be such that $l(s_0 g) < l(g)$. Then, by (D), there exist $i,\ j \in [0,\ k]$, $i \neq j$, such that $s_0 g = s_0 s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. If $i \neq 0$, then $g = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ and $l(g) < k$, a contadiction. So, $i = 0$ and $s_0 g = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. Thus (E) holds in $(G,\ S)$. □

Following [1], we, now show that, if (G, S) satisfies (E), then (G, S) is a Coxeter system.

Assume that $(G,\ S)$ satisfies (E). For $g \in G$, let $\mathcal{D}_g$ denote the set of all reduced words f in S such that $r_f = g$.

Lemma 2.30. *Let A be a set and θ be a map from $\mathcal{D}_g$ to A such that $\theta(f) = \theta(f')$ for $f,\ f' \in \mathcal{D}_g$ whenever either*

(a) f and f' have the same initial term or the same final term; or

(b) $f = p_k(s,\ s')$ and $f' = p_k(s,\ s')$ for some $s,\ s' \in S$

holds. Then, θ is a constant function on $\mathcal{D}_g$.

Proof. Consider $f = (s_1, \dots, s_k)$, $f' = (s'_1, \dots, s'_k) \in \mathcal{D}_g$ and let $h = (s'_1, s_1, \dots, s_{k-1}) \in S^*$. We first show that

(*) *if $\theta(f) \neq \theta(f')$, then $h \in \mathcal{D}_g$ and $\theta(f) \neq \theta(h)$.*

Proof of (*): Since $s'_1 \dots s'_k$ is a reduced expression of g in S, $l(s'_1 g) < l(g)$. By (E), there exists an index j, $j \in [1,\ k]$, such that $r_{h'}$ is a reduced expression of g in S, where $h' = (s'_1,\ s_1,\ s_2, \dots, \hat{s}_j, \dots, s_k)$; i.e., $h \in \mathcal{D}_g$. By (a), $\theta(f') = \theta(h')$. If $j < k$, then, $\theta(f) = \theta(h')$, by (a) again, and $\theta(f') = \theta(h') = \theta(f)$, contradicting $\theta(f) \neq \theta(f')$. So $j = k$ and $h' = h$. Thus $h \in \mathcal{D}_g$ and $\theta(h) = \theta(h') = \theta(f) \neq \theta(f')$, completing the proof of (*).

Now, suppose that $\mathcal{D}_g$ contains elements f and f' such that $\theta(f) \neq \theta(f')$. Consider the following chain of words in S of length k:

$$f' = f_0 = (s'_1, s'_2, \dots, s'_k)$$
$$f = f_1 = (s_1, s_2, \dots, s_k)$$

and, for $0 \le t \le k-1$,

$$f_{k+1-t} = \begin{cases} (s_1, s_1', s_1, s_1' \dots, s_1, s_1', s_1, \dots, s_t), & \text{if } k-t \text{ is even} \\ \\ (s_1', s_1, s_1', s_1, \dots, s_1', s_1, \dots, s_t), & \text{if } k-t \text{ is odd.} \end{cases}$$

Observe that $f_k = p_k(s_1, s_1')$ and $f_{k+1} = p_k(s_1', s_1)$, if k is odd and $f_k = p_k(s_1',\ s_1)$ and $f_{k+1} = p_k(s_1,\ s_1')$, if k is even.

By (*) and induction on j, $f_j,\ f_{j+1} \in \mathcal{D}_g$ and $\theta(f_j) \neq \theta(f_{j+1})$, for $j = 1, \dots, k$. But the statement for $j = k$ contradicts (b). So θ is constant on $\mathcal{D}_g$. □

Theorem 2.31 (Matsumoto)**.** *Let A be a monoid with unit element 1_A and θ be a map from S to A. For $s, s' \in S$, let $m = m(s, s')$ be the order of ss' in G. Put $\alpha(s, s') = \big(\theta(s)\theta(s')\big)^t$, $\big(\theta(s)\theta(s')\big)^t\theta(s)$, 1_A according as $m = 2t$, $m = 2t+1$ or $m = \infty$. Assume that $\alpha(s, s') = \alpha(s', s)$ for $s, s' \in S$ with $s \neq s'$. Then, θ extends to a map $\hat\theta$ from G to A such that $\hat\theta(g) = \theta(s_1)\dots\theta(s_k)$ for all $g \in G$ and each reduced expression $s_1 \dots s_k$ of g in S.*

Proof. For $g \in G$, we show that the map $\hat\theta_g : \mathcal{D}_g \to A$ defined by $\hat\theta_g(f) = \theta(s_1)\dots\theta(s_k)$, where $f = (s_1, \dots, s_k) \in \mathcal{D}_g$, is a constant function on $\mathcal{D}_g$ by verifying the conditions (a) and (b) of Lemma 2.30, using induction on $l(g)$. This is trivial if $l(g) = 0$ or 1. Now, let $l(g) = k \ge 2$.

(a) Let $f_1,\ f_2$ (respectively, $f_1',\ f_2'$) be obtained from $f,\ f'$ by deleting their first (respectively, last) term. Let $g_i = r_{f_i}$ and $g_i' = r_{f_i'}$, $i = 1, 2$. Then r_{f_i} and $r_{f_i'}$ are reduced expresions in S of g_i and g_i' respectively. By induction hypothesis, the definition of $\hat\theta(g_i)$ and $\hat\theta(g_i')$ is independent of the choice of the reduced expressionsfor g_i and g_i' in S. Further

$$\hat\theta_g(f) = \hat\theta(s_1)\hat\theta(g_1) = \hat\theta_g(g_2)\theta(s_k),$$
$$\text{and } \hat\theta_g(f_1) = \hat\theta(s_1')\hat\theta(g_1') = \hat\theta(g_2'')\theta(s_k').$$

If $s_1 = s_1'$ (respectively, $s_k = s_k'$), then $g_1 = g_1' = s_1 g$ (respectively, $g_2 = g_2' = g s_k$) and $\hat\theta(g_1) = \hat\theta(g_1')$ (respectively, $\hat\theta(g_2) = \hat\theta(g_2')$). So $\hat\theta_g(f) = \hat\theta_g(f')$.

(b) Suppose that, there exist $s, s' \in S$ such that $p_k(s, s'), p_k(s', s) \in \mathcal{D}_g$. Since $r_{p_k}(s, s')$ and $r_{p_k}(s', s)$ are distinct reduced expressions of the same element, namely g, of the dihedral group generated by s and s', $\hat{\theta}_g(r_{p_k}(s,s') = \hat{\theta}_g(r_{p_k}(s',s)$ by hypothesis that $\alpha(s,s') = \alpha(s',s)$. So (b) of Lemma 2.30 holds. □

Theorem 2.32 (Matsumoto)**.** *Let M be a Coxeter matrix and $(G,\ S)$ be a pre-Coxeter system of type M satisfying (E). Then, $(G,\ S)$ is a Coxeter system of type M.*

Proof. Let H be a group and $\theta : S \to H$ be a map such that $\big(\theta(s)\theta(s')\big)^{m_{s,s'}} = 1_H$ for all $s,\ s' \in S$ such that $m_{s,s'}$ is finite. By Theorem 2.31, θ extends to a map $\hat{\theta}$ from G to H taking $g \in G$ to $\hat{\theta}(g) = \theta(s_1)\dots\theta(s_k)$, where $s_1 \dots s_k$ is any reduced expression of g in S. To verify that $\hat{\theta}$ is a group homomorphism, and thus show that G has Coxeter presentation given by M, it is enough we show that $\hat{\theta}(sg) = \theta(s)\hat{\theta}(g)$ for all $s \in S$ and $g \in G$, because S generates G.

Let $s \in S$ and $g = s_1 \dots s_k$ be a reduced expression of g in S. If $l(sg) = l(g) + 1$, then $ss_1 \dots s_k$ is a reduced expression of sg and $\hat{\theta}(sg) = \theta(s)\theta(s_1)\dots\theta(s_k) = \theta(s)\hat{\theta}(g)$. If $l(sg) = l(g) - 1$, then, if $g = sg$, $l(sg') = l(g') + 1$ and, by the previous case, $\hat{\theta}(sg') = \theta(s)\hat{\theta}(g')$; i.e., $\hat{\theta}(sg) = \theta(s)\hat{\theta}(g)$, completing the proof of the theorem. □

2.12. Some consequences of the exchange conditions: Assume that $(G,\ S)$ satisfies (E). For $S' \subseteq S$, the subgroup $G_{S'} = \langle S' \rangle$ of G is called a *standard parabolic subgroup* of G, any conjugate of it is called a *parabolic subgroup* of G. We denote by $\mathcal{S}(G)$ the set of all standard parabolic subgroups of G.

(1) *Let $s_1 \dots s_k$ be a reduced expression of g in S. Then, the set $S_g = \{s_1, \dots, s_k\} \subseteq S$ is independent of the reduced expression of $g \in S$ and every reduced expression of g in S contains precisely the elements of $S_{g'}$. Consequently, the number of reduced expressions in S for any element of G is finite. Further, S_g is the smallest subset S' of S such that $g \in G_{S'} \leq G$.* (see Proposition 2.24)

Proof. Let A be the monoid consisting of all subsets of A with "union of sets" as the binary operation. The identity element of this monoid is the empty set. By Theorem 2.31, the map θ from S to A taking $s \in S$ to $\{s\} \in A$ extends to a map $\hat{\theta}$ from G to A taking $g \in G$ with reduced expression $g = s_1 \dots s_k$, $s_i \in S$, in S to $\hat{\theta}(g) = \theta(s_1) \cup \dots \cup \theta(s_k) = \{s_1, \dots, s_k\}$. Further, $\hat{\theta}(g)$ is independent of the reduced expression of g in S. So the first statement in (1) follows. Rest is clear. □

By (1), for $S' \subseteq S$, each S'-residual in the chamber complex $\mathcal{C}_G(S)$ is a convex set. The following also is a consequence of (1).

(2) Let $\{S_i\}_{i\in I}$ be a family of subsets of S and $S_0 = \bigcap_{i\in I} S_i$. Then, $G_{S_0} = \bigcap_{i\in I} G_{S_i}$.

(3) *Corollary of (1): For $S' \subseteq S$ and $g \in G_{S'}$, $S_g \subseteq S'$.*

Proof. The set H of all elements g of G such that $S_g \subseteq S'$ is a subgroup of G; because, for all $s \in S$ and $g,\ g' \in G$, $S_{g^{-1}} = S_g$; and $S_{sg} \subseteq \{s\} \cup \{S_g\}$ and so, by induction, $S_{g'g} \subseteq S_{g'} \cup S_g$. Further, since $S' \subseteq H \subseteq G_{S'}$, $H = G_{S'}$. □

(4) *Corollary of (2): For $S' \subseteq S$, $G_{S'} \cap S = S'$. Consequently, the map $S \to G_{S'}$ from the power set $\mathcal{P}(S)$ of all the subsets of S to $\mathcal{S}(G)$ is an isomorphism of posets with inclusion as partial order in both $\mathcal{P}(S)$ and $\mathcal{S}(G)$.*

Proof. For $s \in G_{S'} \cap S$, $S_s = \{s\}$. By (2), $s \in S'$ and so $G_{s'} \cap S \subseteq S'$. Thus, for $S' \subseteq S'' \subseteq S$, $G_{S'} \subseteq G_{S''}$ (respectively, $G_{S'} = G_{S''}$) if, and only if, $S' \subseteq S''$ (respectively $S' = S$). So, the isomorphism of posets follows. □

(5) *For $g \in G_{S'}$, $S' \subseteq S$, $l_{S'}(g) = l_S(g)$.*

Proof. If $s_1 \dots s_k$, $s_i \in S$, is a reduced expression for $g \in G_{S'}$, then, by (1), $l_{S'}(g) \leq l_S(g) = k$. But g can not be written as a product of fewer than k elements of S'. So, $l_{S'}(g) = l_S(g)$. □

Thus, for each $g \in G_{S'}$, the terms 'reduced expression of g in S' and 'the reduced expression in S'' are the same.

(6) For any subset S' of S, $(G_{S'}, S')$ is a Coxeter system of type $M_{S'}$, where $M_{S'}$ is the restriction of M to $S' \times S'$.

Proof. Elements of S' are of order 2 and generate $G_{S'}$. Let $s \in S'$ and $g \in G_{S'}$ be such that $l_{S'}(sg) \leq l_{S'}(g)$. By (5), $l_S(sg) \leq l_S(g)$ and, if $g = s_1 \dots s_k$ is a reduced expression of g in S, then $s_i \in S'$ by (1) and it is a reduced expression in S' also. Since (G, S) satisfies (E), $sg = ss_1 \dots \hat{s}_i \dots s_k$ for some $i \in [1,\ k]$. But this means that (E) holds for $(G_{S'},\ S')$ also. So $(G_{S'},\ S')$ is a Coxeter system of type M', by Theorem 2.32.

□

(7) *For $g \in G$, define $\Delta(g) = \{s \in S : l(sg) \leq l(w)\}$. Then, $\Delta(g)$ is finite.*

Proof. Since (G, S) satisfies (E), g has a reduced expression in S starting from s. Since the number of reduced expression of g in S is finite, $\Delta(g)$ is finite. □

The following consequence of the condition (E) leads to the important notion of *Chavelley-Bruhat order* (or just *Bruhat order* on G, as it is usually referred to in the literature). For $g \in G$, let $s_1 \dots s_k$, $s_i \in G$, be a reduced expression for g in S and let

$$B(g) = \{s_{i_1} \dots s_{i_k} : 1 \leq i_1 < \dots i_t \leq k\} \cup \{1_G\} \subseteq G.$$

(8) *$B(g)$ is independent of the choice of the reduced expression of g in S.*

Proof. Let A be the monoid whose elements are the subsets of G with the monoid composition of $X, Y \in A$ defined as $XY = \{xy : x \in X, y \in Y\}$. Then, the hypothesis of Theorem 2.31 holds for the map θ from S to A taking $s \in S$ to $\{1_G, s\}$ and (8) follows. □

Bruhat order on $(G,\ S)$ **satisfying** (E)**:** For $g, g' \in G$, define $g' \preceq g$ if $g' \in B(g)$. Clearly 1_G is the smallest element of G relative to this partial order. In any expression of an element g of G as a product of elements of S, we can bring together elements of S appearing in that expression in the same connected component of the Coxeter diagram of (G, S) (see The0rem 2.32). Thus, the Bruhat order on G is the product of the Bruhat orders on the subgroups of G defined by the connected components.

(9) *Let $S' \subseteq S$, $g \in G_{S'}$ and $s \in S \setminus S'$. Then, $l(sg) = l(g) + 1$.*

Proof. Let $g = s_1 \dots s_k$, $s_i \in S$, be a reduced expression of g in S. Suppose that $l(sg) \leq l(g)$. Then, by (E), $g = ss_1 \dots \hat{s}_i \dots s_k$ for some $i \in [1,\ k]$ and this is also a reduced expression of g in S. So, by (1), $s \in S'$, a contradiction. So, $l(sg) > l(g)$ and (9) follows. □

(10) **Generalized exchange condition:** *For $g \in G$, let $\Delta(g) = \{s \in S : l(sg) < l(g)\}$ as in (7). Then, every reduced expression of an element $G_{\Delta(G)}$ appears as the initial segment of a reduced expression of g in S. Hence,*

$$l(g'g) = l(g) - l(g') \tag{2.7}$$

for each $g' \in G_{\Delta(g)}$. In particular, the length function l_S is bounded on $G_{\Delta(g)}$.

Proof. Let $s'_1 \dots s'_t$, $s'_i \in \Delta(g)$, be a reduced expression of some element g' of $G_{\Delta(g)}$. By induction hypothesis, we may assume that

$$g = s'_2 \dots s'_t s_1 \dots s_k, \quad s_i \in S,$$

is a reduced expression of g in S. Since $l(s'g) < l(g)$, by the exchange condition (E), we can get a reduced expression for g in S from this expression by putting s'_1 in the front and removing some s'_i or s_j. The removed term can not be an s'_i because $s'_1 \dots s'_t$ is a reduced expression. So, we get a reduced expression for g in S with $s'_1 \dots s'_t$ as the initial segment. The equality (2.7) follows by getting a reduced expression for g with g'^{-1} as the initial segment. Finally, by (2.7), the length function on $G_{\Delta(g)}$ is bounded by $l(g)$. □

(11) *$G_{\Delta(g)}$ is a finite group for each $g \in G$.*

Proof. $\Delta(g)$ is finite by (8) and the length function l_S is bounded on $G_{\Delta(G)}$ by (10). So, $G_{\Delta(g)}$ is finite. □

(12) *G is finite if, and only if, it has an element g_0 such that $l(sg_0) \leq l(g_0)$ for each $s \in S$. In this case, g_0 is the unique element of G of maximum length and is of order 2. Further,*

$$l(gg_0) = l(g_0) - l(g) \tag{2.8}$$

for each $g \in G$.

Proof. If G is finite, it clearly has an element g_0 of maximal length and $l(sg_0) \leq l(g_0)$ for each $s \in S$.

Conversely, if $l(sg_0) \leq l(g_0)$, for each $s \in S$, then, by remarks preceeding Proposition 2.23, $l(sg_0) < l(g_0)$ and, by (6) and (10) G is finite. Taking g to be g^{-1} in (2.8), we see that $g_0^2 = 1$. Taking $g \neq g_0$ in (2.8), we see that g_0 is the unique element of maximum length. □

2.13. Equivalence of reduced words in S and Tits Theorem.

(a) M-homotopy and M-equivalance in S^*: Let M be a Coxeter matrix over a set S. Interchanging a word $f = f_1 P_m(s, s') f_2 \in S^*,\ f_1,\ f_2 \in S^*,\ s,\ s' \in S$ with $m = m_{s,s'} \neq \infty$, with the word $f' = f_1 P_m(s', s) f_2$ is called an *M-elementary homotopy.* We say that $f,\ f' \in S^*$ are *M-homotopic* if there is a finite sequence $(f = f_0, \dots, f_t = f')$ of elements of S^* such that f_{i-1} and f_i are M-elementary homotopic for each $i \in [1, n]$. Note that M-homotopy of words in S neither changes the length of the word nor the set of the elements of S appearing in it. Replacing a word $f = f_1 s s f_2,\ f_1, f_2 \in S^*, s \in S$ (respectively, $g_1 g_2,\ \ g_1, g_2 \in S^*$) with $f_1 f_2$(respectively, $g_1 s s g_2,\ \ s \in S$) is called an *elementary contraction* (respectively, *elementary expansion*). Two words $f, f' \in S$ are said to be *M-equivalent* if there is a finite sequence $(f = f_0, ..., f_t = f')$ of words in S

such that, for each $i \in [1,\ t]$, there is either an M-elementary homotopy, a contraction or an expansion between f_{i-1} and f_i. A word in S is said to be M- *reduced* if it is not M-equivalent to a word in S of shorter length.

Lemma 2.33. *Let $(W,\ S)$ be Coxeter system of type M. Then two words f, f' in S are M-equivalent if, and only if, $r_f = r_{f'}$.*

Proof. Since $s^2 = 1_W$ and $r_{p_m(s,s')} = r_{p_m(s,s')}$ for all $s, s' \in S$ and $m = m_{s,s'} \neq \infty$ are the only relations in the presentation of W, the lemma follows. □

(b) For a pre-Coxeter system (G, S) of type M, the following theorem due to Tits describes the elements of G in terms of the matrix M, solving Dehn's word problem for Coxeter groups.

Theorem 2.34 (Tits, see [1], Exc 1.13)**.** *Let (G, S) be a pre-Coxeter system of type M satisfying the condition (E). Let $f = (s_1, ..., s_d)$ and $f' = (s'_1, .., s'_d)$ be words in S.*

- *(a) If r_f and $r_{f'}$ are reduced expressions in S for elements of G, then f and f' are M-homotopic if, and only if, $r_f = r_{f'}$.*
- *(b) If r_f is not a reduced expression in S of an element of W, then f is M-homotopic to a word of the form f_1ssf_2, where $f_1, f_2 \in S^*$ and $s \in S$.*
- *(c) The word f is reduced in S if, and only if, it is M-reduced.*

Proof. (a) By the proof of Lemma 2.33, if f and f' are M-homotopic, then $r_f = r_{f'}$.

Now, assume that $r_f = r_{f'}(=: g \in G$, say). We show, by induction on $l(f) = d$, that f and f' are M-homotopic. Write $s_1 = s$ and $s'_1 = s'$.

If $s = s'$, the M-homotopy of f and f' follows by induction hypothesis. So, let $s \neq s'$. Then, $l(sg) < l(g)$ and $l(s'g) < l(g)$. By (2.11)(6), $m = m(s, t)$ is finite and g has a reduced expression in S of the form $g = r_{p_m}(s, s')r_{f_1}$, for some $f_1 \in S^*$ of length $l(f) - m$. By the previous case, f and $p_m(s, s')f_1$ are M-homotopic, and also so are f' and $p_m(s, s')f_1$. So, f and f' are M-homotopic.

(b) We prove (b) by induction on $l(f) = d$. Assume that r_f is not a reduced expression in S. Let $f_1 = (s_2, \ldots, s_d) \in S^*$ and $g_1 = r_{f_1}$. By induction hypothesis, we may assume that r_{f_1} is a reduced expression of g_1 in S. Since $l(s_1g_1) \leq l(g_1)$ by (E), $g_1 = r_{f_2}$ for some $f_2 = (s_1, t_n, ...t_{d-2}) \in S^*$. Since r_{f_2} is also a reduced expression of g_1, by (a), f_1 and f_2 are M-homotopic. So, f and $(s_1, s_1, t_2, \ldots, t_{d-2})$ are M-homotopic.

(c) follows from (a) and (b). □

Corollary 2.35. *If (G, S) is pre-Coxeter system of type M, then G has a presentation*

$$G = \langle S | (ss')^{m_{s,s'}} = 1 \rangle,$$

where there is only one relation for each pair s, s' with $m_{s,s'} \neq \infty$.

Proof. Let W be the Coxeter group defined by the presentation above. Consider the canonical homomorphism η from W onto G. By Theorem 2.34, an element w in the kernel of η can be represented by a word f that is M-equivalent to the empty word. But M-equivalent words in S represent the same element of W. So, $w = 1_W$ and η is an isomorphism. □

Corollary 2.36. *Let (G, S) satisfy the exchange condition (E). Let $g \in G$ and $s \in S \setminus S_g$ satisfy $l(sgs) < l(g) + 2$. Then, s commutes with each element of S_g.*

Proof. Since $s \notin S_g$, by (2.11) (8), $l(sg) = l(g)+1 = l(gs)$. By the condition (F), $l(sgs) < l(g) + 2$ implies $sg = gs$. We show, by induction on $k = l(g)$, that s commutes with all elements of S_g. If $g = s_1 \dots s_k$ is reduced expression of g, since $l(sg) = l(g) + 1 = l(gs)$, $ss_1 \dots s_k$ and $s_1 \dots s_k s$ are reduced expression in S of the same element sg and so by Theorem 2.34 (a), the first can be converetd to the second by a finite sequences of M-homotopies. One these opereations involves s. Prior to this operation, we have a word of the form $ss'_1 \dots s'_k$, where $s'_1 \dots s'_k$ also is a reduced expression for $g \in S$ and $s'_i \neq s$ for each i. This operation is possible only if $m_{s,s'} = 2$, i.e, s and s' commute. Now, s commute with $s'_1 w = s'_2 \dots s'_k$ and the induction hypothesis implies the proof. □

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