Role of initial states, open system quantum dynamics, Markovian and non-Markovian avataras

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Open system quantum dynamics stands on four pillars:

- ❖ Structure of the initial state of a composite system evolving unitarily (initial state: direct product, separable, entangled etc)
- ❖ Subsystem evolution – completely positive (CP) and not-completely positive (NCP) maps
- ❖ Memory of the initial state in the evolved state – Markov and non-Markov avataras
- ❖ Master equation for subsystem evolution (?)

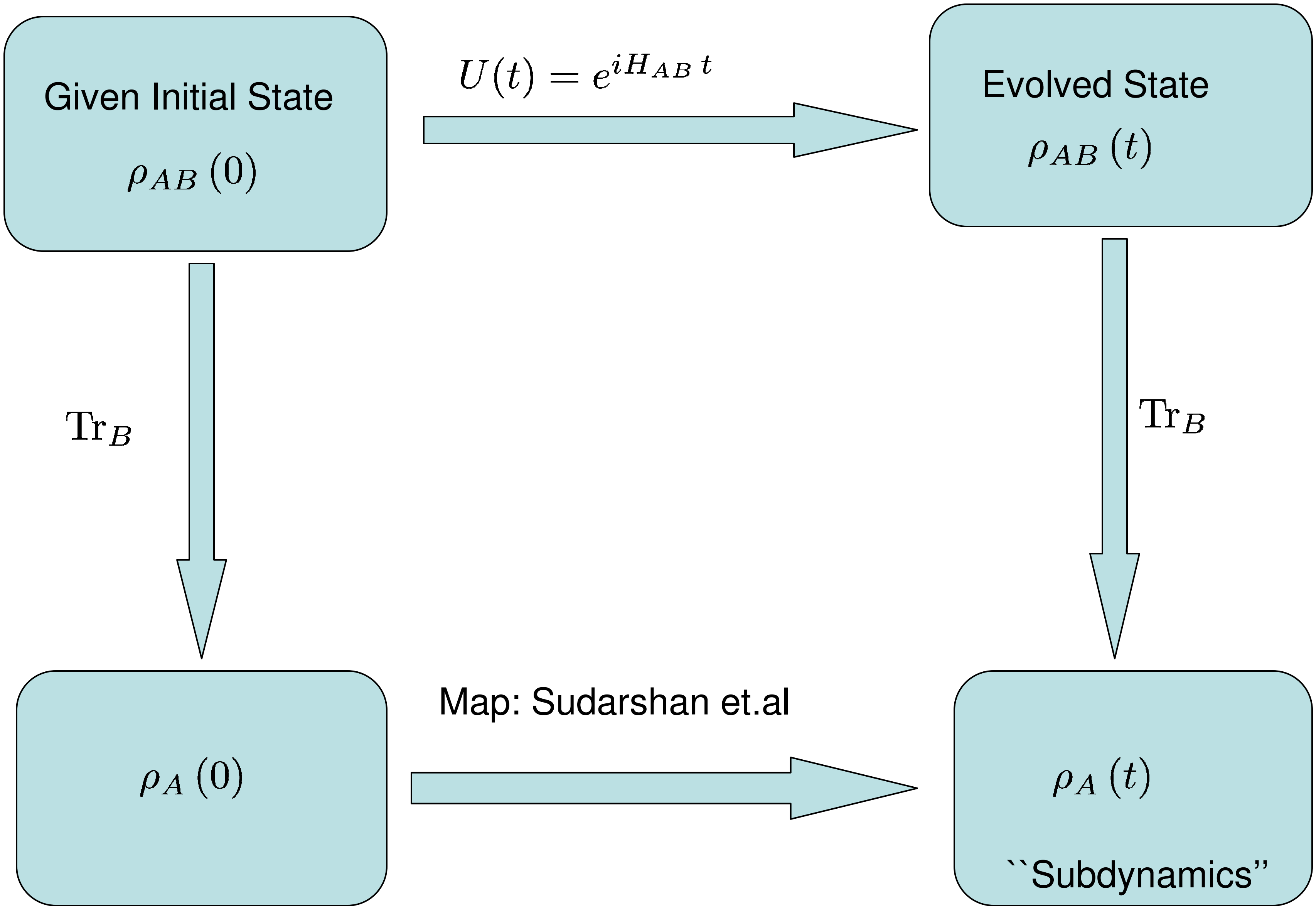
(defining equation for subsystem evolution)

These issues will be discussed based on our recent works:

[1] ***Kraus representation of quantum evolution and fidelity as manifestations of Markovian and non-Markovian forms,*** (AKR, A.R.Usha Devi, and R.W.Rendell), PRA **82**, 042107 (2010)); ArXiv: 1007.4498 (quant-ph)

[2] ***Open system quantum dynamics with correlated initial states,not completely positive maps and non-Markovianity,*** (A.R.Usha Devi, AKR, and Sudha), To appear in PRA, ArXiv: 1011.0621

… and our ongoing work.


Given Initial State
$\rho_{AB}(0)$
$U(t) = e^{iH_{AB}t}$
Evolved State
$\rho_{AB}(t)$
Tr_B
Tr_B
$\rho_A(0)$
Map: Sudarshan et.al
$\rho_A(t)$
``Subdynamics''

Given $U(t) = e^{iH_{AB}\,t}$ and $\rho_{AB}\,(0)$, what can we say about $\rho_A\,(t)$, in relation with $\rho_A\,(0)$?

One can ask: How much of $\rho_A\,(0)$ is remembered, after evolution, in $\rho_A\,(t)$?

Many approaches to this question – Breuer, Plenio,…

We use (1) Fidelity: $\{\mathrm{Tr}\,\sqrt{\rho(t)\rho(t+\tau)\rho(t)}\}^2$ (Propensity)

(2) Relative Entropy: $\mathrm{Tr}\,\left[\rho(t)(\ln\rho(t) - \ln\rho(t+\tau))\right]$

(Distinguishability)

Part A:

(a) Which $\rho_{AB}(0)$ and $U(t)$ will lead to completely positive (CP) maps?

Necessary and Sufficient Conditions:

$$\text{(i)}\ \rho_{AB} = \rho_A \otimes \rho_B$$

$$\text{(ii)}\ \rho_{AB} = \sum_i p_i \left(\rho_{iA} \otimes \rho_{iB}\right)$$ (Separable states with zero discord)

(b) Master equation for $\rho_A(t)$ exists (Stinespring, ECGS Kraus..)

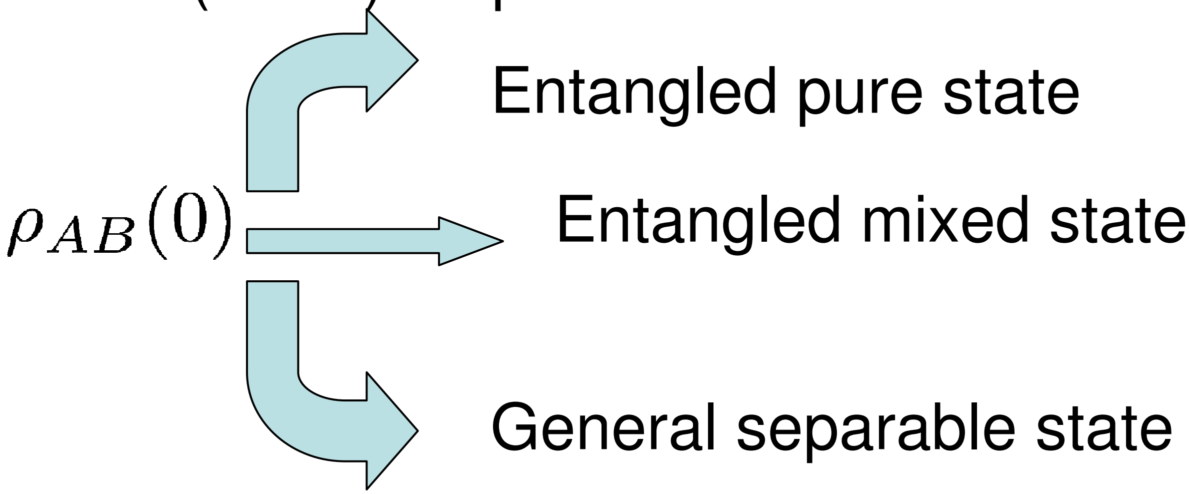
$\Rightarrow$ LGKS master equation

$\Rightarrow$ Markov (t-independent coefficients in LGKS)

$\Rightarrow$ Non-Markov (t-dependent coefficients in LGKS) (Kossakowski et al)

Part B:

(a) Which $\rho_{AB}(0)$ and $U(t)$ will lead to Not-completely positive (NCP) maps?



(b) Nature of memory -- all Non-Markovian! (Determined from fidelity and relative entropy)

(c) No master equation: (Rodriguez & ECGS, quant-ph arxiv 0803.1183)

(d) Examples: Ref. [2] ARU, AKR, S, Phys. Rev. A. (To appear).

Remarks

Initial state plays a crucial role in open system dynamics. Whether memory of an initial state is retained or not depends on the CP/NCP nature of the subdynamics map. This is a fascinating area yet to be explored

Mapping ideas:
Physics based – Sudarshan et al (1961)
continues till today!
Kraus (1971)

Mapping theorems
Mathematics based -- – Stinespring (1955)
Choi (1972,1977)

General math-phys considerations leading to semigroup evolution of the system
(NOT SUBDYNAMICS)--
Lindblad (1976),
Gorini, Kossakowski, Sudarshan, (1976)

$$i\frac{\partial\rho(t)}{\partial t}=[H,\rho(t)]+\sum_i\left(L_i\rho L_i^+-\frac{1}{2}\left(L_i^+L_i\rho+\rho L_i^+L_i\right)\right)$$

$\{L_i\}\rightarrow$ Time independent operators
See also Banks, Suskind & Peskins (1984)

Short-time analysis (1998)
Preskill,
Sudarshan and Rodriguez
(subdynamics)

Relates $\{L_i\}$ to ME of H_{int}

Kossakowski et al
(2006 to present date)
$\{L_i\}$ Time dependent
and variations thereof

General theory showing A – map in canonical form leading to the representation

$$\rho(t) = \sum_{\mu} \lambda_{\mu} C_{\mu} \rho(0) C_{\mu}^{+}, \quad I = \sum_{\mu} \lambda_{\mu} C_{\mu}^{+} C_{\mu} \qquad (1)$$

$$= A(t)\rho(0)$$

Here the C-operators are t-dependent and the eigenvalues are constants. The second relation here expresses the normalization $\mathrm{Tr}\,\rho(t) = 1$ for all times.

From this to obtain the following LGKS operator structure

$$\rho(t) - \rho(0) = \sum_{\mu} \lambda_{\mu} L_{\mu} \rho(0) \qquad (2)$$

$$L_{\mu} \rho(0) = C_{\mu} \rho(0) C_{\mu}^{+} - \frac{1}{2}\left(C_{\mu}^{+} C_{\mu} \rho(0) + \rho(0) C_{\mu}^{+} C_{\mu} \right) \qquad (3)$$

This equation has the appearance of the LGKS equation except the left hand side is NOT time derivative!

When all the eigenvalues are non-negative, this map in CP, otherwise it is NCP

Case (A): CP map is Markovian if it forms a one-parameter semi-group which corresponds to

$$A(t+\tau)=A(t)A(\tau) \qquad (4)$$

Consequences of semi-group property time evolution equation for the density matrix has the LGKS form:

$$i\frac{\partial \rho(t)}{\partial t}=[H,\rho(t)]+\sum_i\left(L_i\,\rho\,L_i^{+}-\frac{1}{2}\left(L_i^{+}L_i\,\rho+\rho\,L_i^{+}L_i\right)\right) \qquad (5)$$

where the operators are t-independent and arbitrary.
This is a generalization of the celebrated Stone's theorem for unitary group of time evolution for closed systems, where the L-terms in eq.(3) are absent.

- The Kraus representation where the positive eigenvalues are absorbed into the C-operators in eq.(1).
- Assuming the composite system is closed with the Hamiltonian containing interaction between system and environment, under weak coupling and short time regimes, one obtains LGKS form (Eq.(4)), with the L-operators expressed in terms of matrix elements of interaction.
- When the short time regime leads to time dependent LGKS form, it is an indication of non-Markof behavior. This provides an added signature of non-Makovianity -- but keeping the CP map structure.

- **Conditions for CP map** Initial state of the composite density matrix is a direct product of the density matrices of the system and its environment OR if the composite density matrix has zero discord

- **Markov property** means that there is memory of the initial state in the subsequent time evolved state. This is here stated in terms of Fidelity, $F[\rho(t), \rho(t+\tau)]$, a measure of propensity of the evolved state $\rho(t+\tau)$ in the initial state $\rho(t)$. And the relative entropy, $S(\rho(t)|\rho(t+\tau))$, a measure of distinguishability of the evolved state with the initial state.

Signature of Markovian dynamics: An important consequence of the semi-group property, the (CP) map, is it places a condition on both Fidelity and Relative entropy

$$F[\rho(t),\rho(t+\tau)] \geq F[\rho(0),\rho(\tau)] \qquad (6)$$

$$S(\rho(t)|\rho(t+\tau)) \leq S(\rho(0)|\rho(\tau)) \qquad (7)$$

Examples to examine these features based on several dynamical models where exact Kraus representations are available for which short time behavior can be evaluated and Markov and non-Markov processes could be discerned by the tests devised above.

a) **Markov model** (Yu and Eberly, PRL 97, 140403 (2006)

b) **Non-Markov model (**Yu and Eberly, Opt. Commun. 283, 676(2010)

c) **Jaynes-Cummings model** (version a la AKR et al, PLA 259, 285 (1999); PRA 67, 062110 (2003); arXiv: 0709.1212)

Yu and Eberly model for two qubits (Opt. Commun. **283**, 676 (2010))

Kraus operators:

$$K_0 = \begin{pmatrix} p(t) & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} p(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} p(t) & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q(t) & 0 \\ 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} q(t) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} p(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} q(t) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q(t) & 0 \\ 0 & 0 \end{pmatrix};$$

$$p(t) = e^{-f(t)}, \quad q(t) = \sqrt{1 - p^2(t)}; \quad f(t) = \frac{\Gamma}{2}\left[t + \frac{1}{\gamma}(e^{-\gamma t} - 1) \right]$$

Non-Markovian limit: $\gamma \to \infty$

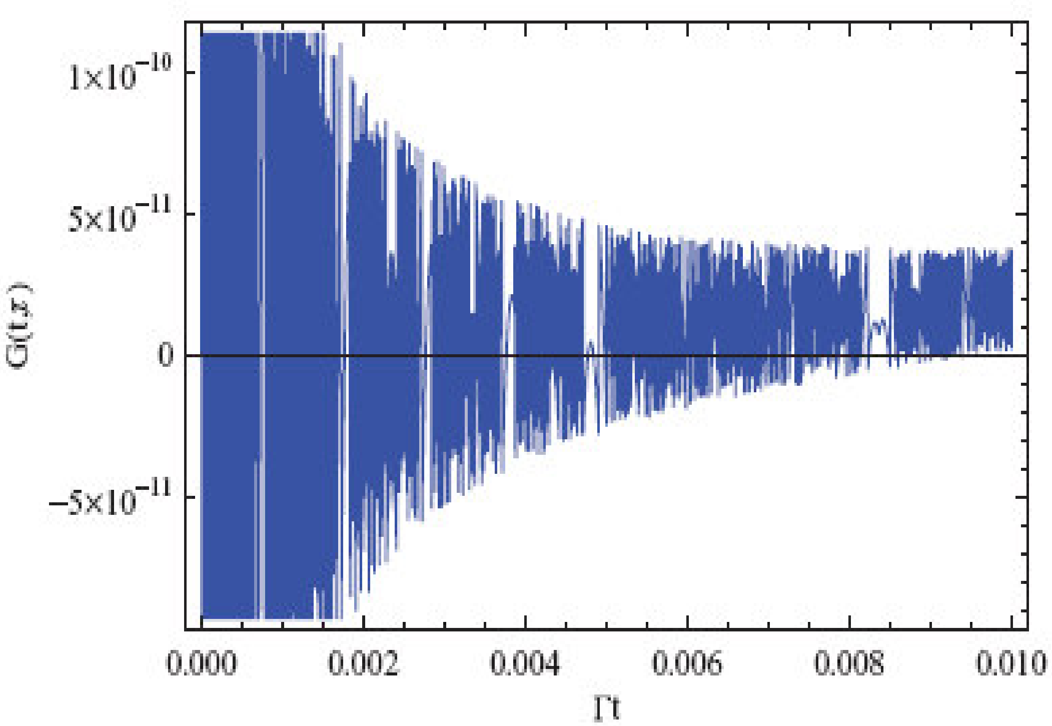
Two qubit density matrix at t=0:

$$\rho(0) = \frac{1}{3}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}$$

and $$\rho(t) = \sum_{i=0,1,2,3} K_i \, \rho(0) K_i^{+}$$

Fidelity difference in the non-Markovian limit $\boxed{\gamma << 1}$

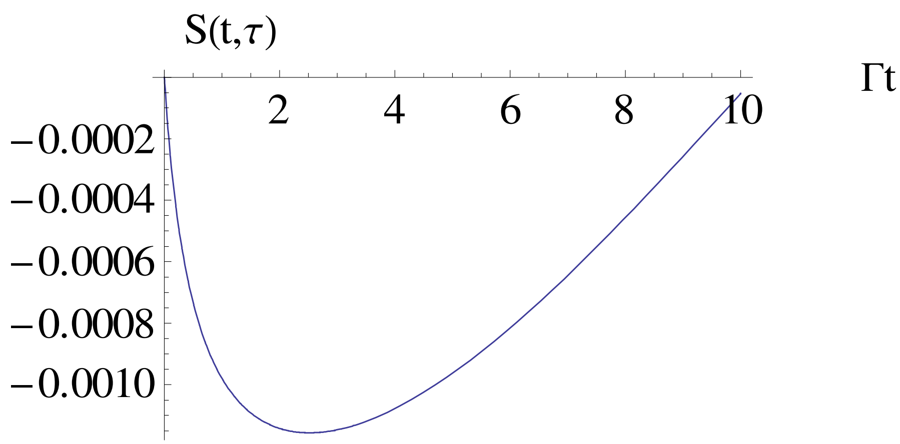
$(\Gamma = 1, \tau = 1, \gamma = 10^{-4})$



Relative entropy difference $S(t,\tau)$

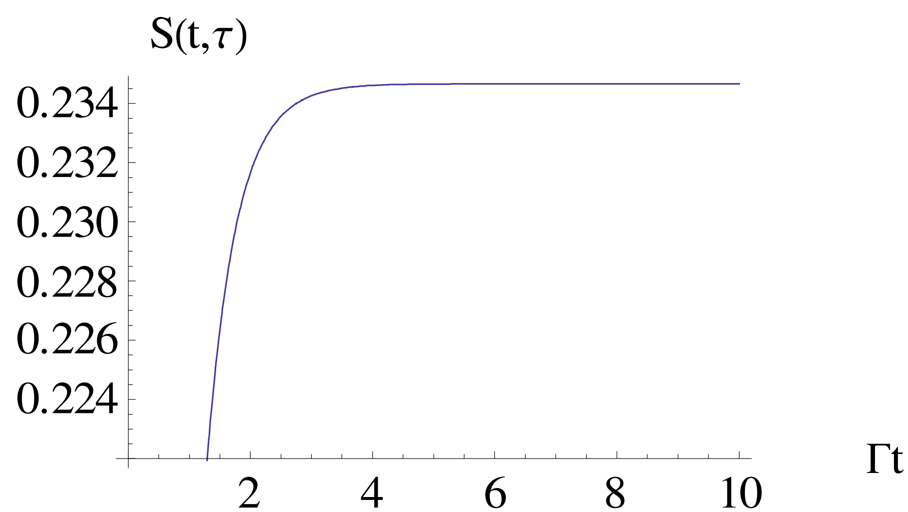
Non-Markovian limit

γ=0.01



Markovian limit

γ=10



Case B: NCP Dynamics

Necessary and sufficient conditions for NCP map: Initial state of the composite density matrix is correlated.

We illustrate this with three examples of correlated initial states evolving under NCP as constructed by the same given Hamiltonian and use the criteria (6) and (7) to check the status of initial state memory at later times. This set of examples are different from the first set of examples, in that the unitary dynamics of the composite state is given, but the initial states chosen have different types of quantum correlations. This tells us the importance of the nature of the initial state is significant in the time evolution of the system.

Canonical structure of the A-map

$$A = \sum_{\alpha\beta} \mathcal{A}_{\alpha\beta}\, T_\alpha \otimes T_\beta^*,$$
$$\mathcal{A}_{\alpha\beta} = \mathrm{Tr}[A(T_\alpha^\dagger \otimes T_\beta^T)].$$

$$\mathrm{Tr}[T_\alpha^\dagger T_\beta] = \delta_{\alpha,\beta},$$

$$A_{r's';rs} = \sum_{\alpha,\beta=1}^{n^2} \mathcal{A}_{\alpha\beta}\, [T_\alpha]_{r'r}\, [T_\beta^*]_{s's}.$$

$$\mathcal{A}_{\alpha\beta} = \mathcal{A}_{\beta\alpha}^*$$

i.e., the coefficients $\mathcal{A}_{\alpha\beta}$ form a $n^2 \times n^2$ hermitian matrix $\mathcal{A}$. Denoting $\mathcal{U}$ as the matrix diagonalizing $\mathcal{A}$ and $\{\lambda_\mu\}$, the real eigenvalues of $\mathcal{A}$, so that $\sum_{\alpha,\beta} \mathcal{U}_{\mu\alpha}\, \mathcal{A}_{\alpha\beta} \mathcal{U}_{\mu\beta}^* = \lambda_\mu$, we finally obtain the canonical structure of the A-map:

$$A = \sum \lambda_\mu\, \mathcal{U}_{\mu\beta} \mathcal{U}_{\mu\alpha}^*\, T_\alpha \otimes T_\beta^*$$
$$= \sum \lambda_\mu\, \mathcal{C}_\mu \otimes \mathcal{C}_\mu^*,$$

$$\rho(t) = \sum_\mu \lambda_\mu\, \mathcal{C}_\mu\, \rho(0)\, \mathcal{C}_\mu^\dagger. \Longrightarrow \sum_\mu \lambda_\mu\, \mathcal{C}_\mu^\dagger\, \mathcal{C}_\mu = I$$

The dynamical evolution is the one used by Jordan et al PRA 70, 052110 (2004):

$$U(t) = \exp - i t H, \quad H = \frac{1}{2}\omega \sigma_{1z}\sigma_{2x} \qquad (8)$$

$$U(t) = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & -i\sin\left(\frac{\omega t}{2}\right) & 0 & 0 \\ -i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ 0 & 0 & i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix}.$$

where the second qubit acts as the environment on the first qubit.

Initial correlations:

$$\langle U^\dagger(t)\sigma_{1x}U(t)\rangle = \langle\sigma_{1x}\rangle\cos(\omega\, t) - \langle\sigma_{1y}\sigma_{2x}\rangle\sin(\omega\, t)$$

$$\langle U^\dagger(t)\sigma_{1y}U(t)\rangle = \langle\sigma_{1y}\rangle\cos(\omega\, t) + \langle\sigma_{1x}\sigma_{2x}\rangle\sin(\omega\, t)$$

$$\langle U^\dagger(t)\sigma_{1z}U(t)\rangle = \langle\sigma_{1z}\rangle,$$

$$a_1 = -\langle\sigma_{1y}\sigma_{2x}\rangle,\ a_2 = \langle\sigma_{1x}\sigma_{2x}\rangle$$

the fixed initial system-environment parameters governing the dynamics of the system qubit.

Dynamical A-map

$$\boxed{[\rho(t)]_{r's'} = \sum_{r,s=1}^{n} A_{r's';rs}(t)\ [\rho(0)]_{rs}\ ,}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}S\,a^* & C & 0 & \frac{1}{2}S\,a^* \\ \frac{1}{2}S\,a & 0 & C & \frac{1}{2}S\,a \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a = a_1 + ia_2; \quad C = \cos(\omega t), \quad S = \sin(\omega t)$

Define

$$A(t_1+t_2)-A(t_1)A(t_2)=SG$$

which is explicitly found to be

$$SG=S_1\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}a^*(C_2-1) & -S_2 & 0 & \frac{1}{2}a^*(C_2-1) \\ \frac{1}{2}a(C_2-1) & 0 & -S_2 & \frac{1}{2}a(C_2-1) \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

- the map does not have a semigroup structure except for small times $\omega\, t_i \ll 1$

Dynamical A-map characterizing the two qubit unitary dynamics of Jordan et al -- with intially correlated states:

$$A = \frac{1}{2}\sum_{\alpha\beta} \mathcal{A}_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta^\star,$$

$$\mathcal{A}_{\alpha\beta} = \frac{1}{2}\mathrm{Tr}[A(\sigma_\alpha \otimes \sigma_\beta^\star)].$$

with the hermitian coefficient matrix $\mathcal{A}$ given by,

$$\mathcal{A} = \frac{1}{2}\mathrm{Tr}[A(t)\sigma_\alpha \otimes \sigma_\beta^\star]$$

$$= \frac{1}{2}\begin{pmatrix} 2(1+C) & a_1\,S & a_2\,S & 0 \\ a_1\,S & 0 & 0 & i\,a_2\,S \\ a_2\,S & 0 & 0 & -i\,a_1\,S \\ 0 & -i\,a_2\,S & i\,a_1\,S & 2(1-C) \end{pmatrix}$$

Eigenvalues:

$$\lambda_{1\pm} = \frac{1}{2}\left\{[1+\cos(\omega t)] \pm \sqrt{[1+\cos(\omega t)]^2 + |a|^2\sin^2(\omega t)}\right\}$$

$$\lambda_{2\pm} = \frac{1}{2}\left\{[1-\cos(\omega t)] \pm \sqrt{[1-\cos(\omega t)]^2 + |a|^2\sin^2(\omega t)}\right\}$$

$\lambda_{1-}, \lambda_{2-}$ negative $\Rightarrow$ NCP dynamics

Example: Two qubit Werner state

$$\rho_W(t=0) = \frac{x}{4} I_1 \otimes I_2 + (1-x)\,|\Psi_-\rangle\langle\Psi_-|$$

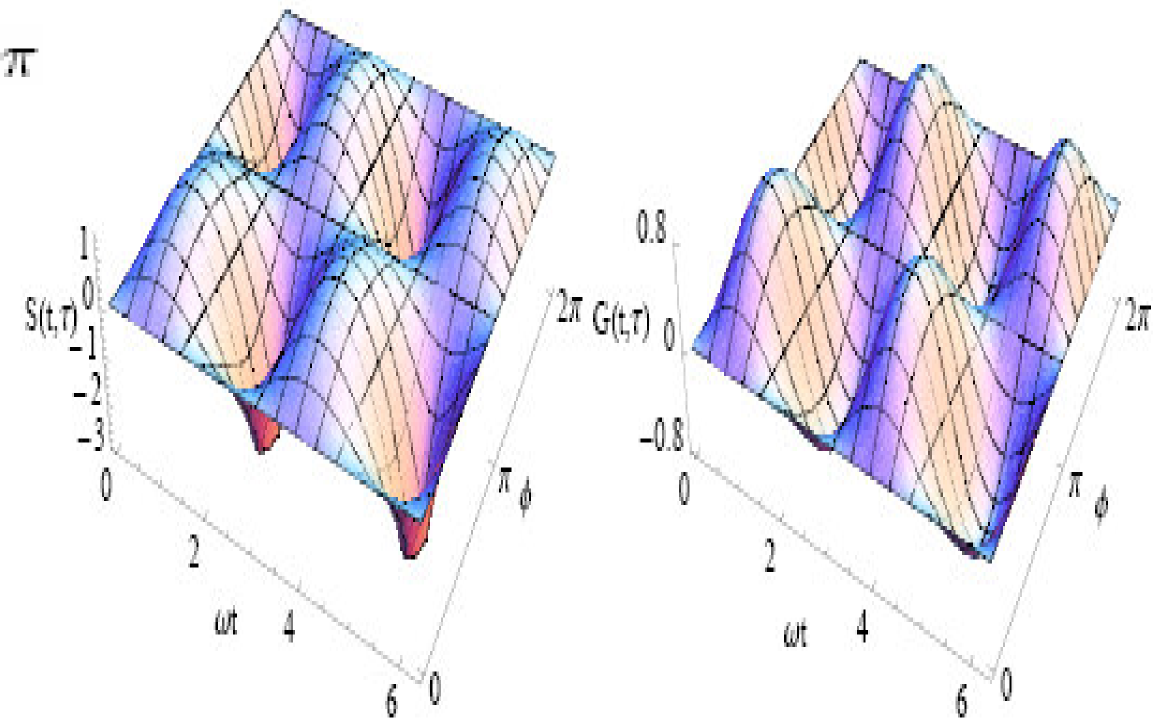
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|0_1, 1_2\rangle - |1_1, 0_2|\rangle\right)$$

Evolution of the first qubit under this NCP dynamics with initial parameters: $\boxed{a_1 = 0, a_2 = (1-x)}$

$$\begin{aligned}\rho_{W1}(t) &= \mathrm{Tr}_2[U(t)\rho_W(0)U^\dagger(t)] \\ &= \frac{1}{2}\begin{pmatrix} 1 & -i\,(1-x)\,\sin(\omega t) \\ i\,(1-x)\,\sin(\omega t) & 1 \end{pmatrix}\end{aligned}$$

The relative entropy difference $S(t, \tau)$ and the fidelity difference $G(t, \tau)$

$\omega\tau = \pi$

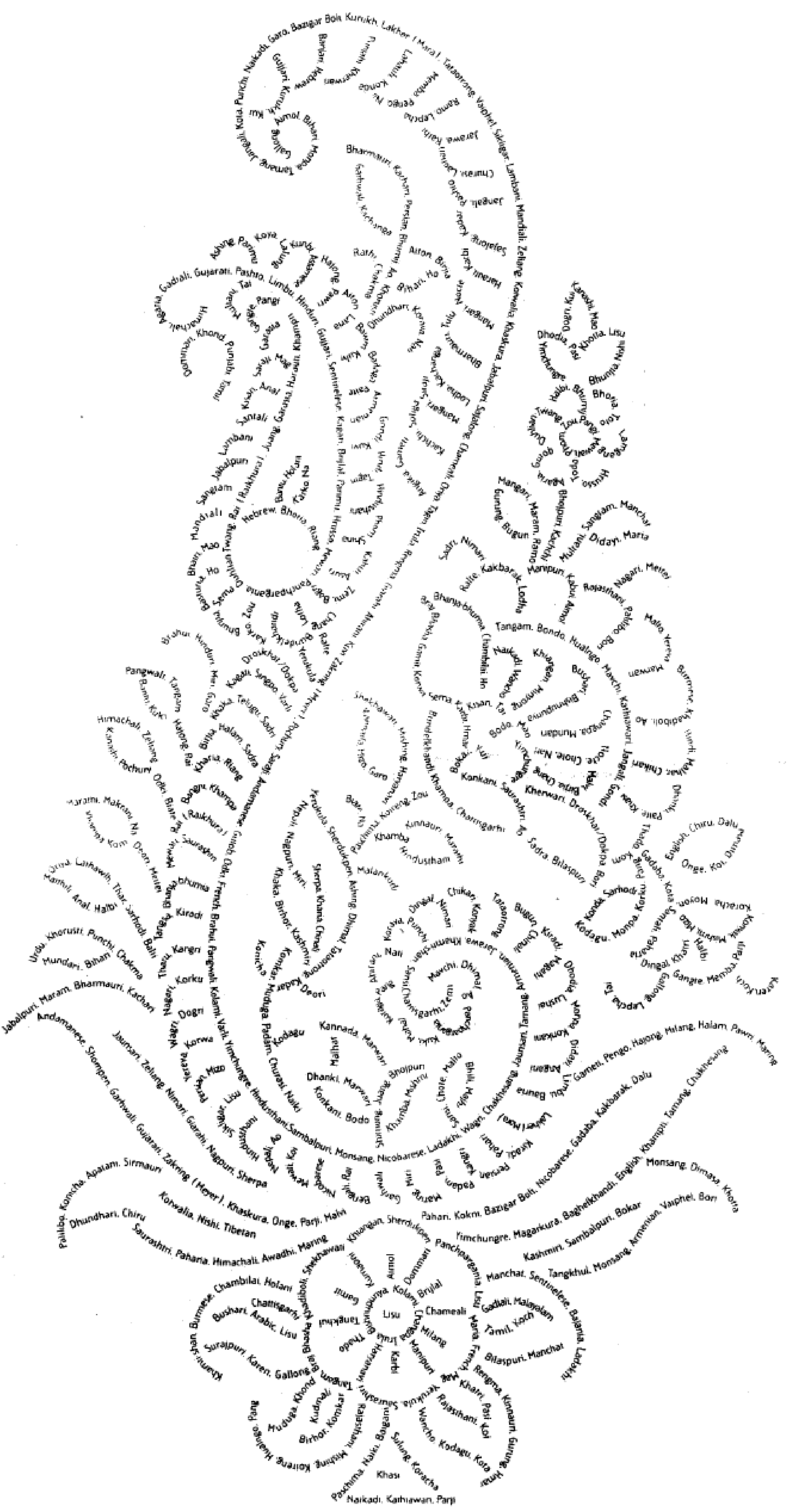


Negative regions point towards non-Markovian evolution

Summary

- CP map, semi-group evolution, direct product/zero discord initial state, Kraus-Sudarshan Rep., LGKS equation with no t-dependence – Markov, with t-dep. nonMarkov.
- NCP map, no semi-group evolution, correlated initial state, canonical representation of dynamical A-map, no LGKS, non-Markov signatures. (All in the local time framework).

- These topics are open ended questions for new research areas: The NCP structure may have to be investigated in other situations where one has traditionally used CP structure as in measurement theory, tomography, quantumness, density functional theory of condensed matter etc.



Naikadi, Kathiawari, Parji
Khasi
Pahari, Kokni, Bazigar Boli, Nicobarese, Gadaba, Kakbarak, Dalu
Yimchungre, Magarkura, Baghelkhandi, Bokar
Kashmiri, Sambalpuri, Bokar
Tangkhul, Monsang, Armenian, Vaiphei, Bori
Konwalia, Nishi, Tibetan
Paiibo, Koncha, Apatani, Sirmauri
Dhundhari, Chiru
Saurashtri, Pahana, Himachali, Awadhi, Maring
Jabalpuri, Maram, Bharmauri, Kachari
Gamett, Pengo, Hajong, Hmang, Halam, Pawri, Maring
Gadiali, Malayalam
Tamil, Koch
Bilaspuri, Manchat
Chamaeli
Lisu
Mangari, Maram, Ramo
Multani, Sangtam, Manchar
Rajasthani
English, Chiru, Dalu
Onge, Koi, Dimasa