Quantum chaos and entanglement

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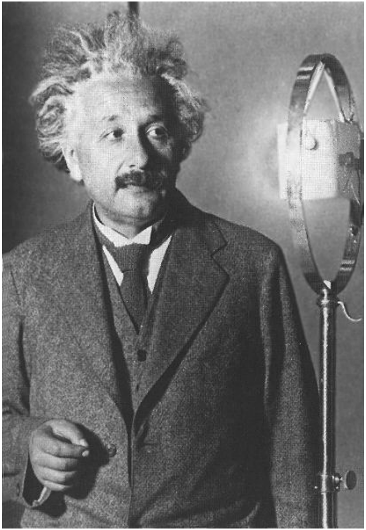
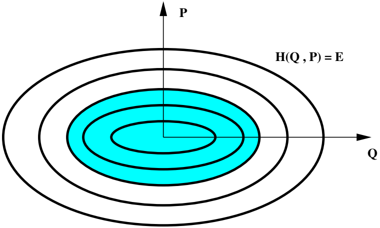
Chaos on top of Einstein's head



Einstein and Quantum Chaos

1913-1917 Bohr-Sommerfeld: Quantize Action

$I = \oint p\, dq \;=\; 2\pi\hbar\, n$. $n = 1, 2, 3, \ldots$

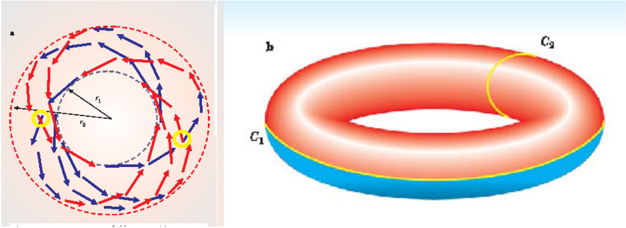


For separable d freedoms: $I_j = \oint p_j\, dq_j \;=\; 2\pi\hbar\, n_j$, $j = 1, \ldots, d$.

11 May 1917

Paper presented by AE at the meeting of the German Physical Society.
"On the quantum theory of Sommerfeld and Epstein".

- The Sommerfeld-Epstein rules are not invariant under general coordinate transformations if the system is not separable.
- $\oint_{C_j} \mathbf{p} \cdot \mathbf{dq} = n_j \hbar$
- one particle in a central force field: generally double-valued momentum field. Lift to a 2-torus.



Einstein-Brillioun-Keller quantization: $\oint_{C_j} \mathbf{p} \cdot \mathbf{dq} = (n_j + \alpha_j/4)\hbar$

- *What if at any point in space there exists an infinite number of possible momentum directions? Ergodic systems?*
- *"If there exists fewer than d constants of the motion, for example, according to Poincaré in the three-body problem, then the p_j are not expressible as functions of q_j and the quantum condition of Sommerfeld-Epstein fails also in the slightly generalized form that has been given here."*

Chaos

Poincare had found "homoclinic chaos" in the restricted three-body problem. 1900?

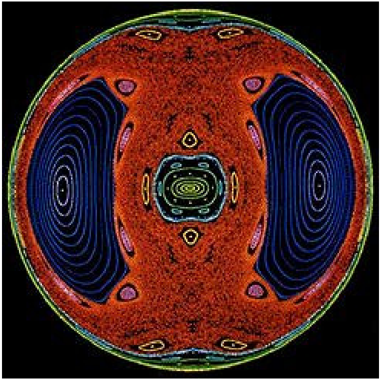
Period-3 implies Chaos: 1976 : Li and Yorke.

Chaos is exponential sensitivity to initial conditions even with a *bounded* phase space.

Implies linear growth of information with time: Kolmogorov Sinai Entropy.

Simplest quantum 3-body problem: Pauli's thesis problem: H_2^+ stable?

More recent experiments with just the hydrogen atom:
The hydrogen atom in a strong magnetic field.



"It turns out that an eerie type of chaos can lurk just behind a facade of order - and yet, deep inside the chaos lurks an even eerier type of order."
–D. Hofstadter.

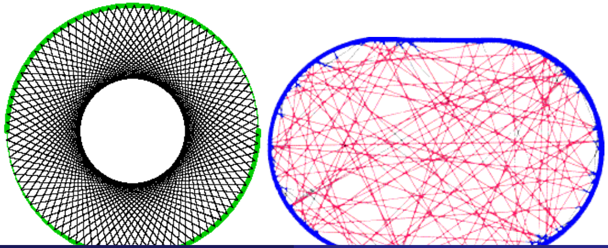
Quantum Chaos

The analysis of quantum systems whose classical limit corresponds to a chaotic Hamiltonian system.

Eigenvalues: The Gutzwiller Trace Formula / Periodic orbit sum generalizes the EBK rules. (Gutzwiller, 1970, M V Berry, A Voros, 1980s)

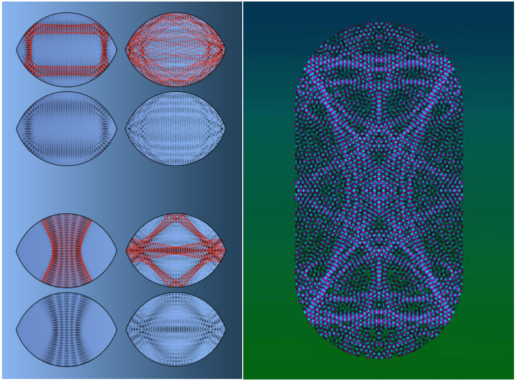
Eigenfunctions: Analytical structure little known. "Ergodic" (Schnirelman 70s) "Scarring" (Eric Heller 1984),

Statistical modelling: Random Matrix Theory.

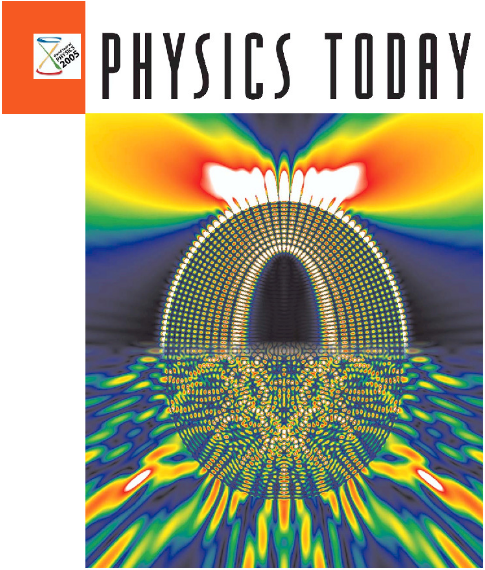


Quantum Billiards

$\nabla^2\psi(x,y) + k^2\psi(x,y) = 0$; $\psi(x,y) = 0$ on billiard boundary
$k^2 = 2mE/\hbar^2$



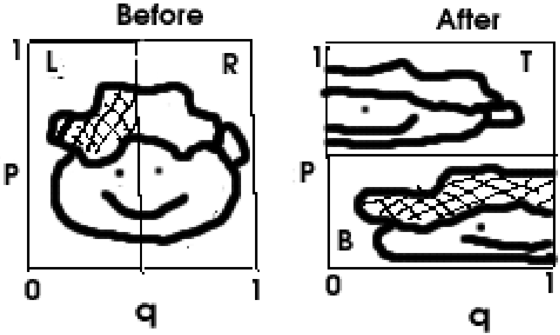
Scars Eric Heller (1984).



Einstein versus chaos

The baker's map: A paradigm of classical chaos

On the lightly floured surface, flatten the dough slightly into a disk-shape. Use the heels of your hands to PUSH the dough away. Pick up the edge furthest away from you and FOLD it toward you, sliding the dough back to its original spot on the counter. TURN the dough a quarter-turn. Vigorously repeat "push, fold, and turn" steps. — From www.baking911.com.

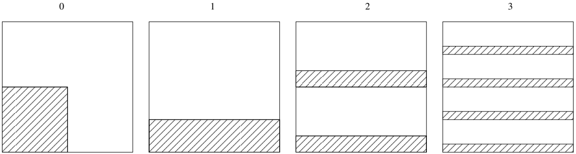


The baker mixes chaotically

$$q \to 2\,q(\mathrm{mod})\,1, \quad p \to (p + [2q])/2$$

The Left shift. $a_i = \{0, 1\}$.

$$\ldots a_{-3}a_{-2}a_{-1} \bullet a_0 a_1 a_2 \ldots \longrightarrow \ldots a_{-2}a_{-1}a_0 \bullet a_1 a_2 a_3 \ldots$$



Lyapunov exponent, Topological Entropy= ln(2). Completely hyperbolic and "as random as a coin toss".

The quantum baker's map

Balazs-Voros Baker (1989):

$$B \;=\; G_N^{-1} \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}$$

$$(G_N)_{mn} = \langle p_m | q_n \rangle = \frac{1}{\sqrt{N}} \exp\left(-2\pi i \, nm/N\right)$$

$$0 \le m, n \le N-1, \; N = 1/h.$$

For L qubits, $N = 2^L$.

- Product of two non-commuting matrices.
- Spectrum analytically unknown.
- Semiclassical ($N \to \infty$) analysis implies Gutzwiller-like trace formula.
- Has no degeneracies, spectral fluctuations close to RMT.
- Efficient implementation on quantum computers, including recent experimental implementation on a 3-qubit NMR processor (quant-ph/0201064).

Entangling power of the quantum baker's map

A. J. Scott* and Carlton M. Caves†
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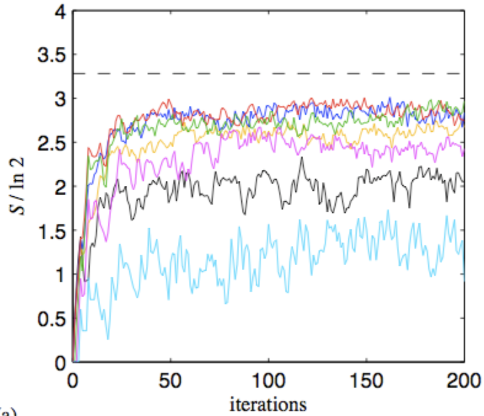
We investigate entanglement production in a class of quantum baker's maps. The dynamics of these maps is constructed using strings of qubits, providing a natural tensor-product structure for application of various entanglement measures. We find that, in general, the quantum baker's maps are good at generating entanglement, producing multipartite entanglement amongst the qubits close to that expected in random states. We investigate the evolution of several entanglement measures: the subsystem linear entropy, the concurrence to characterize entanglement between pairs of qubits, and two proposals for a measure of multipartite entanglement. Also derived are some new analytical formulae describing the levels of entanglement expected in random pure states.

(quant-ph/0305046)



(a)

($L = 8$ qubits. Entanglement between 4+4 in $B^n|00000000\rangle$)

Entangling power of quantized chaotic systems

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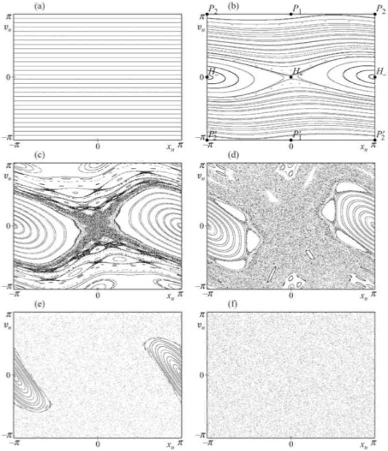
(Received 4 December 2000; published 20 August 2001)

We study the quantum entanglement caused by unitary operators that have classical limits that can range from the near integrable to the completely chaotic. Entanglement in the eigenstates and time-evolving arbitrary states is studied through the von Neumann entropy of the reduced density matrices. We demonstrate that classical chaos can lead to substantially enhanced entanglement. Conversely, entanglement provides a useful characterization of quantum states in higher-dimensional chaotic or complex systems. Information about eigenfunction localization is stored in a graded manner in the Schmidt vectors, and the principal Schmidt vectors can be scarred by the projections of classical periodic orbits onto subspaces. The eigenvalues of the reduced density matrices is sensitive to the degree of wave-function localization, and is roughly exponentially arranged. We also point out the analogy with decoherence, as reduced density matrices corresponding to subsystems of fully chaotic systems, are diagonally dominant.

(Phys. Rev. E. 2001)

One kicked rotor



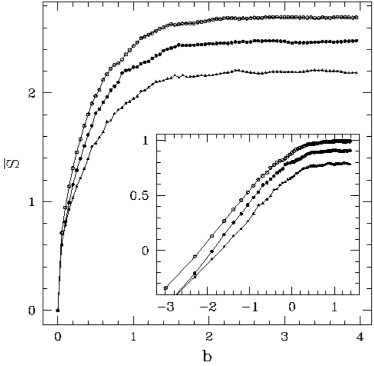
Standard Map: $(q, p) \to (q + p, p + K \sin(q + p))$

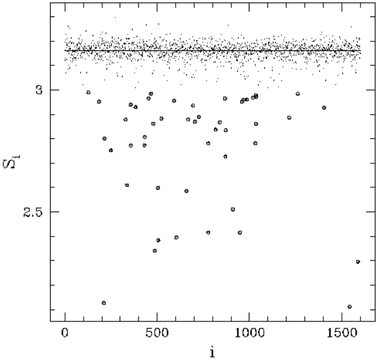
Entanglement in two coupled kicked rotors

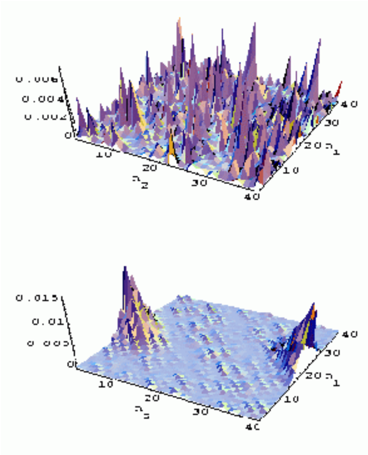
$$\begin{aligned}
q_1' &= q_1 + p_1' \\
p_1' &= p_1 + \frac{K_1}{2\pi}\sin(2\pi q_1) + \frac{\mathbf{b}}{2\pi}\sin(2\pi(q_1+q_2)) \\
q_2' &= q_2 + p_2' \\
p_2' &= p_2 + \frac{K_2}{2\pi}\sin(2\pi q_2) + \frac{\mathbf{b}}{2\pi}\sin(2\pi(q_1+q_2)).
\end{aligned}$$

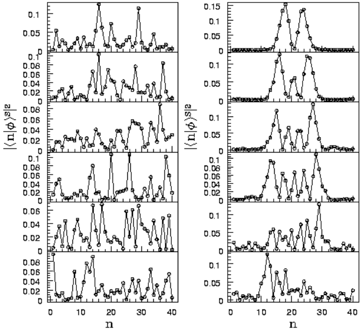
Quantize this to get a Unitary operator U. What are the entanglement properties of the eigenstates of U? What connection does it have to classical chaos?

Use $K_1 = 0.1, K_2 = 0.15$ Left: $N = 15, 20, 25$. Right: $N = 40, b = 0.2$.



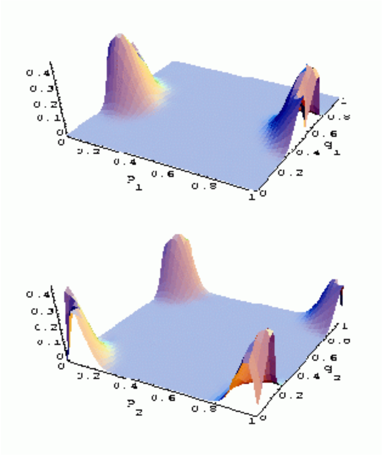






Two Eigenfunctions $|\langle q_1 q_2|\psi\rangle|^2$ and their Principal Schmidt vectors.

Scarring and Schmidt vectors



Husimi of the principal Schmidt vector of the scarred state (orbit (0.5,0,0,0))
(AL, Phys. Rev. E, vol. 64, 2001)

Why is chaos a friend of entanglement?

Classical chaos is ergodic and mixing.

Quantum chaos leads to eigenstates and time evolving states that are very well modeled by *generic* or random states.

Generic random states have very large, nearly maximal, entanglement.

States, at Random

$$\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_M, \quad M \geq N. \; |\psi\rangle = \sum_i \sum_\alpha a_{i\alpha} |i\alpha\rangle$$

Random states: choose uniformly from $2NM - 1$ dimensional unit sphere.

$$P(\{a_{i\alpha}\}) = C\,\delta\left(\sum_{i\alpha} |a_{i\alpha}|^2 - 1\right)$$

Measure: Unitarily invariant Haar measure: Usual geometric hypersurface volume on the unit sphere S^{2NM-1}.

$$\langle \mathrm{Tr}(\rho_A^2) \rangle = \frac{N+M}{NM+1} \approx \frac{1}{N} + \frac{1}{K}$$

$$\langle E \rangle \approx \log(N) - \frac{N^2 - 1}{2NM + 2}, \quad N \ll M \quad (\text{ Lubkin 1978})$$

Induced Measure on λ: spectrum of RDM

j.p.d.f. ($\beta = 1, 2$ for real, complex states)

$$P_\beta(\lambda_1, \cdots, \lambda_N) = B_{M,N}\delta\left(\sum_{i=1}^{N} \lambda_i - 1\right) \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

S. Lloyd, H. Pagels, "Complexity as Thermodynamic Depth" Ann. Phys. 1988.
K. Zyczkowski, H-J Sommers, J. Phys. A. 2001.
Entanglement in Random States:

$$\langle E \rangle = -\int d\lambda_1, .., d\lambda_N \sum_i \lambda_i \log(\lambda_i) P_2(\lambda_1, .., \lambda_N) = -N \int \lambda \log(\lambda) f(\lambda) d\lambda$$

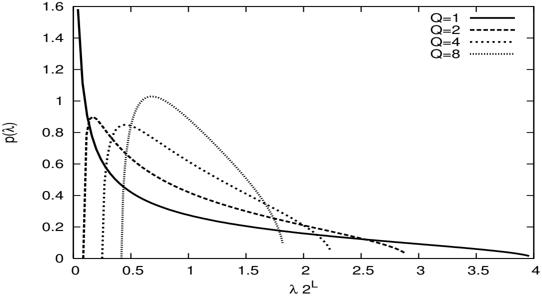
$$f(\lambda) = \int d\lambda_2 \cdots \int d\lambda_N P_2(\lambda, \lambda_2, \cdots, \lambda_N)$$

Distribution of eigenvalues of RDM

$Q = M/N$. For large M and N and finite Q the distribution of $f(\lambda)$ is that of Marcenko and Pastur (1967).

$$f(\lambda) = \frac{Q}{2\pi} \frac{\sqrt{(\lambda - \lambda_{min})(\lambda_{max} - \lambda)}}{\lambda}$$

$$\lambda_{max,min} = \frac{1}{N}(1 \pm \sqrt{Q})^2$$



Testing Statistical Bounds on Entanglement Using Quantum Chaos

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Physical Research Laboratory, Navrangpura, Ahmedabad 380009, India

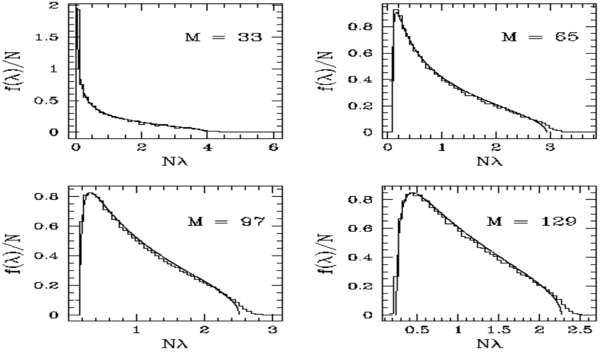
(Received 20 March 2002; published 22 July 2002)

Previous results indicate that while chaos can lead to substantial entropy production, thereby maximizing dynamical entanglement, this still falls short of maximality. Random matrix theory modeling of composite quantum systems, investigated recently, entails a universal distribution of the eigenvalues of the reduced density matrices. We demonstrate that these distributions are realized in quantized chaotic systems by using a model of two coupled and kicked tops. We derive an explicit statistical universal bound on entanglement, which is also valid for the case of unequal dimensionality of the Hilbert spaces involved, and show that this describes well the bounds observed using composite quantized chaotic systems such as coupled tops.

(Phys. Rev. Lett. 2002)

Two Coupled Quantum Chaotic Tops

Quantum chaos leads to universal bipartite pure state entanglement:



The entanglement is nearly maximal and $S = \ln(\gamma N)$. For $N = M$, $S = \ln(N) - 1/2$. As $M \to \infty$, $\gamma \to 1$.
J Bandyopadhyay, AL *PRL 02*)

And it is seen in *Nature*

LETTERS

Quantum signatures of chaos in a kicked top

S. Chaudhury[1], A. Smith[1], B. E. Anderson[1], S. Ghose[2] & P. S. Jessen[1]

The 'butterfly effect' has now been seen at the quantum level. They've brought together two sexy concepts in physics that are usually thought to operate in completely different regimes. (Nature News, 7 Oct. 2009)

And it is seen in *Nature*

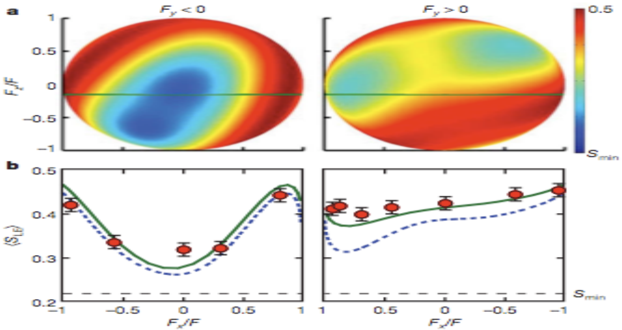


Figure 4 | Entanglement as a quantum signature of chaos. Entanglement between the electron and nuclear spins is quantified by the linear entropy, $S_{LE} = 1 - \mathrm{Tr}[\rho_e^2]$, of the electron reduced density operator. It is averaged over 40 periods of the kicked-top Hamiltonian, and shown as a function of the centre of the initial spin coherent state $|\theta,\phi\rangle$. **a**, Theoretical prediction for Schrödinger evolution, corresponding to an ideal situation without perturbations (no decoherence or κ variation). Colours indicate the value of $\langle S_{LE} \rangle$. **b**, Experimental measurements performed for states lying along the green cross-section in **a**. Also shown are the predictions of a full model (solid green line) and the perturbation free model (dashed blue line) used in **a**. The black dashed line is the linear entropy of a minimally entangled pure state in the $F = 3$ manifold. A marked contrast in dynamically generated entanglement can be seen between regular and chaotic regions. Experimental error bars, ± 1 s.d.

The minimum eigenvalue in a RDM ...

... of a typical pure state.

$$Q_N(x) = \text{Prob}\left[\lambda_{min} \geq x\right] = \text{Prob}\left[\lambda_1 \geq x, \lambda_2 \geq x, \ldots, \lambda_N \geq x\right].$$

$\beta = 2$. Symmetric Case of $M = N$:

$$Q_N(x) = B_{N,N} \int_x^{\infty} \cdots \int_x^{\infty} \delta\left(\sum_{i=1}^{N} \lambda_i - 1\right) \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{i=1}^{N} d\lambda_i$$

Laplace transfrom, Change of variables, Selberg integral and an inverse LT: (Majumdar, Bohigas, AL, J. Stat. Phys. 2008.)

$$Q_N(x) = \text{Prob}\left[\lambda_N \geq x\right] = (1 - Nx)^{N^2-1}\, \Theta\left(t - Nx\right).$$

$$\langle \lambda_N \rangle = -\int_0^{\infty} x \frac{dQ_N(x)}{dx} dx = \int_0^{1/N} (1 - Nx)^{N^2-1} dx = \frac{1}{N^3},$$

first conjectured by Znidaric (J. Math. Phys., 2007).

Entanglement in RDM of a typical pure state

L qubits in a typical pure state. What is the entanglement between two blocks having L_1 and L_2 number of qubits, when $L_1 + L_2 < L$?

If ρ_{12} is not entangled, then its PT is necessarily positive. Negative partial transpose implies entanglement. If

$$\rho_{12} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)}, \quad \rho_{12}^{PT} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)\ T}.$$

If $L_1 + L_2 < L/2$ then ρ_{12} has a minimum eigenvalue $\sim 1/N$. If $L_1 + L_2 = L/2$ the minimum eigenvalue $\sim 1/N^3$. If $L_1 + L_2 > L/2$ there are zero eigenvalues. ($N = 2^{L_1+L_2}$).

Digression: density of states of a random matrix

Wigner, Dyson, Mehta, ... 1950s, 60s, 70s.

A_{ij} elements are random i.i.d.

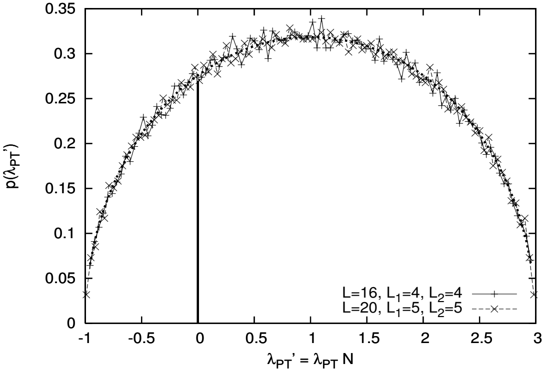
If the distribution for a Hermitian matrix is gaussian, the Gaussian random ensembles result.

The density of states of such matrices have been known to be the "Wigner Semicircle".

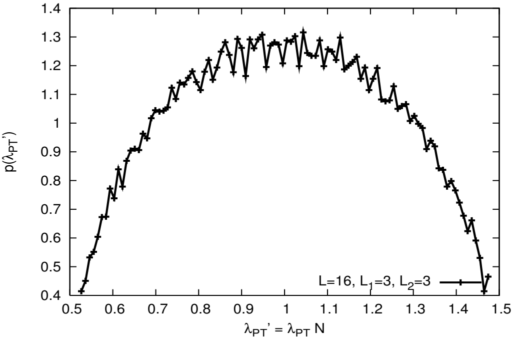
Back to Entanglement in RDM of a typical pure state

If $L_1 = L_2$ the spectrum of the ρ_{12}^{PT} fits the Wigner semicircle law! The Partial Transpose is NPT.

$$x = \lambda_{PT} N, \;\; p(\lambda_{PT}) = \frac{1}{2\pi}\sqrt{4-(x-1)^2}$$



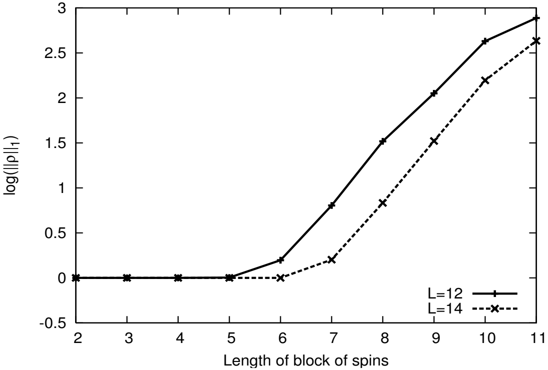
$L_1 + L_2 < L/2$: PPT



1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$p(\lambda_{PT}')$
$\lambda_{PT}' = \lambda_{PT}$ N
L=16, L_1=3, L_2=3

Negativity of sub-blocks of a random state

$$\text{Log-Negativity} = \log_2 \left(||\rho_{12}||_1 \right) = \log_2 \left(\sum_i |\lambda_i^{PT}| \right)$$



If the subsystem size is less than half of that of the system, it is typically PPT, else it is NPT and hence entangled. (Udaysinh Bhosale, AL, 2011)

Concurrence and other entanglement measures

Consider a pure state of N qubits or spin-1/2 particles.

- **1. Concurrence between any 2 spins A and B**
  If Spec$\{\ \rho_{AB}\,\sigma^y \otimes \sigma^y \rho^*_{AB} \sigma^y \otimes \sigma^y\} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$,
  Concurrence $C_{AB} = \max\left(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\right)$.
  $0 \le C_{AB} \le 1$, entanglement of formation of the two qubits is known to be a monotonic function of C_{AB} (Hill, Wootters, 1997)
- **2. Residual tangle** :
  Concurrence in a pure state: $|\langle \psi_{AB} | \sigma^y \otimes \sigma^y | \psi^*_{AB} \rangle|$. Tangle :
  $\tau_{AB} = C^2_{AB}$.
  We can also define the tangle between one spin (say the k-th) and the rest: $\tau_{k,(\mathrm{rest})} = 4 \det(\rho_k)$.
  This was used to define a purely three-way entanglement measure in a pure state of three qubits as $\tau_{1,(23)} - \tau_{1,2} - \tau_{1,3}$. (Coffman, Kundu, Wootters, 2000).

- **3.** ***n*-tangle: multipartite measure**
  $|\langle\Psi|\sigma^{y\,\otimes N}|\Psi^*\rangle|^2$. This is evidently the tangle for $N=2$, for $N=3$ this is the residual tangle, while for $N>3$ and odd this vanishes. (Wong, Christensen, 2001)
- **4. The Meyer, Wallach and Brennen Q measure: multipartite measure**

$$Q(\psi) = 2\left(1 - \frac{1}{L}\sum_{k=1}^{L} \mathrm{Tr}(\rho_k^2)\right) = \frac{1}{L}\sum_{k=1}^{L} \tau_k.$$

$1 - \mathrm{Tr}(\rho_k^2) = 2\det(\rho_k)$

Entanglement measures must not increase under LOCC and must vanish for separable states.

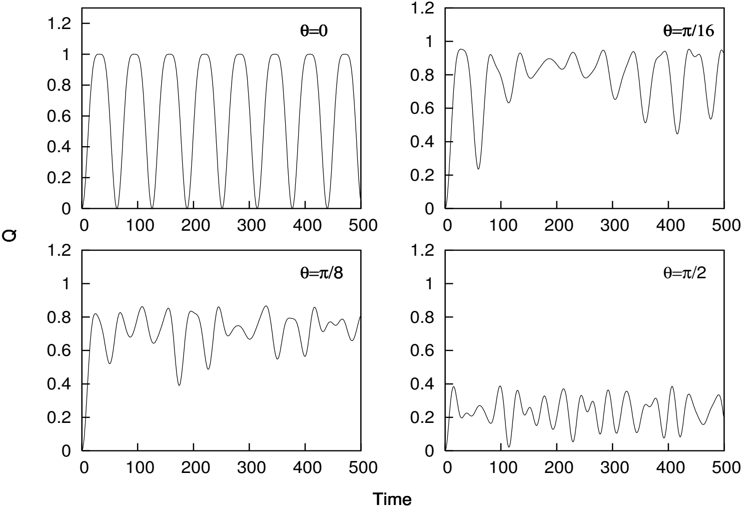
Tilted magnetic field: Nonintegrability

$$U = \exp\left(-i\frac{J}{4}\sum_{k=1}^{L}\sigma_k^x\sigma_{k+1}^x\right)\,\exp\left(-i\frac{B}{2}\sum_{k=1}^{L}(\cos(\theta)\sigma_k^x + \sin(\theta)\sigma_k^z)\right).$$

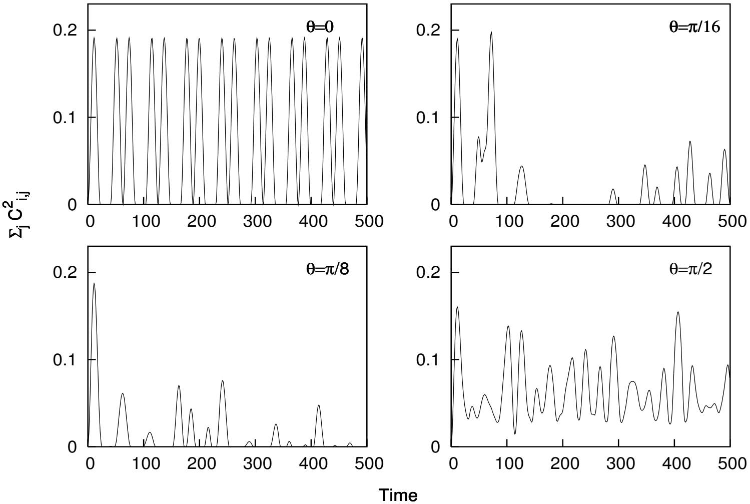
$$|\psi_L(n)\rangle = U^n|11\cdots 1\rangle$$

For $J, B \neq 0$ integrable only for $\theta = 0$ or $\pi/2$. Evidence of quantum chaos for intermediate angles of tilt. Jordan-Wigner fermions are interacting.

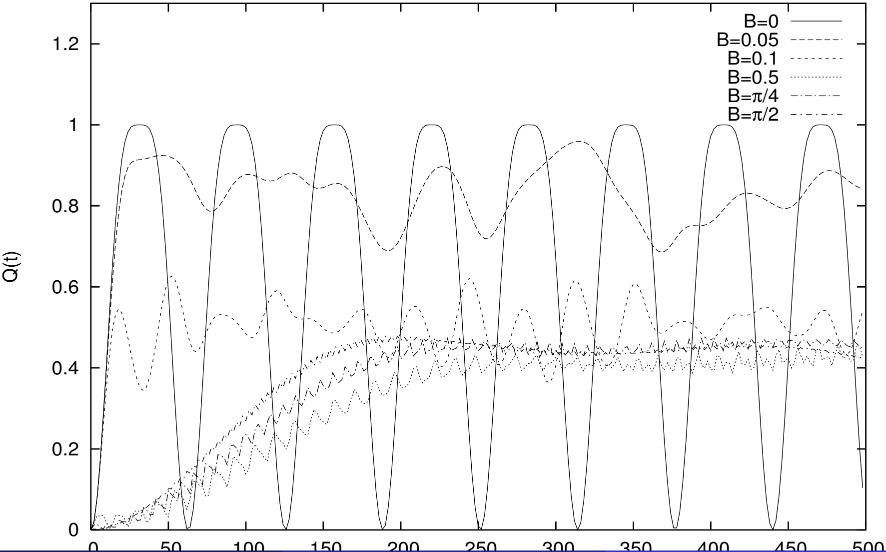
Q vs time. $J = 0.1$, $B = 0.1$, $L = 10$.



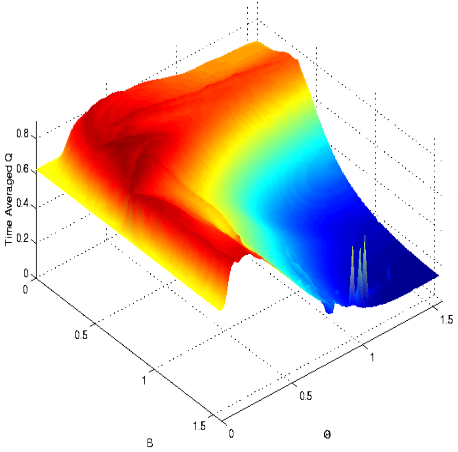
Concurrence or Tangle

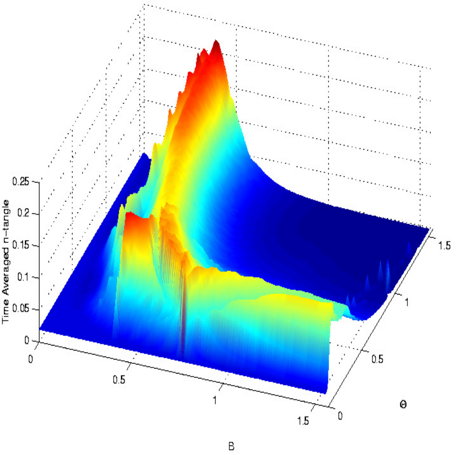


It pays to tilt. $\theta = \pi/4$. $L = 10$ $J = 0.1$.



$L = 6$. $J = \pi/4$. Average Q and tangle,





Ising Model in a tilted field: Autonomous

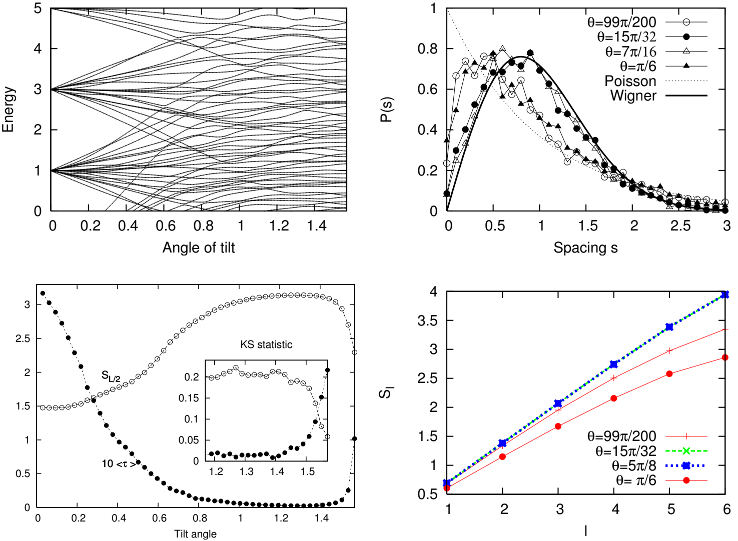
$$H(J,B,\theta) = J\sum_{n=1}^{L-1}\sigma_n^z\sigma_{n+1}^z + B\sum_{n=1}^{L}(\sin(\theta)\sigma_n^x + \cos(\theta)\sigma_n^z)$$

- Direct correlation between quantum chaos markers and entanglement.
- Stationary states.
- Transport of entanglement.
- Avoided crossings and entanglement.

$$\left(\prod_{i=1}^{L}\otimes\sigma_i^y\right) H(J,B,\theta) \left(\prod_{i=1}^{L}\otimes\sigma_i^y\right) = -H(-J,B,\theta).$$

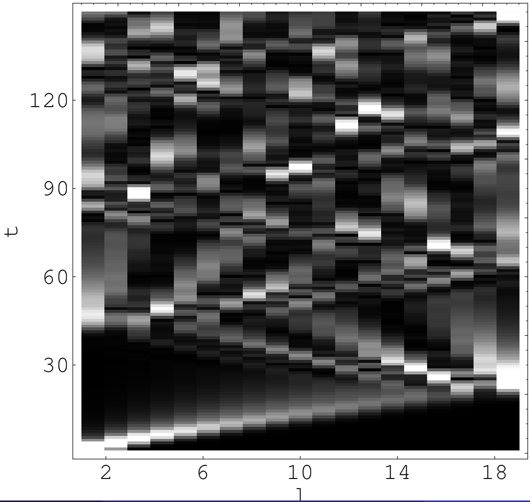
$$\mathcal{B}|s_1 s_2 \ldots s_N\rangle = |s_N \ldots s_2 s_1\rangle, \quad [H,\,\mathcal{B}] = 0$$

Quantum Chaos: tilted field . $J = B = 1$



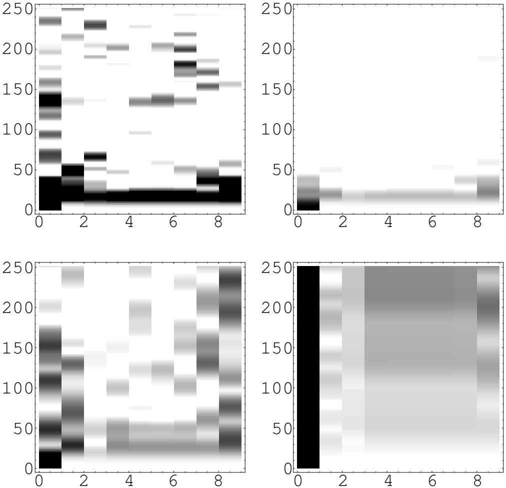
Entanglement Transport: Heisenberg chain

$$|\psi_e\,(t=0)\rangle = \frac{1}{\sqrt{2}}|\,(11+00)\,111111111111111111\rangle$$



Entanglement Transport: Ising chains

$$|\psi_e\,(t=0)\rangle = \frac{1}{\sqrt{2}}|\,(11+00)\,11111111\rangle \quad L=10, J=B=1,\ \frac{\theta}{\pi}=1/2, 5/12$$



Summary

There is a "complex link" between quantum entanglement and classical chaos.

- In the case of two coupled high-dimensional quantum chaotic systems in a pure state, chaos leads to nearly maximal entanglement via the von Neumann entropy.
- Multipartite entanglement can be enhanced with increasing nonitegrability often it seems at the expense of bipartite entanglement.
- Studied spin models such as the Ising model in a tilted magnetic field and showed that quantum chaos arises for even small longitudinal fields. Avoided crossings can lead to local enhanced multipartite entanglement.
- In "single-particle" states of many qubits there is an enhanced two-spin correlation (concurrence) with chaos. This is describable with RMT and leads to universal concurrence distributions. TR violating states are more entangled. Two-particle states have concurrence $1/L^2$, three and more $\exp(-L\log(L))$, exponentially small

Some Relevant Publications

1. Entangling Power of Quantized Chaotic Systems,
   AL, Phys. Rev. E., **64** (2001) 036207.
2. Testing Statistical Bounds on Entanglement Using Quantum Chaos,
   J. N. Bandyopadhyay, AL, Phys. Rev. Lett. **89** (2002) 060402.
3. Entanglement sharing in one particle states
   AL, V. Subrahmanyam, Phys. Rev. A **67** (2003) 052304.
4. Entanglement Production in Coupled Chaotic Systems: Case of the kicked tops,
   J. N. Bandyopadhyay, AL, Phys. Rev. E, **69** (2004) 016201.
5. Multipartite Entanglement in a One-Dimensional Time Dependent Ising Model,
   AL, V. Subrahmanyam, Phys. Rev. A., **71**, 062334 (2005).

6. Transport of Entanglement Through a Heisenberg-XY Spin Chain, V. Subrahmanyam, AL, Phys. Lett. A **349**, 164 (2006).
7. Quantum chaos in the spectrum of operators used in Shor's algorithm, Krishnendu Maity, AL, Phys. Rev. E. **74**, 035203 (Rapid Comm.) (2006).
8. Entanglement, avoided crossings, and quantum chaos in an Ising model with a tilted magnetic field
   J. Karthik, A. Sharma, and AL, Phys. Rev. A **75**, 022304 (2007).
9. Modular multiplication operator and quantized baker's maps
   AL, Phys. Rev. A **76**, 042330 (2007).
10. Exact Minimum Eigenvalue Distribution of an Entangled Random Pure State,
    Satya N. Majumdar, Oriol Bohigas, AL, J. Stat. Phys. **131**, 33 (2008).
11. Effect of classical bifurcations on the quantum entanglement of two coupled quartic oscillators
    M. S. Santhanam, V. B. Sheorey, AL, Phys. Rev. E. **77**, 026213 (2008).