Quantum target detection using entangled photons

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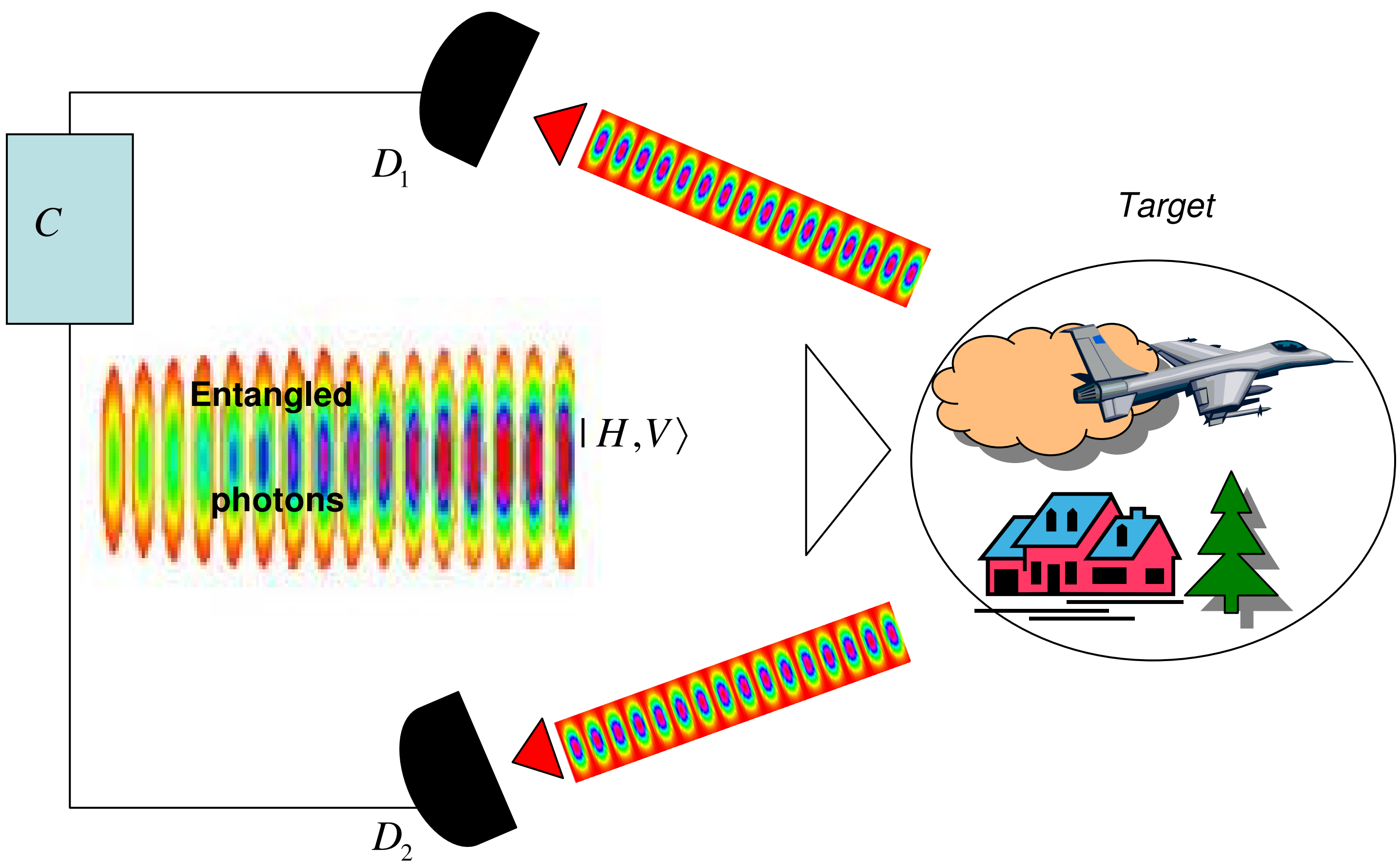
Quantum Target Detection



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Enhanced Sensitivity of Photodetection via Quantum Illumination

Seth Lloyd

The use of quantum-mechanically entangled light to illuminate objects can provide substantial enhancements over unentangled light for detecting and imaging those objects in the presence of high levels of noise and loss.

Let ρ_{in} be the input radiation state used to illuminate the target.

Hypothesis H_0 $\Longrightarrow$ **Target present:**

The state of the radiation, received at the detector: ρ_0

Hypothesis H_1 $\Longrightarrow$ **Target absent:**

The state of the radiation, received at the detector: ρ_1

Quantum target Detection $\Longrightarrow$ Ability to distinguish between the states ρ_0, ρ_1
(i.e., choosing between two hypotheses H_0, H_1)

Hypothesis Testing $\Longrightarrow$ **fundamental issue in statistical inference**

Classical Chernoff Bound

Asymptotic error rates in hypothesis testing: Chernoff (1952), Sanov (1957) and Hoeffding (1965)

- Question is to choose between two possible explanations (or models) called Hypothesis $\Longrightarrow H_0, H_1$
- Decision is based on a set of data collected from observations.

Example:

Deciding whether a patient is healthy (hypothesis H_0) or has certain disease (hypothesis H_1) based on some clinical tests.

$H_0 \Longrightarrow$ working hypothesis or null hypothesis;

$H_1 \Longrightarrow$ the alternative hypothesis.

Two types of errors:

(1) the rejection of a true null hypothesis (wrongly concluding that a healthy patient has the disease)

error probability $\Longrightarrow$ $p(1 \mid H_0) = p_0(1)$

(2) the acceptance of a false null hypothesis (failure to diagnose the disease)

error probability $\Longrightarrow$ $p(0 \mid H_1) = p_1(0)$

Minimizing the errors

- Common approach: Minimize one of the errors by keeping the other bounded by a constant (depending on the number of observations)
- Another approach (Baysean-like) : Minimize a linear combination of two error probabilities

$$
\begin{aligned}
P_e &= \min[\pi_0 \; p(1 \mid H_0) + \pi_1 \; p(0 \mid H_1)] \\
&= \min[\pi_0 \; p_0(1) + \pi_1 \; p_1(0)]
\end{aligned}
$$

$\pi_0, \quad \pi_1 \quad \Longrightarrow \quad$ *a priori* probabilities assigned to the occurrence of each hypothesis.

With N (large N) optimal tests, the probability P_e **of error declines exponentially as (considering equal *a priori* probabilities)**

$$P_e \approx \mathrm{Exp}[- N\, C(p_0,p_1)],$$

$$C(p_0,p_1) = -\min_{s=[0,1]} \log \sum_{b=0,1} p_0^s(b)\, p_1^{1-s}(b) \qquad (1)$$

The so called "Chernoff information" or Chernoff distance $C(p_0,p_1)$ **is expressed in terms of the Kullback-Leibler divergence**

$$C(p_0, p_1) = K(p_{s*} \,\|\, p_0) = K(p_{s*} \,\|\, p_1)$$

$$p_{s*}(b) = \frac{p_0^s(b)\, p_1^{1-s}(b)}{\sum_b p_0^s(b)\, p_1^{1-s}(b)},$$

$$K(p_{s*} \,\|\, p_0) = \sum_b p_{s*}(b) \log[\; p_0(b) / p_{s*}(b)]$$

$s*$ **is the value of** $s=[0,1]$ **that minimizes the right-hand side of (1).**

Quantum Scenario

Suppose we are given a sample of N identical quantum states, which are either ρ_0 or ρ_1 with the prior probability 1/2. Task is to minimize the average probability of making an incorrect decision about the state by devising a system of measurements and a decision rule.

Take a two element POVM set: $\{E_0, E_1; E_0 + E_1 = I\}$

Single copy minimum error probability is given by

$$
\begin{aligned}
P_{e,Q}^{(1)} &= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(\mathrm{Tr}[\rho_0 E_1] + \mathrm{Tr}[\rho_1 E_0]\right) \\
&= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 - \mathrm{Tr}[(\rho_0 - \rho_1)E_0]\right) \\
&= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 + \mathrm{Tr}[(\rho_0 - \rho_1)E_1]\right) \\
&= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 - \frac{1}{2}\mathrm{Tr}[(\rho_0 - \rho_1)(E_0 - E_1)]\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2}\max_{\{\mathrm{E}_0,\mathrm{E}_1\}}\mathrm{Tr}[(\rho_0 - \rho_1)(E_0 - E_1)]\right)
\end{aligned}
$$

Choose

$$E_0 = \sum_{\alpha} \left| \psi_{+\alpha} \right\rangle \left\langle \psi_{+\alpha} \right|,$$

Eigenvectors of $(\rho_0 - \rho_1)$ corresponding to positive/negative eigenvalues

$$E_1 = \sum_{\beta} \left| \psi_{-\beta} \right\rangle \left\langle \psi_{-\beta} \right|$$

…………. so that

the maximum of

$$\mathrm{Max}_{\{\mathrm{E}_0,\mathrm{E}_1\}}\mathrm{Tr}[(\rho_0-\rho_1)(E_0-E_1)] = \| \rho_0-\rho_1 \|$$

is obtained.

Tracenorm

Therefore, the minimum error probability in distinguishing the two states $\rho_0, \quad \rho_1$ takes the form

$$P_{e,Q}^{(1)} = \frac{1}{2}\left[1-\frac{1}{2}\| \rho_0-\rho_1 \|\right]$$

(Holevo-Helstrom result)

C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

Channel Discrimination

Distinguishing two channels Φ_0, Φ_1 with a input state ρ :

Single copy error probability

$$P_e^{(1)} = \frac{1}{2}\left(1 - \frac{1}{2}||\Phi_0(\rho) - \Phi_1(\rho)||_1\right)$$
$$= \frac{1}{2}\left(1 - \frac{1}{2}||\rho_0 - \rho_1||_1\right).$$

When the input state is a composite bipartite quantum system, with the channel affecting only one part of the state, the single-shot error-probability is expressed as,

$$P_e^{(1)} = \frac{1}{2}\left(1 - \frac{1}{2}||(\Phi_0 \otimes I)\rho - (\Phi_1 \otimes I)\rho||_1\right).$$

Discriminating depolarizing (Φ_0)**and identity** (Φ_1) **channels:**
(using d dimensional pure input states)

Input state: $\rho = |\psi\rangle\langle\psi|$

Output states:

$$\rho_0 = \Phi_0(\rho) = \frac{I}{d}$$

$$\rho_1 = \Phi_1(\rho) = |\psi\rangle\langle\psi|.$$

Error Probability:

$$P^{(1)}_{e,|\psi\rangle} = \frac{1}{2}\left(1 - \frac{1}{2}\left\|\frac{I}{d} - |\psi\rangle\langle\psi|\right\|_1\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\left[\left|\frac{1}{d} - 1\right| + \frac{d-1}{d}\right]\right) = \frac{1}{2d}.$$

Use of a maximally entangled $d \times d$ input state:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k_A, k_B\rangle,$$

Output states:

$$\begin{aligned} \rho_0 &= (\Phi_0 \otimes I)|\Psi_{AB}\rangle\langle\Psi_{AB}| \\ &= \frac{I}{d} \otimes \mathrm{Tr}_A[|\Psi_{AB}\rangle\langle\Psi_{AB}] = \frac{I \otimes I}{d^2} \\ \text{and} \quad \rho_1 &= (\Phi_1 \otimes I)|\Psi_{AB}\rangle\langle\Psi_{AB}| = |\Psi_{AB}\rangle\langle\Psi_{AB}|. \end{aligned}$$

Error-probability:

$$P^{(1)}_{e,|\Psi_{AB}\rangle} = \frac{1}{2d^2}.$$

Maximally entangled states $\Longrightarrow$ **enhanced sensitivity of channel discrimination**

M. F. Sacchi, Phys. Rev. A, 71 062340 (2005)

N copy error probability:

$$\boxed{P_{e,Q}^{(N)} = \frac{1}{2}\left[1 - \frac{1}{2} \| \rho_0{}^{\otimes N} - \rho_1{}^{\otimes N} \|\right]}$$

How does the error decline as N grows??

Finding the eigenvalues of $\rho_0^{\otimes N} - \rho_1^{\otimes N}$ is a hard computational task --- as the dimensionality of the states grows rapidly with increasing sample size N

Some special cases

V. Kargin, Ann. Stat. **33**, 959 (2005).

(1) Both the states to be discriminated are pure:

$$\rho_0 = |\psi_0\rangle\langle\psi_0|, \quad \rho_1 = |\psi_1\rangle\langle\psi_1|$$

N copy error probability is given by

$$P_{e,Q,\text{pure}}^{(N)} = \frac{1}{2}\left[1 - \frac{1}{2}\sqrt{1 - |\langle\psi_0|\psi_1\rangle|^{2N}}\right]$$

Asymptotical decline:

$$\lim_{N\to\infty} \frac{1}{N} \log P_{e,Q,\text{pure}}^{(N)} \approx 2\log|\langle\psi_0|\psi_1\rangle|$$

(2) If the states ρ_0 and ρ_1 commute, then classical error decline rate holds.

Bounds on error: (Audenaert et. al., Phys. Rev. Lett. 98, 160501 (2007))

Any two positive operators A, B satisfy the inequality

$$[A^s B^{1-s}] \geq \frac{1}{2}\left[\mathrm{Tr}[A+B] - \| A - B \|_1\right], \quad 0 \leq s \leq 1$$

Choosing

$$A = \frac{1}{2}\rho_0^{\otimes N}, B = \frac{1}{2}\rho_1^{\otimes N}, \quad \text{we get}$$

$$\frac{1}{2}\mathrm{Tr}\left[\left(\rho_0^{\otimes N}\right)^s \left(\rho_1^{\otimes N}\right)^{1-s}\right] \geq \frac{1}{2}\left[1 - \frac{1}{2}\| \rho_0^{\otimes N} - \rho_1^{\otimes N} \|_1\right]$$

or $$P^{(N)}{}_{e,Q} \leq P^{(N)}{}_{e,QCB} \equiv \min_{0\leq s\leq 1}\left(\frac{1}{2}\mathrm{Tr}\left[\rho_0{}^s \rho_1{}^{1-s}\right]^N\right)$$

For s=1/2, gives Bhattacharya bound on error

$$P^{(N)}{}_{e,QCB} \equiv \min_{0 \le s \le 1} \left(\frac{1}{2} \mathrm{Tr} \left[\rho_0{}^{s} \rho_1{}^{1-s} \right]^N \right)$$

Quantum Chernoff Bound

Reduces to the results on error in special cases

When only one of the states is pure, i.e., $\rho_1 = |\psi_1\rangle\langle\psi_1|$

$$P^{(N)}{}_{e,QCB} \equiv \frac{1}{2}\langle\psi_1|\rho_0|\psi_1\rangle^N = \frac{1}{2}[F(\rho_0,\rho_1)]^N$$

Fidelity

$$F(\rho_0,\rho_1) = \left(\mathrm{Tr}[\sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}}]\right)^2$$

Upper and lower bounds on N-copy error probability

Fuch-Graaf *IEEE Trans. Inform. Theory* **45** 1216–1227.

$$[1-\sqrt{F(\rho_0,\rho_1)}]\le\frac{1}{2}\|\rho_0-\rho_1\|_1\le\sqrt{1-F(\rho_0,\rho_1)};$$

$$F({\rho_0}^{\otimes N},{\rho_1}^{\otimes N})=F^N(\rho_0,\rho_1)$$

$$\Rightarrow\frac{1}{2}[1-\sqrt{1-F^N(\rho_0,\rho_1)}]\le P_{e,Q}^{(N)}\le\sqrt{F^N(\rho_0,\rho_1)}$$

Quantum Bhattacharya Bounds

$$\frac{1}{2}\left(1-\sqrt{1-\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{2N}}\right) \le P_{e,Q}^{(N)} \le \frac{1}{2}\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{N}$$

$$\frac{1}{2}\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{N} \ge P_{e,QCB}^{(N)}$$

weaker upper bound

Quantum Illumination with Gaussian States

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An optical transmitter irradiates a target region containing a bright thermal-noise bath in which a low-reflectivity object might be embedded. The light received from this region is used to decide whether the object is present or absent. The performance achieved using a coherent-state transmitter is compared with that of a quantum-illumination transmitter, i.e., one that employs the signal beam obtained from spontaneous parametric down-conversion. By making the optimum joint measurement on the light received from the target region together with the retained spontaneous parametric down-conversion idler beam, the quantum-illumination system realizes a 6 dB advantage in the error-probability exponent over the optimum reception coherent-state system. This advantage accrues despite there being no entanglement between the light collected from the target region and the retained idler beam.

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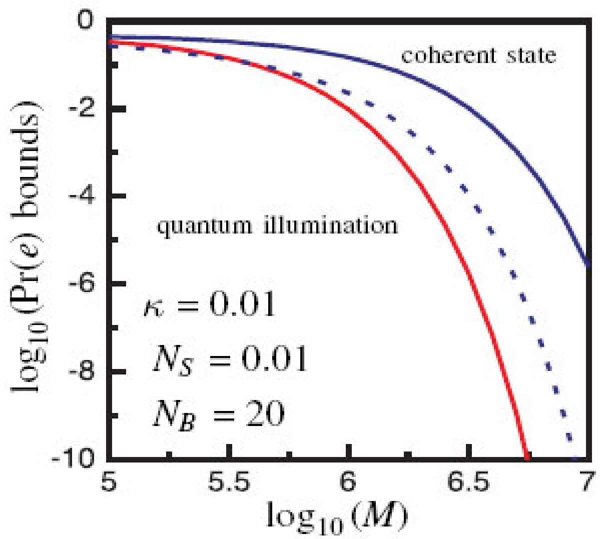

FIG. 1 (color online). Upper bounds (solid curves) on the target-detection error probabilities for coherent-state (Chernoff bound) and quantum-illumination (Bhattacharyya bound) transmitters with M transmitted modes each with $N_S = 0.01$ photons on average when $\kappa = 0.01$ and $N_B = 20$. Also shown is the lower bound (dashed curve) for the coherent-state case, which (see below) also applies to *all* classical-state transmitters with $\sum_{m=1}^{M}\langle \hat{a}_{S_m}^{\dagger}\hat{a}_{S_m}\rangle = MN_S$. For large M, the classical-state lower bound exceeds the quantum-illumination upper bound.

This work motivates us to explore a simpler mathematical model that captures and elucidates the role of continuous variable entanglement in quantum target detection.

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Quantum target detection using entangled photons

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We investigate performances of pure continuous variable states in discriminating thermal and identity channels by comparing their M-copy error-probability bounds. This offers us a simplified mathematical analysis for quantum target detection with slightly modified features: the object—if it is present—perfectly reflects the signal beam irradiating it, while thermal noise photons are returned to the receiver in its absence. This model facilitates us to obtain analytic results on error-probability bounds, i.e., the quantum Chernoff bound and the lower bound constructed from the Bhattacharya bound on M-copy discrimination error probabilities of some important quantum states, like photon number states, N-photon maximally entangled (N00N) states, coherent states and the entangled photons obtained from spontaneous parametric down conversion (SPDC). Comparing the M-copy error-bounds, we identify that path-entangled states indeed offer enhanced sensitivity than the photon number state system, when average signal photon number is small compared to the thermal noise level. However, in the high signal-to-noise scenario, N00N states fail to be advantageous than the photon number states. Entangled SPDC photon pairs too outperform conventional coherent state system in the low signal-to-noise case. On the other hand, conventional coherent state system surpasses the performance sensitivity offered by entangled photon pair, when the signal intensity is much above that of thermal noise. We find an analogous performance regime in the lossy target detection (where the target is modeled as a weakly reflecting object) in a high signal-to-noise scenario.

Simple model of target detection:

- An optical transmitter sends light toward a region where a perfectly reflecting object is suspected to be present.
- The object, when present, reflects light falling on it to the receiver end.
- When the object is absent, the signal light passes through the region undeflected and a thermal noise radiation is returned to the receiver.
- The returned light is processed by the receiver to decide between the two hypotheses

  H_0 : *object not there,* H_1 : *object there.*
- The receiver has to distinguish between two quantum states of light: (1) output of a thermal channel (object not there) (2) output of an identity channel (object there)

TARGET DETECTION $\Longrightarrow$ DISCRIMINATION OF THERMAL AND IDENTITY CHANNELS USING PHOTONS

Photon number states:

$$\rho_0 = \rho_{\text{th}}(N_B) = \sum_{k=0}^{\infty} \frac{N_B^k}{(N_B+1)^{k+1}} |k\rangle\langle k|,$$

$$=(1-e^{-\beta})\sum_{k=0}^{\infty} e^{-k\beta}|k\rangle\langle k|,$$

$$\text{where} \quad N_B = \frac{e^{-\beta}}{(1-e^{-\beta})},$$

Target absent

$$\rho_1 = |n\rangle\langle n|$$

Target present

N-copy Error probability:

$$P_{e,n}^{(N)} = \frac{1}{2}\left[\langle n|\rho_{\text{th}}(N_B)|n\rangle\right]^N$$

$$= \frac{1}{2}\left(1-e^{-\beta_B}\right)^N e^{-nN\beta_B} = \frac{1}{2}\left[\frac{N_B^n}{(1+N_B)^{n+1}}\right]^N$$

N00N states

$$|\Psi^{SI}_{\mathrm{N00N}}\rangle = \frac{1}{\sqrt{2}}[|2n,0\rangle + |0,2n\rangle]$$

Intensity: $\langle a_S^\dagger a_S\rangle = \langle a_I^\dagger a_I\rangle = n$

Photon states to be discriminated:

$$\rho_0 = \rho_{\mathrm{th}}(N_B) \otimes \mathrm{Tr}_S[|\Psi^{SI}_{\mathrm{N00N}}\rangle\langle\Psi^{SI}_{\mathrm{N00N}}|]$$
$$= \rho_{\mathrm{th}}(N_B) \otimes \frac{1}{2}[|0\rangle\langle 0| + |2n\rangle\langle 2n|],$$

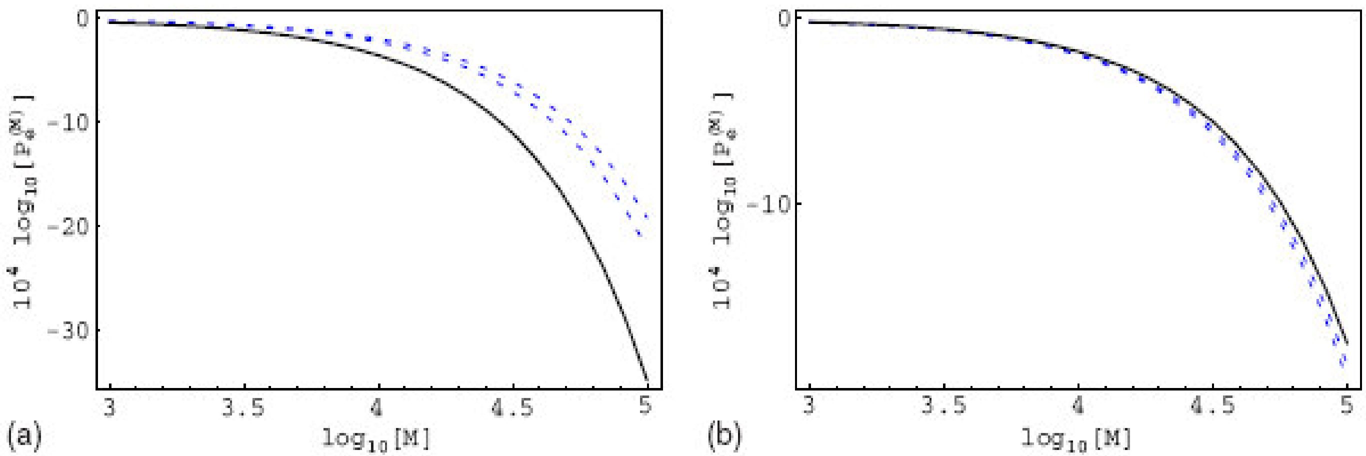
$$\rho_1 = |\Psi^{SI}_{\mathrm{N00N}}\rangle\langle\Psi^{SI}_{\mathrm{N00N}}|.$$

Quantum Chernoff bound on N-shot error-probability

$$
\begin{aligned}
P_{e,QCB,N00N}^{(N)} &= \frac{1}{2}\left[\langle\Psi_{N00N}^{SI}|\left\{\rho_{\mathrm{th}}(N_B)\otimes\frac{1}{2}[|0\rangle\langle 0| + |2n\rangle\langle 2n|]\right\}|\Psi_{N00N}^{SI}\rangle\right]^N \\
&= \frac{1}{2}\left(\frac{N_B^n}{(1+N_B)^{n+1}}\left[\frac{1}{4}\left\{\left(\frac{1+N_B}{N_B}\right)^n + \left(\frac{1+N_B}{N_B}\right)^{-n}\right\}\right]\right)^N \\
&= \frac{1}{2}(1-e^{-\beta})^N\, e^{-Nn\beta}\left(\frac{\cosh(n\beta)}{2}\right)^N
\end{aligned}
$$

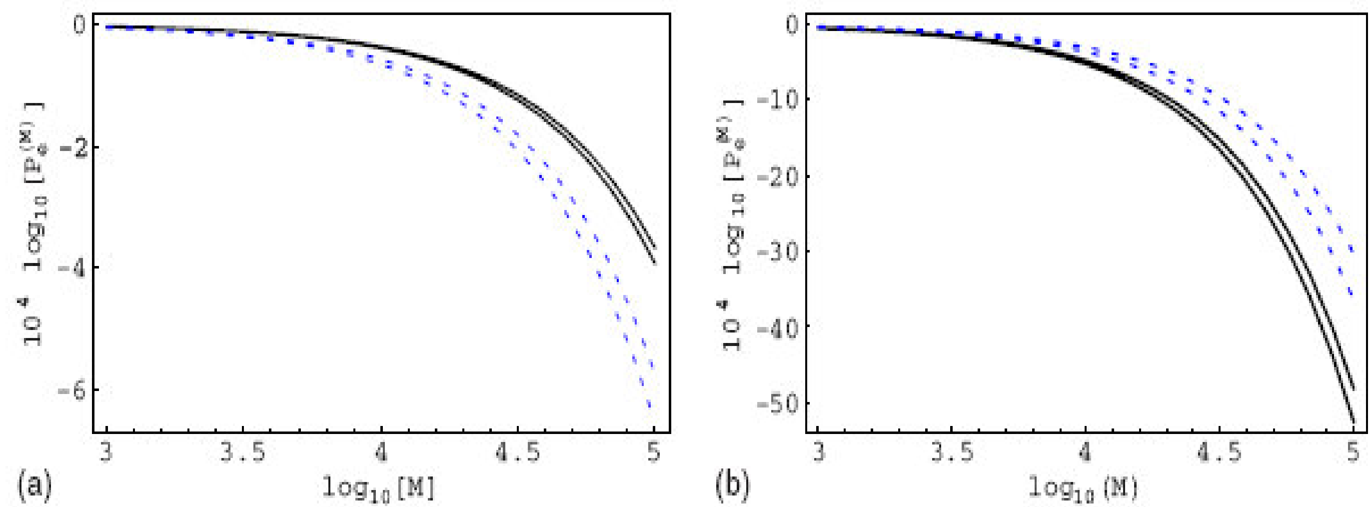
Lower bound (Bhattacharya bound) on N-shot error-probability

$$
P_{e,LB,\mathrm{N00N}}^{(N)} = \frac{1}{2}\left[1-\sqrt{1-\left(\sqrt{\frac{e^{-n\beta}(1-e^{-\beta})}{2}}\cosh(n\beta/2)\right)^{2N}}\right]
$$



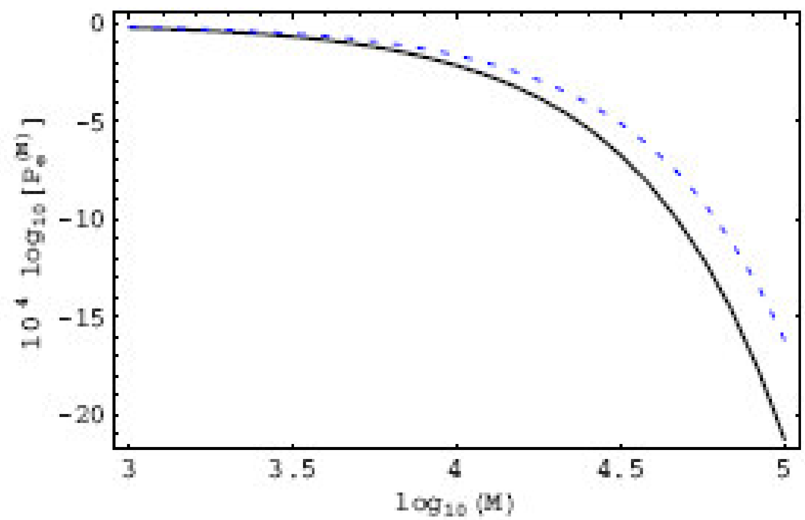
Upper, lower bounds (dashed curves) on N-copy error-probability with N00N states and photon number state's error-probability (solid curve) for a thermal noise $\beta = 0.05$; photon numbers in (a) n=100 and in (b) n=20. The lower bound lies above the number state error-probability in (a) implying that N00N states are *not* advantageous over photon number states. But, with smaller number of photons (as illustrated in (b)), entangled N00N states indeed offer an enhanced sensitivity over number state system.

Coherent light vs two mode entangled photons from SPDC process



Logarithms of upper and lower bounds (dashed curves) on N-shot error-probability with entangled photon pairs from SPDC source and that of coherent state system (solid curves) for (a) thermal noise $N_B = 0.75$ and $N_S = 0.5$ and in (b) $N_B = 2$, $N_S = 30$, plotted as a function of $\log_{10}[N]$. The target detection with $N_S < N_B$ in (a) is illustrative of the regime where entangled photon pairs show enhanced performance sensitivity over coherent light. But, it is seen from (b) that when $N_S >> N_B$ coherent state system is more advantageous than entangled SPDC photon pairs.

Performance sensitivity of coherent states vs SPDC entangled photon pairs in a lossy, noisy case



A comparison of N-shot quantum Chernoff bound on error-probability achievable with coherent state system (solid curve) with the corresponding lower bound (dashed curve) associated with entangled photon pair system in the lossy (relectivity $\kappa = 0.01$), noisy (average thermal noise photons $N_B = 20$) target detection scenario using highly intense signals (with signal to noise ratio $N_S/N_B = 2000$). It may be seen that the lower bound on entangled photon error-probability lies above the upper bound on coherent state error-probability revealing that coherent state system turns out to be more advantageous compared to entangled photon pair system, with very high signal to noise ratio.

Illuminating a target with entangled/unentangled light

Signal intensity << Thermal noise intensity: *Entangled photons are advantageous over the conventional ones.*

Signal intensity >>Thermal noise intensity: Entangled photons loose over their unentangled counterparts

Summary

Entangled photon states do offer enhanced sensitivity than the photon number state system, when average signal photon number is small compared to the thermal noise level. However in the high signal-to-noise scenario, they fail to be advantageous in quantum target detection.

Thank you