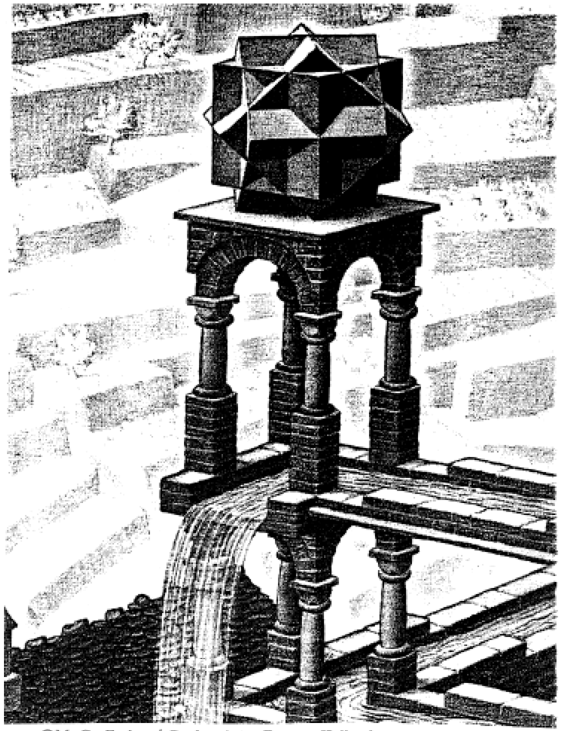

Harnessing Quantum No-Go Theorems

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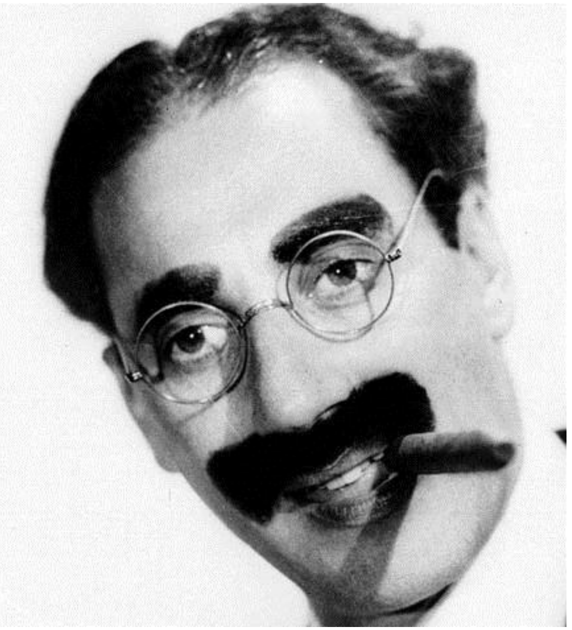
Grouche Marx on quantum theory:



"Very interesting theory – it makes no sense at all."

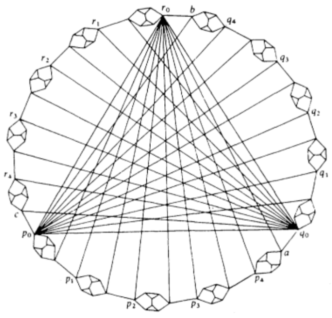
Motivation

- Quantum Mechanics (QM): statistical predictions
- Can we reproduce predictions of QM using (deterministic) Hidden Variables?
- Bell, Kochen & Specker: *Only if they are non contextual or non local!*
- *These features are useful resource!*

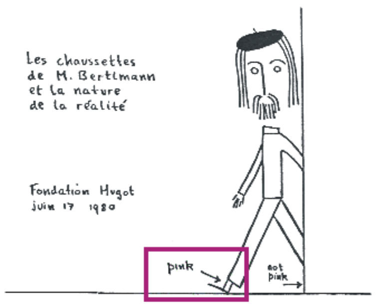
Bell, J. S. *Rev. Mod. Phys.* **38**, 447-452 (1966).
Kochen, S. & Specker, E. P. *J. Math. Mech.* **17**, 59–87 (1967).

Outline

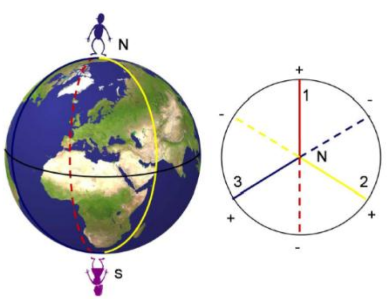
1. Kochen-Specker Theorem
   1.1 Non-contextual hidden variable models
   1.1 Kochen-Specker theorem for two qubits
   1.2 Kochen-Specker theorem for a qutrit
   1.3 Klyachko et al. inequality



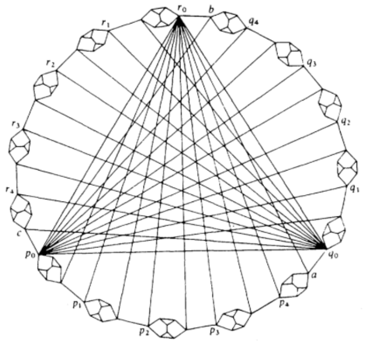
2. Bell's Theorem
   2.1 Local hidden variable models
   2.2 Bell's theorem
   2.3 WWZB inequalites



3. Quantum Communication Complexity
   3.1 Distributed computation
   3.2 Coordination games

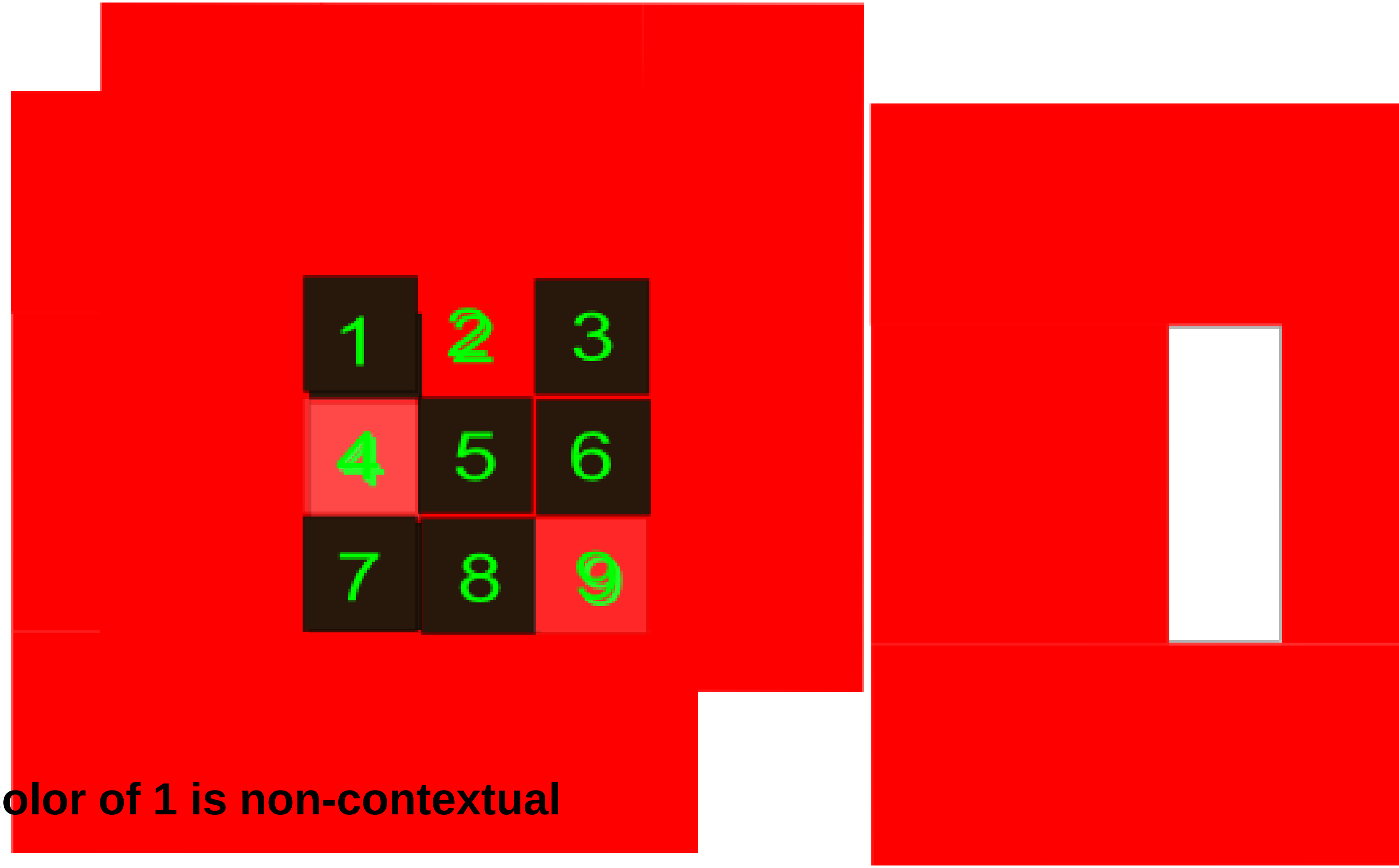


1. Kochen-Specker Theorem

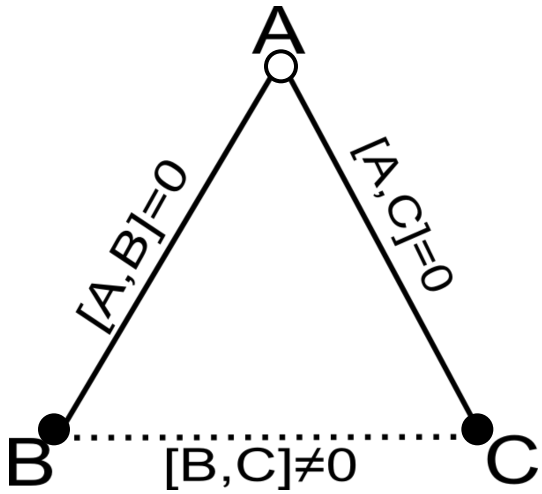


Hidden colors

color of 1 checked (measured) in two different ways (contexts)



color of 1 is non-contextual



Non-Contextuality:

$$v_B(A) = v_C(A) \equiv v(A)$$

Non-contextuality: The value assigned to observable *A* of an individual system is independent of the experimental context in which it is measured, in particular of any observable that is measured jointly with that one.

Kochen-Specker theorem (**KS**) =>

Non-Contextual Hidden Variable theories (**NCHV**) are incompatible with Quantum Mechanics (**QM**)

Correspondence

Toy	**Quantum Mechanics**
fields	observables
colors (■/□)	results (+/-)
slit	commutativity (comeasurability)

No underlying coloring can explain what we 'observe through the slit' for the following observables (2-qubits, Mermin-Peres square):

$1\otimes\sigma_z$	$\sigma_z\otimes 1$	$\sigma_z\otimes\sigma_z$
$\sigma_x\otimes 1$	$1\otimes\sigma_x$	$\sigma_x\otimes\sigma_x$
$\sigma_x\otimes\sigma_z$	$\sigma_z\otimes\sigma_x$	$\sigma_y\otimes\sigma_y$

tested experimentally with

ions,

neutrons,

photons

$$1 = Z_2 Z_1 (Z_1 Z_2) X_1 X_2 (X_1 X_2) (X_1 Z_2)(Z_1 X_2)(Y_1 Y_2)$$

$$= Z_2 X_1 (X_1 Z_2) Z_1 X_2 (Z_1 X_2) (Z_1 Z_2)(X_1 X_2)(Y_1 Y_2)$$

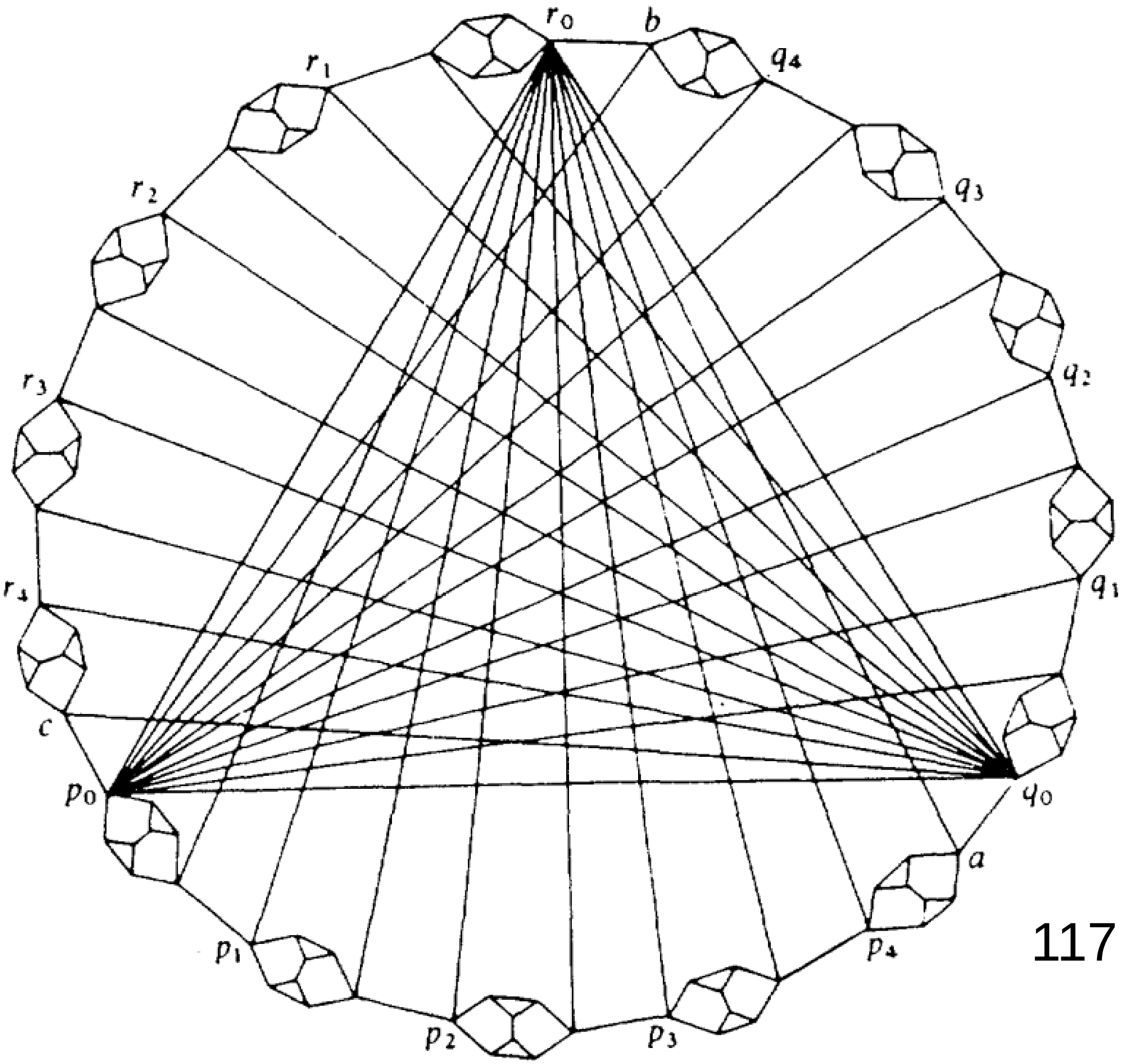
$$= (Z_1 Z_2)(X_1 X_2)(Y_1 Y_2)$$

Non-contextual hidden variable

$$(\sigma_z \sigma_z)(\sigma_x \sigma_x)(\sigma_y \sigma_y) = -1$$ **QM!**

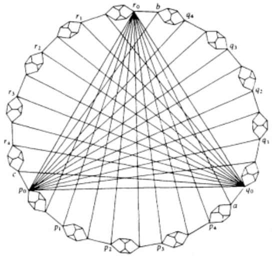
Mermin, N. D. *Phys. Rev. Lett. 65, 3373–3376 (1990).*; Peres, A. *Journal of Physics A 24, L175-L178 (1991);* Cabello, A. *Phys. Rev. Lett.* ***101****, 210401 (2008).*

Kochen-Specker proof for qutrits

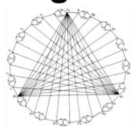
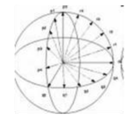


117 directions!

Kochen, S. & Specker, E. P. *J. Math. Mech.* **17**, 59–87 (1967).

Diagram 	**Spin-1 system**	

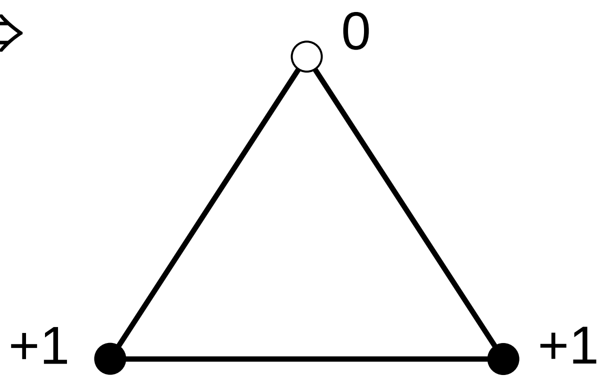
From math to coloring rules

Diagram 	Spin-1 system	$\mathbf{R}^3$ 
vertex	observable $S_{\vec{u}}^2$	direction $\vec{u}$
straight line	$\left[S_{\vec{u}}^2, S_{\vec{v}}^2\right] = 0$	$\vec{u} \perp \vec{v}$
color	$v\left(S_{\vec{u}}^2\right)$	

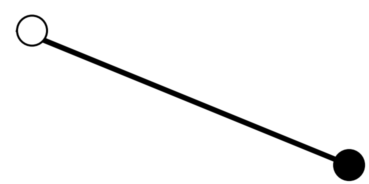
for orthogonal directions x,y,z:

$$S_x^2 + S_y^2 + S_z^2 = s(s+1) = 2I$$

$\Rightarrow$

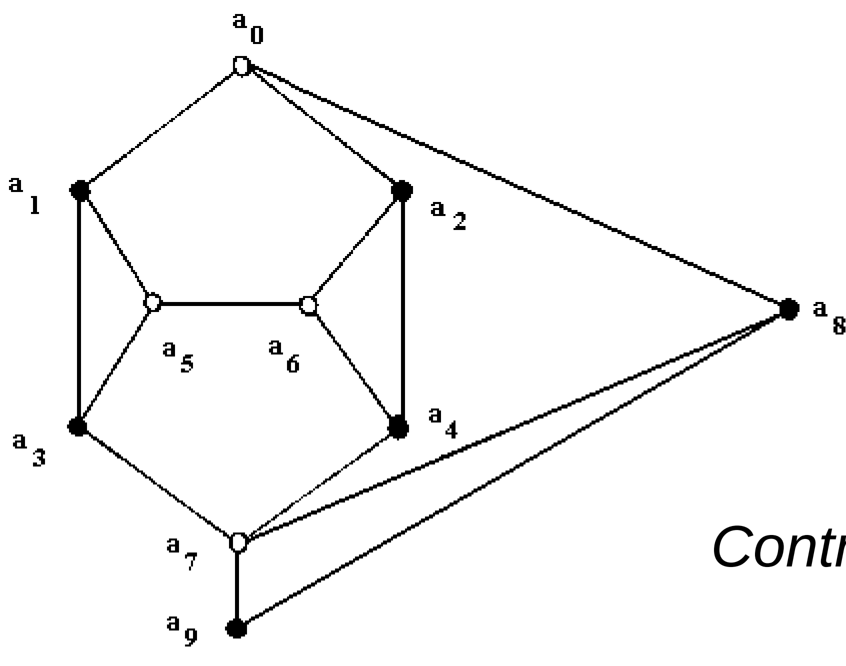


$\Rightarrow$



Kochen-Specker proof for qutrits

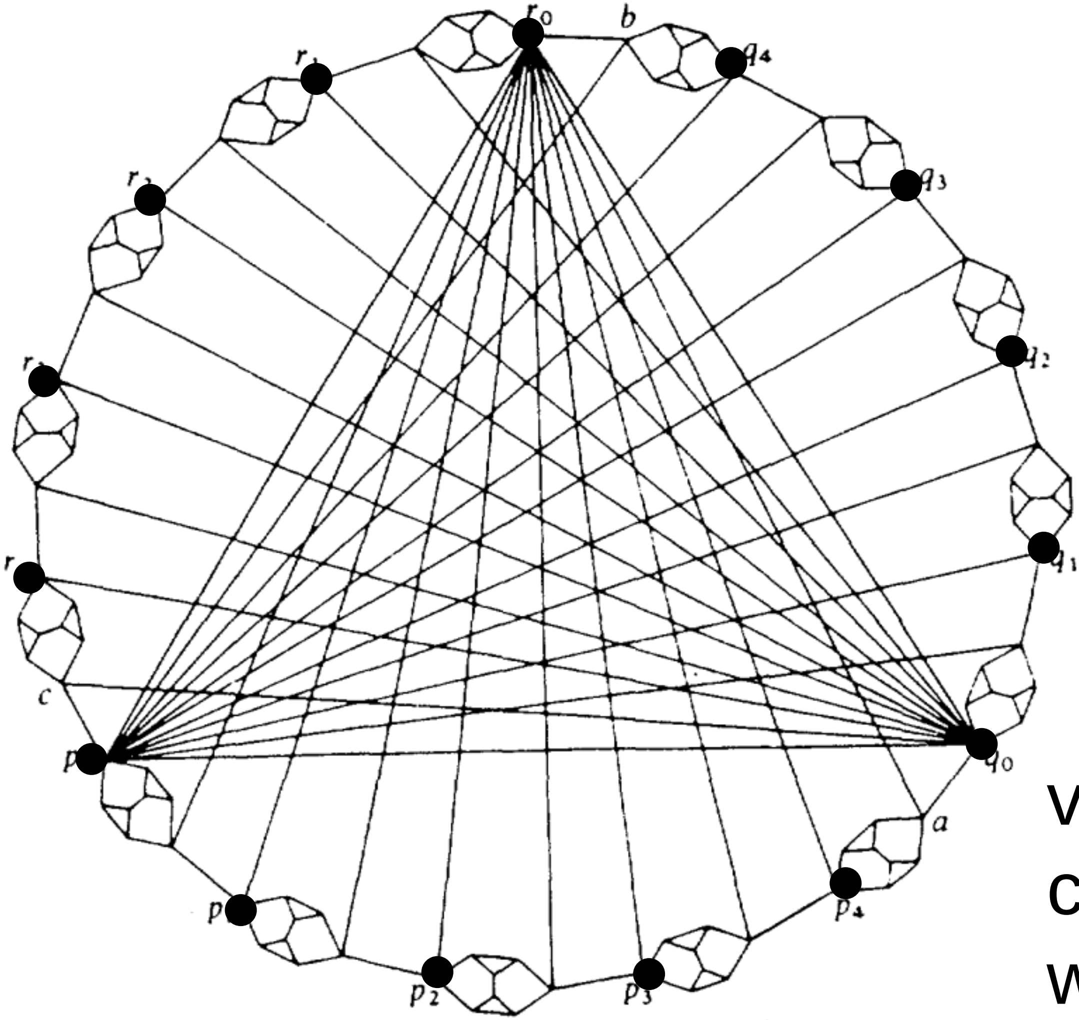
Vertices a0 and a9 must have the same color (*Reducio ad absurdum*):



Contradiction

http://plato.stanford.edu/entries/kochen-specker/

The point of the proof is that



vertices cannot be colored in agreement with QM

Simpler?

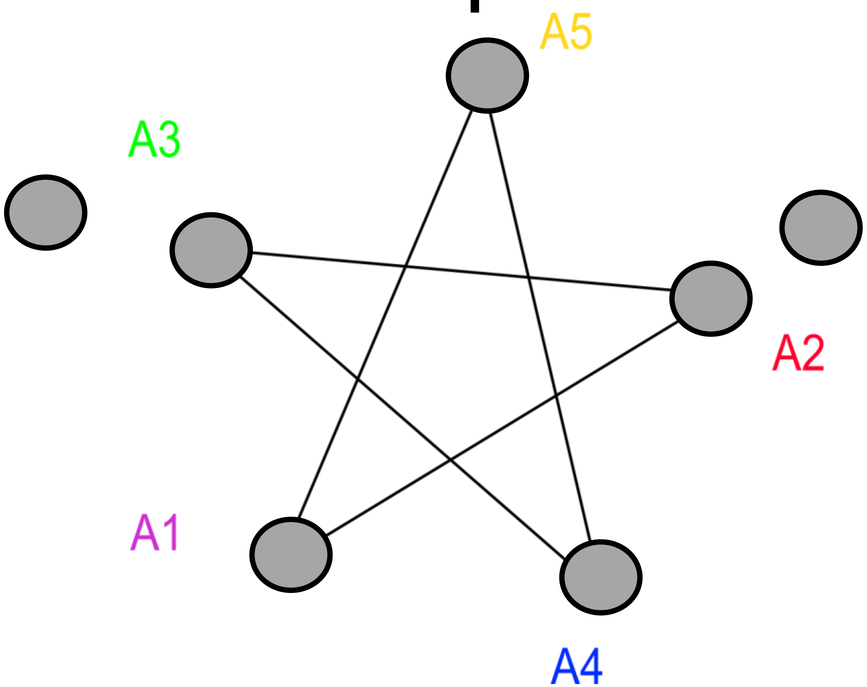
- too many observables (117) =>
  inequalities hard to violate

- but a proof involving only 5 observables was recently found *(Klyashko et. al)*

- 5 is provable to be the minimum

Klyachko, A., Can, M. A., Binicioğlu, S., Shumovsky, A., *Phys. Rev. Lett. 101, 020403 (2008)*

Five cups – rules of the game



1) We close our eyes, while the magician puts stones under some of the cups.

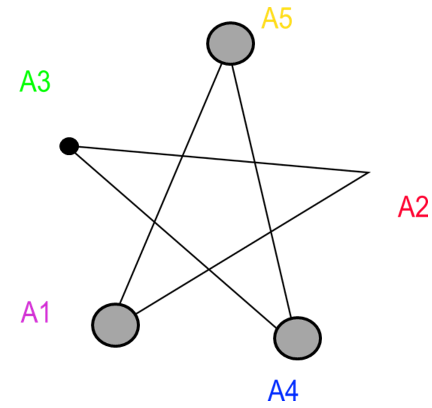
2) We open the eyes and lift two cups.
(only two cups and only the ones lying on one edge can be lifted at a time)

We assign values to the outcomes:
'a stone at A_i ' - $a_i = -1$
'no stone at A_i ' - $a_i = +1$

3) back to 1), check different pair of cups

Classical bound

Compute:



The magician wins if

$$\langle A_1A_2\rangle+\langle A_2A_3\rangle+\langle A_3A_4\rangle+\langle A_4A_5\rangle+\langle A_5A_1\rangle < -3$$

How can the magician win?

$$\overset{+}{a_1}\overset{-}{a_2}+\overset{-}{a_2}\overset{+}{a_3}+\overset{+}{a_3}\overset{-}{a_4}+\overset{-}{a_4}\overset{+}{a_5}+\overset{+}{a_5}\overset{+}{a_1} \geq -3 \qquad a_i = \pm 1$$

assume the **Joint Probability Distribution** for the 2^5 possible measurement outcomes **exists, then**

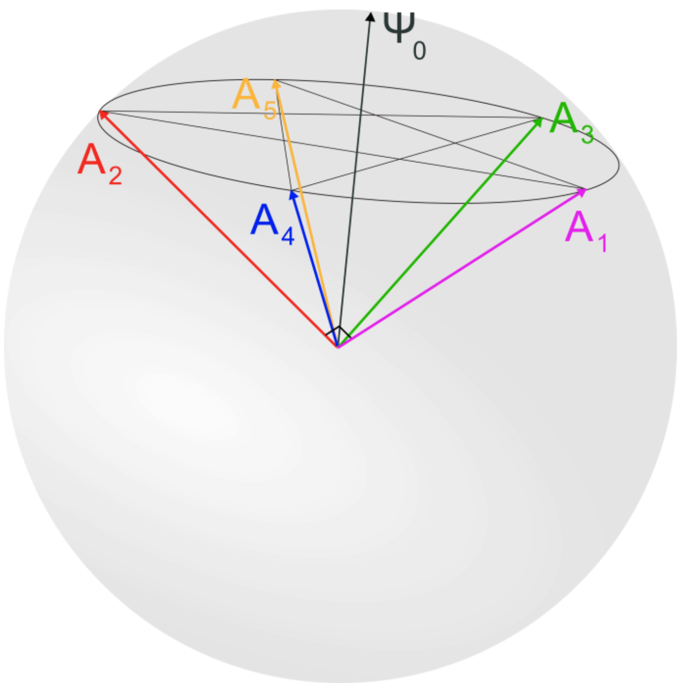
$$\langle A_1A_2\rangle+\langle A_2A_3\rangle+\langle A_3A_4\rangle+\langle A_4A_5\rangle+\langle A_5A_1\rangle \geq -3$$

If the classical magician doesn't rearrange the stones after he learns which cups we choose, he cannot win.

Klyachko, A., Can, M. A., Binicioğlu, S., Shumovsky, A., *Phys. Rev. Lett. 101, 020403 (2008)*

Quantum violation

$$\left|\Psi\right\rangle \equiv \left|S_{\Psi}^{2}=0\right\rangle$$



Measurements on a spin-1 particle

Squared spin projections on five directions

$$A_i = 2\hat{S}_i^2 - 1$$

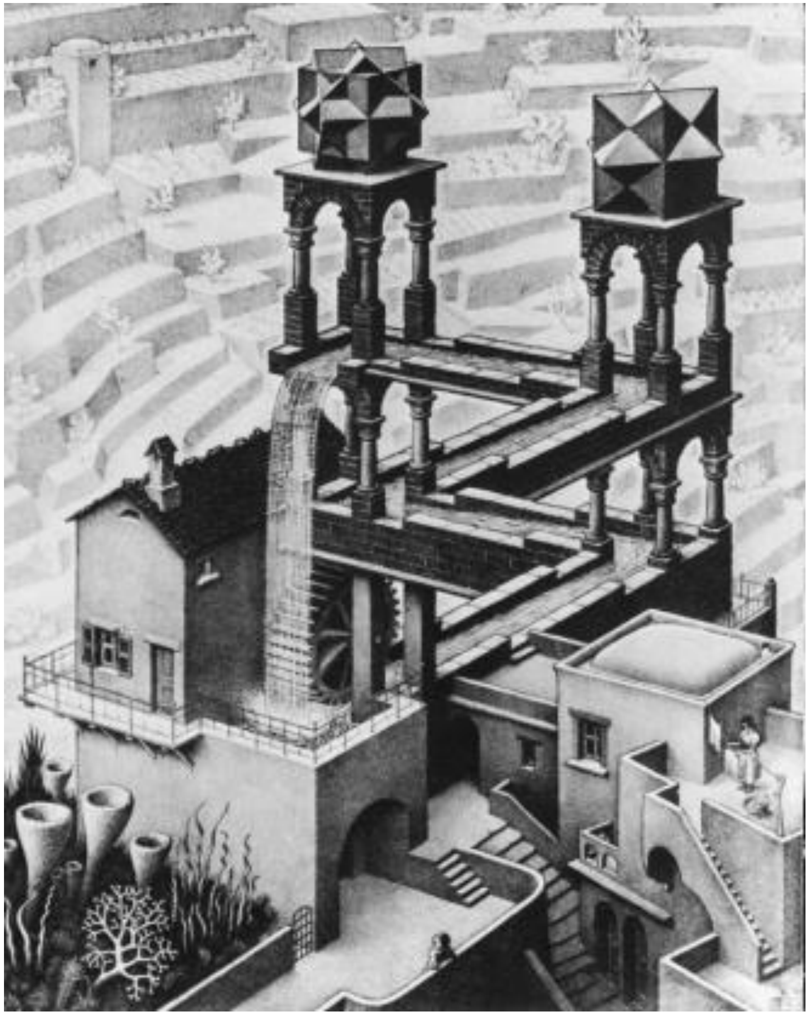
$$\left[A_i, A_{(i+1)\,\mathrm{mod}\,5}\right] = 0$$

$$\langle A_1 A_2\rangle_\Psi + \langle A_2 A_3\rangle_\Psi + \langle A_3 A_4\rangle_\Psi + \langle A_4 A_5\rangle_\Psi + \langle A_5 A_1\rangle_\Psi = 5 - 4\sqrt{5} \approx -3.94$$

Quantum magician can win the game, if he uses proper measurements on a spin-1 particle.

Klyachko, A., Ali Can M., Binicioğlu, S., Shumovsky, A., *Phys. Rev. Lett. 101, 020403 (2008)*

Analog to “Waterfall”

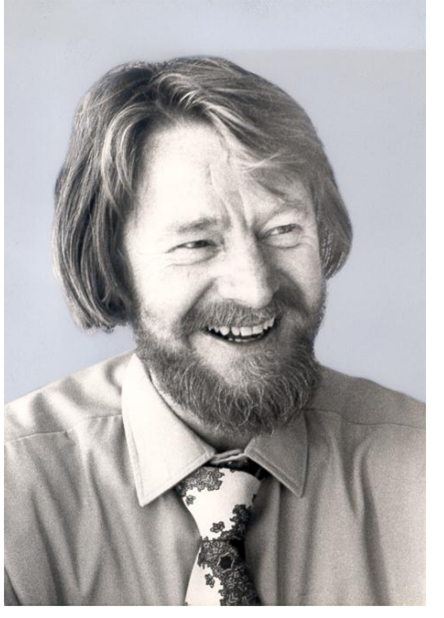


M. C. Escher, *Waterfall*, (1961) (from wikipedia)

Peres, A. *Quantum Theory: Concepts and Methods,* Ch. 7, Kluwer, Dordrecht (1993).

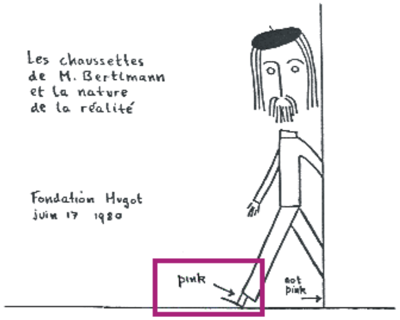
„Non-contextuality“ – reasonable assumption?

“‘These different possibilities require different experimental arrangements; there is no a priori reason to believe that the results … should be the same. The result of observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of apparatus.”

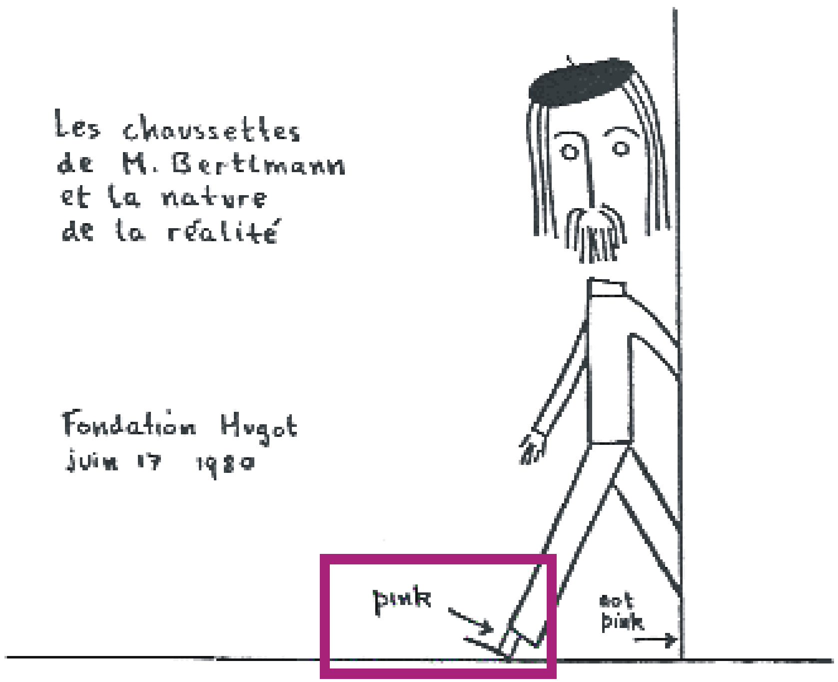


J. S. Bell

2. Bell's Theorem



„Bertlmann‘s Socks and the Nature of Reality“



Are entangled systems like
Berlmann‘s socks?

Local Realism

1. Reality („Hidden-variables“)

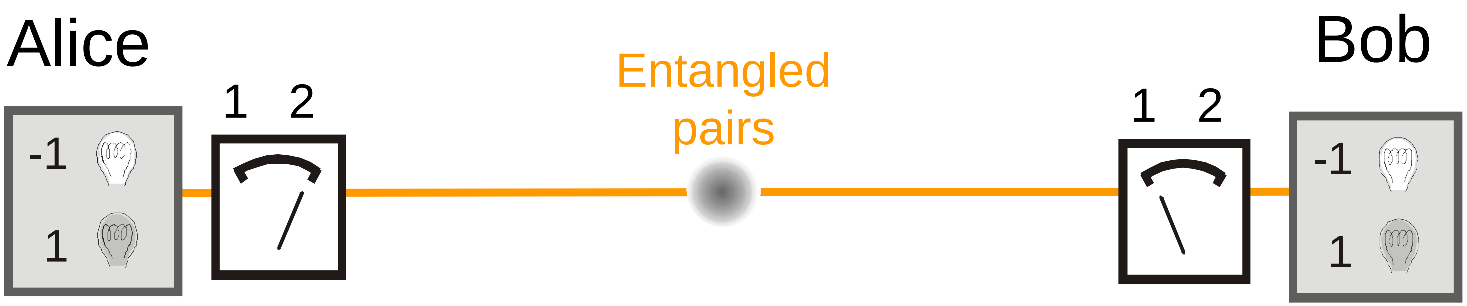
The measurement results are determined by properties the particles carry **prior to and independent** of observation.

2. Locality

“Since at the time of measurements the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.”
(in words of Einstein-Podolsky-Rosen paper).

Bell's inequality

(Clauser-Horne-Shimony-Holt)



$A_1=\pm1$, $A_2=\pm1$

$B_1=\pm1$, $B_2=\pm1$

$$A_1(B_1+B_2)+A_2(B_1-B_2)=\pm2$$

$$\left|\langle A_1B_1\rangle+\langle A_1B_2\rangle+\langle A_2B_1\rangle-\langle A_2B_2\rangle\right|\le 2$$

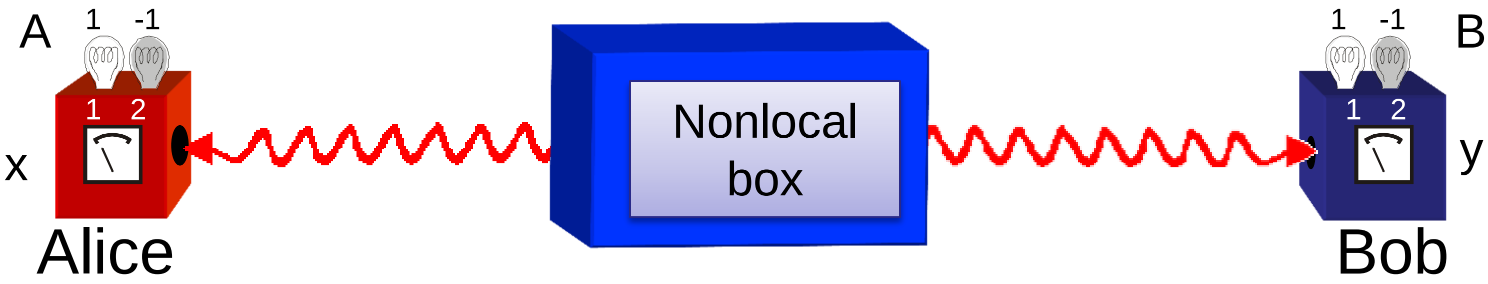
Average over all runs

$\boxed{2\sqrt{2}}$ But, QM!

Why not 4?

What went wrong? Locality?, Realism?, Both?

Non-signalling limits quantum correlations?



No signalling:

$$\sum_{B=-1,1} p(A,B \mid x,y) = p(A \mid x)$$

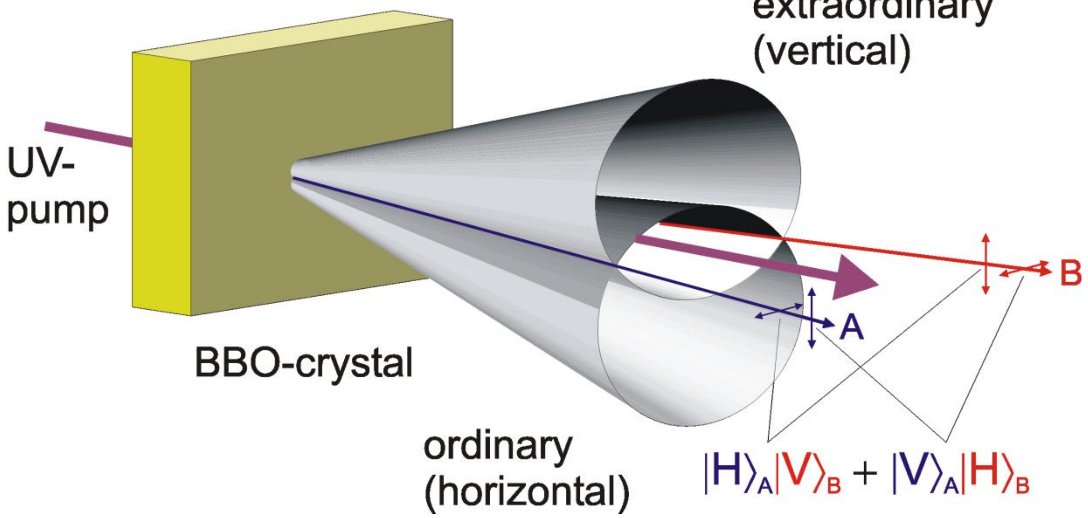
Does **not** depend on y

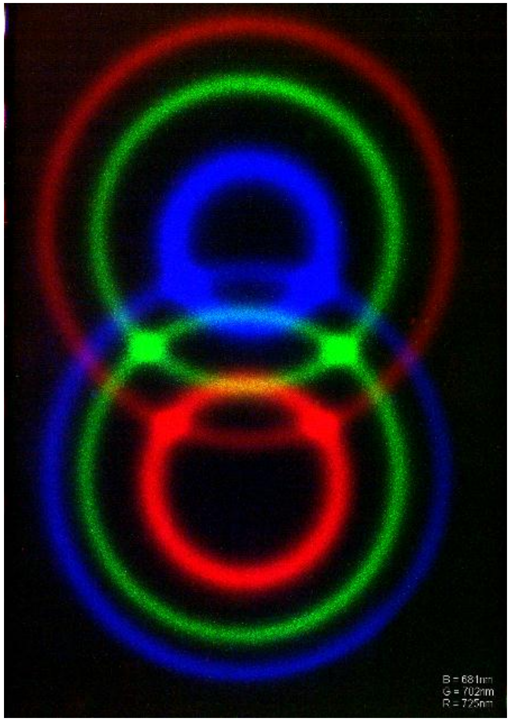
$$p(A=1,B=1 \mid 1,1) = p(A=-1,B=-1 \mid 1,1) = 1/2$$
$$p(A=1,B=1 \mid 1,2) = p(A=-1,B=-1 \mid 1,2) = 1/2$$
$$p(A=1,B=1 \mid 2,1) = p(A=-1,B=-1 \mid 2,1) = 1/2$$
$$p(A=1,B=-1 \mid 2,2) = p(A=-1,B=1 \mid 2,2) = 1/2$$

$$p(A=B \mid 1,1) + p(A=B \mid 1,2) + p(A=B \mid 2,1) + p(A=-B \mid 2,2) = 4$$

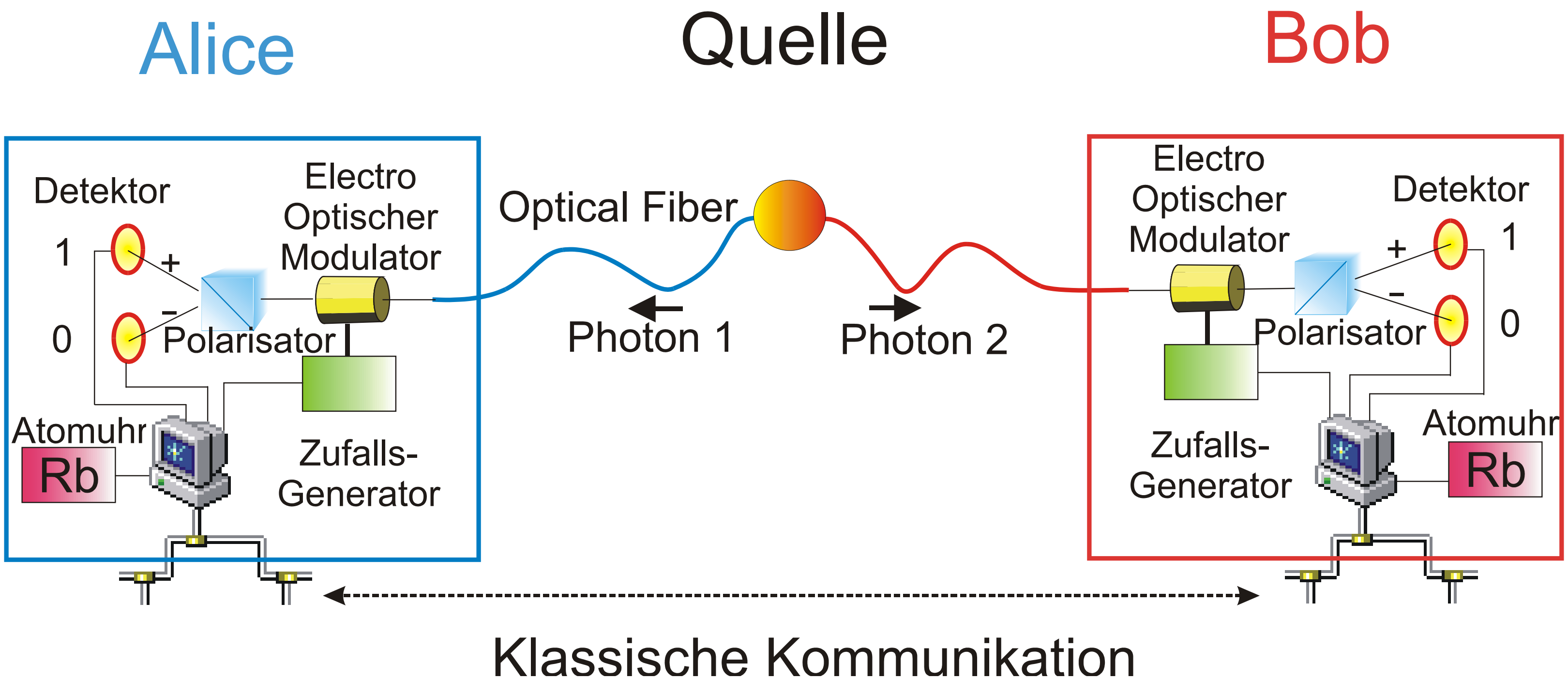
Popescu & Rohrlich, Found. Phys. 1994

Spontaneous Parametric Down-Conversion Type II





Innsbruck Bell's Experiment

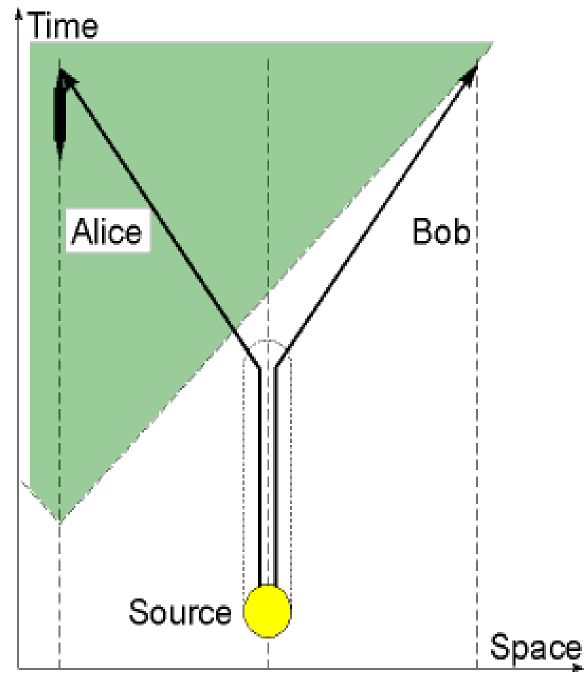


Closing Locality Loophole

Weihs *et al.*, 1998

„Of more importance, in my opinion, is the complete absence of the vital time factor in existing experiments. The analyzers are not rotated during the flight of the particles. Even if one is obliged to admit some long-range influence, it need not travel faster than light - and so would be much less indigestible."

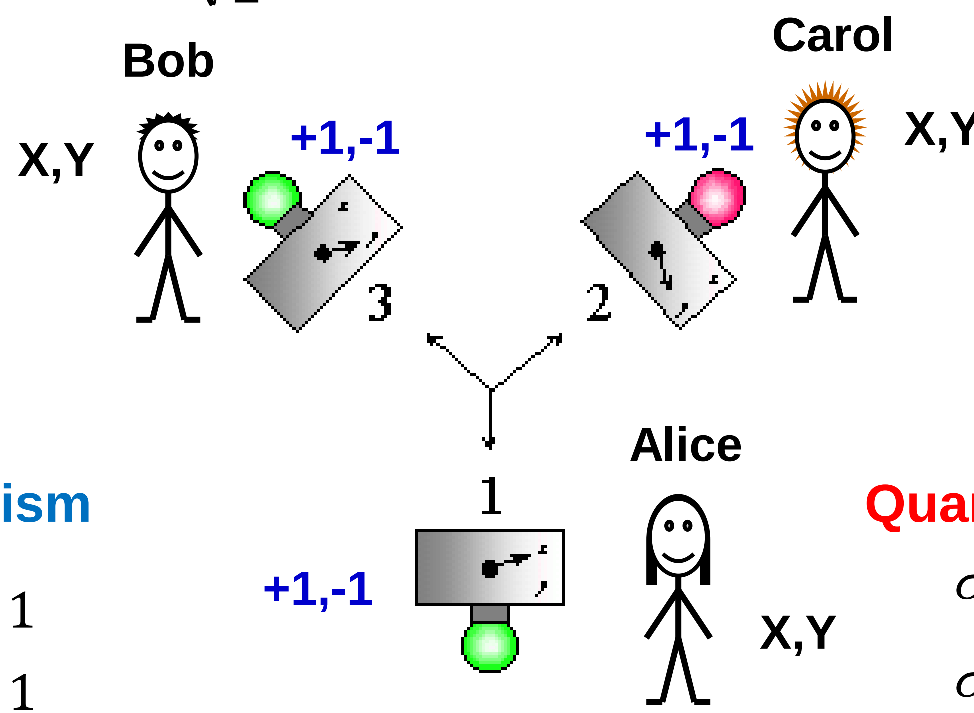
J. S. Bell



Innsbruck Experiment

Greenberger-Horne-Zeilinger argument

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}\left(|z+\rangle|z+\rangle|z+\rangle - |z-\rangle|z-\rangle|z-\rangle\right)$$



Local Realism

$$X_A Y_B Y_C = 1$$
$$Y_A X_B Y_C = 1$$
$$Y_A Y_B X_C = 1$$
$$\overline{}$$
$$\Rightarrow X_A X_B X_C = 1$$

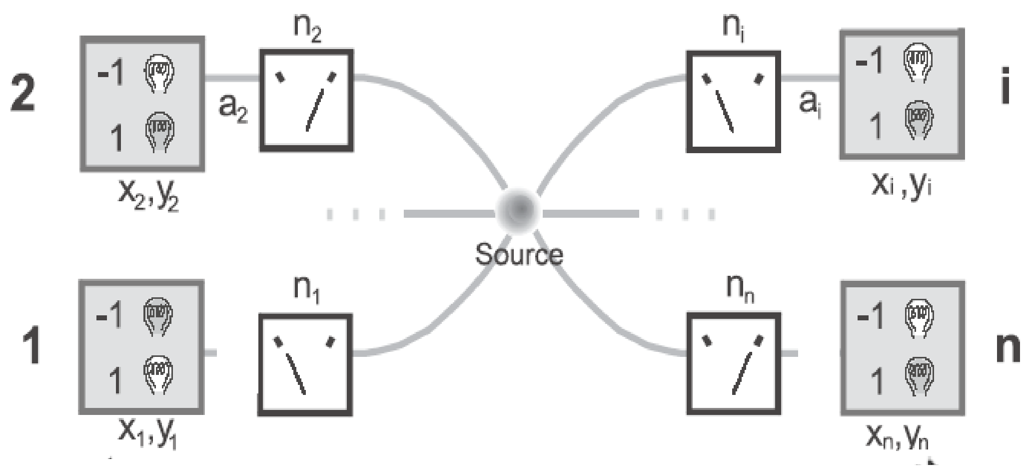
Contradiction!

Quantum Mechanics

$$\sigma_x^{(A)}\sigma_y^{(B)}\sigma_y^{(C)} = 1$$
$$\sigma_y^{(A)}\sigma_x^{(B)}\sigma_y^{(C)} = 1$$
$$\sigma_y^{(A)}\sigma_y^{(B)}\sigma_x^{(C)} = 1$$
$$\overline{\phantom{\sigma_x^{(A)}\sigma_x^{(B)}\sigma_x^{(C)} = -1}}$$
$$\sigma_x^{(A)}\sigma_x^{(B)}\sigma_x^{(C)} = -1$$

Bell's inequalities for N partners

(WWZB inequalities)



$$| \sum_{s_1,\dots,s_N=\pm 1} S(s_1,\dots,s_N) \sum_{k_1,\dots,k_N=1}^{2} s_1^{k_1}\cdots s_N^{k_N} E(\vec{n}_{k_1},\dots,\vec{n}_{k_1}) | \leq 2^N$$

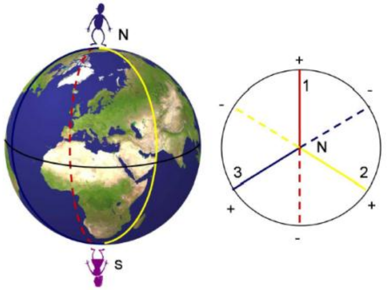
$$\left\langle \prod_{j=1}^{N} A(\vec{n}_{k_j}) \right\rangle = E\left(\vec{n}_{k_1},\dots,\vec{n}_{k_N}\right)$$

QM Bound / LR Bound ~ $\sqrt{2^{N-1}}$

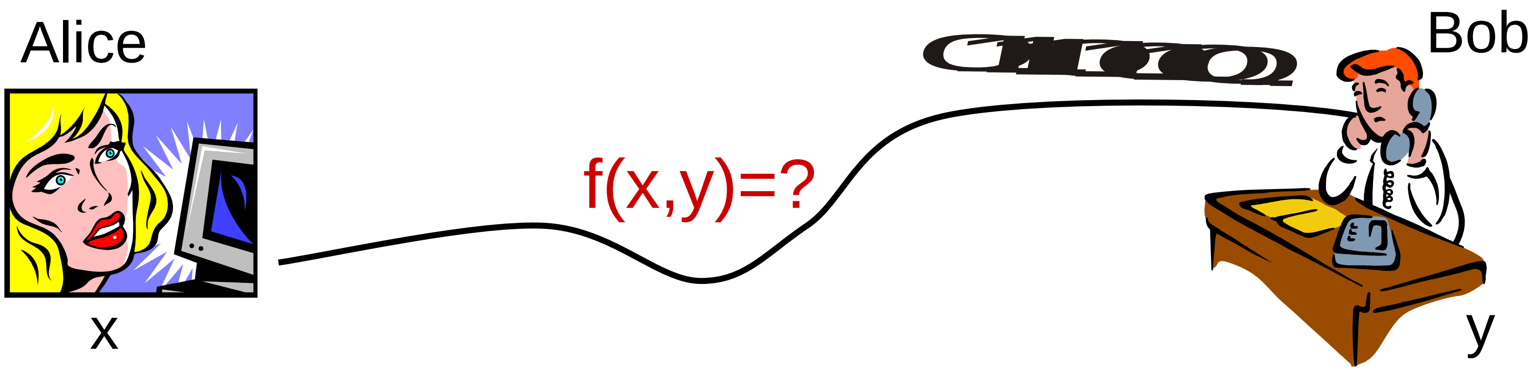
Exponentially larger *(Mermin)*

Werner, R. and Wolf, M., *Phys. Rev. A, 64, 032112 (2001),*
Zukowski, M and B. Č., *Phys. Rev. Lett. 88, 210401 (2002)*

3. Quantum Communication Complexity



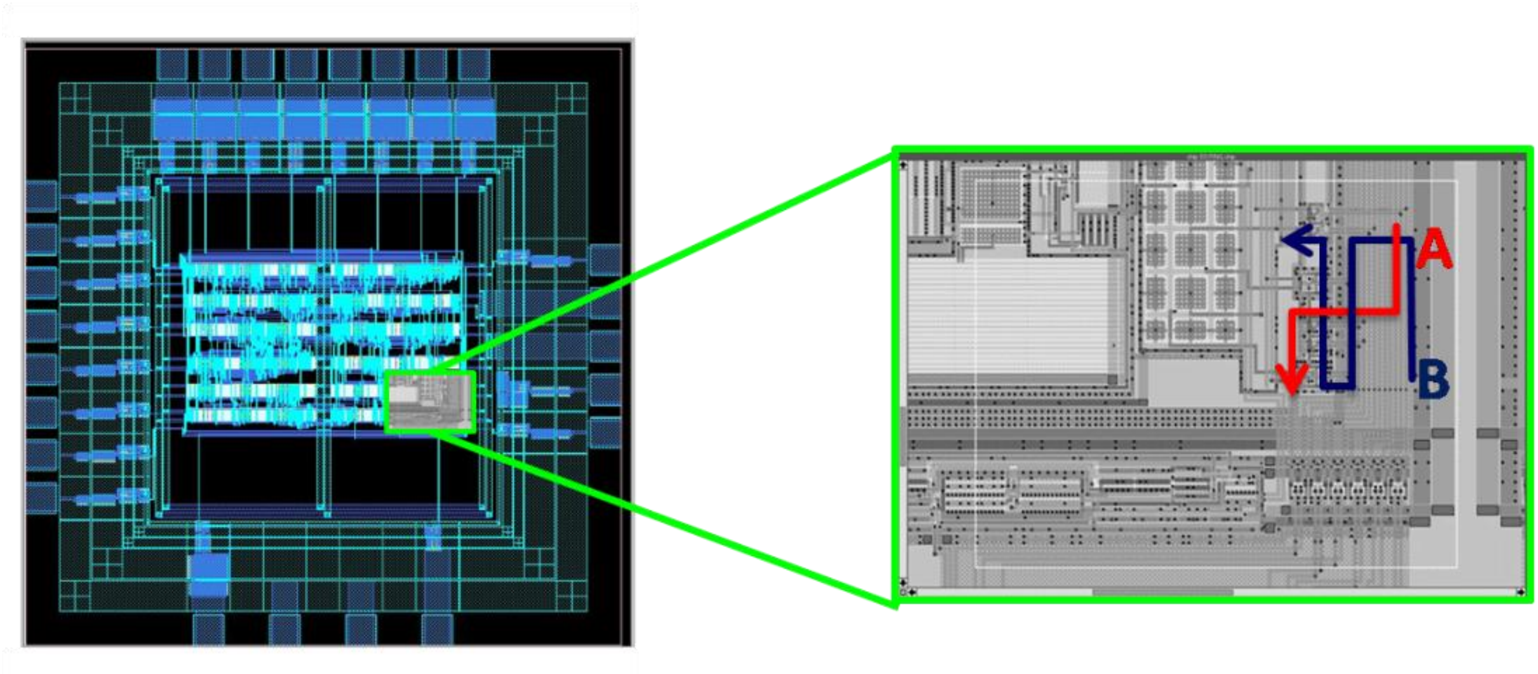
Communication Complexity Problems …



<u>Problem</u>: Both partners need to compute f(x,y), but only a **limited** communication is allowed.

<u>Question:</u> What is the highest possible **probability** of arriving at the correct value?

…are important problems.

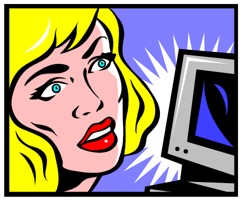


- Distributed computation
- Optimization of computers networks
- VLCI chips
- Communication in deep Space

Important whenever **classical communication is expensive.**

Can entanglement help?

Alice



0

1

Bob

x_1=0,1
y_1=1,-1

$$f = y_1 y_2 \cdot (-1)^{x_1 x_2} \quad \textbf{?}$$

x_2=0,1
y_2=1,-1

- Each partner receives two bits (altogether **four** bits)
- Only **two** bits of communication are allowed!

What is the highest probability that they both arrive at correct result?

Buhrman H., Cleve R., and van Dam W., *e-print quant-ph/9705033.*

Guessing Method

$$f = \underbrace{y_1 y_2}_{\text{„random“}} \cdot \underbrace{(-1)^{x_1 x_2}}_{\text{„biased“}}$$

X_1	X_2	$(-1)^{x_1 x_2}$
0	0	1
0	1	1
1	0	1
1	1	-1

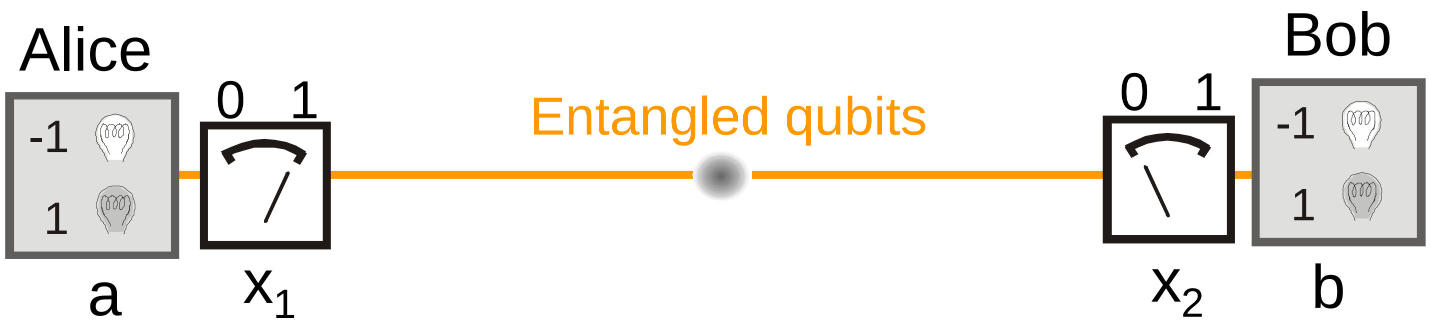
1. Exchange y_1 and y_2
2. Guess 1 for $(-1)^{x_1 x_2}$

75% success

Can one achieve higher success?

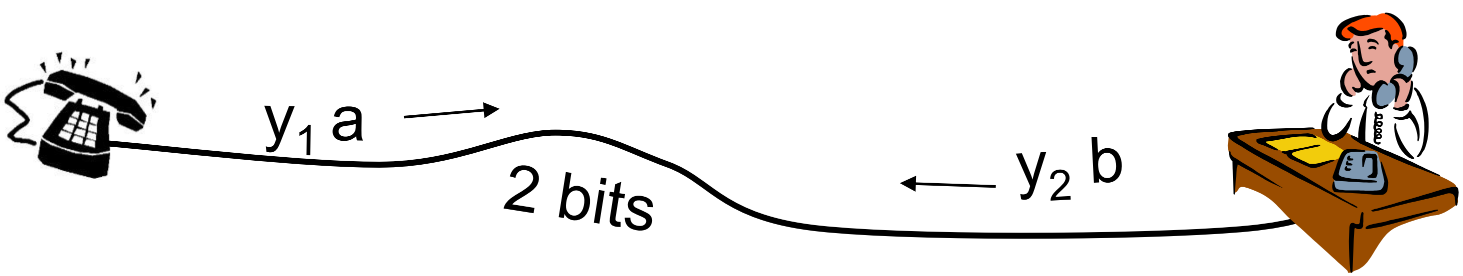
Quantum Protocol

1.



Choose **measurement setting** according to x_i.
Local measurement results are a (b).

2.



Send the product of **measurement result** and y_i.

3. Guess: y_1 y_2 a b.

How often is $ab=(-1)^{x_1 x_2}$?

x_1	x_2	$(-1)^{x_1 x_2}$
0	0	1
0	1	1
1	0	1
1	1	-1

x_1 x_2

$$P = \tfrac{1}{4}\,[P_{00}(ab=1)+P_{01}(ab=1)+P_{10}(ab=1)+P_{11}(ab=-1)]$$

Clauser-Horne-Shimony-Holt inequality

$$P_{00}(ab=1)+P_{01}(ab=1)+P_{10}(ab=1)+P_{11}(ab=-1) \leq 3$$

Classically: **75%**

Quantum: **85%**

Non-local Box: **100%**

Classical Protocol

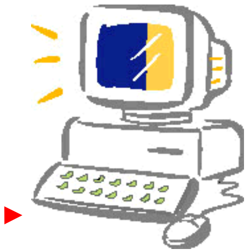
Alice

Bob

1.

Alice calculates $a(x_1,\lambda)$

Bob calculates $b(x_2,\lambda)$



Choose a **local operation** according to x_i

2.

y_1 a

2 bits

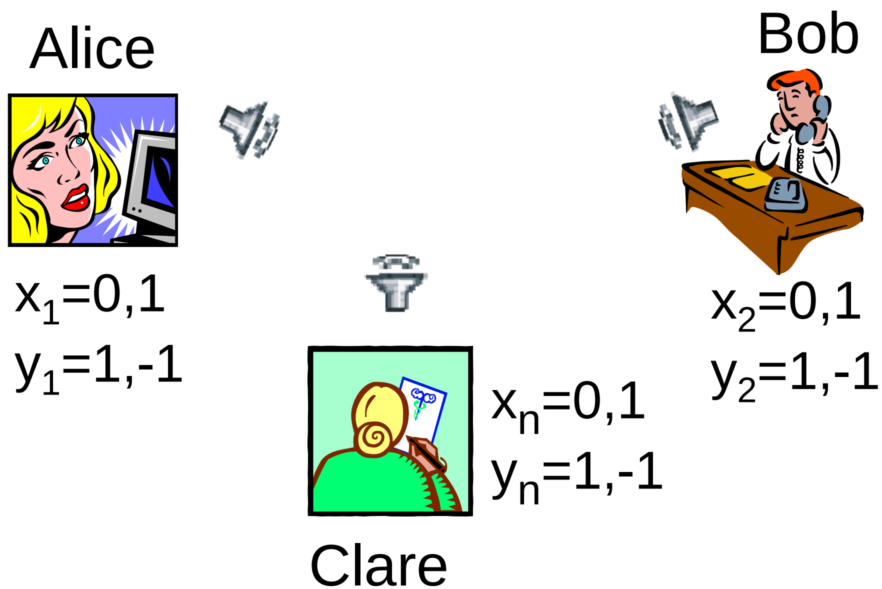
y_2 b

Send the product of **local output** and y_i

3. Guess: $y_1\ y_2$ a b

Many Parties (n≥3)

$$f = y_1 \cdot \ldots \cdot y_n \cdot g(x_1, \ldots, x_n) = \pm 1 \ ?$$



- Each partner receives two bits
- y_i are distributed randomly
- Each party is allowed to broadcast only 1 bit, but can receive bits from all partners

What is the probability that all partners arrive at the correct value?

Solutions

$$P_{classical} \to 50\% \qquad P_{quantum} = 100\%$$
$$n \to \infty$$

as for the **random** choice!

On the basis Mermin (WWZB) inequalities

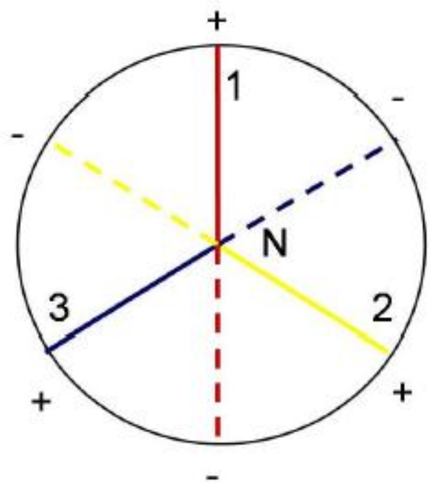
New Advanced Results:
Quantum communication complexity protocols require **exponentially less** communication than any classical protocol for accomplishing the same task: the subgroup membership problem, the „vector in subspace problem”

B.Č., Zukowski M., Pan J.-W. and Zeilinger A., *Phys. Rev. Lett.* ***92****, 127901 (2004).*

B. Č., Zukowski M. and Zeilinger A., *Phys. Rev. Lett.* ***89****, 197901 (2002).*

Coordination without communication

Under angle of ± 60°angle
they can see each other



Alice	Bob
+	+
-	-
+	+
-	-
+	+
-	-

Alice	Bob
+	-
+	-
-	+
-	+
+	-
+	-
-	+
-	+
+	-
+	-
-	+
-	+

Paths have **same** colour, take the **same** direction
Paths have **different** colour, take **opposite** direction

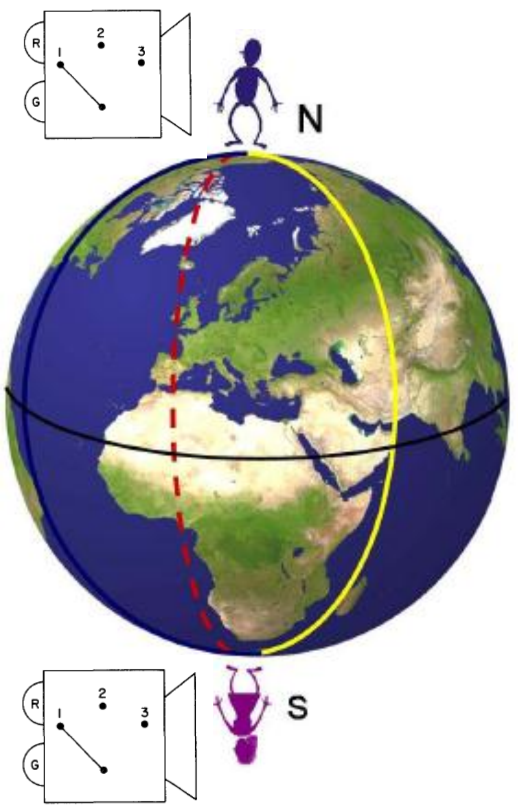
B. Č., Paunkovic N., Rudolph T., Vedral V., *Int. J. Quant. Inf. 4, (2006) 365*

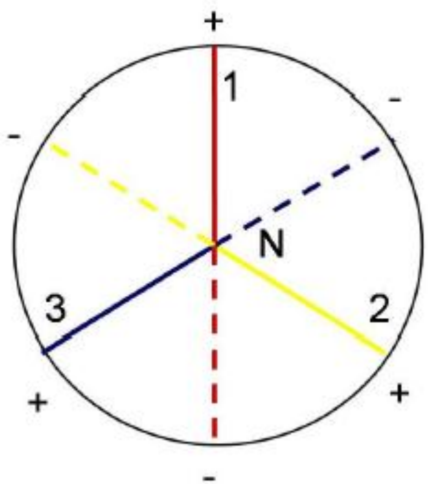
Entanglement helps

Directions: +1,-1

$$P_{success} = \frac{1}{9}\left(\sum_{i=1}^{3} P(A = B \,|\, \mathrm{ii}) + \sum_{i \neq j = 1}^{3} P(A = -B \,|\, \mathrm{i,j}) \right)$$

Paths: blue, red, yellow





LR: $\frac{7}{9} \approx 78\%$

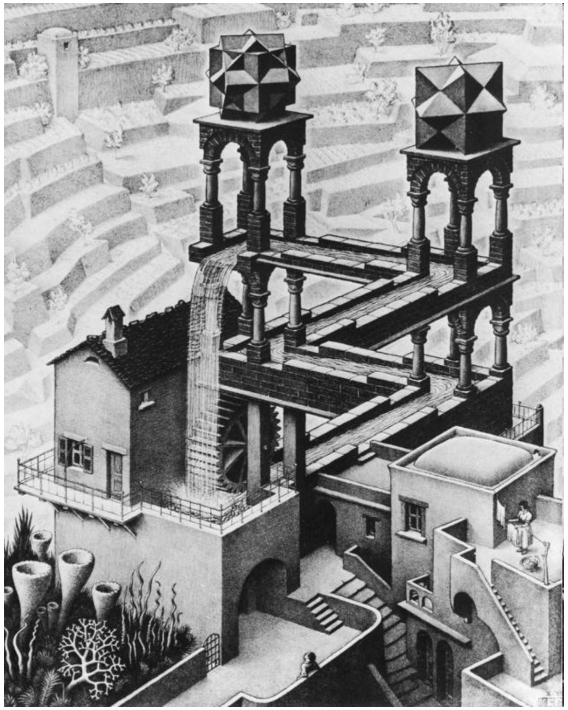
QM: 83%

$$|\phi^{+}\rangle \frac{1}{\sqrt{2}}\left(|H\rangle|H\rangle + |V\rangle|V\rangle\right)$$

B. Č., Paunkovic N., Rudolph T., Vedral V., *Int. J. of Quant. Inf. 4, (2006) 365*

Summary

1. Non-contextual Hidden Variables (NCHV)
2. **Bell-Kochen-Specker Theorem: (BKS) => NCHV ≠ QM**
3. Proofs of BKS: Mermin-Peres, Kochen-Specker 117, Klyachko 5
4. **Bell's Theorem => Local Realism ≠ QM**
5. CHSH inequalities
6. GHZ, Mermin and WWBZ inequalities
7. Non-local Box
8. **Quantum Communication Complexity**
9. CHSH: 75% versus 85%
10. Mermin (WWBZ): 50% versus 100%
11. "Date without a phone call"



Thank you for your attention!

CoQuS Complex Quantum Systems FWF FQXi Foundational Questions Institute