QUANTUM CLONING AND BEYOND

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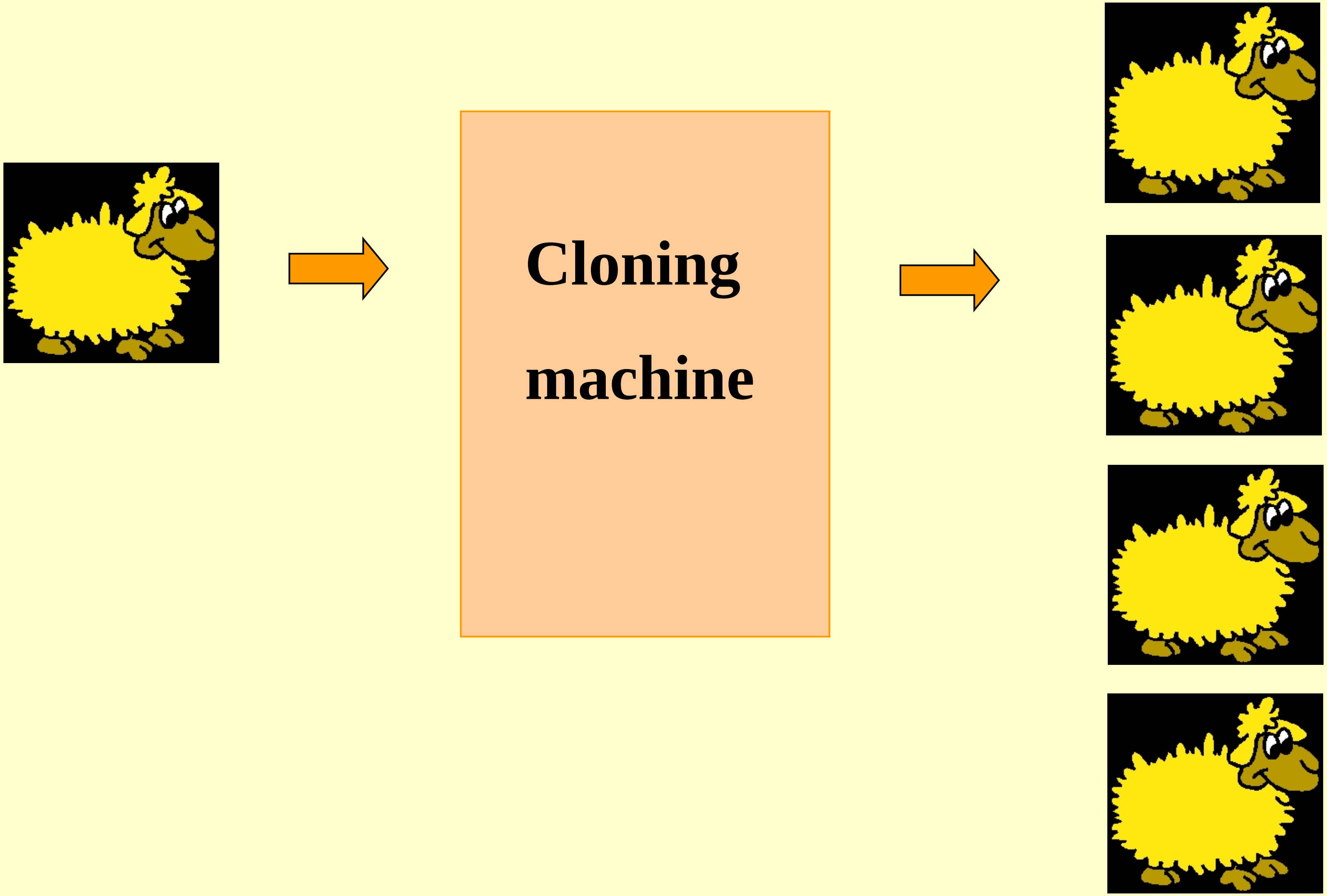
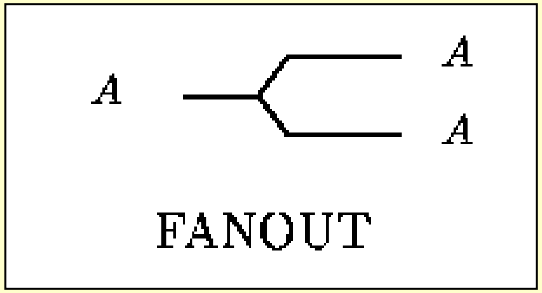
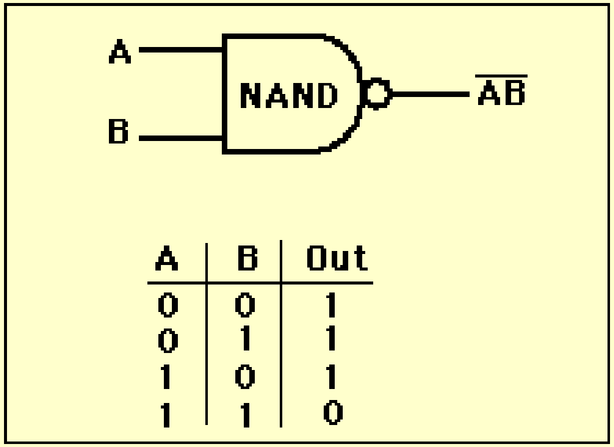
CLONING



IN CLASSICAL INFORMATION

Computing with bits (0 and 1)





FANOUT and **NAND** are **universal gates for classical computation**

Measurements: bits can be measured perfectly

IN QUANTUM INFORMATION

Qubits: $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ $\quad |a|^2 + |b|^2 = 1$

NO CLONING THEOREM:

an unknown quantum state cannot be perfectly cloned

NO CLONING THEOREM

PERFECT CLONING TRANSFORMATION:

$$U\left(|\psi\rangle|0\rangle|Q\rangle\right) = |\psi\rangle|\psi\rangle|Q_\psi\rangle$$

$$U\left(|\phi\rangle|0\rangle|Q\rangle\right) = |\phi\rangle|\phi\rangle|Q_\phi\rangle \qquad |Q_i\rangle \text{ NORM.}$$

UNITARITY OF Q. MECHANICAL TRANSFORMATIONS:

$$\langle\psi,0,Q|\phi,0,Q\rangle = \langle\psi,\psi,Q_\psi|\phi,\phi,Q_\phi\rangle$$

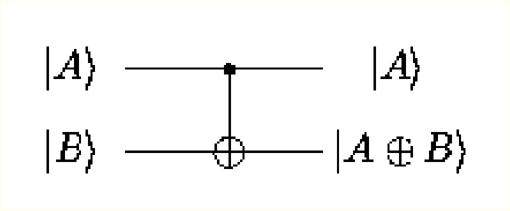
$$\Rightarrow \quad \langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2\langle Q_\psi|Q_\phi\rangle$$

OK IF $\langle\psi|\phi\rangle = 0, 1$

OTHERWISE: $\langle\psi|\phi\rangle = \dfrac{1}{\langle Q_\psi|Q_\phi\rangle}$ IMPOSSIBLE !

CONTROLLED NOT

Universal gate for quantum computation (with single qubit transformations)



$$|0\rangle\,|0\rangle \longrightarrow |0\rangle\,|0\rangle$$
$$|0\rangle\,|1\rangle \longrightarrow |0\rangle\,|1\rangle$$
$$|1\rangle\,|0\rangle \longrightarrow |1\rangle\,|1\rangle$$
$$|1\rangle\,|1\rangle \longrightarrow |1\rangle\,|0\rangle$$

With B=0 perfect cloning for a classical bit, but:

$$(a\,|0\rangle + b\,|1\rangle)\,|0\rangle \longrightarrow a\,|0\rangle\,|0\rangle + b\,|1\rangle|1\rangle \neq |\psi\rangle\,|\psi\rangle$$

NO CLONING THEOREM

Fundamental theorem in quantum information, with consequences for:

Quantum measurements

Quantum error correction

Superluminal communications

Secure communications

……………..

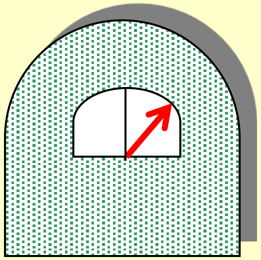
QUANTUM MEASUREMENTS

No cloning ⟺ **No perfect measurability of a single quantum system**

Perfect cloning ⟹ **Perfect measurements**

$|\psi\rangle$ ⟹ **PERFECT CLONER** ⟹

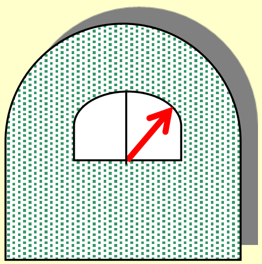
$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$
....
....
....



QUANTUM MEASUREMENTS

Perfect measurements $\Rightarrow$ **Perfect cloning**

$|\psi\rangle$



$\Rightarrow$

Full information on the state

$\Rightarrow$

Repreparation of $|\psi\rangle$

$\Rightarrow$

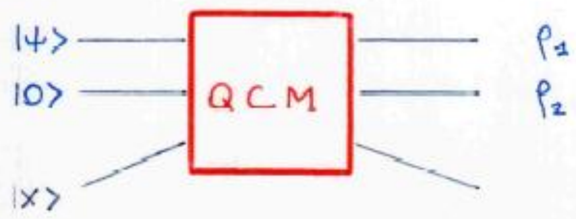
$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$

How well can we approximate perfect cloning?

It depends on the form of the input states we wish to clone

- **Universal cloning**
- **Phase covariant cloning**
- **State dependent cloning**

Universal cloning



$$|\psi\rangle|0\rangle|X\rangle \longrightarrow |\Phi\rangle = U\left(|\psi\rangle|0\rangle|X\rangle\right)$$

BLOCH VECTOR REPRESENTATION:

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left[\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right]$$

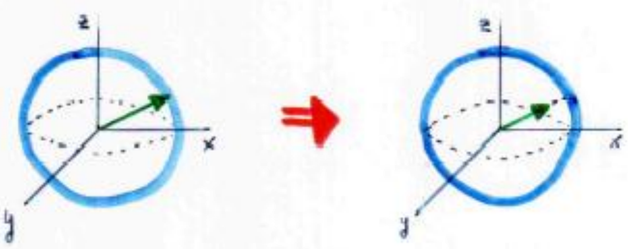
UNIVERSAL CLONER:

1) $\rho_1 = \rho_2$ SYMMETRY

2a) $\vec{s}_1 = \eta_\psi \vec{s}_\psi$ ORIENTATION INVARIANCE

2b) $F = \langle\psi|\rho_1|\psi\rangle = \text{CONST}$ ISOTROPY

$$F = \frac{1}{2}(\eta + 1) \Rightarrow \boxed{\eta_\psi = \eta}$$



$$\rho_1 = \frac{1}{2}\left[\mathbb{1} + \eta\, \vec{s}\cdot\vec{\sigma}\right]$$

- MAXIMISING F:

$$\boxed{F_{max} = \frac{5}{6} \qquad \eta_{max} = \frac{2}{3}}$$

- OPTIMAL TRANSFORMATION:

$$U\,|0\rangle|0\rangle|X\rangle = \sqrt{\frac{2}{3}}\, e^{i\delta_1}\,|00\rangle|A\rangle + \sqrt{\frac{1}{6}}\, e^{i\delta_2}\left(|01\rangle+|10\rangle\right)|A_\perp\rangle$$

$$U\,|1\rangle|0\rangle|X\rangle = \sqrt{\frac{2}{3}}\, e^{i\delta_2}\,|11\rangle|A_\perp\rangle + \sqrt{\frac{1}{6}}\, e^{i\delta_1}\left(|01\rangle+|10\rangle\right)|A\rangle$$

A TWO-DIMENSIONAL ANCILLA IS ENOUGH

UNIVERSAL CLONING AND STATE ESTIMATION

- **UNIVERSAL $N \rightarrow M$ CLONER:**

$$\rho^{in} = \frac{1}{2}\left[\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right] \longrightarrow \rho^{out} = \mathrm{Tr}_{M-1}\left[\rho_M\right] = \frac{1}{2}\left[\mathbb{1} + \eta_{QCM}^{(N,M)}\, \vec{s}\cdot\vec{\sigma}\right]$$

- **UNIVERSAL STATE ESTIMATION:**

POVM: $\{\hat{P}_\mu\}$

$\hat{P}_\mu \geq 0 \qquad \sum_\mu \hat{P}_\mu = \mathbb{1}$

$p_\mu(\psi) = \mathrm{Tr}\left[\hat{P}_\mu\, |\psi\rangle\langle\psi|^{\otimes N}\right] \longrightarrow$ CANDIDATE $|\psi_\mu\rangle$ FOR $|\psi\rangle$

$$\bar{\rho} = \sum_\mu p_\mu(\psi)\, |\psi_\mu\rangle\langle\psi_\mu| \overset{\text{UNIV.}}{=} \frac{1}{2}\left[\mathbb{1} + \eta_{EST}(N)\, \vec{s}\cdot\vec{\sigma}\right]$$

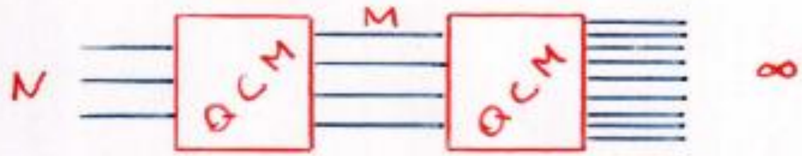
$$F_{EST}(N) = \langle\psi|\bar{\rho}|\psi\rangle = \frac{1}{2}\left[1 + \eta_{EST}(N)\right]$$

LINK:

- AFTER OPTIMISATION OF BOTH PROCEDURES:

$$\boxed{\eta^{opt}_{QCM}(N,\infty) = \eta^{opt}_{EST}(N)}$$

OPTIMAL UNIVERSAL CLONING



VIEW THE GLOBAL PROCESS AS AN $N \to \infty$ UNIV. CLONER:

$$\eta_{QCM}(N,M)\cdot\eta_{QCM}(M,\infty) \leq \eta^{opt}_{QCM}(N,\infty)$$

$$\Rightarrow \quad \eta_{QCM}(N,M) \leq \frac{\eta^{opt}_{QCM}(N,\infty)}{\eta^{opt}_{QCM}(M,\infty)} = \frac{\eta^{opt}_{EST}(N)}{\eta^{opt}_{EST}(M)}$$

$$\eta^{opt}_{EST}(N) = \frac{N}{N+2}$$ (MASSAR + POPESCU)

$$\Rightarrow \quad \boxed{\begin{aligned} \eta^{opt}_{QCM}(N,M) &\leq \frac{N}{M}\cdot\frac{M+2}{N+2} \\ F^{opt}_{QCM}(N,M) &\leq \frac{NM+N+M}{M(N+2)} \end{aligned}}$$

UNIVERSAL CLONING AND STATE ESTIMATION IN d>2

- GENERALISED BLOCH VECTOR REPRESENTATION FOR d-DIMENSIONAL SYSTEMS:

$$\boxed{\rho_d = \frac{1}{d}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{d^2-1} \lambda_i \tau_i}$$

$\mathrm{Tr}[\tau_i] = 0$

$\mathrm{Tr}[\tau_i \tau_j] = 2\delta_{ij}$

d LEVELS

- UNIVERSAL OPERATION:

$$\boxed{\rho_d \longrightarrow \frac{1}{d}\mathbb{1} + \frac{1}{2}\eta_d \sum_{i=1}^{d^2-1} \lambda_i \tau_i}$$

- LINK BETWEEN OPTIMAL CLONING AND OPTIMAL STATE ESTI

$$\boxed{\eta^{opt}_{d,EST}(N) = \eta^{opt}_{d,QCM}(N,\infty)}$$

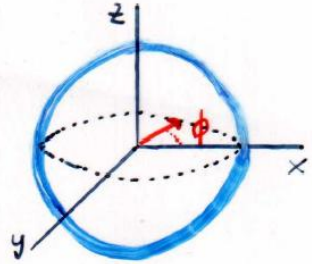
$$\eta^{opt}_{d,QCM}(N,M) = \frac{N(M+d)}{M(N+d)}$$ (Werner

OPTIMAL STATE ESTIMATION:

$$\boxed{F^{opt}_{d,EST}(N) = \frac{N+1}{N+d}}$$

PHASE COVARIANT CLONING

- INPUT STATES : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$



- PHASE COVARIANCE ➡ FIDELITY INDEPENDENT OF ϕ

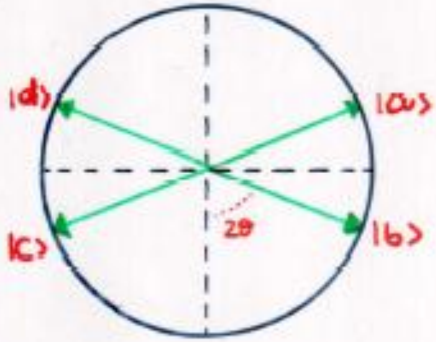
- LINK WITH PHASE ESTIMATION :

$$\boxed{\eta^{opt}_{PCC}(N,\infty) = \eta^{opt}_{ph.est}(N)}$$

$\eta^{opt}_{PCC}(1,2) = \frac{1}{\sqrt{2}}$ ACHIEVED FOR $1 \rightarrow 2$ PCC

STATE DEPENDENT CLONING

- TWO PAIRS OF ORTHOGONAL STATES:



$$|a\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

$$|b\rangle = \sin\theta|0\rangle + \cos\theta|1\rangle$$

$$|c\rangle = \sin\theta|0\rangle - \cos\theta|1\rangle$$

$$|d\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle$$

$1 \to 2$ CASE: SYMMETRY CONDITION

FIDELITY INDEPENDENT OF THE INPUT STATE

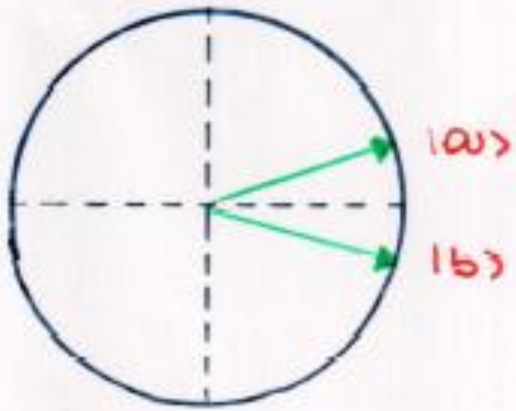
BEST FIDELITY:

$$\boxed{F = \frac{1}{2}\left(1 + \sqrt{1 - 2S^2 + 2S^4}\right)}$$

$S = \langle a|b\rangle = \sin 2\theta$

STATE DEPENDENT CLONING

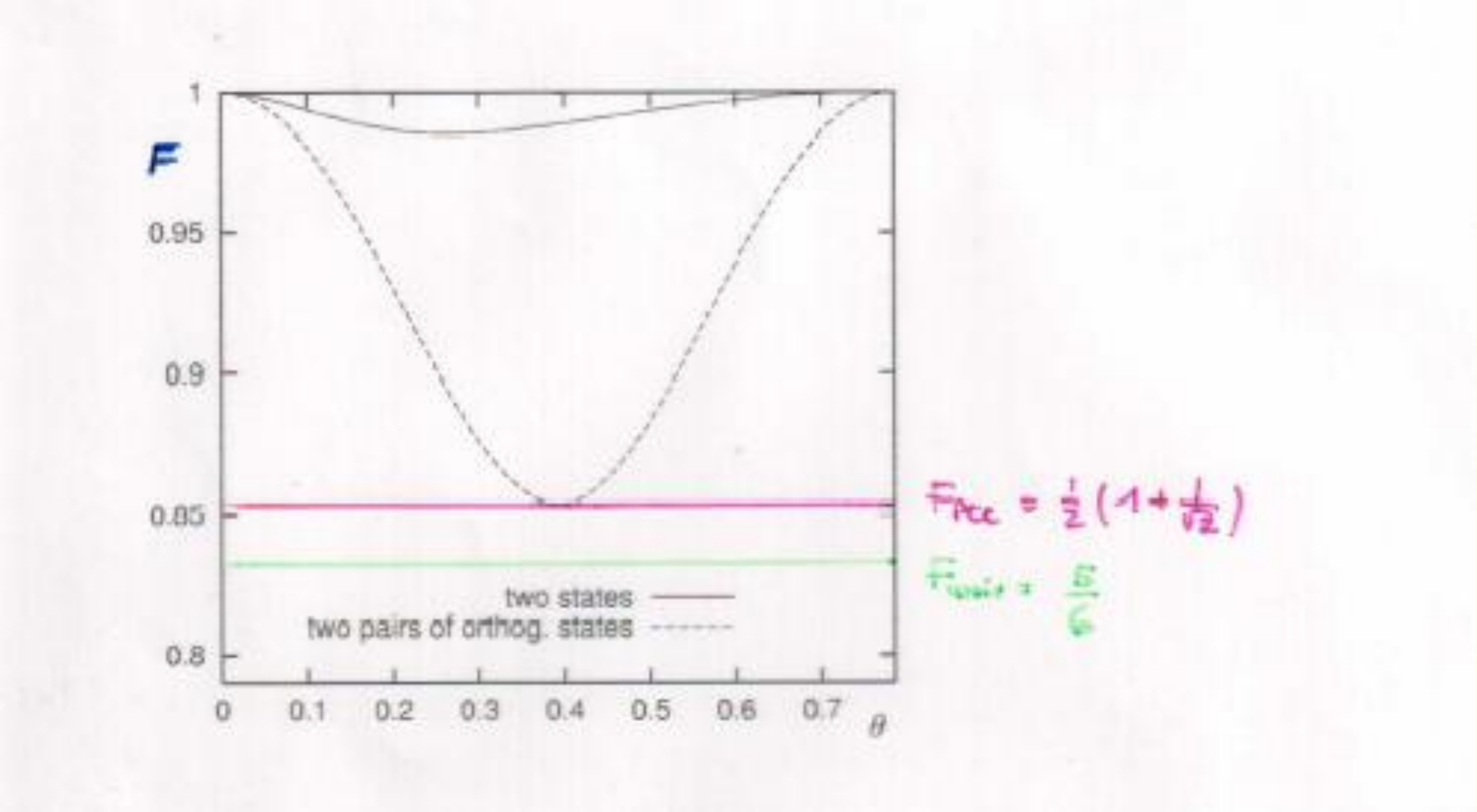
- TWO NON ORTHOGONAL STATES:



BEST FIDELITY:

$$\boxed{F = \frac{1}{2}\left[1 + \frac{1-S^2}{\sqrt{1+S^2}} + \frac{S^2(1+S)}{1+S^2}\right]}$$

COMPARISON



Fidelity increases by restricting the class of input states

PPC = two pairs of orthogonal states for Pi/8, related to eavesdroppiung strategies in the BB84 cryptographic scheme

SUPERLUMINAL COMMUNICATIONS

Perfect cloning would allow superluminal communications:

Alice $\longleftarrow$ $|\uparrow\rightarrow\rangle - |\rightarrow\uparrow\rangle$ $\longrightarrow$ **Bob**

Information encoded into the type of measurement

Perfect cloner

Measure $\oplus$ $\Longrightarrow$ $\uparrow$ or $\rightarrow$

Measure $\otimes$ $\Longrightarrow$ $\nearrow$ or $\searrow$

QUESTIONS:

Does approximate cloning lead to superluminal communications?

What properties are needed in order to have no-signalling?

ENTANGLEMENT SCHEME

- $\mathcal{A}\otimes \mathbb{1}$: ALICE'S LOCAL MAP
- ALICE ENCODES THE MESSAGE "m" IN THE LOCAL MAP $\mathcal{A}$
- BOB PERFORMS A LOCAL TRANSFORMATION $\mathbb{1}\otimes\beta$ AND THEN A LOCAL MEASUREMENT $\mathbb{1}\otimes\Pi_R$ TO DECODE THE MESSAGE

$$\rho_{AB} \longrightarrow \mathbb{1}\otimes\Pi_R\left(\mathcal{A}_m\otimes\beta(\rho_{AB})\right)$$

- CONDITIONAL PROBABILITY:

$$p(R|m) = Tr_{AB}\left[\mathbb{1}\otimes\Pi_R\left(\mathcal{A}_m\otimes\beta(\rho_{AB})\right)\right]$$

ASSUMPTIONS

- THE MAP $\mathcal{A}$ IS TRACE PRESERVING (COMPLETE for a POV)

$$\mathrm{Tr}\left[\mathcal{A}(\rho_A)\right] = \mathrm{Tr}\left[\rho_A\right] \qquad \text{for any } \rho_A$$

- $\mathcal{A}$ IS LINEAR:

$$\mathrm{Tr}_A\left[\mathcal{A}\otimes\mathbb{1}(\rho_{AB})\right] = \mathrm{Tr}_A\left[\rho_{AB}\right]$$

- β IS LINEAR:

$$\mathrm{Tr}_A\left[\mathcal{A}\otimes\beta(\rho_{AB})\right] = \beta\,\mathrm{Tr}_A\left[\mathcal{A}\otimes\mathbb{1}(\rho_{AB})\right]$$

⇒ CONDITIONAL PROBABILITY:

$$p(r|m) \equiv \mathrm{Tr}_{AB}\left[\mathbb{1}\otimes\Pi_r\left(\mathcal{A}_m\otimes\mathcal{B}(\rho_{AB})\right)\right]$$

$$= \mathrm{Tr}_B\left[\Pi_r\left(\mathcal{B}\,\mathrm{Tr}_A\left[\mathcal{A}_m\otimes\mathbb{1}(\rho_{AB})\right]\right)\right]$$

$$= \mathrm{Tr}_B\left[\Pi_r\left(\mathcal{B}\,\underbrace{\mathrm{Tr}_A(\rho_{AB})}_{\rho_B}\right)\right] \equiv p(r)$$

INDEPENDENT OF THE ENCODED MESSAGE "m" !

⇒ LINEARITY AND PRESERVATION OF TRACE FOR LOCAL MAP GUARANTEE THE IMPOSSIBILITY OF SUPERLUM. COMM.

EXAMPLE

LINEAR $1 \to 2$ CLONING TRANSFORMATION BY BOB

$$|\psi(\vec{s})\rangle\langle\psi(\vec{s})| \to \rho_{out}(\vec{s}) = \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + \eta\,(\vec{s}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{s}\cdot\vec{\sigma}) + t\sum \sigma_j\otimes\sigma_j\right]$$

NO - SUPERLUMINAL SIGNALLING CONDITION:

$$p(\vec{m}_1)\rho_{out}(\vec{m}_1) + p(-\vec{m}_1)\rho_{out}(-\vec{m}_1) = p(\vec{m}_2)\rho_{out}(\vec{m}_2) + p(-\vec{m}_2)\rho_{out}(-\vec{m}_2)$$

It is always fulfilled.

Any (cloning) transformation compatible with quantum mechanics does not allow superluminal communications

CLONING versus BROADCASTING

$|\psi\rangle$
$|\psi\rangle$

$\Rightarrow$ ~~PERFECT CLONER~~ $\Rightarrow$

$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$
$|\psi\rangle$

Pure states:
NO CLONING THEOREM:
Two non orthogonal states cannot be perfectly cloned

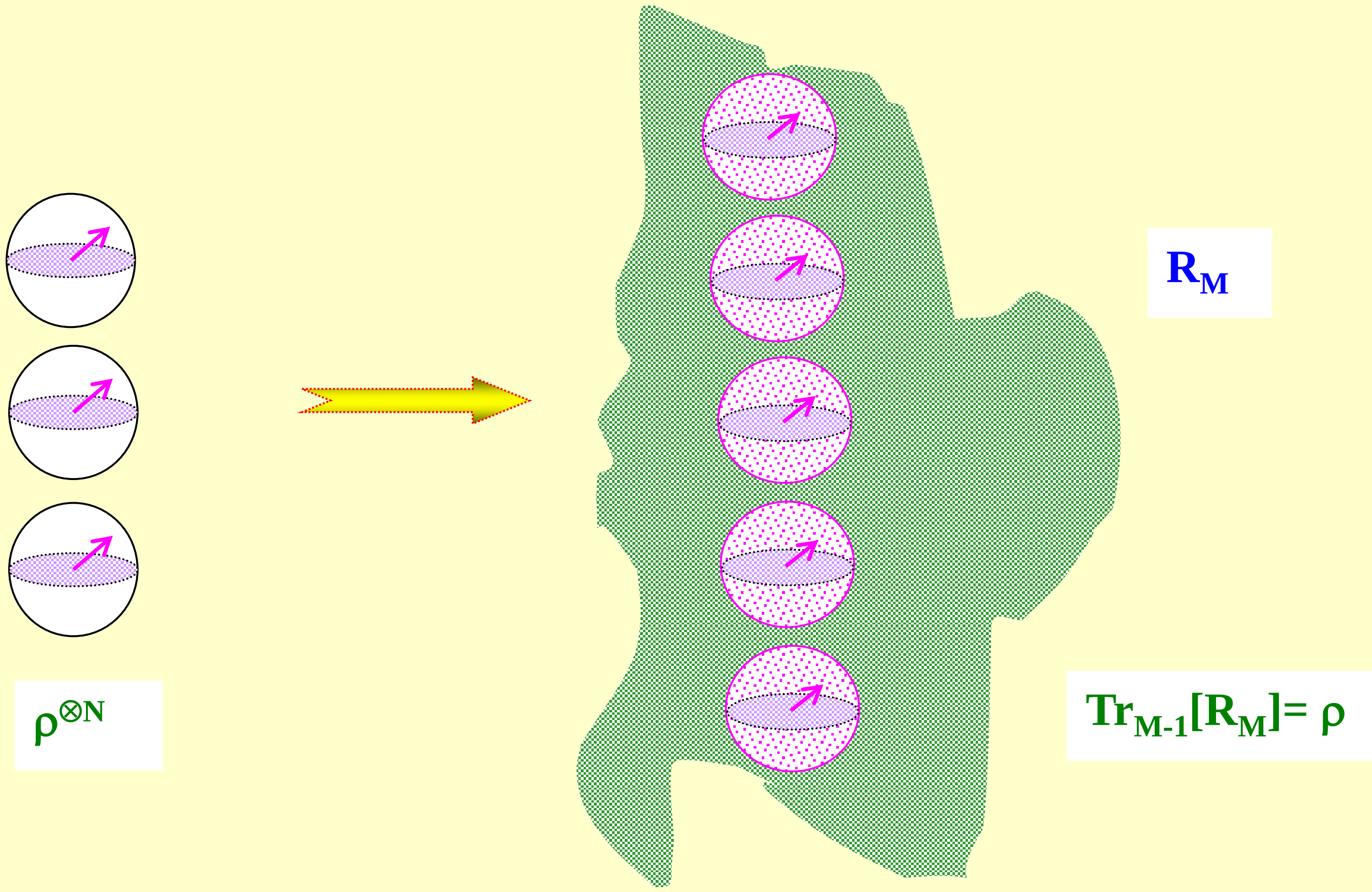
This holds also for mixed states:

$$\cancel{\rho^{\otimes N} \rightarrow \rho^{\otimes M}}$$

$\rho_0\rho_1 \neq 0$

Cloning is a particular form of broadcasting:
Pure states: CLONING=BROADCASTING
Mixed states: Broadcasting is more general

BROADCASTING MIXED STATES

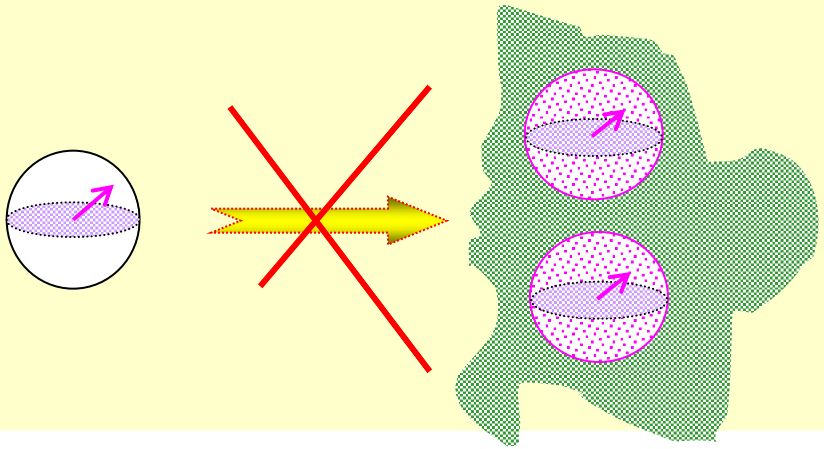


BROADCASTING MIXED STATES

No-broadcasting theorem:

Two noncommuting states cannot be perfectly broadcast onto two quantum systems **(Barnum et al., 96)**

Extension of the *no cloning theorem* to mixed states for N=1 and M=2



Does it hold also for N>1 and M>2?

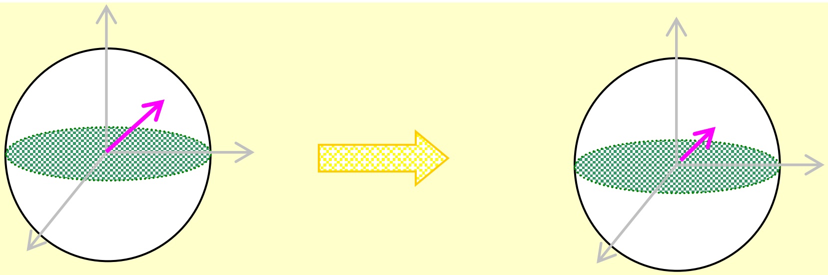
UNIVERSAL SUPERBROADCASTING

Input: N mixed qubits $\rho^{\otimes N}$ (e.g. after depolarising noise)

Symmetry: invariance under permutation of input and output qubits

Universality: quality of the map independent of the orientation of the input Bloch vector (covariance under all qubit transformations)

Universal single qubit broadcasting transformations:



$\rho_{in}=[I+s\bullet\sigma]/2$ ⟹ $\rho_{out}=[I+p(r)s\bullet\sigma]/2$ $|s|=r<1$

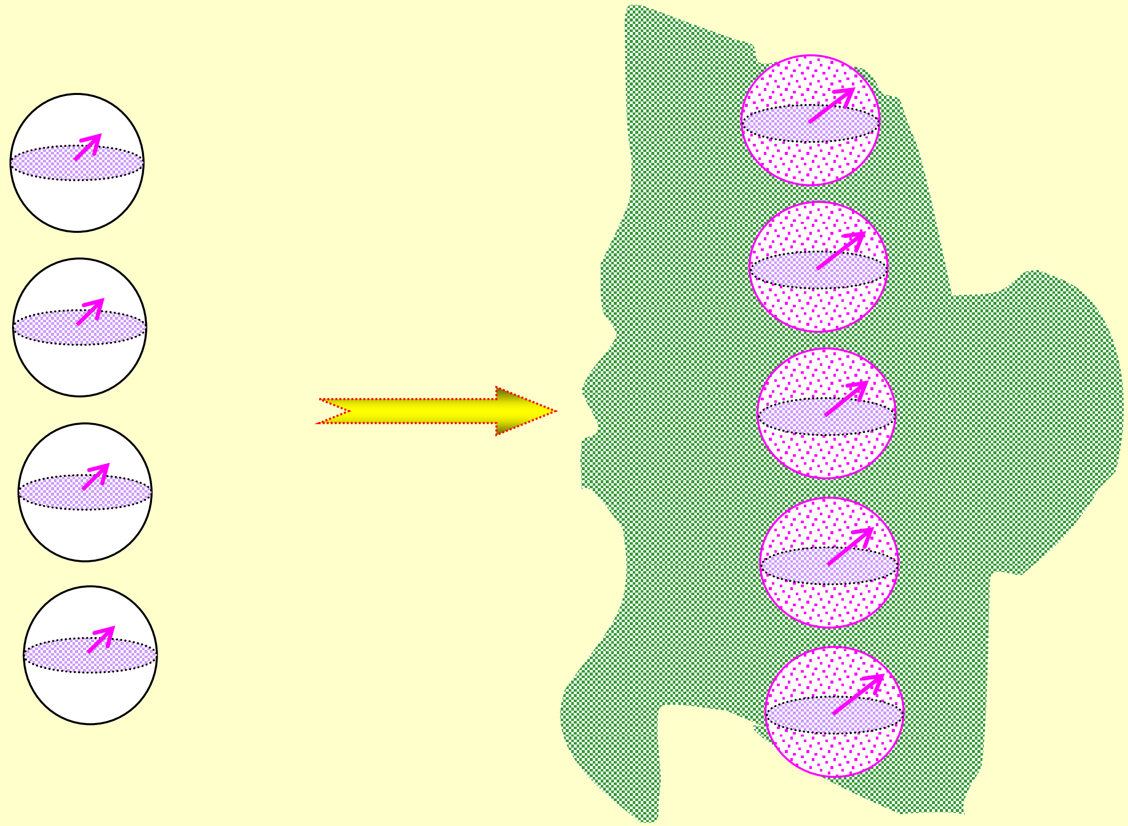
p(r): scaling factor, not necessarily <1

Maximise p(r)!

Equivalent to maximise the purity: $Tr[\rho_{out}^2] = [1+r^2\ p(r)^2]/2$

UNIVERSAL SUPERBROADCASTING

It is possible to have $p(r)>1$ for $N \geq 4$, i.e.



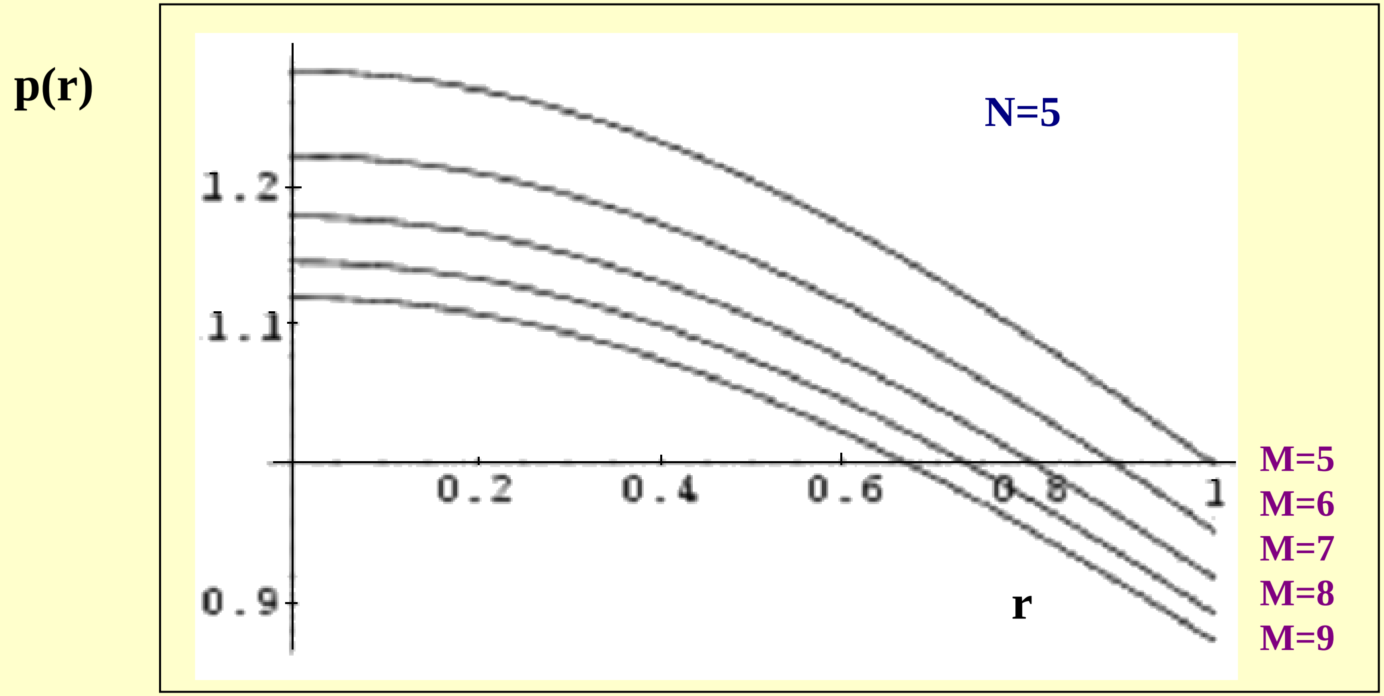
BROADCASTING+PURIFICATION = SUPERBROADCASTING!!

The local purity is increased by introducing correlations among the output qubits

The optimal superbroadcasting map is independent of the input state

UNIVERSAL SUPERBROADCASTING

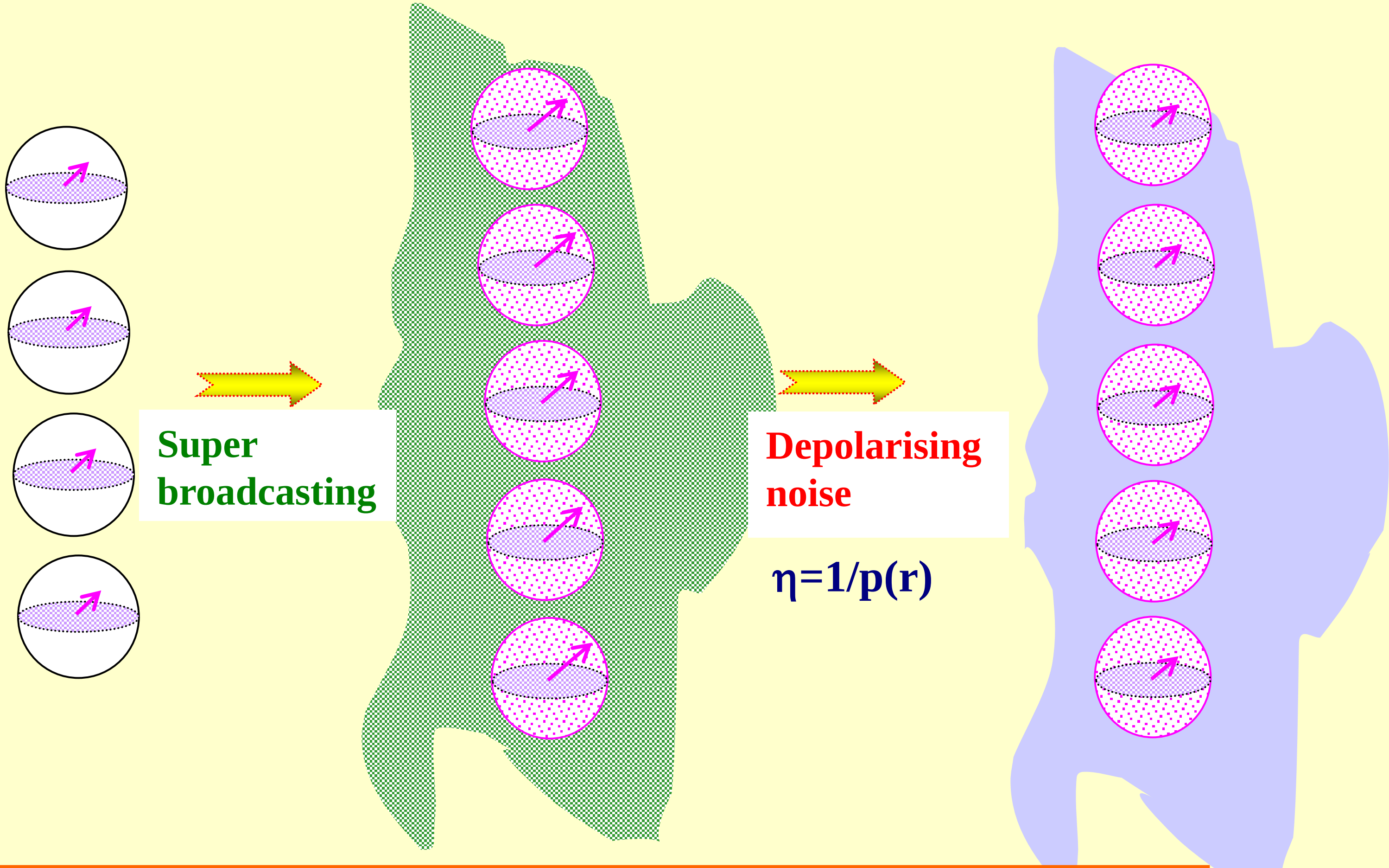
It is possible to superbroadcast up to certain values of r



For N=M superbroadcasting is possible for any value of r

IDEAL BROADCASTING

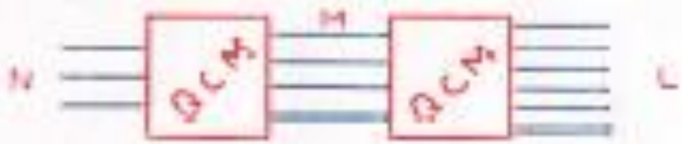
Whenever $p(r)>1$ we can achieve a final $p_f(r)=1$ for $N\geq 4$ by adding depolarising noise



Violation of the no-broadcasting theorem!!

The ideal broadcasting map depends explicitly on the value of r!

CONCATENATION OF UNIV. CLONERS



THE GLOBAL PROCESS IS A UNIVERSAL CLONER WITH:

$$\boxed{\eta^{\text{tot}}_{UCM}(N,L) = \eta_{UCM}(N,M)\cdot\eta_{UCM}(M,L)}$$

PROOF:

$$\mathrm{Tr}_{M-1}\left[C_{NM}\left(|\psi\rangle\langle\psi|^{\otimes N}\right)\right] = \eta(N,M)\,|\psi\rangle\langle\psi| + \left[1-\eta(N,M)\right]\tfrac{1}{2}\mathbb{1}$$

BY LINEARITY, FOR $\rho_N = \sum_i \alpha_i |\psi_i\rangle\langle\psi_i|^{\otimes N}$:

$$\mathrm{Tr}_{M-1}\left[C_{NM}(\rho_N)\right] = \eta(N,M)\,\rho + \left[1-\eta(N,M)\right]\tfrac{1}{2}\mathbb{1}$$

$\rho = \sum_i \alpha_i$ ...

$$\Rightarrow \quad \rho^{out} = \mathrm{Tr}_{L-1}\left[\rho_L\right] = \eta(N,M)\cdot\eta(M,L)\,|\psi\rangle\langle\psi| + \left[1-\eta(N,M)\cdot\eta(M,L)\right]\tfrac{1}{2}\mathbb{1}$$

PROOF OF THE EQUALITY $\eta_{EST}^{opt}(N) = \eta_{QCM}^{opt}(N,\infty)$

a) VIEW STATE ESTIMATION AS AN $N \to L$ CLONER:

$$\eta_{EST}^{opt}(N) \leq \eta_{QCM}^{opt}(N,L)$$

b)

VIEW THIS GLOBAL PROCESS AS A STATE ESTIMATION:

$$F_{EST}(N) = \langle\psi| \sum_{\mu} \mathrm{Tr}\left[P_{\mu}\, \rho_L\right] |\psi_{\mu}\rangle\langle\psi_{\mu}| \psi\rangle \qquad \rho_L = \sum_i \alpha_i |\psi_i\rangle\langle\psi_i|^{\otimes L}, \quad \sum_i \alpha_i = 1$$

$$\overset{\text{LINEARITY}}{=} \langle\psi| \sum_i \alpha_i \left[\eta_{EST}^{opt}(L)\, |\psi_i\rangle\langle\psi_i| + \left(1-\eta_{EST}^{opt}(L)\right)\tfrac{1}{2}\mathbb{1}\right] |\psi\rangle$$

$$\eta_{EST}^{opt}(L) \to 1 \quad \text{for } L \to \infty$$

$$\Rightarrow \quad F_{EST}(N) = \langle\psi| \textstyle\sum_i \alpha_i |\psi_i\rangle\langle\psi_i| \psi\rangle = \frac{1}{2}\left[1 + \eta_{QCM}(N,\infty)\right]$$

$$\Rightarrow \quad \eta_{QCM}^{opt}(N,\infty) \leq \eta_{EST}^{opt}(N)$$