Permutationally invariant quantum tomography

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Outline

1 Motivation
   - Why quantum tomography is important?

2 Quantum experiments with multi-qubit systems
   - Physical systems
   - Local measurements

3 Full quantum state tomography
   - Basic ideas and scaling
   - Experiments

4 Permutationally invariant tomography
   - Main results
   - Example: XY PI tomography
   - Example: Experiment with a 4-qubit Dicke state

5 Extra slide 1: Number of settings
Why tomography is important?

- Many experiments aiming to create many-body entangled states
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales exponentially with the number of qubits.
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Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions

- 10 qubits with photons
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Only local measurements are possible

**Definition**

A single **local measurement setting** is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit $k$ for all qubits.

All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)} A^{(2)} \rangle, \langle A^{(1)} A^{(3)} \rangle, \langle A^{(1)} A^{(2)} A^{(3)} \rangle \ldots$$
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The density matrix can be reconstructed from $3^N$ measurement settings.

Example

For $N = 4$, the measurements are

1. X X X X X
2. X X X X Y
3. X X X Z
...
3^4. Z Z Z Z Z

Note again that the number of measurements scales exponentially in $N$. 
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Experiments with ions and photons


Approaches to solve the scalability problems

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.

- If the state is of low rank, we need fewer measurements.

- We make tomography in a subspace of the density matrices (our approach).
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Permutationally invariant part of the density matrix:

\[ \varrho_{\text{PI}} = \frac{1}{N!} \sum \Pi_k \varrho \Pi_k, \]

where \( \Pi_k \) are all the permutations of the qubits.


Main results

Features of our method:

1. Is for spatially separated qubits.
2. Needs the minimal number of measurement settings.
3. Uses the measurements that lead to the smallest uncertainty possible of the elements of $\rho_{PI}$.
4. Gives an uncertainty for the recovered expectation values and density matrix elements.
5. If $\rho_{PI}$ is entangled, so is $\rho$. Can be used for entanglement detection!
Measurements

- We measure the same observable $A_j$ on all qubits. (Necessary for optimality.)

Each qubit observable is defined by the measurement directions $\vec{a}_j$ using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$D_N = \binom{N+2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$
What do we get from the measurements?

We obtain the expectation values for

$$\langle (A_j^{\otimes(N-n)} \otimes 1^{\otimes n})_{\text{PI}} \rangle$$

for $j = 1, 2, \ldots, D_N$ and $n = 0, 1, \ldots, N$. 
How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

\[
\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \rangle = \sum_{j=1}^{D_N} c_j^{(k,l,m)} \times \langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \rangle.
\]

- Coefficients are not unique if \( n > 0 \).
The uncertainty of the reconstructed Bloch vector element is

\[ \mathcal{E}^2[(X^k \otimes Y^l \otimes Z^m \otimes 1^n)_{\text{PI}}] = \sum_{j=1}^{D_N} |c_j^{(k,l,m)}|^2 \mathcal{E}^2[(A_j^{(N-n)} \otimes 1^n)_{\text{PI}}]. \]

For a fixed set of \( A_j \), we have a formula to find the \( c_j^{(k,l,m)} \)'s giving the minimal uncertainty.
Optimization for $A_j$

- We have to find $D_N$ measurement directions $\vec{a}_j$ on the Bloch sphere minimizing the variance

$$\left(\mathcal{E}_{\text{total}}\right)^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[ (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{PI} \right] \times \left( \frac{N!}{k!l!m!n!} \right).$$
Summary of algorithm

Obtaining the "total uncertainty" for given measurements

\[ \rho_0, \text{ the state we expect} \]
\[ A_j, \text{ what we measure} \]
\[ \Rightarrow \text{ BOX #1 } \Rightarrow (\epsilon_{\text{total}})^2 \]

Evaluating the experimental results

measurement results \[ A_j \]
\[ \Rightarrow \text{ BOX #2 } \Rightarrow \left\{ \begin{array}{l}
\text{Bloch vector elements} \\
\text{variances}
\end{array} \right. \]
How much is the information loss?

Estimation of the fidelity $F(\rho, \rho_{PI})$:

$$F(\rho, \rho_{PI}) \geq \langle P_s \rangle_\rho^2 \equiv \langle P_s \rangle_{\rho_{PI}}^2,$$

where $P_s$ is the projector to the $N$-qubit symmetric subspace.

- $F(\rho, \rho_{PI})$ can be estimated only from $\rho_{PI}$!
- Proof: using the theory of angular momentum for qubits.
- Similar formalism appear concerning handling multi-copy qubit states:
  

  [ E. Bagan et al., PRA 2006;

⇒ TALK TODAY BY RAMON MUÑOZ-TAPIA.
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Let us assume that we want to know only the expectation values of operators of the form

\[ \langle A(\phi)^{\otimes N} \rangle \]

where

\[ A(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y. \]

The space of such operators has dimension \( N + 1 \). We have to choose \( \{\phi_j\}^N_{j=1} \) angles for the \( \{A_j\}^N_{j=1} \) operators we have to measure.
Let us assume that we measure

\[ \langle A_j \otimes N \rangle \]

for \( j = 1, 2, ..., N + 1 \).

Reconstructed values and uncertainties

\[ \langle A(\phi) \otimes^N \rangle = \sum_{j=1}^{N+1} c_j(\phi) \times \langle A_j \otimes^N \rangle. \]

Reconstructed coefficients Measured data

\[ E^2[A(\phi)] = \sum_{j=1}^{N+1} |c_j(\phi)|^2 E^2(A_j \otimes^N). \]

Let us assume that all of these measurements have a variance \( \Delta^2 \).
Simple example III

- Numerical example for $N = 6$.

Random directions $\phi_j$  
Uncertainty of $A(\phi)^{\otimes N}$  
Uniform directions
Numerical example for $N = 6$. This random choice is even worse ...

Random directions $\phi_j$  Uncertainty of $A(\phi)^\otimes N$  Uniform directions
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The measured correlations \( \vec{\alpha}_j \) measurement directions
Random settings (exp.)

The measured correlations

$\vec{a}_j$ measurement directions
Density matrices (exp.)
We determined the optimal $A_j$ for p.i. tomography for $N = 4, 6, ..., 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon_{\text{max}}^2 = \max_{k,l,m,n} \mathcal{E}^2[(X^\otimes k \otimes Y^\otimes l \otimes Z^\otimes m \otimes I^\otimes n)_{\text{PI}}]$$

(Total count is the same as in the experiment: 2050.)
Expectation values directly from measured data

- Operator expectation values can be recovered directly from the measurement data as

\[ \langle Op \rangle = \sum_{j=1}^{D_N} \sum_{n=1}^{N} c_{j,n}^{Op} \langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^{\otimes n}) \rangle_{PI}, \]

where the \( c_{j,n}^{Op} \) are constants.

- \( Op = |D_{N}^{(N/2)}\rangle\langle D_{N}^{(N/2)}| \), blue: \( \rho_0 \propto \mathbb{1} \), red: upper bound for any \( \rho_0 \).
Comparison with other methods for efficient tomography

- If a state is detected as entangled, it is surely entangled. No assumption is used concerning the form of the quantum state.

- Expectation values of all permutationally invariant operators are the same for $\rho$ and $\rho_{\text{PI}}$.

- Typically, it can be used in experiments in which permutationally invariant states are created.
We discussed permutationally invariant tomography for large multi-qubits systems.

It paves the way for quantum experiments with more than 6 – 8 qubits.

See:

THANK YOU FOR YOUR ATTENTION!
How many settings we need?

- Expectation values of \((X^\otimes k \otimes Y^\otimes l \otimes Z^\otimes m \otimes \mathbb{1}^\otimes n)_{PI}\) are needed.

- For a given \(n\), the dimension of this subspace is \(D_{(N-n)}\) (simple counting).

- Operators with different \(n\) are orthogonal to each other.

- Every measurement setting gives a single real degree of freedom for each subspace.

- Hence the number of settings cannot be smaller than the largest dimension, which is \(D_N\).