Permutationally invariant quantum tomography

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18 February 2011


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Murmansk
Tromsø
Kiruna
White Sea
Arkhangel
Norwegian Sea
Reykjavík
ICELAND
Arctic Circle
Luleå
Oulu
NORWAY
Umeå
FINLAND
Trondheim
SWEDEN
Gulf of Bothnia
Tampere
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
Turku
Helsinki
Gulf of Finland
St. Petersburg
SHETLAND ISLANDS
Bergen
Oslo
ALAND ISLANDS
Tallinn
ESTONIA
RUSSIA
Rockall (U.K.)
ORKNEY ISLANDS
Stockholm
HEBRIDES
Stavanger
North Atlantic Ocean
Aberdeen
Göteborg
Gotland
Riga
LATVIA
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
UNITED KINGDOM
Malmö
Kaliningrad
RUSSIA
Mahilyow
Minsk
BELARUS
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
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Liverpool
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LUX.
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Vienna
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Turin
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Bilbao
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Genoa
BOSNIA AND HERZEGOVINA
Sarajevo
SERBIA AND MONTENEGRO
Varna
BULGARIA
Sofia
Toulouse
Andorra la Vella
MONACO
SAN MARINO
Florence
Marseille
Ligurian Sea
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Skopje
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ALB.
Thessaloniki
Lisbon
SPAIN
Valencia
Balearic Sea
Naples
TURK
Sardinia
Tyrrhenian Sea
GREECE
Aegean Sea
Athens
BALEARIC ISLANDS
Sevilla
Cagliari
Mediterranean Sea
Palermo
Ionian Sea
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Alborán Sea
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Algiers
Tunis
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles
Crete
Valletta
MALTA
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA

Outline

1. **Motivation**
   - Why quantum tomography is important?
2. **Quantum experiments with multi-qubit systems**
   - Physical systems
   - Local measurements
3. **Full quantum state tomography**
   - Basic ideas and scaling
   - Experiments
4. **Permutationally invariant tomography**
   - Main results
   - Example: XY PI tomography
   - Example: Experiment with a 4-qubit Dicke state
5. **Extra slide 1: Number of settings**

Why tomography is important?

- Many experiments aiming to create many-body entangled states
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales exponentially with the number of qubits.

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Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions
  T. Monz, P. Schindler, J.T. Barreiro, M. Chwalla, D. Nigg, W.A. Coish, M. Harlander, W. Haensel, M. Hennrich, R. Blatt, arxiv:1009.6126, 2010.

- 10 qubits with photons
  W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, J.-W. Pan, Nature Physics, 6, 331 (2010).

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Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.

$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	...	$A^{(N)}$

- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

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Full quantum state tomography

- The density matrix can be reconstructed from 3^N measurement settings.

Example

For $N = 4$, the measurements are

1.	X	X	X	X
2.	X	X	X	Y
3.	X	X	X	Z
		...		
3^4.	Z	Z	Z	Z

- Note again that the number of measurements scales exponentially in N.

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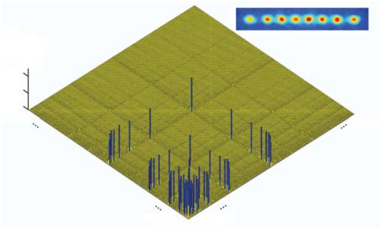
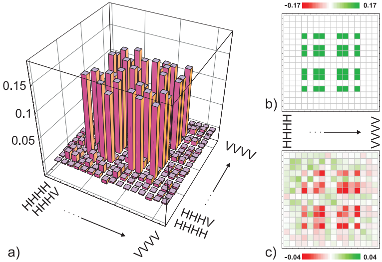
Experiments with ions and photons





- H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).
- N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. 98, 063604 (2007).

Approaches to solve the scalability problems

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.
  S.T. Flammia et al., arxiv:1002.3839; M. Cramer, M.B. Plenio, arxiv:1002.3780.

- If the state is of low rank, we need fewer measurements.
  D. Gross et al., Phys. Rev. Lett. 105, 150401 (2010).

- We make tomography in a subspace of the density matrices (our approach).

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Permutationally invariant part of the density matrix

Permutationally invariant part of the density matrix:

$$\varrho_{\mathrm{PI}} = \frac{1}{N!} \sum \Pi_k \varrho \Pi_k,$$

where Π_k are all the permutations of the qubits.

- Related literature: Reconstructing ϱ_{PI} for spin systems.
  [G. M. D'Ariano *et al.*, J. Opt. B **5**, 77 (2003).]
- Photons in a single mode optical fiber are always in a permutationally invariant state. Small set of measurements are needed for their characterization (experiments).
  [R.B.A. Adamson *et al.*, Phys. Rev. Lett. **98**, 043601 (2007); R.B.A. Adamson *et al.*, Phys. Rev. A 2008; L. K. Shalm *et al.*, Nature **457**, 67 (2009).]

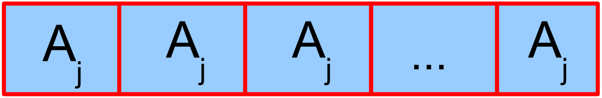
Main results

Features of our method:

1. Is for spatially separated qubits.
2. Needs the minimal number of measurement settings.
3. Uses the measurements that lead to the smallest uncertainty possible of the elements of $\varrho_{\rm PI}$.
4. Gives an uncertainty for the recovered expectation values and density matrix elements.
5. If $\varrho_{\rm PI}$ is entangled, so is ϱ. Can be used for entanglement detection!

Measurements

- We measure the same observable A_j on all qubits. (Necessary for optimality.)



- Each qubit observable is defined by the measurement directions $\vec{a}_j$ using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$\mathcal{D}_N = \binom{N+2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$

What do we get from the measurements?

We obtain the expectation values for

$$\langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle$$

for $j = 1, 2, .., \mathcal{D}_N$ and $n = 0, 1, ..., N$.

How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

$$\underbrace{\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle}_{\text{Bloch vector elements}} = \sum_{j=1}^{\mathcal{D}_N} \underbrace{c_j^{(k,l,m)}}_{\text{coefficients}} \times \underbrace{\langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle.}_{\text{Measured data}}$$

- Coefficients are not unique if $n > 0$.

Uncertainties

The uncertainty of the reconstructed Bloch vector element is

$$\mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}] = \sum_{j=1}^{\mathcal{D}_N} |c_j^{(k,l,m)}|^2 \mathcal{E}^2[(A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}].$$

- For a fixed set of A_j, we have a formula to find the $c_j^{(k,l,m)}$'s giving the minimal uncertainty.

Optimization for A_j

- We have to find $\mathcal{D}_N$ measurement directions $\vec{a}_j$ on the Bloch sphere minimizing the variance

$$(\mathcal{E}_{\text{total}})^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \right] \times \left(\frac{N!}{k!l!m!n!} \right).$$

Summary of algorithm

Obtaining the "total uncertainty" for given measurements

$$\left.\begin{array}{ll} \varrho_0, & \text{the state we expect} \\ A_j, & \text{what we measure} \end{array}\right\} \Rightarrow \text{BOX \#1} \Rightarrow (\mathcal{E}_{\mathrm{total}})^2$$

Evaluating the experimental results

$$\left.\begin{array}{c} \text{measurement results} \\ A_j \end{array}\right\} \Rightarrow \text{BOX \#2} \Rightarrow \left\{\begin{array}{c} \text{Bloch vector elements} \\ \text{variances} \end{array}\right.$$

How much is the information loss?

Estimation of the fidelity $F(\varrho, \varrho_{\mathrm{PI}})$:

$$F(\varrho, \varrho_{\mathrm{PI}}) \geq \langle P_{\mathrm{s}} \rangle_{\varrho}^{2} \equiv \langle P_{\mathrm{s}} \rangle_{\varrho_{\mathrm{PI}}}^{2},$$

where P_{s} is the projector to the N-qubit symmetric subspace.

- $F(\varrho, \varrho_{\mathrm{PI}})$ can be estimated only from ϱ_{PI}!
- Proof: using the theory of angular momentum for qubits.
- Similar formalism appear concerning handling multi-copy qubit states:

  [J. I. Cirac, A. K. Ekert, C. Macchiavello, Optimal purification of single qubits PRL 1999.]

  [E. Bagan *et al.*, PRA 2006;

  G. Sentís, E. Bagan, J. Calsamiglia, R. Muñoz-Tapia, Multi-copy programmable discrimination of general qubit states, PRA 2010.]
  $\Rightarrow$ TALK TODAY BY RAMON MUÑOZ-TAPIA.

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Simple example

- Let us assume that we want to know only the expectation values of operators of the form
$$\langle A(\phi)^{\otimes N} \rangle$$
where
$$A(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y.$$

- The space of such operators has dimension $N + 1$. We have to choose $\{\phi_j\}_{j=1}^{N+1}$ angles for the $\{A_j\}_{j=1}^{N+1}$ operators we have to measure.

Simple example II

- Let us assume that we measure

$$\langle A_j^{\otimes N} \rangle$$

for $j = 1, 2, ..., N+1$.

- Reconstructed values and uncertainties

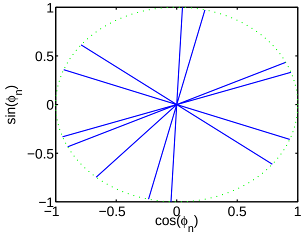
$$\underbrace{\langle A(\phi)^{\otimes N} \rangle}_{\text{Reconstructed}} = \sum_{j=1}^{N+1} \underbrace{c_j^{(\phi)}}_{\text{coefficients}} \times \underbrace{\langle A_j^{\otimes N} \rangle.}_{\text{Measured data}}$$

$$\mathcal{E}^2[A(\phi)] = \sum_{j=1}^{N+1} |c_j^{(\phi)}|^2 \mathcal{E}^2(A_j^{\otimes N}).$$

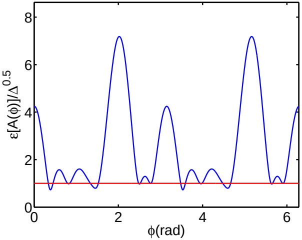
- Let us assume that all of these measurements have a variance Δ^2.

Simple example III

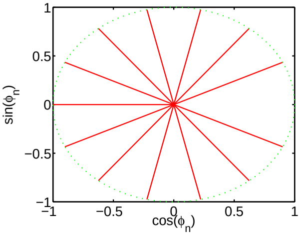
- Numerical example for $N = 6$.



Random directions ϕ_j



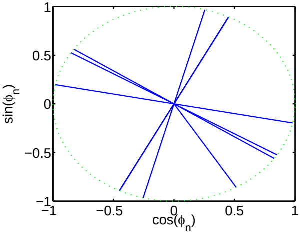
Uncertainty of $A(\phi)^{\otimes N}$



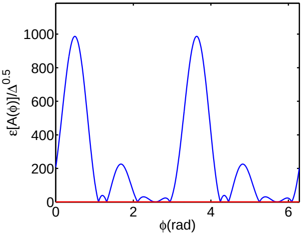
Uniform directions

Simple example IV

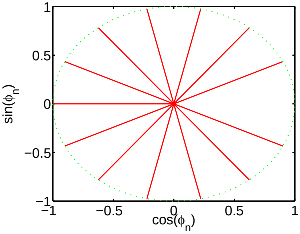
- Numerical example for $N = 6$. This random choice is even worse ...



Random directions ϕ_j



Uncertainty of $A(\phi)^{\otimes N}$



Uniform directions

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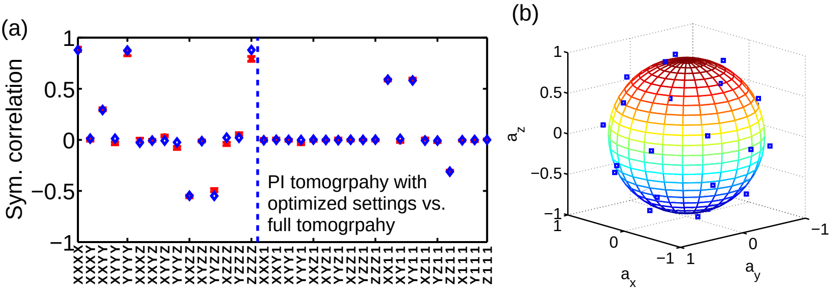
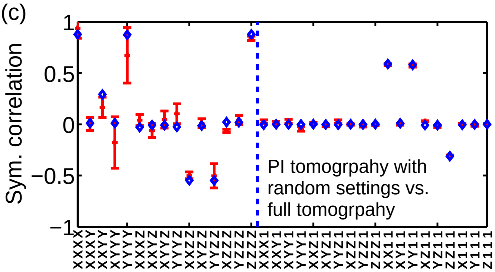
4-qubit Dicke state, optimized settings (exp.)

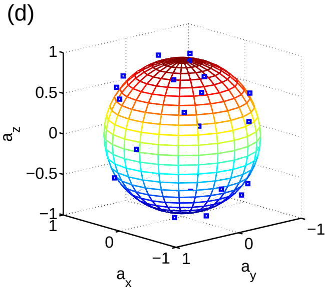


The measured correlations

$\vec{a}_j$ measurement directions

Random settings (exp.)

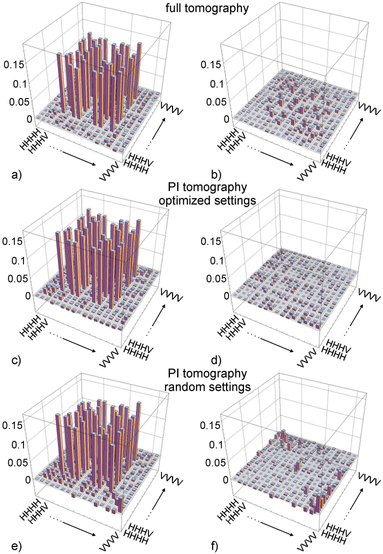




The measured correlations

$\vec{a}_j$ measurement directions

Density matrices (exp.)

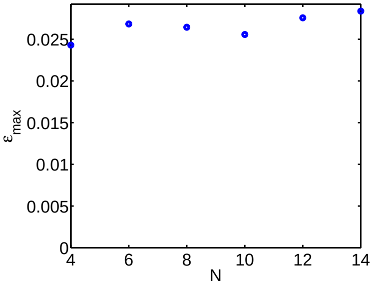


PI tomography for larger systems

- We determined the optimal A_j for p.i. tomography for $N = 4, 6, ..., 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon^2_{\max} = \max_{k,l,m,n} \mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}]$$

(Total count is the same as in the experiment: 2050.)



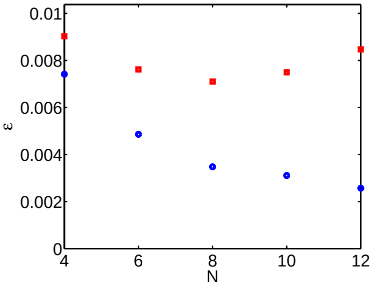
Expectation values directly from measured data

- Operator expectation values can be recovered directly from the measurement data as

$$\langle Op \rangle = \sum_{j=1}^{\mathcal{D}_N} \sum_{n=1}^{N} c_{j,n}^{Op} \langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle,$$

  where the $c_{j,n}^{Op}$ are constants.

- $Op = |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|$, blue: $\varrho_0 \propto \mathbb{1}$, red: upper bound for any ϱ_0.



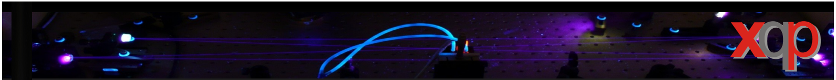
Comparison with other methods for efficient tomography

- If a state is detected as entangled, it is surely entangled. No assumption is used concerning the form of the quantum state.

- Expectation values of all permutationally invariant operators are the same for ϱ and $\varrho_{\rm PI}$.

- Typically, it can be used in experiments in which permutationally invariant states are created.

Participants in the project



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MPQ, Munich
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Christian Schwemmer
MPQ, Munich

Géza Tóth
Bilbao

Summary

- We discussed permutationally invariant tomography for large multi-qubits systems.
- It paves the way for quantum experiments with more than 6 – 8 qubits.

See:
G. Tóth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Permutationally invariant quantum tomography, Phys. Rev. Lett. 105, 250403 (2010).

THANK YOU FOR YOUR ATTENTION!

How many settings we need?

- Expectation values of $(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}$ are needed.
- For a given n, the dimension of this subspace is $\mathcal{D}_{(N-n)}$ (simple counting).
- Operators with different n are orthogonal to each other.
- Every measurement setting gives a single real degree of freedom for each subspace
- Hence the number of settings cannot be smaller than the largest dimension, which is $\mathcal{D}_N$.