Topological codes, quantum memory and communication

Michał Horodecki

University of Gdansk
KCIK, IFTiA

QIPA'11

Supported by:
- Polish Ministry of Science and Higher Education
- EC IP Q-ESSENCE

PART I

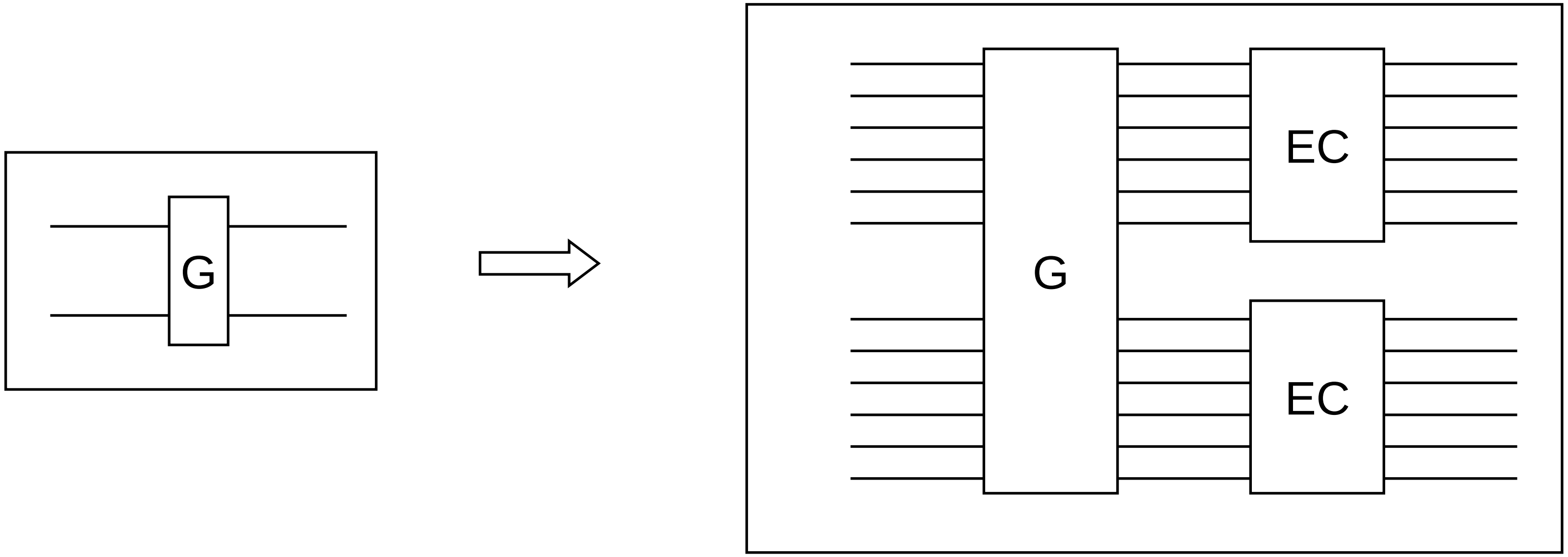
Thermal stability of self-correcting Kitaev-like models

Fault tolerant quantum computing

Threshold theorem:
Let **p** be error rate (probability that a gate is faulty). If it satisfies

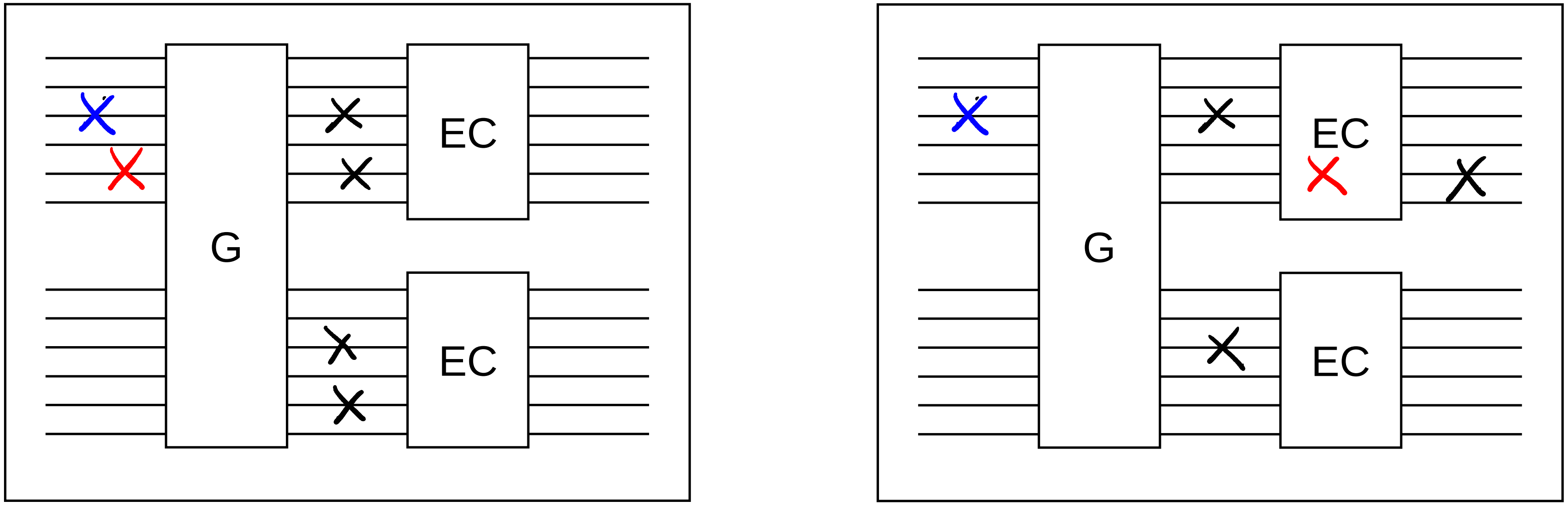
$$p_0 \leqslant p_{th}$$

Then any function can be computed with accuracy ε
with polylogarythmic overhead in time and space.



Property: if a single physical error hits the box, the state remains correctable.

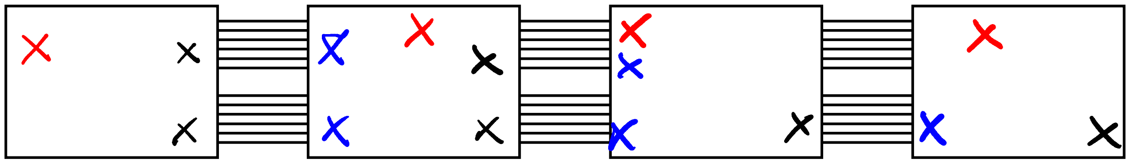
Fault tolerant quantum computing



- fault received by a box

- error inherited from a previous box

- error resulting from propagation of x and x



Remark: the state will almost never belong to the code. Most time, it will have some errors.

If every box received at most single fault x then the state of either of logical qubits will have always no more than two errors $\Rightarrow$ There is no logical error

Threshold result

Aharonov Ben-Or,
Knill, Laflamme, Zurek,
Kitaev]

Thus, the logical fault can occur only if two physical faults occur in a single box.

$$p_{(1)} \leq \text{Prob}(\text{two faults}) \leq c\, p_0^2$$

where c is number of pairs of locations.

We concatenate the scheme:

$$p_{(1)} \leq c\, p_0^2$$

$$p_{(2)} \leq c\, p_{(1)}^2 = c\,(c\, p_0^2)^2$$

$$\vdots$$

$$p_{(r)} \leq \frac{1}{c}(c\, p_0)^{2^r}$$

$$\boxed{p_0 < \frac{1}{c} \equiv p_{th} \Rightarrow p_{eff} \simeq e^{-\mathrm{poly}(\#\,\text{qubits})}}$$

High fidelity FT scheme

[Alicki, Horodeckis, OSID 2010]

In FT scheme, we never have a pure state.

Can we say, that we can preserve a state with high fidelity?

YES: one has to single out a proper SUBSYSTEM. A qubit on the subsystem will be preserved with arbitrarily high fidelity.

How to find the relevant subsystem?

To define qubit subsystem, it is enough to fix two observables X and Z.

Note: X_L and Z_L act only on a code, we need observables acting on the whole space.

$$X = \sum_s Corr^{-1}(s)\, X_L\, Corr(s)\, P_s$$

$$Z = \sum_s Corr^{-1}(s)\, Z_L\, Corr(s)\, P_s$$

- P_s is projector onto syndrom s
- *Corr (s)* is correction procedure, which returns erronous states to the code subspace

Threshold result within Hamiltonian description

Main assumptions of original threshold theorem:

1. Phenomenological model of noise (not Hamiltonian one).
2. Active error correction realized by special circuits.

Problem [Loss & DiVincenzo 1998, Alicki & Horodeckis 2004]:

In Hamiltonian description of decoherence the noise is not independent

Partial progress: [Tehral & Burkard 2005]

Arbitrary long computing is possible, provided that

$$\lambda_0 t_0 < c'$$

Here:
- t_0 is time duration of gate,
- λ_0 is norm of Hamiltonian of interaction with the bath.

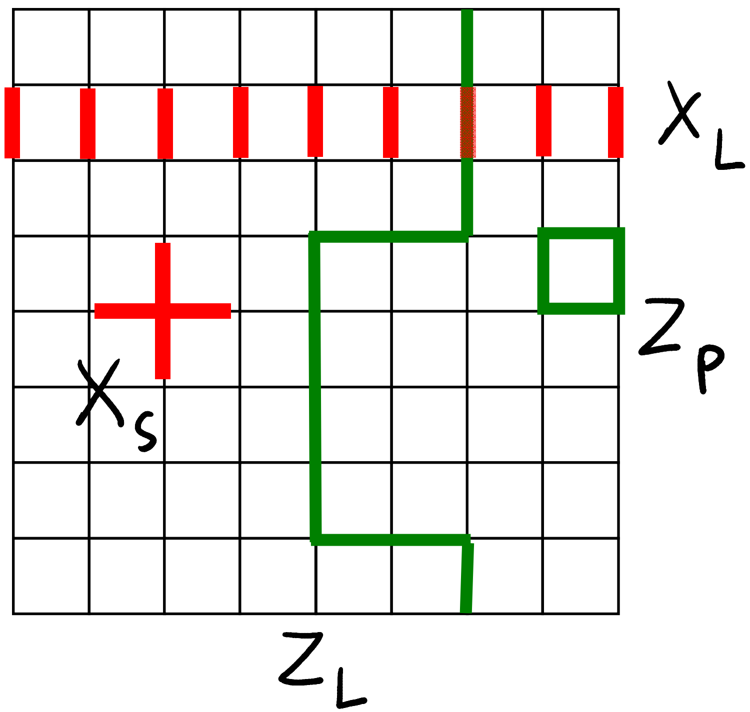
Open problem:
- λ_0 is area below spectral density (big, in principle infinite)

Topological protection

[Kitaev]

Topological codes seem more physical:
- the correction can be performed by the system itself, as magnetization is maintained in Ising 2D model.

2D Kitaev code on torus:



- Qubits are situated on edges
- code is a ground state of the Hamiltonian:

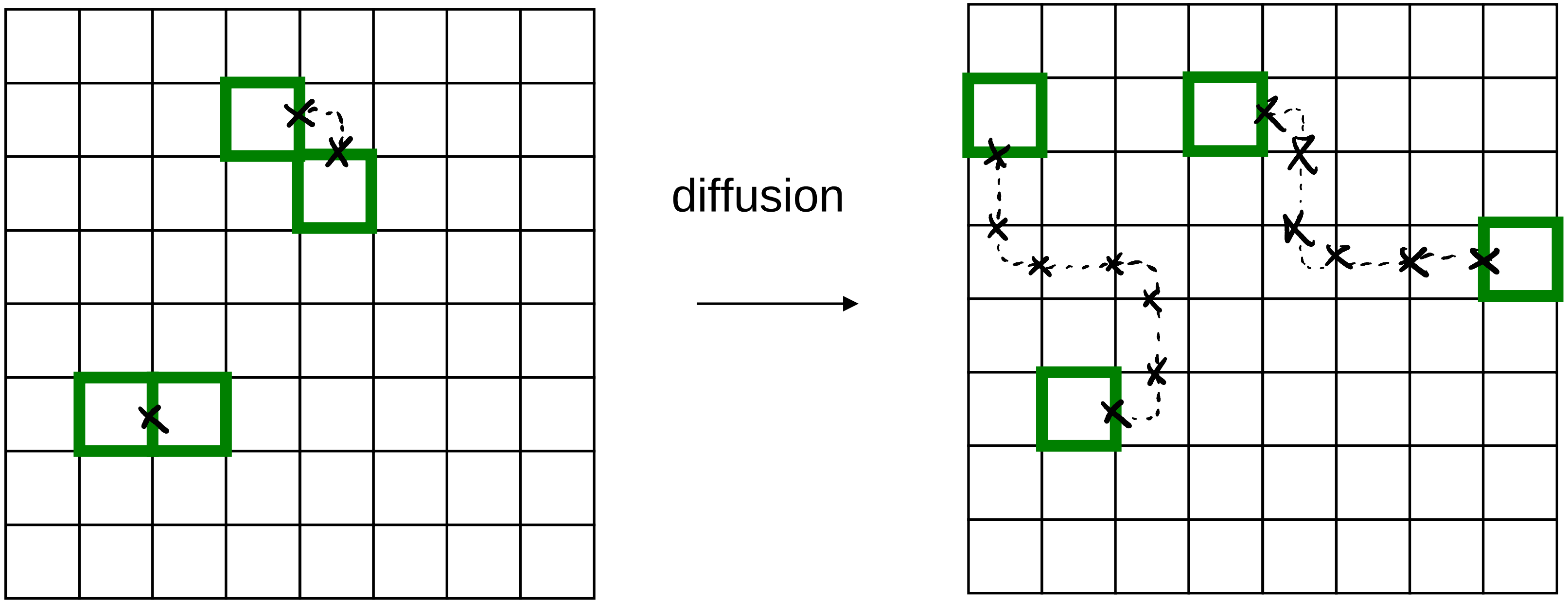
$$H = -\sum_s X_s - \sum_p Z_p$$

s – star, p – plaquette

| = σ_x | = σ_z

Long path of errors corresponds to error on logical qubit

Is 2D Kitaev code self-correcting against thermal noise?



Prob (long path) = # pairs x Prob(a pair makes long path)

N – average number of pairs
p – probability that a pair makes a long path

N ~ volume
p ~ volume $^{-1}$

Prob (long path) ~ const

$\Rightarrow$ Kitaev 2D model is not self-correcting for T>0

Rigorous proof of instability of Kitaev 2D model

[Alicki, Fannes, MH J. Phys. A 2009]

$$H_{int} = \sum_\alpha \sigma_x^\alpha \otimes f_\alpha^x + \sum_\alpha \sigma_y^\alpha \otimes f_\alpha^y + \sum_\alpha \sigma_z^\alpha \otimes f_\alpha^y$$

α - runs over qubits, f_α - operators of environment

- Use weak coupling approximation leading to Davies generator:

$$\frac{dA}{dt} = i[H, A] - LA$$

- Terms [H, .] and L commute $\Rightarrow$ it is enough to consider:

$$\frac{dA}{dt} = -LA \iff A(t) = e^{-Lt} A$$

- L is of the form

$$L = \sum_{\alpha, k, \omega} L_\omega^{\alpha, k} \qquad k = x, y, z, \qquad \omega \text{ – Bohr freq. of Hamiltonian}$$

Properties of L

- Hermitian in scalar product:

$$\langle X, Y\rangle_\beta = \mathrm{Tr}(\rho_\beta X^\dagger Y)$$

- positive

$$L \geq 0$$

- for our coupling, L has single „ground state”

$$L(I) = 0$$

- L is frustration free:

$$\sum_\omega L_\omega^{\alpha, k}(I) = 0$$

Rigorous proof of instability of Kitaev 2D model

[Alicki, Fannes, MH J. Phys. A 2009]

Thermal instability is related to gap of L:

$$\langle X(t), X \rangle_\beta \leq e^{-\mathrm{gap}(L)\, t}$$

for any X s.t. $\langle X, I \rangle_\beta = 0$, $\langle X, X \rangle_\beta = 1$

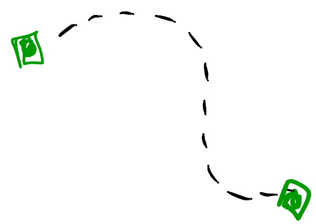
Theorem: Spectral gap of dissipative generator L for Kitaev 2D model satisfies

$$\mathrm{gap}(L) \geq \frac{1}{3} e^{-8\beta}$$

Proof: Boring, technical. Using techiques for estimating gaps, but also explicit calculations of eigenvalues.

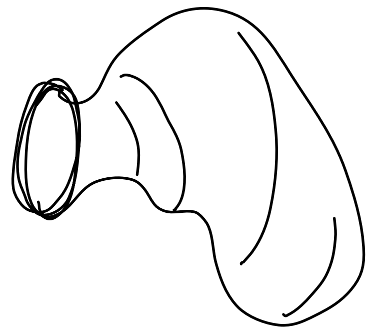
Thermal stability of Kitaev 4D model

[Alicki, Fannes,
MH J. Phys. A 2009]



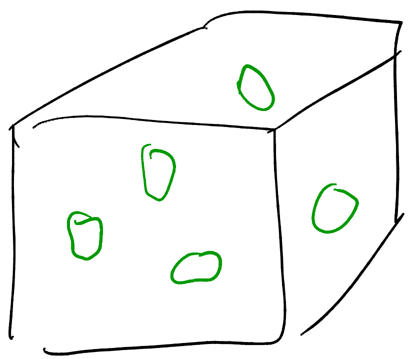
In 2D model:
- defects are particles (point-like)
- error form paths
- spreading of errors does not cause increase of energy

⇒ thermal fluctuations easily produce logical error
(non-trivial loop)



In 4D model,
- errors are situated on surfaces,
- defects are strings (boundaries of error surfaces)
- energy is proportional to length of the strings

⇒ One can try follow Peierls argument for 2D Ising model
and it should give thermal stability



Problem:
- find topological analogue of magnetization,
- prove that it is stable.

Thermal stability of Kitaev 4D model

[Alicki, Horodeckis, OSID 2010]

Observables: „dressed” or „error corrected” observables X and Z mentioned before

Useful inequality:

$$\epsilon_Z \equiv \langle Z, L Z \rangle_\beta \leq \text{const} \sum_\alpha \left(1 - \langle Z, \sigma_\alpha^z Z \sigma_\alpha^z \rangle_\beta \right)$$

Fidelity of logical qubit:

$$F \geq \frac{1}{2}\left(e^{-\epsilon_Z t} + e^{-\epsilon_X t} \right)$$

Main result: for 4D Kitaev model we have

$$\langle Z, L Z \rangle_\beta \leq \text{poly}(l)\, e^{-\frac{1}{8} l (\beta - \beta_c)}$$

l - linear size of torus

β_c - depends only on dimension d and type of lattice

Independent/further resutls on instability of stabilizer models

1) No-go results for 2D stabilizer codes
   - Terhal & Bravyi, New J. Phys. 11 (2009) 043029
   - Kay

2) Some general estimates:
[Chesi, Loss, Bravyi. Terhal, New J. Phys. 12, 025013 (2010)]

3) Analysis of topological models, which allow for universal computing
[H. Bombin, R. Chhajlany, M. H. & M.-A. Martin-Delgado, arXiv:0907.5228]

Summary of PART I

Problems:

- The famous threshold result for fault-tolerant QC assumes phenomenological, independent noise. Hamiltonian description, the noise is not iid anymore.

2) Generalizations of threshold result to Hamiltonian dynamics exist.
   - not quite satisfactory (assume small norm of Hamiltonian of interaction with the bath)

Self-correcting models: more physical, Hamiltonian description more natural

1) 2D Kitaev model thermally **instable** (though by logarythmic scaling temperature, one can prolong time of protection)

2) 4D Kitaev model is thermally **stable** (though in 3D requires non-local interaction)

3) There are more general resutls for stabilizer codes.

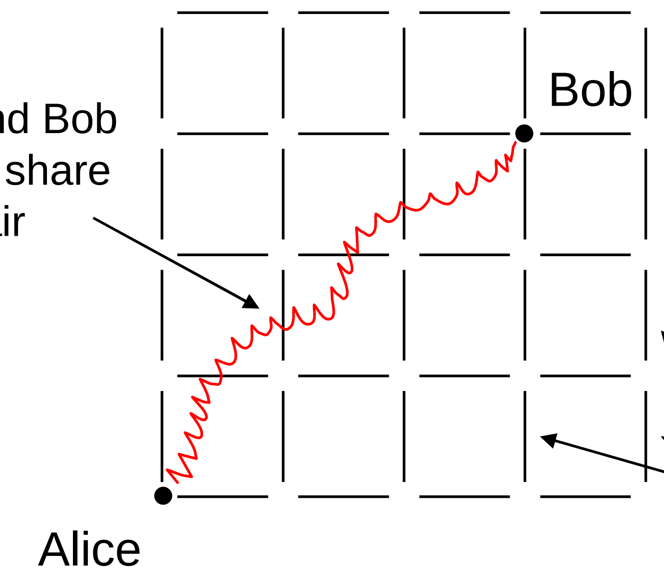
PART II

Quantum communication and cryptography over EPR networks

Quantum communication over EPR networks: basic scheme

[Acin, Cirac, Lewenstein
Nat. Phys. 2006]

Alice and Bob
want to share
EPR pair

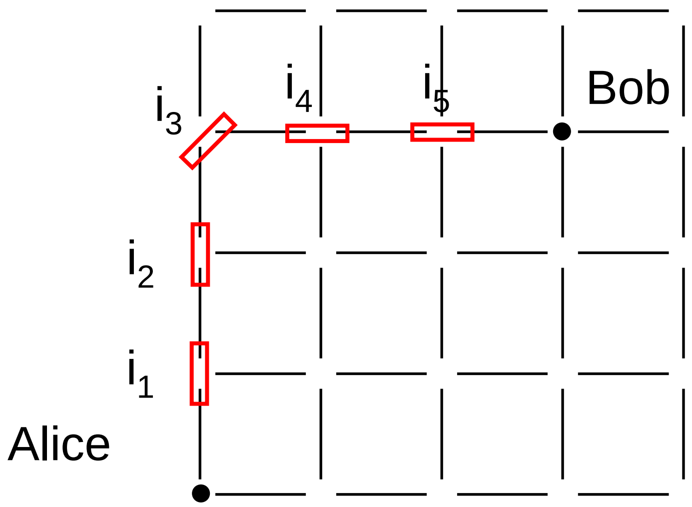


$$\phi_+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

EPR network:
- nodes: EPR pairs
- vertices: labs

Condition: constant number of operations in each vertex

Noiseless case:
- perform Bell measurement
 on a chosen path.
- send the resutls to Bob
- Bob applies a unitary depending
 on some function of the outcomes



Noisy networks and fault-tolerance

Problem:
- Can Alice and Bob share an EPR pair over a noisy network?

Equivalently:
- Can Alice send to Bob an unknown state ?

- Can Alice and Bob perform BB84 protocol?
i.e. send each of the four states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$

Solution: given in Perseguers, Jiang, Schuch, Verstraete, Lukin, Cirac, Vollbrecht, PRA 78, 062324 (2008).

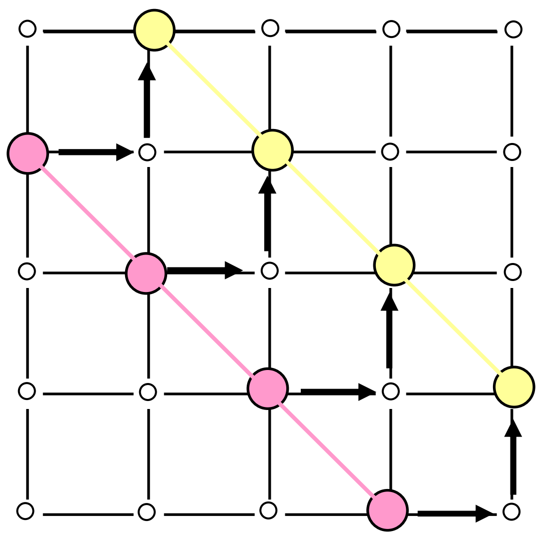
YES: there exists a protocol over 2D EPR network, which allows quantum communication. It uses scheme for quantum fault-tolerant computing.

We like this solution, but think it is incomplete.

▪ consider fault tolerant quantum computing (FT QC) on line with local gates
▪ translate it into teleportation protocol in two-dimensional array

t=1

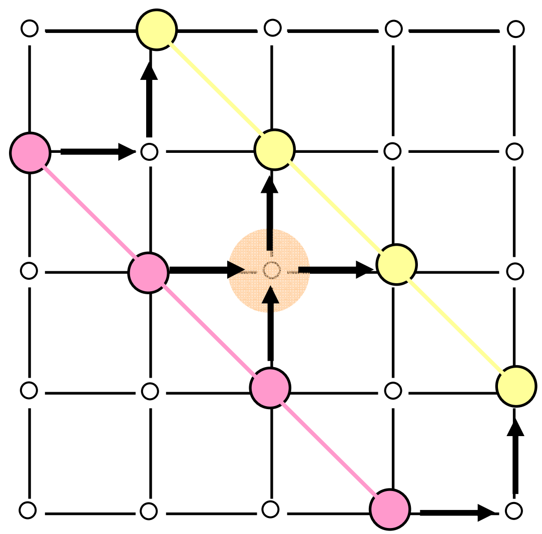
t=0



Two teleportations transport quantum computer between time t=0 and t=1

t=1

t=0



Two qubit gates can be applied to neighboring qubits

▪ since FT QC allows to transmit qubit in time, the network version will transmit qubit in space
▪ combining QC forward in time, and backward in time, will allow to share e-bit in space

Noisy networks and fault-tolerance: remaining problem

Fault tolerant threshold theorem says that we can have fidelity arbitrarily close to 1.

However, this is the fidelity of an apriori **known** state.

How to deal with **unknown** state?

Computation on line: to our knowledge there is no explicit estimate in literature, what is the fidelity of storing a qubit in unknown state.

Proposition: Suppose that error rate p_0 satisfies

$$p_0 \leqslant \frac{1}{3} p_{th}$$

Then it is possible to maintain a qubit in unknown state with fidelity

$$F \geq F_0 (1 - O(2^{-N})) \qquad \text{where} \qquad F_0 \geq e^{-p_0 v}$$

N - number of qubits,
v – volume of physical encoding circuit (a **constant**)

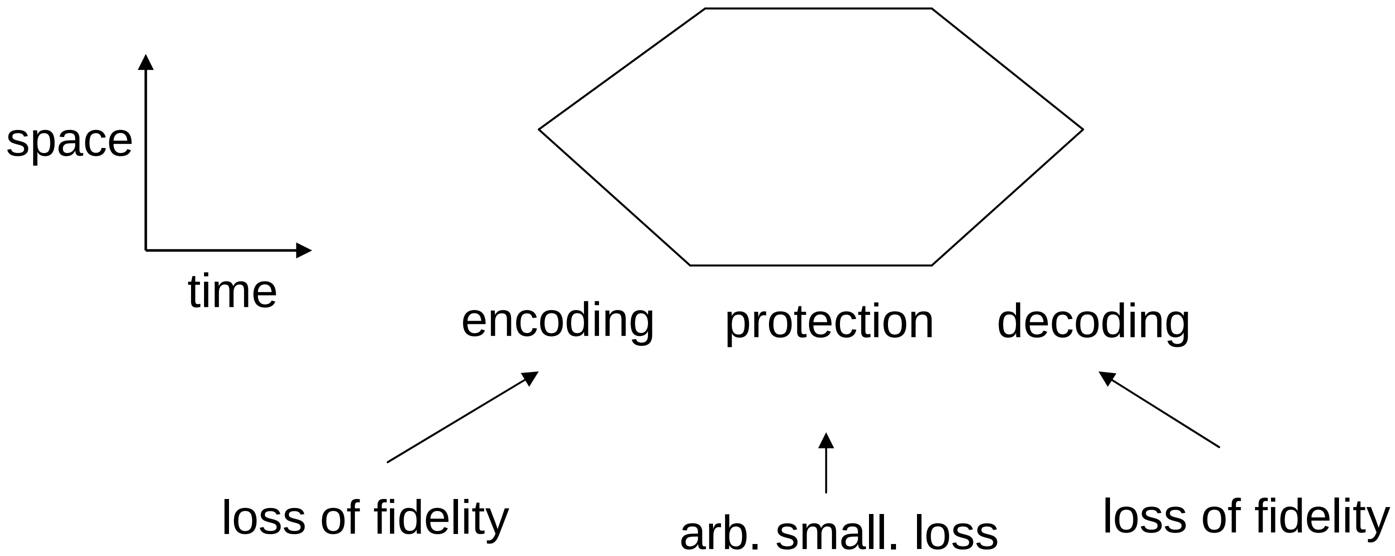
3D networks with quantum communication

The 1D fault-tolerant quantum computing is extremely complicated.
Much simpler scheme is one based on 2D Kitaev model.

Quantum communication in 3D = maintaining qubit in 2D.

How to protect a qubit in unknown state in 2D?

- **Encode** a qubit in an unknown state
- **Protect** the encoded state
- **Decode** the qubit



Encoding, Protecting, Decoding

Protecting encoded state:
- measure repeatedly syndrom (i.e. the star and plaquette operators)
- collect outcomes

Encoding/Decoding a qubit:
- Dennis et al. provided a procedure of enlarging/diminishing code

Problems:
- encoding is in a form of circuit
- decoding is just a reverse of the circuit

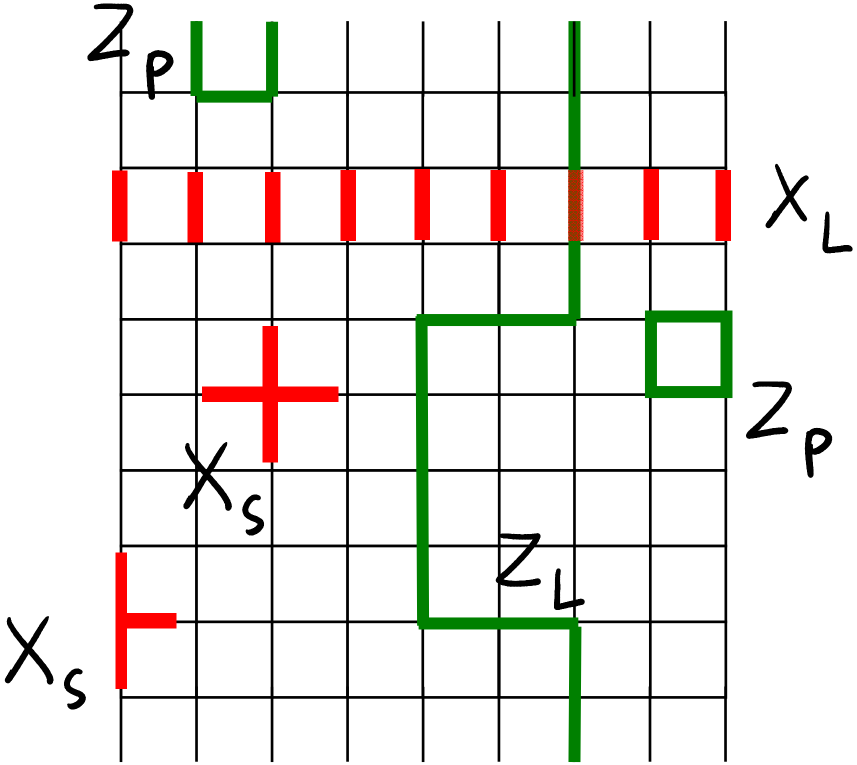
Our goal: provide
- Unify the encoding stage with protecting stage
  $\Rightarrow$ encodigng by measuring star X_s and plaquette Z_p operators
- Provide single shot decoding.

Grudka et al. arXiv/????.????

Planar Kitaev code

[Dennis et al. 2001]

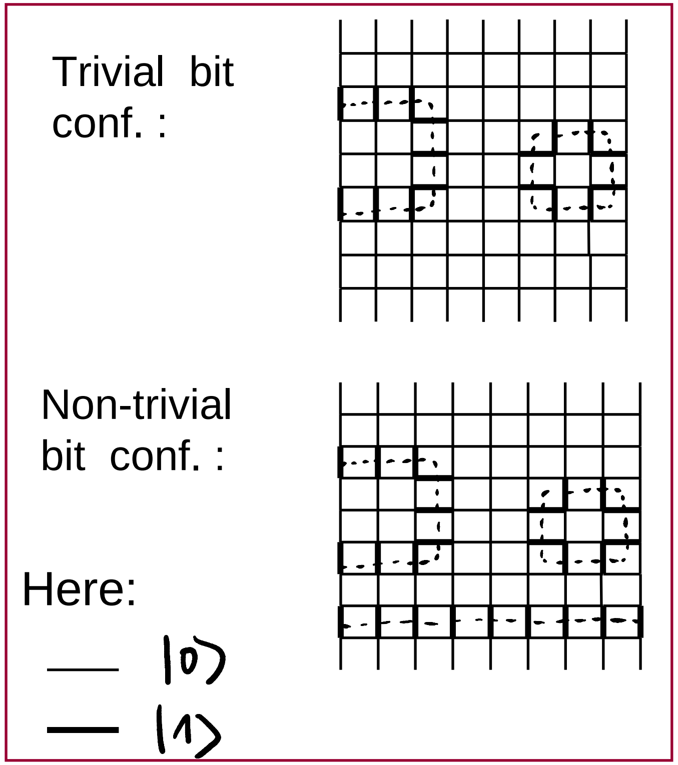
Torus is not so practical. Here is planar version of toric code encodes one logical qubit.



- qubits are situated on edges
- code is given by all X_s and Z_p equal to +1

s – star, p – plaquette

| $= \sigma_x$ | $= \sigma_z$



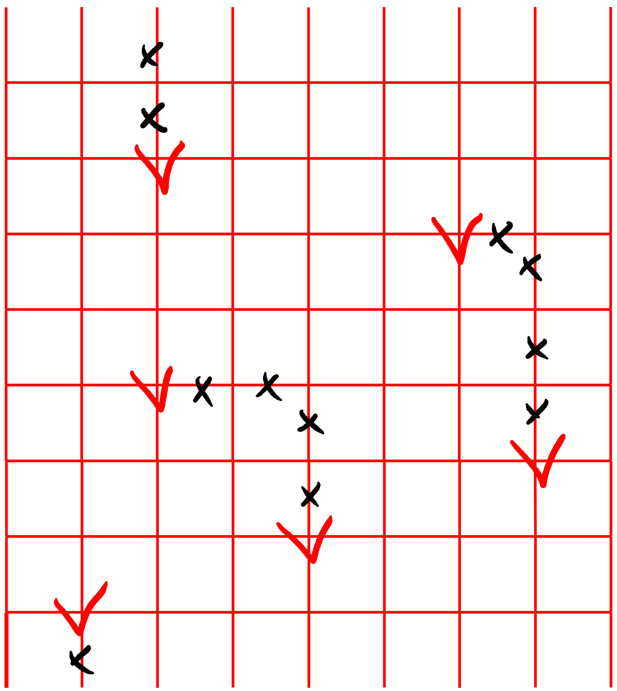
Codewords:

$$|0\rangle_L = \sum_{\text{trivial hom.}} |\text{bit configurations}\rangle$$

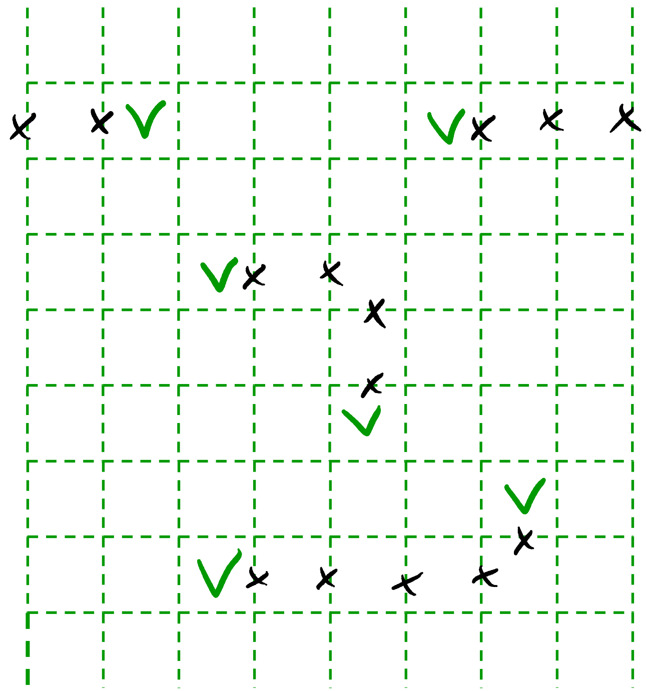
$$|1\rangle_L = \sum_{\text{nontrivial hom.}} |\text{bit configurations}\rangle$$

Encoding known state in absence of noise

Encoding $|0\rangle_L$



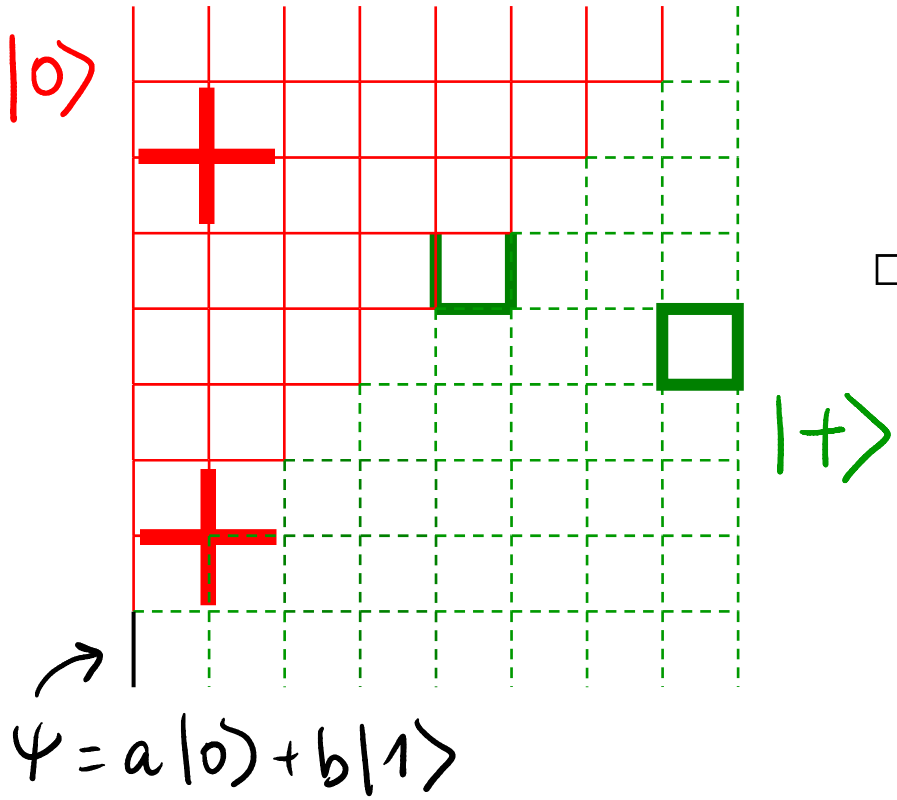
Encoding $|+\rangle_L$

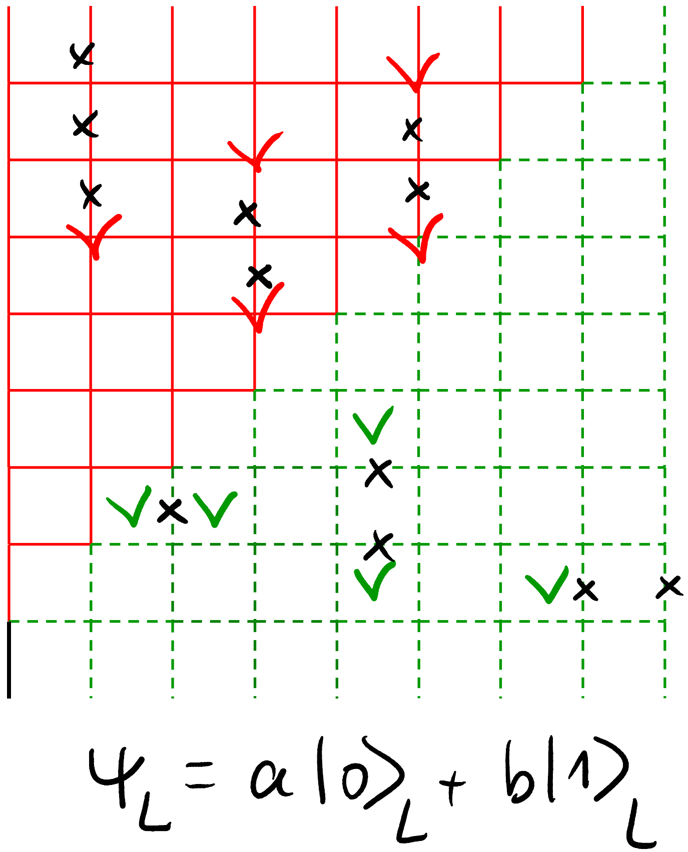


- Prepare every qubit in **|0⟩** state
- Measure all $\mathbf{X}_s$
- Anihilate defects by joining them with **phase-flips** in arbitrary way

- Prepare every qubit in **|+⟩** state
- Measure all $\mathbf{Z}_s$
- Anihilate defects by joining them with **bit-flips** in arbitrary way

Encoding unknown state in absence of noise





- prepare:
  - $|0\rangle$ (red dotted)
  - $|+\rangle$ (green dashed)
  - $a|0\rangle + b|1\rangle$ (black solid)

- measure:

  all Z_p touching green qubits

  all X_s touching red qubits

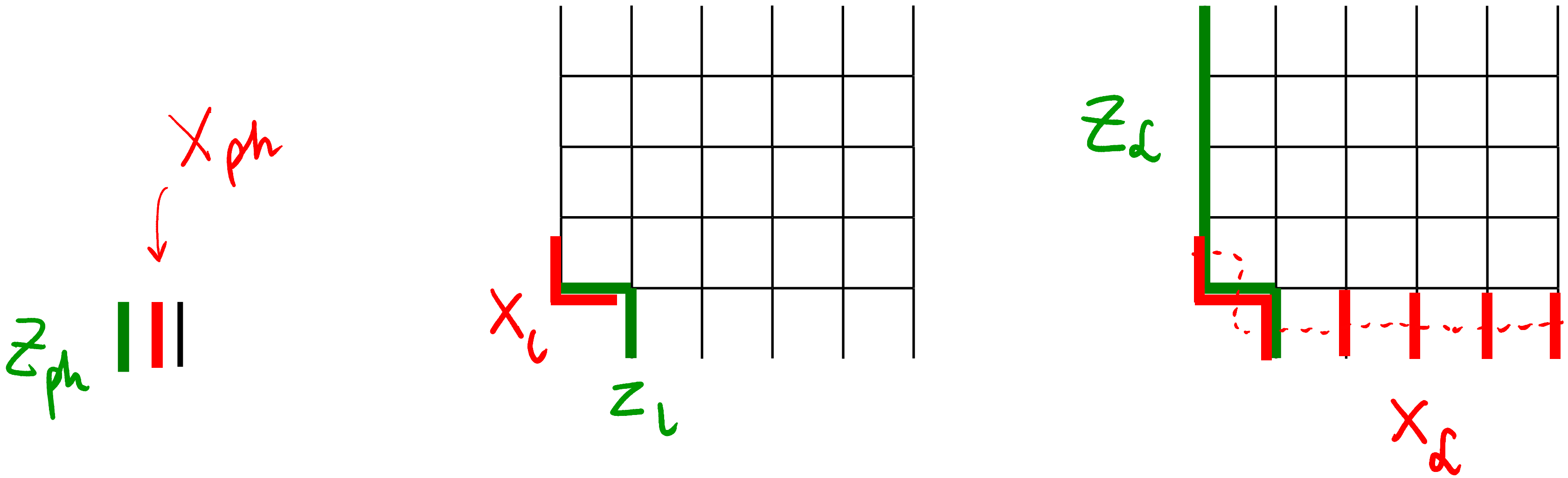
- anihilate defects:

  Z-type defects move right

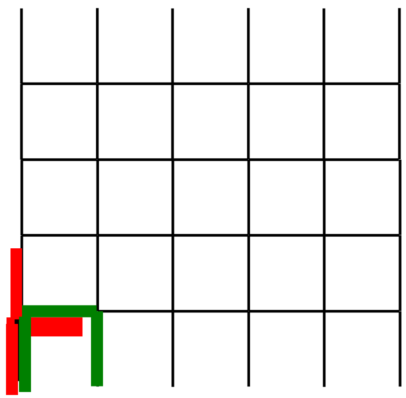
  X-type defects move up

Encoding unknown state as teleportation

$$\mathcal{H}_{ph} \otimes \mathcal{H}_{L} \otimes \mathcal{H}_{\mathcal{L}}$$



- Prepare a code without our qubit we want to encode
- Attach the qubit and measure the only left X_s and Z_p:



But:

$$X_s = X_{ph} \otimes X_L$$

$$Z_p = Z_{ph} \otimes Z_L$$

Bell measurement

on $\mathcal{H}_{ph} \otimes \mathcal{H}_L$

Summary of Part II

- **Important problem:** communicate quantum information over EPR networks with constant complexity for each node

- **Previous result:** Fault tolerant quantum computing in dimension d $\Rightarrow$ quantum communication over EPR networks of dimension d+1

- **Problem:** One element was lacking – encoding unknown state

Our resutls:

- we have provided explicit estimate for fidelity of encoding a qubit : within standard FT scheme

- we proposed an encoding scheme for Kitaev code, basing on measuring star and plaquette operators in ideal case

Work in progress:

- currently we are making simulations of our encoding for noisy case