Topological codes, quantum memory and communication

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PART I

Thermal stability of self-correcting Kitaev-like models
**Fault tolerant quantum computing**

**Threshold theorem:**
Let $p$ be error rate (probability that a gate is faulty). If it satisfies

$$p_0 \leq P_{th}$$

Then any function can be computed with accuracy $\varepsilon$ with polylogarythmic overhead in time and space.

**Property:** if a single physical error hits the box, the state remains correctable.
If every box received at most single fault $\times$ then the state of either of logical qubits will have always no more than two errors $\Rightarrow$ There is no logical error.

Remark: the state will almost never belong to the code. Most time, it will have some errors.
Thus, the logical fault can occur only if two physical faults occur in a single box.

\[ p_{(1)} \leq \text{Prob}(\text{two faults}) \leq c p_0^2 \]

where \( c \) is number of pairs of locations.

We concatenate the scheme:

\[ p_{(1)} \leq c p_0^2 \]

\[ p_{(2)} \leq c p_{(1)}^2 = c (c p_0^2)^2 \]

\[ \vdots \]

\[ p_{(r)} \leq \frac{1}{c} (c p_0^2)^2 \]

\[ p_0 < \frac{1}{c} \equiv p_m \Rightarrow \text{Eff} \leq e^{-\text{poly} \left( \text{# qubits} \right)} \]
In FT scheme, we never have a pure state.

Can we say, that we can preserve a state with high fidelity?

**YES:** one has to single out a proper **SUBSYSTEM**. A qubit on the subsystem will be preserved with arbitrarily high fidelity.

**How to find the relevant subsystem?**

To define qubit subsystem, it is enough to fix two observables $X$ and $Z$.

**Note:** $X_L$ and $Z_L$ act only on a code, we need observables acting on the whole space.

\[ X = \sum_s \text{Corr}^{-1}(s) X_L \text{Corr}(s) P_s \]
\[ Z = \sum_s \text{Corr}^{-1}(s) Z_L \text{Corr}(s) P_s \]

- $P_s$ is projector onto syndrom $s$
- $\text{Corr}(s)$ is correction procedure, which returns erroneous states to the code subspace
Main assumptions of original threshold theorem:

1. Phenomenological model of noise (not Hamiltonian one).
2. Active error correction realized by special circuits.

**Problem** [Loss & DiVincenzo 1998, Alicki & Horodeckis 2004]:

In Hamiltonian description of decoherence the noise is not independent

**Partial progress:** [Tehral & Burkard 2005]

Arbitrary long computing is possible, provided that

\[ \lambda_0 t_0 < c' \]

Here:
- \( t_0 \) is time duration of gate,
- \( \lambda_0 \) is norm of Hamiltonian of interaction with the bath.

**Open problem:**
- \( \lambda_0 \) is area below spectral density (big, in principle infinite)
Topological protection

Topological codes seem more physical:
- the correction can be performed by the system itself, as magnetization is maintained in Ising 2D model.

2D Kitaev code on torus:

- Qubits are situated on edges
- code is a ground state of the Hamiltonian:

\[ H = - \sum_s X_s - \sum_p Z_p \]

\( s \) – star, \( p \) – plaquette

\[ | = \sigma_x \quad \_ = \sigma_z \]

Long path of errors corresponds to error on logical qubit
Is 2D Kitaev code self-correcting against thermal noise?

Prob (long path) = # pairs \times \text{Prob(a pair makes long path)}

N – average number of pairs
p – probability that a pair makes a long path

N \sim \text{volume}
p \sim \text{volume}^{-1}

\Rightarrow \text{Kitaev 2D model is not self-correcting for } T>0
Rigorous proof of instability of Kitaev 2D model

\[ H_{\text{int}} = \sum_\alpha \sigma_\alpha^x \otimes f_\alpha^x + \sum_\alpha \sigma_\alpha^y \otimes f_\alpha^y + \sum_\alpha \sigma_\alpha^z \otimes f_\alpha^z \]

\( \alpha \) - runs over qubits, \( f_\alpha \) - operators of environment

- Use weak coupling approximation leading to Davies generator:

\[ \frac{dA}{dt} = i [H, A] - LA \]

- Terms \([H, .] \) and \( L \) commute \( \Rightarrow \) it is enough to consider:

\[ \frac{dA}{dt} = -LA \quad \Rightarrow \quad A(t) = e^{-Lt} A \]

- \( L \) is of the form

\[ L = \sum_{\alpha, k, \omega} L_{\alpha, k, \omega} \]

\( k = x, y, z \), \( \omega \) - Bohr freq. of Hamiltonian

Properties of \( L \)

- Hermitian in scalar product:

\[ \langle x_1 y_1 \rangle = \text{Tr}(\rho x_1^T y_1) \]

- Positive:

\[ L \geq 0 \]

- For our coupling, \( L \) has single "ground state":

\[ L(I) = 0 \]

- \( L \) is frustration free:

\[ \sum_\omega L_{\alpha, k, \omega}^*(I) = 0 \]

Thermal instability is related to gap of L:

\[ \langle X(t)X \rangle_\beta \leq e^{-\text{gap}(L)t} \]

for any $X$ s.t. $\langle X, I \rangle_\beta = 0$, $\langle X, X \rangle_\beta = 1$

**Theorem**: Spectral gap of dissipative generator $L$ for Kitaev 2D model satisfies

\[ \text{gap}(L) \geq \frac{1}{3} e^{-8\beta} \]

**Proof**: Boring, technical. Using techniques for estimating gaps, but also explicit calculations of eigenvalues.
Thermal stability of Kitaev 4D model

In 2D model:
- defects are particles (point-like)
- error form paths
- spreading of errors does not cause increase of energy

⇒ thermal fluctuations easily produce logical error
   (non-trivial loop)

In 4D model,
- errors are situated on surfaces,
- defects are strings (boundaries of error surfaces)
- energy is proportional to length of the strings

⇒ One can try follow Peierls argument for 2D Ising model
   and it should give thermal stability

Problem:
- find topological analogue of magnetization,
- prove that it is stable.

Thermal stability of Kitaev 4D model

**Observables**: „dressed” or „error corrected” observables $X$ and $Z$ mentioned before

**Useful inequality:**

$$
\epsilon_Z \equiv \langle Z \mid L \mid Z \rangle_{\beta} \leq \text{const} \sum_{\alpha} (1 - \langle Z, \sigma^z_{\alpha} Z \sigma^z_{\alpha} \rangle_{\beta})
$$

**Fidelity of logical qubit:**

$$
F \geq \frac{1}{2} \left( e^{-\epsilon_Z t} + e^{-\epsilon_X t} \right)
$$

**Main result**: for 4D Kitaev model we have

$$
\langle Z \mid L \mid Z \rangle_{\beta} \leq \text{poly}(l) \ e^{-\frac{1}{8} l (\beta - \beta_c)}
$$

- $l$ - linear size of torus
- $\beta_c$ - depends only on dimension $d$ and type of lattice

[Alicki, Horodeckis, OSID 2010]
Independent/further results on instability of stabilizer models

1) No-go results for 2D stabilizer codes
   - Kay

2) Some general estimates:
   [Chesi, Loss, Bravyi, Terhal, New J. Phys. 12, 025013 (2010)]

3) Analysis of topological models, which allow for universal computing
Summary of PART I

Problems:

• The famous threshold result for fault-tolerant QC assumes phenomenological, independent noise. Hamiltonian description, the noise is not iid anymore.

2) Generalizations of threshold result to Hamiltonian dynamics exist.
   - not quite satisfactory (assume small norm of Hamiltonian of interaction with the bath)

Self-correcting models: more physical, Hamiltonian description more natural

1) 2D Kitaev model thermally **instable** (though by logarithmic scaling temperature, one can prolong time of protection)

2) 4D Kitaev model is thermally **stable** (though in 3D requires non-local interaction)

3) There are more general results for stabilizer codes.
PART II

Quantum communication and cryptography over EPR networks
Quantum communication over EPR networks: basic scheme

[Acin, Cirac, Lewenstein Nat. Phys. 2006]

EPR network:
- nodes: EPR pairs
- vertices: labs

Alice and Bob want to share EPR pair

**Condition:** constant number of operations in each vertex

**Noiseless case:**
- perform Bell measurement on a chosen path.
- send the results to Bob
- Bob applies a unitary depending on some function of the outcomes

\[ \phi_+ = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]
Problem:
- Can Alice and Bob share an EPR pair over a noisy network?

Equivalently:
- Can Alice send to Bob an unknown state?
- Can Alice and Bob perform BB84 protocol?
  i.e. send each of the four states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$


YES: there exists a protocol over 2D EPR network, which allows quantum communication. It uses scheme for quantum fault-tolerant computing.

We like this solution, but think it is incomplete.
- consider fault tolerant quantum computing (FT QC) on line with local gates
- translate it into teleportation protocol in two-dimensional array

Two teleportations transport quantum computer between time $t=0$ and $t=1$

Two qubit gates can be applied to neighboring qubits

- since FT QC allows to transmit qubit in time, the network version will transmit qubit in space
- combining QC forward in time, and backward in time, will allow to share e-bit in space
Fault tolerant threshold theorem says that we can have fidelity arbitrarily close to 1.

However, this is the fidelity of an apriori known state.

How to deal with unknown state?

**Computation on line:** to our knowledge there is no explicit estimate in literature, what is the fidelity of storing a qubit in unknown state.

**Proposition:** Suppose that error rate $p_0$ satisfies

$$p_0 \leq \frac{1}{3} p_{th}$$

Then it is possible to maintain a qubit in unknown state with fidelity

$$F \geq F_0 (1 - O(2^{-N})) \quad \text{where} \quad F_0 \geq e^{-p_0 v}$$

N - number of qubits,
$v$ – volume of physical encoding circuit (a **constant**).
3D networks with quantum communication

The 1D fault-tolerant quantum computing is extremely complicated. Much simpler scheme is one based on 2D Kitaev model.

Quantum communication in 3D = maintaining qubit in 2D.

How to protect a qubit in unknown state in 2D?

- **Encode** a qubit in an unknown state
- **Protect** the encoded state
- **Decode** the qubit

![Diagram showing space-time encoding, protection, and decoding with loss of fidelity and arbitrary small loss](image)
**Protecting encoded state:**
- measure repeatedly syndrom (i.e. the star and plaquette operators)
- collect outcomes

**Encoding/Decoding a qubit:**
- Dennis et al. provided a procedure of enlarging/diminishing code

**Problems:**
- encoding is in a form of circuit
- decoding is just a reverse of the circuit

**Our goal:** provide
- Unify the encoding stage with protecting stage
  ⇒ encoding by measuring star $X_s$ and plaquette $Z_p$ operators
- Provide single shot decoding.

Grudka et al. arXiv/???? .????
Planar Kitaev code [Dennis et al. 2001]

Torus is not so practical. Here is planar version of toric code encodes one logical qubit.

- qubits are situated on edges
- code is given by all $X_s$ and $Z_p$ equal to +1
  
  \[ s \text{ – star, } p \text{ – plaquette} \]

\[ | = \sigma_x \quad | = \sigma_z \]

Codewords:

\[ |0\rangle_L = \sum_\text{trivial config.} |1\text{ bit configurations}\rangle \]

\[ |1\rangle_L = \sum_\text{nontrivial config.} |1\text{ bit configurations}\rangle \]

Here:

\[ \begin{array}{ll}
10 & \quad |0\rangle \\
11 & \quad |1\rangle
\end{array} \]
Encoding known state in absence of noise

- Prepare every qubit in $|0\rangle$ state
- Measure all $X_s$
- Anihilate defects by joining them with **phase-flips** in arbitrary way

- Prepare every qubit in $|+\rangle$ state
- Measure all $Z_s$
- Anihilate defects by joining them with **bit-flips** in arbitrary way
Encoding unknown state in absence of noise

\[ |\psi\rangle = a|0\rangle + b|1\rangle \]

- **Prepare:**
  - \( |0\rangle \)
  - \( |+\rangle \)
  - \( a|0\rangle + b|1\rangle \)

- **Measure:**
  - All \( Z_p \) touching green qubits
  - All \( X_s \) touching red qubits

- **Anihilate defects:**
  - \( Z \)-type defects move right
  - \( X \)-type defects move up
Encoding unknown state as teleportation

- Prepare a code without our qubit we want to encode
- Attach the qubit and measure the only left $X_s$ and $Z_p$:

\[ X_s = X_{ph} \otimes X_l \]
\[ Z_p = Z_{ph} \otimes Z_l \]

Bell measurement on $H_{ph} \otimes H_l$
**Summary of Part II**

- **Important problem:** communicate quantum information over EPR networks with constant complexity for each node

- **Previous result:** Fault tolerant quantum computing in dimension $d \Rightarrow$ quantum communication over EPR networks of dimension $d+1$

- **Problem:** One element was lacking – encoding unknown state

**Our results:**
- we have provided explicit estimate for fidelity of encoding a qubit within standard FT scheme
- we proposed an encoding scheme for Kitaev code, basing on measuring star and plaquette operators in ideal case

**Work in progress:**
- currently we are making simulations of our encoding for noisy case