Quantum Dissension: Generalization of Discord for Three Qubits

Pankaj Agrawal
Institute of Physics
Bhubaneswar

February 19, 2011

Outline

Introduction

- For a long time, it was believed that quantum mechanical framework was well understood except some philosophical issues which may or may not be taken seriously.
- About twenty years ago, it began to change. To use quantum systems to carry out information processing tasks including computing and communication optimally, one had to improve the understanding of the nature and quantification of the quantum correlations.
- We understand how to quantify the correlations of a bipartite system in a pure state. However, there are open issues in understanding the correlations of any mixed state, or a multipartite state.
- For a two-qubit system, Quantum Discord was introduced by Ollivier and Zurek to quantify the correlations.

Introduction

- We have generalized this concept to a multipartite sytem, in particular to a tripartite system. This generalization is based on three-variable mutual information.
- Classically, mutual information quantifies the correlation of three random variables. So, it is quite natural to generalize it to quantum domain to understand the correlations.
- This work has been done in collaboration with Indranil Chakrabarty and Arun Pati.

Outline

Introduction

Quantum Discord

Quantum Dissension

Conclusions

Mutual Information

Discord was introduced by Olivier and Zurek (2002) as a measure of the “ quantumness of correlations”. It is defined in terms of mutual information which is a measure of amount of information one random variable possesses about another. Classically, one can write mutual information in two alternate ways,

$$\begin{aligned} I(X:Y) &= H(X) - H(X|Y), \\ J(X:Y) &= H(X) + H(Y) - H(X,Y). \end{aligned}$$

Here $H(X), H(X,Y)$ and $H(X|Y)$ are the entropy, joint entropy, and conditional entropy for the random variables X and Y. The Joint entropy and conditional entropy are related by the chain rule,

$$H(X|Y) = H(X,Y) - H(Y).$$

Quantum Domain

These expressions for the entropies can be generalized to the quantum domain by substituting random variables by density matrices and Shannon entropies by von Neumann entropies. For example,

$$H(X) = H(\rho_X) = -\mathrm{Tr}[\rho_X \log(\rho_X)].$$

One has to be careful in generalizing conditional entropy to quantum domain. With a specific generalization, the mutual information can be written as,

$$I(X:Y) = H(X) - H(X|\{\pi_i^Y\}).$$

$H(X|\{\pi_j^Y\}) = \sum_j p_j H(\rho_{X|\pi_j^Y})$, $\rho_{X|\pi_j^Y} = \frac{\pi_j^Y \rho_{XY} \pi_j^Y}{Tr(\pi_j^Y \rho_{XY})}$ (where p_j is the probability of obtaining the j th outcome). Here, $H(X|\{\pi_j^Y\})$ is the von Neumann entropy of the qubit X, when the projective measurement is done on Y.

Definition of Quantum Discord

The quantum discord is then defined as,

$$D(X:Y) = J - I = H(Y) - H(X,Y) + H(X|\{\pi_j^Y\}).$$

This is to be minimized over all sets of one dimensional projectors $\{\pi_i^Y\}$.

- It was found that the Werner state has non-zero discord in the domain of the mixing parameter where the entanglement was known to vanish.
- For a pure bipartite state, it reduces to the measure of von Neumann entropy.

Outline

Quantum Dissension

We are generalizing the discord to a multipartite system. To do so we consider three-variable mutual information and take it to quantum domain. For a multipartite system, we can make not only one-particle measurement, but also multiparticle measurements. Different types of measurements may probe different aspects of correlations. So, the correlations will be characterized by multiple numbers. These multiple measurements can be used to define quantities which are being called “Quantum Dissension”.

In particular, for a three-qubit system, one can consider one-particle and two-particle measurements. We have chosen to adopt two different definitions - in one case we only make one-particle measurements while in the second case we only consider two-particle measurements.

Three-Variable Mutual Information

Three-variable mutual information is defined as,

$$I(X:Y:Z) = I(X:Y) - I(X,Y|Z).$$

Here $I(X,Y|Z)$ is the conditional mutual information,

$$I(X,Y|Z) = H(X|Z) + H(Y|Z) - H(X,Y|Z).$$

Both $I(X:Y)$ and $I(X,Y|Z)$ are non-negative. However, there may exist a situation, when the conditional mutual information is greater than the mutual information. It happens when knowing the variable Z enhances the correlation between X and Y. In such a case, the three variable mutual information is negative. This happens quite generally, as we shall see in the case of GHZ and W-states.

Classically Equivalent Expressions

First expression can be obtained which has all possible conditional entropies with respect to one variable only. Its generalization to quantum domain will involve only one-particle measurement. This is,

$$\begin{aligned} I(X:Y:Z) = H(X,Y) - H(Y|X) - H(X|Y) \\ -H(X|Z) - H(Y|Z) + H(X,Y|Z). \end{aligned}$$

One can convert the above expression that involves conditional entropies to that contains only entropies and joint entropies. We obtain,

$$\begin{aligned} J(X:Y:Z) = [H(X) + H(Y) + H(Z)] - [H(X,Y) + H(X,Z) \\ +H(Y,Z)] + H(X,Y,Z). \end{aligned}$$

Classically Equivalent Expressions

Using the chain rule $H(X,Y,Z) = H(Y,Z) + H(X|Y,Z)$, we can define three variable mutual information involving two variable conditional entropies. This gives another equivalent expression,

$$\begin{aligned} K(X:Y:Z) &= [H(X) + H(Y) + H(Z)] \\ &\quad -[H(X,Y) + H(X,Z)] + H(X|Y,Z). \end{aligned}$$

All these three expressions for the three variable mutual information are classically equivalent, but not so in quantum domain. The difference of the three definitions can capture various aspects of the quantum correlations.

One-Particle Measurement

Let us consider a three qubit state ρ_{XYZ}, where X, Y, Z refer to the first, second and the third qubit. The extension of the definition of $J(X:Y:Z)$ is obtained by replacing the random variables by the density matrices and the Shannon entropies by the Von Neumann entropies. In the quantum case, the expression for $I(X:Y:Z)$ is given by,

$$\begin{aligned} I(X:Y:Z) = & H(X,Y) - H(Y|\{\pi_j^X\}) - H(X|\{\pi_j^Y\}) \\ & - H(X|\{\pi_j^Z\}) - H(Y|\{\pi_j^Z\}) + H(X,Y|\{\pi_j^Z\}) \end{aligned}$$

$H(X|\{\pi_j^Y\}$ has been defined earlier. Similarly, one can write down the equivalent expression for $H(X|\{\pi_j^Z\}), H(Y|\{\pi_j^Z\})$, and $H(Y|\{\pi_j^X\})$.

One-Particle Measurement

$H(X,Y|\{\pi_j^Z\}) = \sum_j p_j H(\rho_{X,Y|\pi_j^Z})$,$\rho_{X,Y|\pi_j^Z} = \frac{\pi_j^Y \rho_{XYZ} \pi_j^Z}{Tr(\pi_j^Y \rho_{XYZ})}$ is the Von Neumann entropy of the subsystem ρ_{XY}, when the projective measurement is carried out on the qubit Z.

The dissension for one-particle measurement can be defined as the difference of I(X:Y:Z) and J(X:Y:Z),

$$\begin{aligned} D_1(X:Y:Z) = I(X:Y:Z) - J(X:Y:Z) \\ = H(X,Y|\{\pi_j^Z\}) + [H(X,Z) + H(Y,Z) \\ +2H(X,Y)] - H(X,Y,Z) - [H(X|\{\pi_j^Y\}) \\ +H(X|\{\pi_j^Z\}) + H(Y|\{\pi_j^Z\}) + H(Y|\{\pi_j^X\})] \\ -[H(X) + H(Y) + H(Z)]. \end{aligned}$$

One-Particle Measurement

The projective measurement is done on the subsystem ρ_Y in the general basis
$\{|u_1\rangle = \cos(t)|0\rangle + \sin(t)|1\rangle, |u_2\rangle = \sin(t)|0\rangle - \cos(t)|1\rangle\}$
(where $t \in [0, 2\pi]$).
One minimizes this over all possible one-particle measurement projectors. So mathematically the dissension is given by,
$\delta_1 = min(D_1(X : Y : Z))$. For single-particle measurements there can be a number of classically equivalent expressions for $I(X : Y : Z)$, but the above expression is the most general one in the sense that it includes all possible one-particle measurements. As a consequence, the dissension δ_1, may reveal the maximum possible quantum correlations.

One-Particle Measurement

We note the following,

- The dissension is not symmetric with respect to the permutations of the subsystems X, Y and Z, as in the case of discord.
- For an arbitrary pure three-qubit state $J(X:Y:Z) = 0$. Therefore, $D_1 = I(X:Y:Z)$
- For an arbitrary pure three-qubit state, $H(X,Y|Z) = H(X,Z|Y) = H(Y,Z|X) = 0$.
- Dissension and discord are related,

$$\begin{aligned} D_1(X:Y:Z) &= D(X,Y:Z) - D(X:Z) - D(Y:Z) - \\ &\quad D(X:Y) - D(Y:X). \end{aligned}$$

Two-Particle Measurement

The quantum analogue of the classical mutual information $K(X:Y:Z)$ is,

$$K(X:Y:Z) = [H(X)+H(Y)+H(Z)] \\ -[H(X,Y)+H(X,Z)]+H(X|\{\pi_j^{Y,Z}\}).$$

where $H(X|\{\pi_j^{YZ}\}) = \sum_j p_j H(\rho_{X|\pi_j^{YZ}})$, $\rho_{X|\pi_j^{YZ}} = \frac{\pi_j^{YZ}\rho_{XYZ}\pi_j^{YZ}}{Tr(\pi_j^{YZ}\rho_{XYZ})}$. The projective measurement is carried out on the subsystem ρ_{YZ} in the general basis $\{|v_1\rangle = \cos(t)|00\rangle + \sin(t)|11\rangle, |v_2\rangle = -\sin(t)|00\rangle + \cos(t)|11\rangle, |v_3\rangle = \cos(t)|01\rangle + \sin(t)|10\rangle, |v_4\rangle = -\sin(t)|01\rangle + \cos(t)|10\rangle\}$ (where p_j is the probability of obtaining the j th outcome.). Here, $H(X), H(Y), H(Z), H(X,Y), H(X,Z)$ represents the Von Neumann entropies of the subsystems.

Two-Particle Measurement

To define the dissension, we take

$$D_2(X, Y, Z) = K(X, Y, Z) - J(X, Y, Z)$$
$$= H(X|\{\pi_j^{YZ}\}) + H(Y, Z) - H(X, Y, Z).$$

Like one-particle projective measurement case, we define dissension as, $\delta_2 = min(D_2(X : Y : Z))$. Furthermore, as in the case of discord, this quantity is not symmetric under the permutations of X, Y and Z. We note that in this case of three qubits, $D_2(X, Y, Z)$ is nothing but discord with for the split of the system in X and subsystem 'YZ'.

Two-Particle Measurement

We note the following,

- For an arbitrary pure three-qubit system, $H(X|Y,Z) = H(Y|X,Z) = H(Z|X,Y) = 0$. Therefore, $D_2 = H(X)$ and the dissension is given by the Von Neumann entropy of the bipartite partition.
- The relation between the dissension and discord is $D_2(X:Y:Z) = D(X:Y,Z)$.
- These simple relations exist as we are considering only three-qubit systems. If we go beyond three qubits, then D_2 would not probe bipartite partition only.

GHZ-state

Let us first of all consider pure three-qubit GHZ state

$$\rho_{ABC}^{GHZ} = \frac{1}{2}\{|000\rangle\langle000| + |000\rangle\langle111| + |111\rangle\langle000| + |111\rangle\langle111|\}$$

For this state,
$H(A) = H(B) = H(C) = H(AB) = H(BC) = H(CA) = 1$ and $H(ABC) = 0$. For conditional entropies, we find,
$H(AB|\{\pi_j^C\}) = 0$ and

$$H(A|\{\pi_j^B\}) = H(A|\{\pi_j^C\}) = H(B|\{\pi_j^C\}) = H(B|\{\pi_j^A\}) = (-\frac{1-\cos(2t)}{2})\log_2\frac{1-\cos(2t)}{2} -(\frac{1+\cos(2t)}{2})\log_2\frac{1+\cos(2t)}{2}$$

GHZ State

The expression for D1 is,

$$D_1 = 1 + 4[\frac{1-\cos(2t)}{2}\log_2\frac{1-\cos(2t)}{2} + \frac{1+\cos(2t)}{2}\log_2\frac{1+\cos(2t)}{2}]$$

In the Figure 1 (i) D_1 is plotted as a function of t. It is a oscillating function which varies between $[-3, 1]$. The dissension $\delta_1 = -3$. The calculations here and below were performed using the mathematica package QDENSITY. For GHZ state, in case of two-particle measurement, the conditional entropy is zero and the dissension reduces to the bipartite entanglement present in the system and is equal to one, i.e. $\delta_2 = 1$.

W-State

We now consider W-state,

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}\{|100\rangle + |010\rangle + |001\rangle\}$$

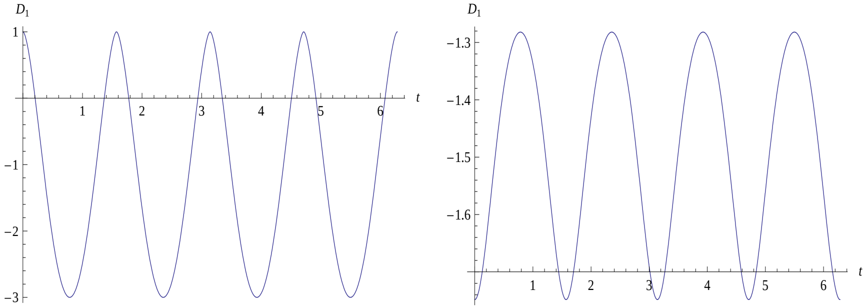
For this state,
$H(A) = H(B) = H(C) = H(AB) = H(BC) = H(CA) = 0.92$ and $H(ABC) = 0$. Also conditional entropy $H(AB|\{\pi_j^C\}) = 0$.
D_1 is given by,

$$D_1 = H(AB) - 4H(A|\{\pi_j^B\})$$

The expression D_1 for the W state is plotted in Figure 1 (ii).
The dissension δ_1 is -1.74.
For the two-particle projective measurements, the conditional entropy is zero and the dissension δ_2 is equal to .92.

Plots for Pure GHZ- and W-states

1



(i) (ii)

Figure: D_1 for (i) three-qubit pure GHZ state and (ii) three-qubit pure W state .

Mixed GHZ-state

Let us consider a three qubit mixed GHZ state,

$$\rho_{GHZ} = (1-a)\frac{I}{8} + a|GHZ\rangle\langle GHZ|$$

The reduced density matrices are,

$$\rho_A = \rho_B = \rho_C = \frac{1}{2}\{|0\rangle\langle 0| + |1\rangle\langle 1|\} = \frac{I}{2}.$$

$$\begin{aligned}\rho_{AB} = \rho_{BC} = \rho_{CA} \quad &= \quad \frac{1+a}{4}[|00\rangle\langle 00| + |11\rangle\langle 11|] \\ &+ \frac{1-a}{4}[|01\rangle\langle 01| + |10\rangle\langle 10|]\end{aligned}$$

D_1 and D_2 are plotted in Figures 2 $(i), (ii)$. Note that for $a = 1$, we get back the D_1 of the GHZ state. The dissensions δ_1 and δ_2 are non zero for any non-zero values of a. This is like the two-qubit Werner state. We also notice that D_2 is independent of t and reduces to that of the pure GHZ state for $a = 1$.

Mixed W-State

The mixed three-qubit W state we consider is,

$$\rho_W = (1-a)\frac{I}{8} + a|W\rangle\langle W|$$

The reduced density matrices are,

$$\rho_A = \rho_B = \rho_C = \frac{3+a}{6}|0\rangle\langle 0| + \frac{3-a}{6}|1\rangle\langle 1|$$

$$\begin{aligned}\rho_{AB} = \rho_{BC} = \rho_{CA} \quad &= \quad [\frac{3+a}{12}][|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|] \\ &+[\frac{1-a}{4}]|11\rangle\langle 11| + \frac{a}{3}[|01\rangle\langle 10| + |10\rangle\langle 01|]\end{aligned}$$

In the Figures 3 (i) and (ii), we show D_1 and D_2 as a function of the classical probability of mixing a as well as the angle t. Figure 3 (i) for $a = 1$, gives Figure 1 (ii). Furthermore, D_2 is independent of t and reduces to that of the pure W-state for $a = 1$.

Plots for Mixed GHZ-state

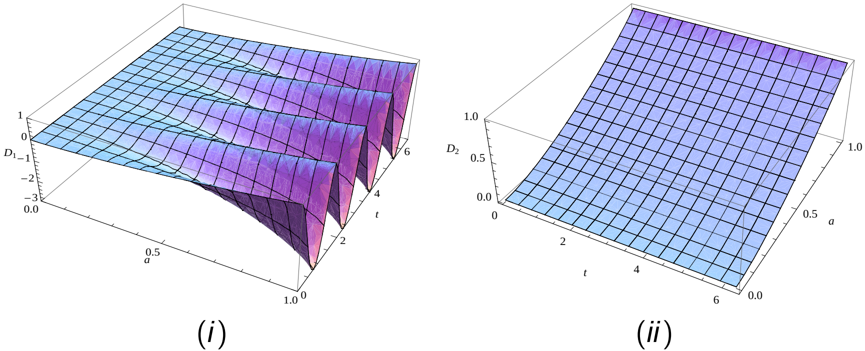


Figure: D_1 and D_2 for three-qubit Mixed GHZ state (figures (i), (ii))

Plots for Mixed W-state

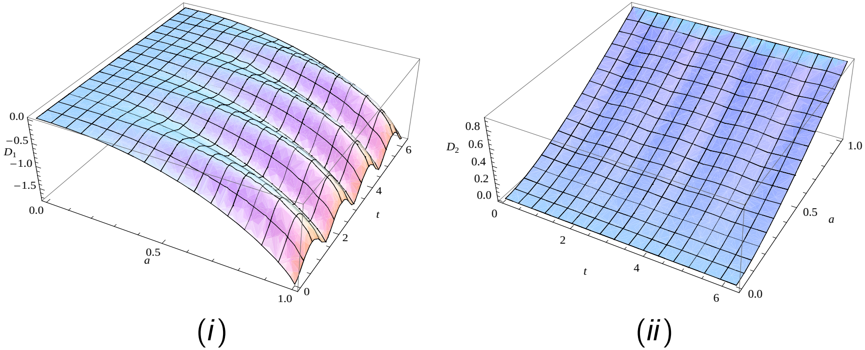


Figure: D_1 and D_2 for three-qubit Mixed W state (figures (i), (ii)).

Outline

Conclusions

- We have generalized discord to a tripartite system using three-variable mutual information in quantum domain.
- We have introduced dissensions δ_1 and δ_2 based on one-particle and two-particle measurements.
- δ_1 can be negative. It reflects the fact that a measurement on a subsystem can enhance the correlations of the rest of the system.
- Dissension is non-zero for all non-zero values of the classical mixing parameter for the mixed GHZ- and W-states.
- In future, one may like to relate dissension to the success of various quantum information processing tasks.