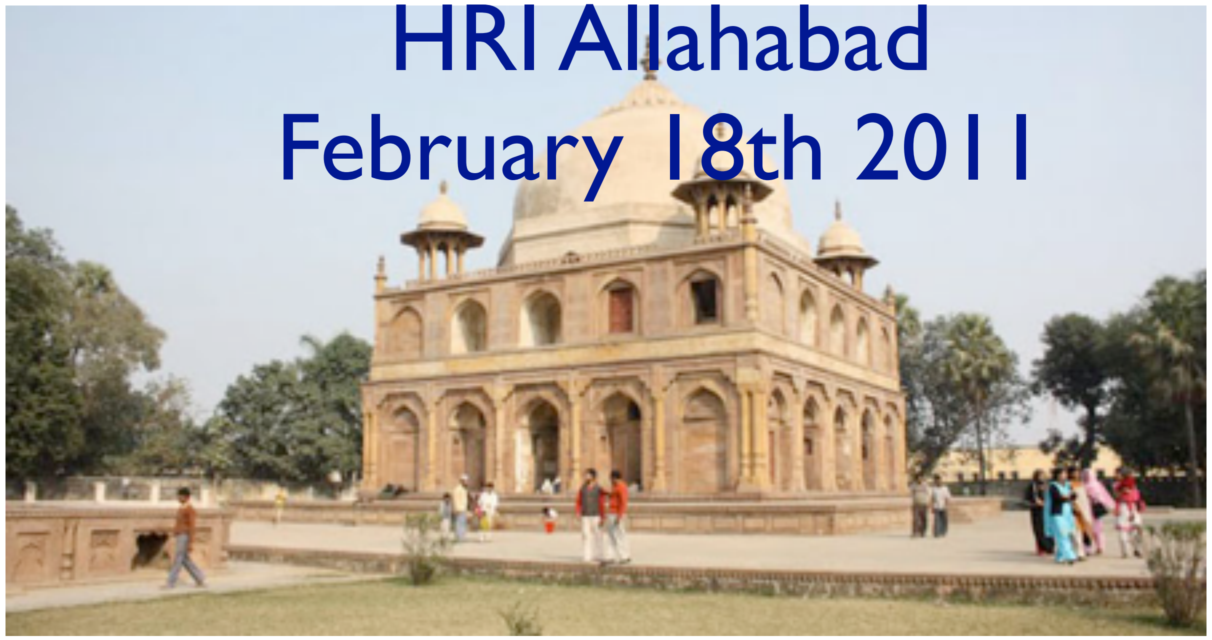
Programmable multi-copy discrimination

HRI Allahabad
February 18th 2011



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Outline

- Introduction
- Pure states
- Mixed states
- MAD protocols
- Conclusions

Introduction

Introduction

Measurements

- Quantum ☛ Probabilistic
- Non-orthogonal states are not perfectly distinguishable

From the outcome take a decision

Unambiguous discrimination

☛Never wrong, sometimes inconclusive

Minimum error

☛Never inconclusive, sometimes wrong

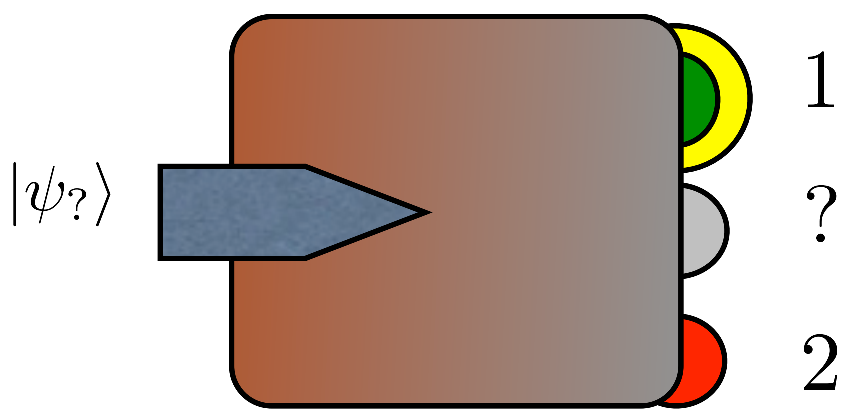
☛ Goal

Maximize the success probability
➜Minimize inconclusive probability
➜or the error probability

Known states

$$\pi_1 = \pi_2 = 1/2$$

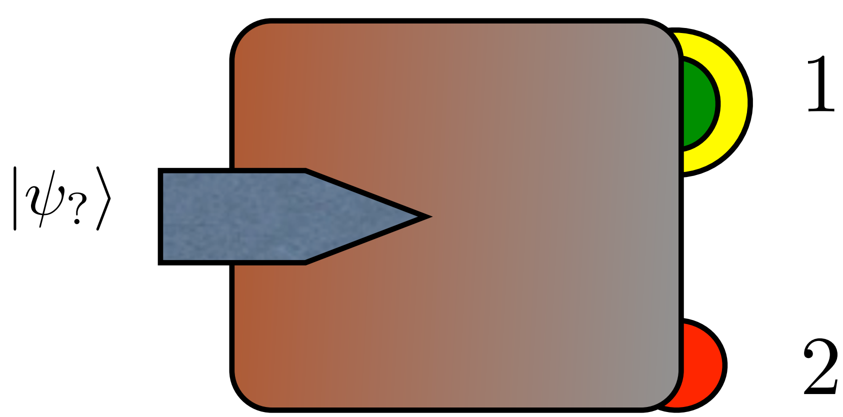
$$E_\alpha \geq 0$$



$$E_1 + E_2 + E_? = \mathbb{I}$$

$$P^? = \min_E \frac{1}{2}\left(\langle\psi_1|\, E_? \,|\psi_1\rangle + \langle\psi_2|\, E_? \,|\psi_2\rangle\right)$$

$$\boxed{P^? = |\langle\psi_1|\,\psi_2\rangle|}$$



$$E_1 + E_2 = \mathbb{I}$$

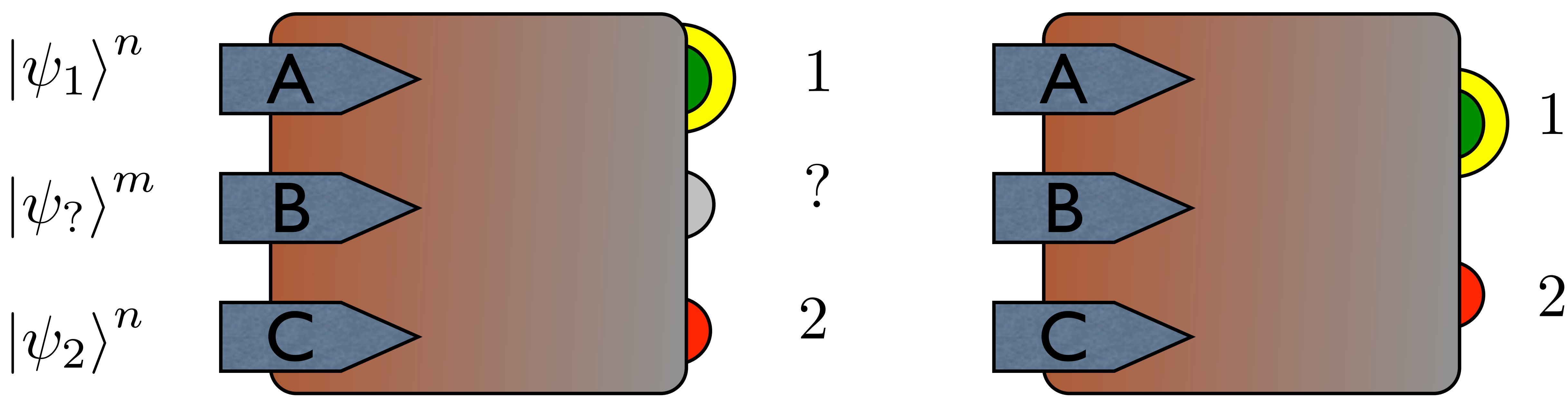
$$P^{\mathrm{ME}} = \min_E \frac{1}{2}\left(\langle\psi_1|\, E_2 \,|\psi_1\rangle + \langle\psi_2|\, E_1 \,|\psi_2\rangle\right)$$

$$P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}\right)$$

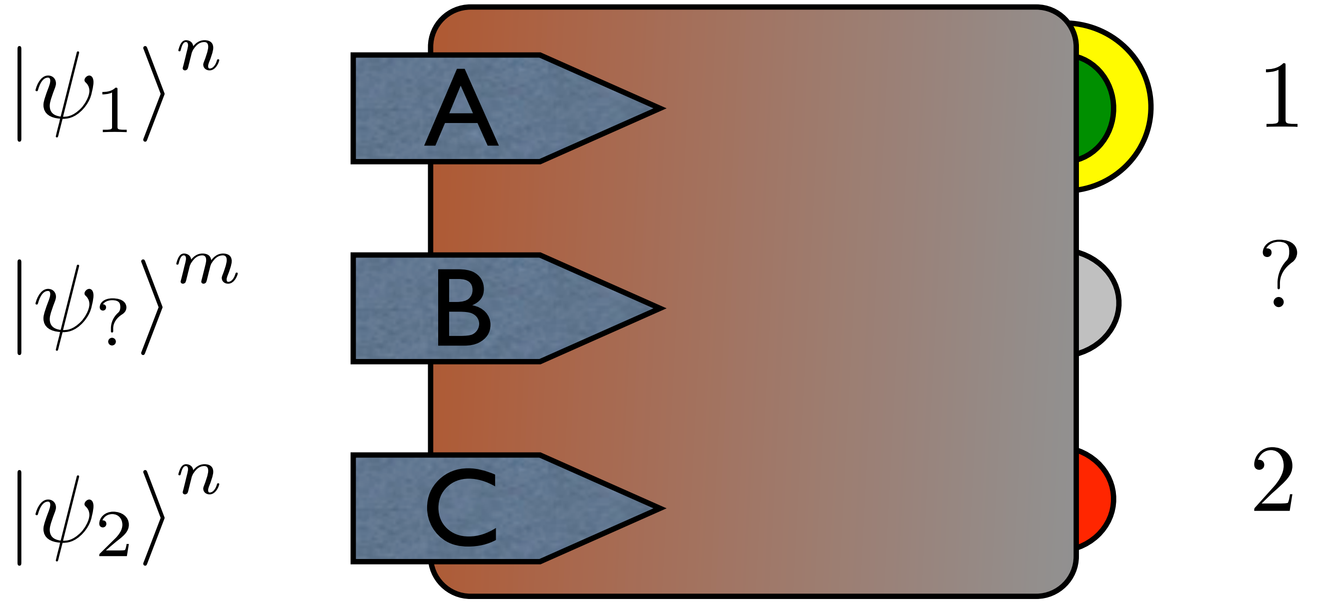
$$\boxed{P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \frac{1}{2}\|\rho_1 - \rho_2\|\right)}$$

Programmable discrimination

- The states to be discriminated given in a quantum way
  - ☛ Analogically
- The protocol must work for any two given states
  - ☛ Programmable



Programmable discrimination



Dušek-Bužek, PRA 66 022112 (2002)
Bergou-Hillery, PRL 94 160501 (2005)

$[\phi] \equiv |\phi\rangle\langle\phi|$

$$\sigma_1 = \int d\psi_1 d\psi_2 [\psi_1^{\otimes n}]_A [\psi_1^{\otimes m}]_B [\psi_2^{\otimes n}]_C$$

$$\sigma_2 = \int d\psi_1 d\psi_2 [\psi_1^{\otimes n}]_A [\psi_2^{\otimes m}]_B [\psi_2^{\otimes n}]_C \,,$$

Schur Lemma

$$\int d\phi [\phi]^{\otimes n} = \frac{1}{d_n} \mathbb{I}_n$$

$\mathbb{I}_n$: projector on the fully symmetric subspace

$d_A = d_C = n + 1$

$d_{AB} = n + m + 1 = d_{BC}$

$$\sigma_1 = \frac{1}{d_{AB} d_C} \mathbb{I}_{AB} \otimes \mathbb{I}_C$$

$$\sigma_2 = \frac{1}{d_A d_{BC}} \mathbb{I}_A \otimes \mathbb{I}_{BC}$$

•Angular momentum basis

$$\sigma_1 = \frac{1}{d_{AB} d_C} \sum_{JM} |j_A, j_B, j_C, j_{AB}; JM\rangle\langle j_A, j_B, j_C, j_{AB}; JM|$$

$$\sigma_2 = \frac{1}{d_A d_{BC}} \sum_{JM} |j_A, j_B, j_C, j_{BC}; JM\rangle\langle j_A, j_B, j_C, j_{BC}; JM|$$

Qubits

$$j_A, = j_C = n/2$$

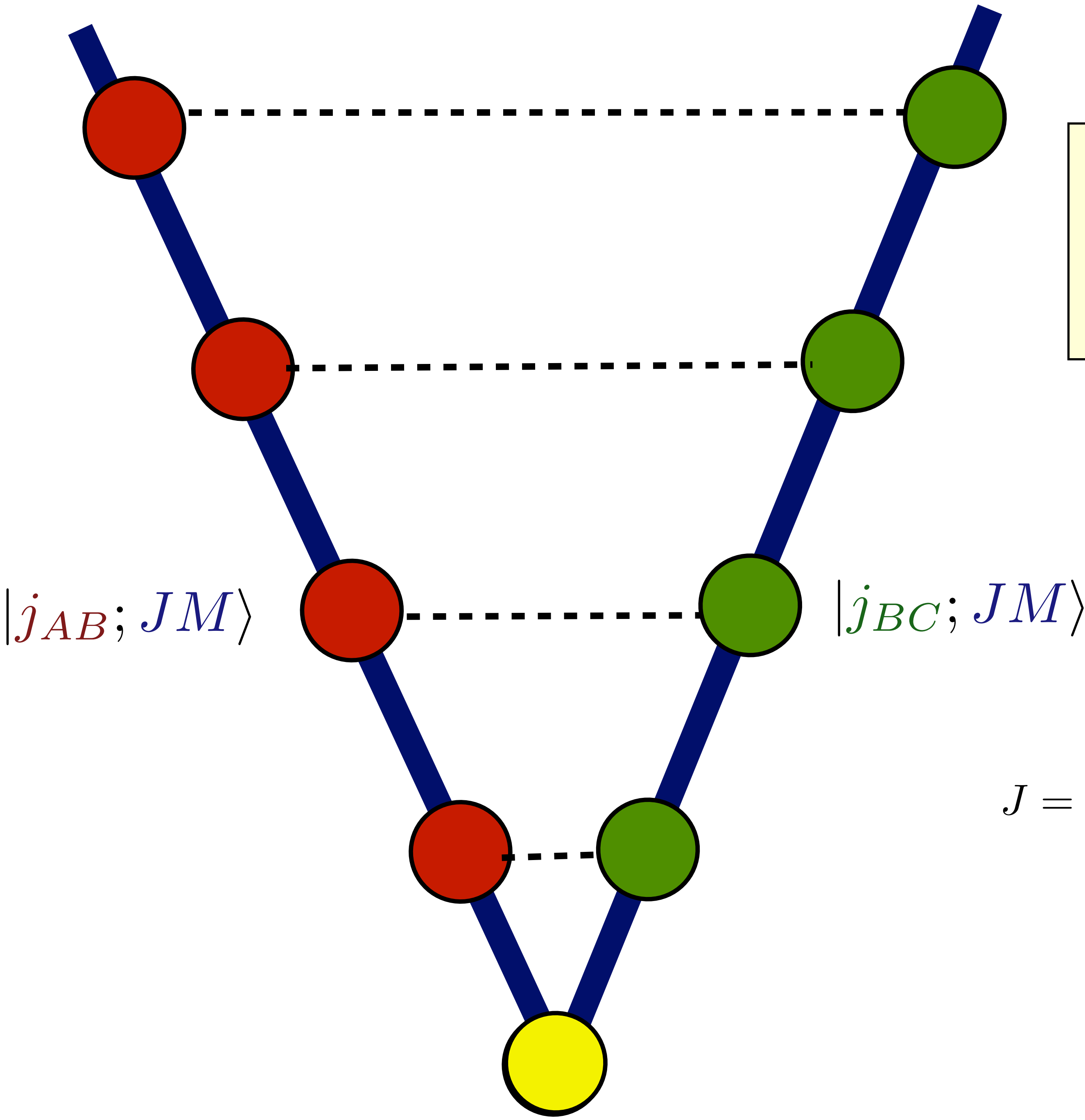
$$j_B = m/2$$

$$j_{AB} = j_{BC} = \frac{n+m}{2}$$

Jordan basis

$$\langle j_{AB}; JM | j_{BC}; J'M' \rangle = 0 \qquad \forall J \neq J' \text{or} M \neq M'$$

•Overlaps



$$\langle j_{AB}; JM | j_{BC}; JM \rangle \propto \begin{Bmatrix} j_A & j_B & j_{AB} \\ j_C & J & j_{BC} \end{Bmatrix}$$

Independent of *M*

$$J = \frac{m}{2} + k$$

$$\langle j_{AB}; J | j_{BC}; J \rangle = \frac{\binom{n}{k}}{\binom{n+m}{n-k}}$$

Pure states results

•Unambiguous discrimination

$$P^{?} = \sum_{k=0}^{n} \frac{m+2k+1}{(n+m+1)(n+1)} \frac{(m+k)!n!}{(m+n)!k!}$$

$$J = \frac{m}{2} + k$$

$$P^{?} = 1 - \frac{nm}{(n+1)(m+2)}$$

$$n = m = 1 \rightarrow P^{?} = \frac{5}{6}$$

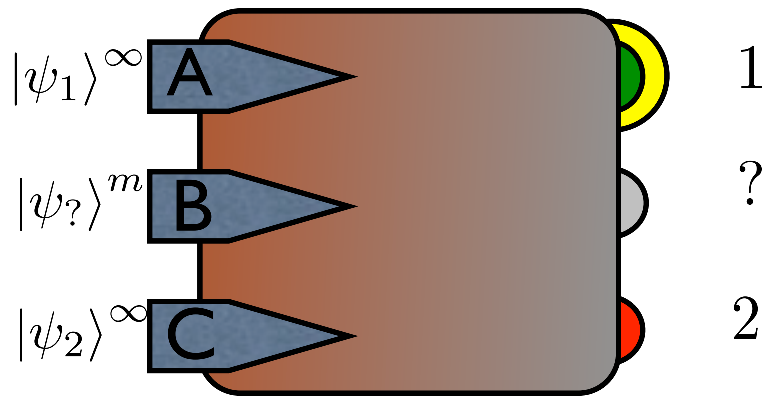
•Minimum Error

$$P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \sum_{k=0}^{n} \frac{m+2k+1}{(n+1)(n+m+1)} \sqrt{1 - \left(\frac{(m+k)!n!}{(m+n)!k!}\right)^{2}}\right)$$

$$n = m = 1 \rightarrow P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \frac{1}{2\sqrt{3}}\right) \simeq 0.356$$

Limiting cases

- $n \rightarrow \infty$, m finite



Equivalent to the average error probabilities of *known* states?

- Unambiguous $\lim_{n\to\infty} P^? = \frac{2}{m+2}$

$$m = 1 \rightarrow P^? = \frac{2}{3}$$

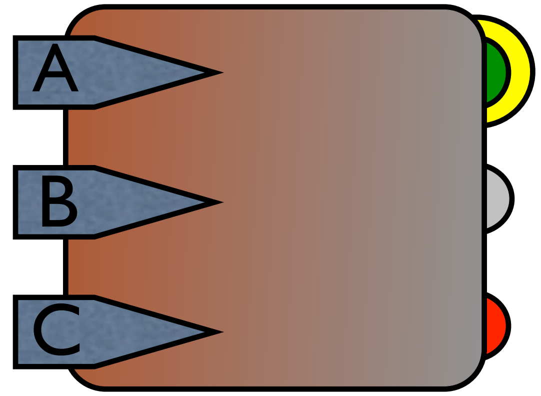
$$P^?(\psi_1, \psi_2) = |\langle\psi_1|\psi_2\rangle|^m \qquad \int d\psi_2\, |\langle\psi_1|\psi_2\rangle|^2)^{m/2} = \frac{1}{d_{m/2}} = \frac{2}{m+2}$$

- Minimum error $\lim_{n\to\infty} P^{\mathrm{ME}} = \frac{1}{2}\left[1 - 2\int_0^1 dx\, x\sqrt{1-x^{2m}}\right] = \frac{1}{2}\left[1 - \frac{\sqrt{\pi}}{2}\frac{\Gamma(1+1/m)}{\Gamma(3/2+1/m)}\right]$

$$P^{\mathrm{ME}}(\psi_1,\psi_2) = \frac{1-\sqrt{1-|\langle\psi_1|\psi_2\rangle|^{2m}}}{2} \qquad m = 1 \rightarrow P^{\mathrm{ME}} = \frac{1}{6}$$

$$\langle P^{\mathrm{ME}}(\psi_1,\psi_2)\rangle = \frac{1}{2}\left[1 - \frac{1}{2}\int_0^{\pi} d\theta \sin\theta \sqrt{1-\cos^{2m}(\theta/2)}\right]$$

• *n* finite, *m*→∞



$|\psi_1\rangle^n$

$|\psi_?\rangle^\infty$

$|\psi_2\rangle^n$

State comparison

•Unambiguous

$$\lim_{m\to\infty} P^? = \frac{1}{n+1}$$

$$\sigma_1 = \frac{1}{d_n}[\Psi^{\otimes n}] \otimes \mathbb{I}_n$$
$$\sigma_2 = \frac{1}{d_n}\mathbb{I}_n \otimes [\Psi^{\otimes n}],$$

$$\mathbb{I} = [\Psi^{\otimes n}] \oplus [\Psi^{\otimes n}]^\perp$$

$$P^? = \frac{1}{d_n} = \frac{1}{n+1}$$

•Minimum error

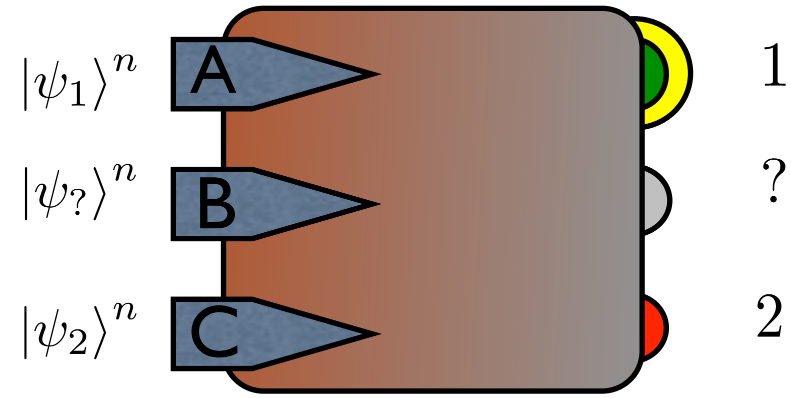
$$\lim_{m\to\infty} P^{\mathrm{M}E} = \frac{1}{2(n+1)}$$

$$P^{\mathrm{M}E} = \frac{1}{2}(1 - \frac{1}{2}\|\sigma_1 - \sigma_2]\|)$$

$$P^{\mathrm{M}E} = \frac{1}{2}\left(1 - \frac{1}{2(n+1)}\|[\Psi^{\otimes n}] \otimes [\Psi^{\otimes n}]^\perp - [\Psi^{\otimes n}]^\perp \otimes [\Psi^{\otimes n}]\|\right)$$

$$P^{\mathrm{M}E} = \frac{1}{2}\left(1 - \frac{2}{2(n+1)}\|[\Psi^{\otimes n}] \otimes [\Psi^{\otimes n}]^\perp\|\right) = \frac{1}{2}\left(1 - \frac{n}{n+1}\right)$$

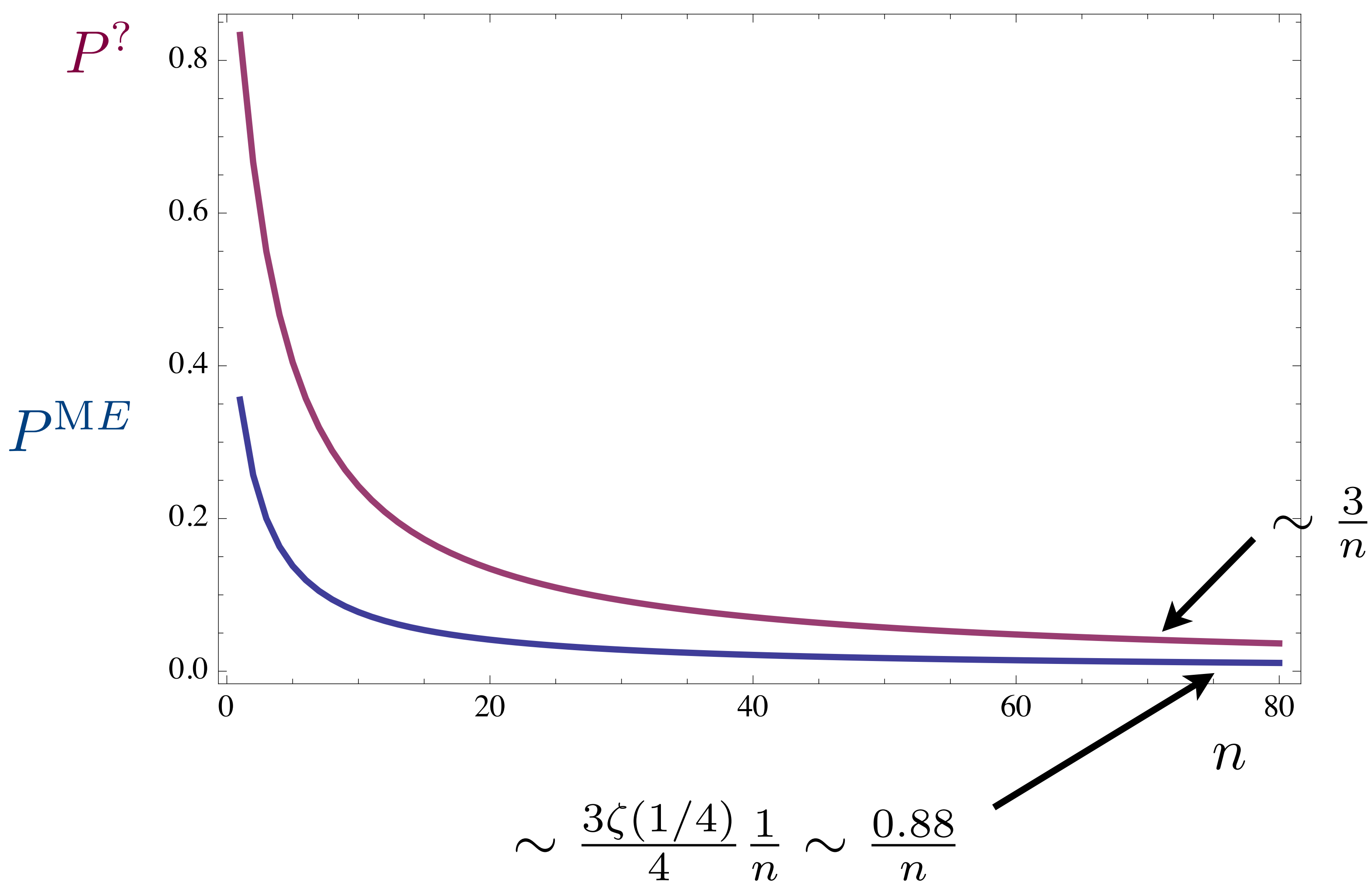
• $n=m \to \infty$



$$\zeta(x) = \sum_{k=0}^{\infty} \left(1 - \sqrt{1 - x^k}\right)$$

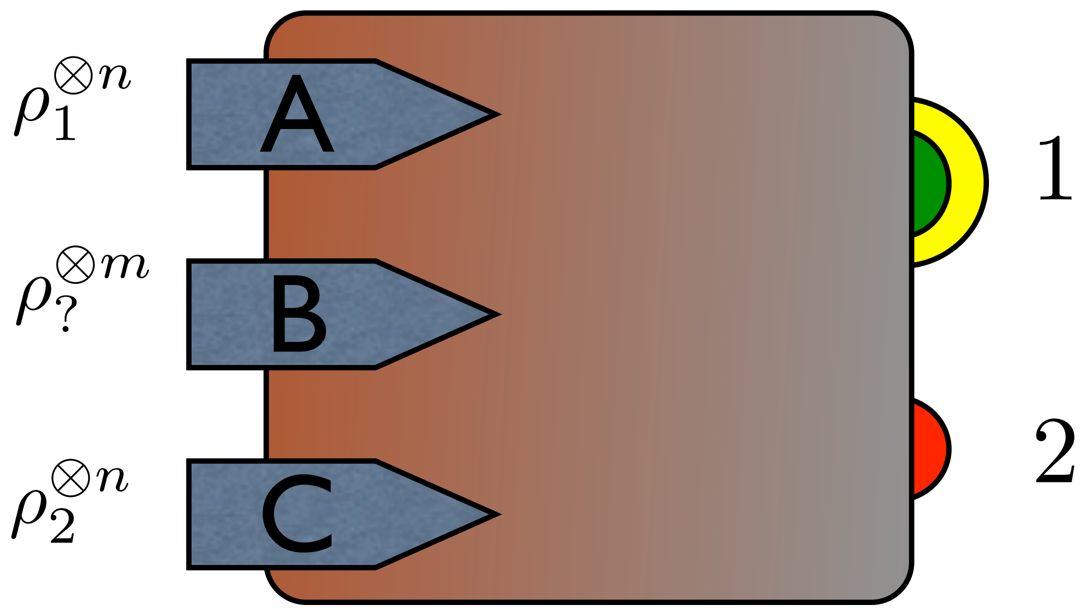
$$\lim_{m=n\to\infty} P^? = \frac{3}{n} + \cdots$$

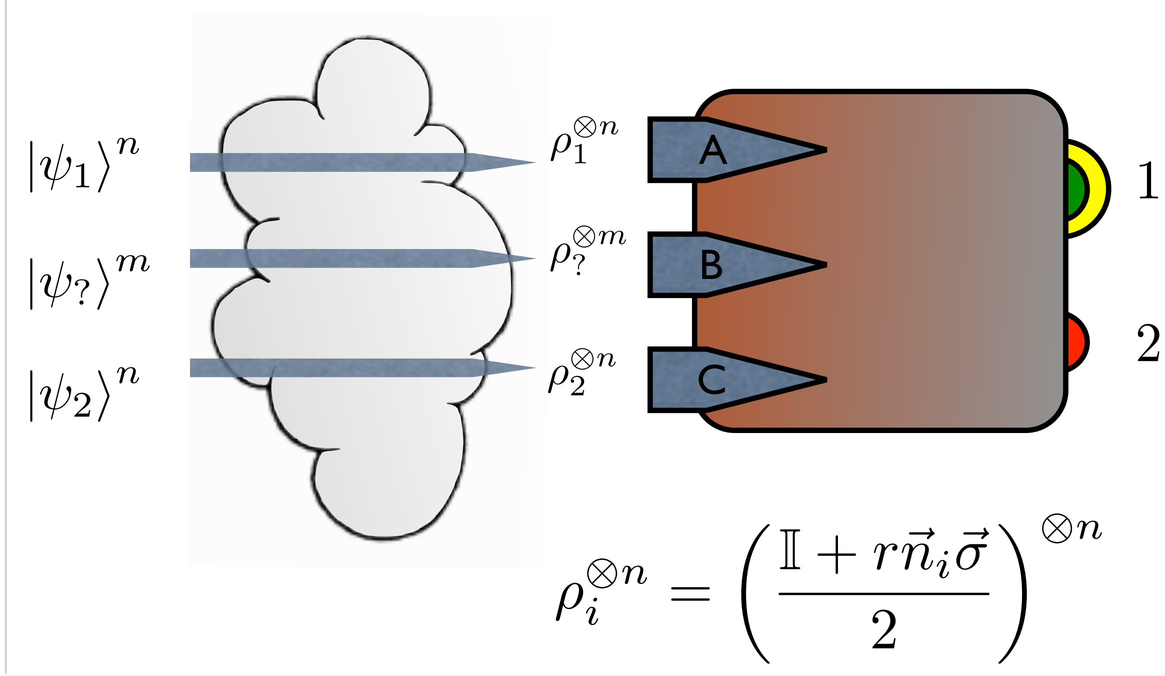
$$\lim_{m=n\to\infty} P^{\mathrm{M}E} = \frac{3\zeta(1/4)}{4n} + \cdots \approx \frac{0.882}{n} + \cdots$$



Mixed states

Minimum error only





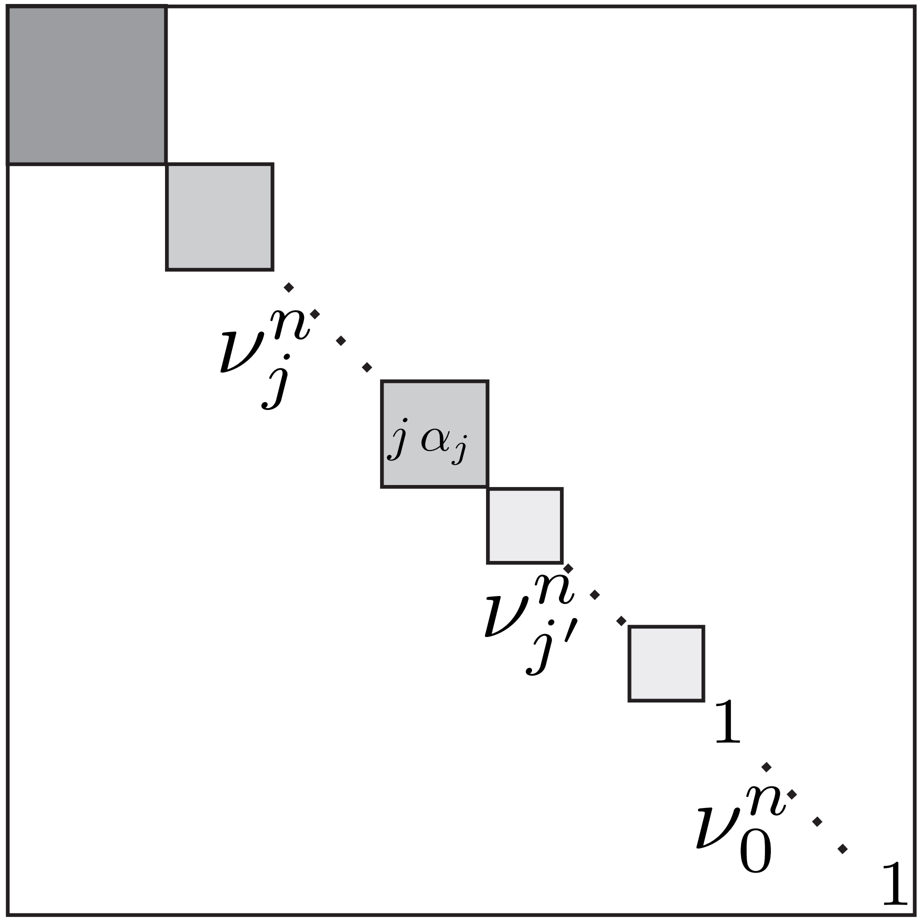
$$\sigma_1 = \int dn_1 dn_2 \rho_{1\,A}^{\otimes n} \otimes \rho_{1\,B}^{\otimes m} \otimes \rho_{2\,C}^{\otimes n}$$

$$\sigma_2 = \int dn_1 dn_2 \rho_{1\,A}^{\otimes n} \otimes \rho_{2\,B}^{\otimes m} \otimes \rho_{2\,C}^{\otimes n}$$

$$P^{\mathrm{ME}}(\sigma_1, \sigma_2) = \frac{1}{2}\left(1 - \frac{1}{2}\|\sigma_1 - \sigma_2\|\right)$$

•Deconstruction

$$\rho^{\otimes n} = \left(\frac{\mathbb{I} + r\vec{n}\vec{\sigma}}{2}\right)^{\otimes n} =$$



$$= \bigoplus \rho^{\alpha_j j}$$

Multiplicity

$$\nu_j^n = \binom{n}{n/2 - j} \frac{2j+1}{n/2 + j + 1}$$

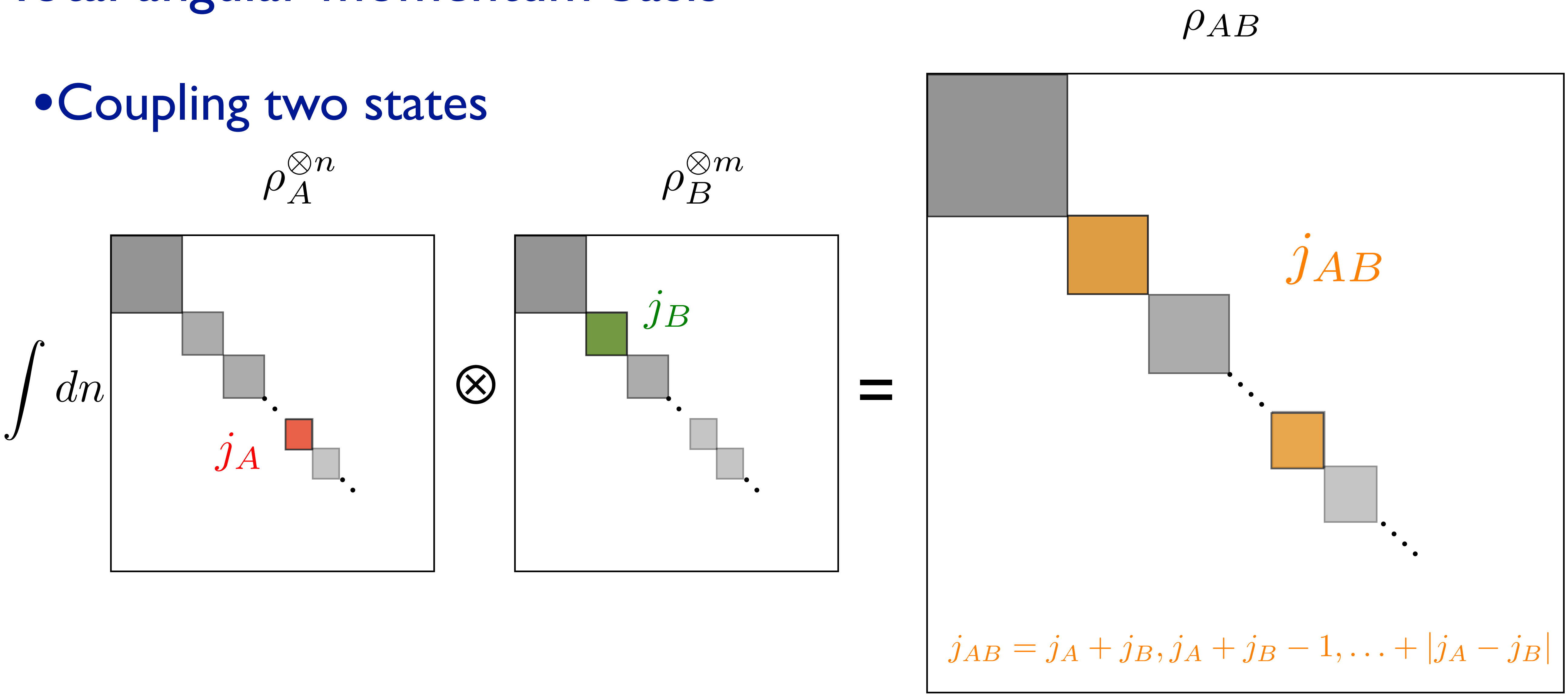
$$\mathrm{tr}\rho^j = \left(\frac{1-r^2}{4}\right)^{n/2-j} \frac{1}{r}\left[\left(\frac{1+r}{2}\right)^{2j+1} - \left(\frac{1-r}{2}\right)^{2j+1}\right]$$

$$C_j^n = \frac{1}{2j+1}\left(\frac{1-r^2}{4}\right)^{n/2-j} \frac{1}{r}\left[\left(\frac{1+r}{2}\right)^{2j+1} - \left(\frac{1-r}{2}\right)^{2j+1}\right]$$

$$\int dn\rho^{\otimes n} = \bigoplus_j \nu_j^n C_j^n \mathbb{I}_j$$

Total angular momentum basis

- Coupling two states



$$\rho_{AB} = \int dn_1 \rho_{1\,A}^{\otimes n} \otimes \rho_{1\,B}^{\otimes m} = \sum_{\xi_A \xi_B} \sum_{j_{AB} m_{AB}} C_{j_{AB}}^{n+m} |\xi_A \xi_B; j_{AB} m_{AB}\rangle\langle \xi_A \xi_B; j_{AB} m_{AB}|$$

$$\xi_X = \{j_X, \alpha_X\}$$

$$\rho_{AB} = \bigoplus_{j_A j_B j_{AB}} \nu_{j_A}^n \nu_{j_B}^m C_{j_{AB}}^{n+m} \, \mathbb{I}_{j_{AB}}^{(j_A j_B)}$$

•Coupling three states: the states σ's

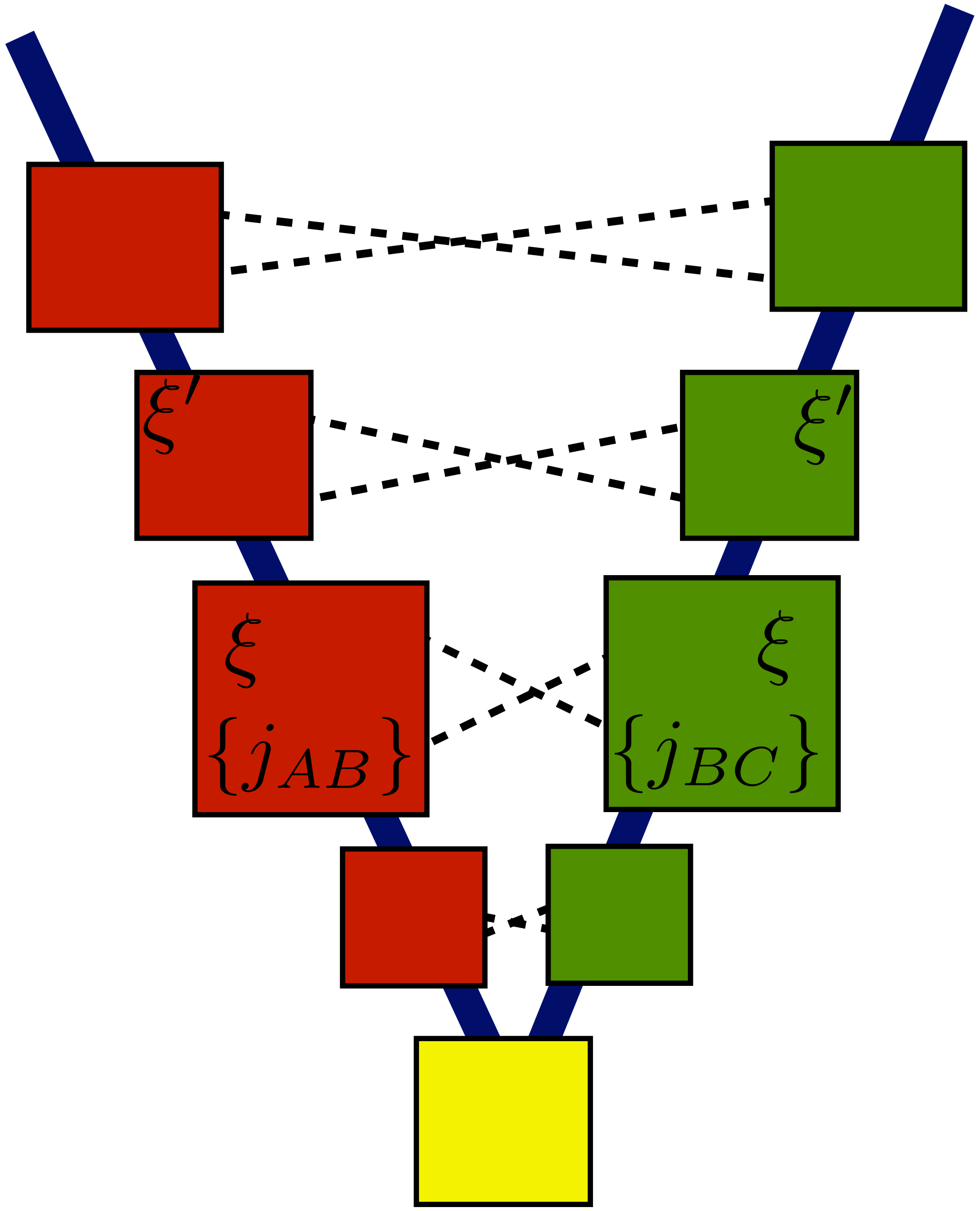
$$\sigma_1 = \sum_{\xi_A \xi_B \xi_C} \sum_{j_{AB}} \sum_{JM} C^{n+m}_{j_{AB}} C^{n}_{j_C} |\xi_A \xi_B \xi_C, j_{AB}; JM\rangle\langle \xi_A \xi_B \xi_C, j_{AB}; JM|$$

$$\sigma_2 = \{A \longleftrightarrow C\}$$

Compute the trace norm in: $P^{\mathrm{ME}}(\sigma_1, \sigma_2) = \frac{1}{2}\left(1 - \frac{1}{2}\|\sigma_1 - \sigma_2\|\right) = \frac{1}{2}(1 - \frac{1}{2}T)$

- ⋆Equivalent representations give identical contributions ⇒ factor $\nu^{n}_{j_A} \nu^{m}_{j_B} \nu^{n}_{j_C}$
- ⋆Overlaps are independent of the quantum number *M* ⇒ multiply by (*2J+1*)
- ⋆Define the set of relevant quantum numbers ⇒ $\xi = \{j_A, j_B, j_C, J\}$

★Change of basis $|\xi_A \xi_B \xi_C, j_{BC}; JM\rangle \rightarrow |\xi_A \xi_B \xi_C, j_{AB}; JM\rangle$ Overlaps



$$\xi = \{j_A, j_B, j_C, J\}$$

$$\gamma_\xi = \nu^n_{j_A} \nu^m_{j_B} \nu^n_{j_C} (2J+1)$$

$$\boxed{\Lambda^{(\xi)}_{j_{AB}, j_{BC}} = \langle \xi, j_{AB} | \xi, j_{BC} \rangle}$$

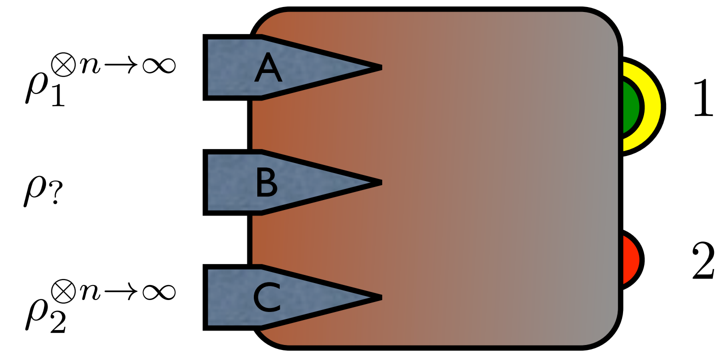
$$\propto \begin{Bmatrix} j_A & j_B & j_{AB} \\ j_C & J & j_{BC} \end{Bmatrix}$$

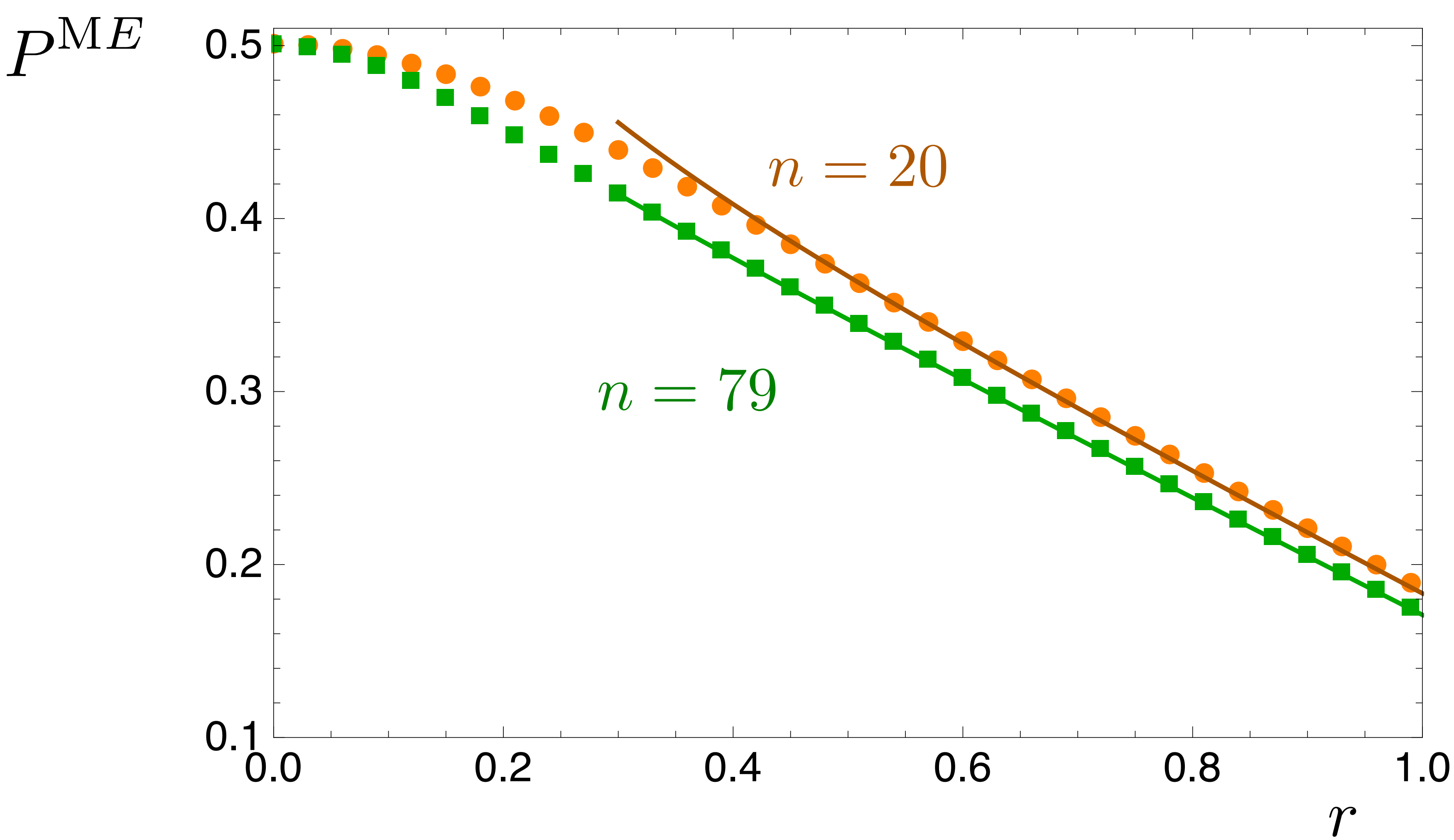
$$T = \sum_\xi \gamma_\xi T^\xi = \sum_\xi \gamma_\xi \| \sigma_1^{(\xi)} - \Lambda^{(\xi)} \sigma_2^{(\xi)} {\Lambda^{(\xi)}}^T \|$$

$$\boxed{P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \frac{1}{2} \sum_\xi \gamma_\xi T^\xi \right)}$$

Results

•n×1×n





•1×1×1

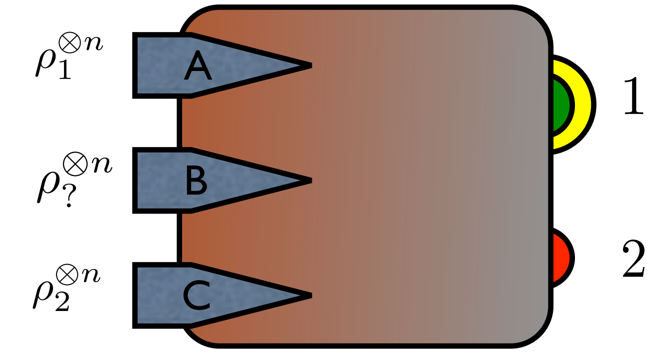
$$P^{\mathrm{ME}} = \frac{1}{2}\left(1 - \frac{r^2}{2\sqrt{3}}\right).$$

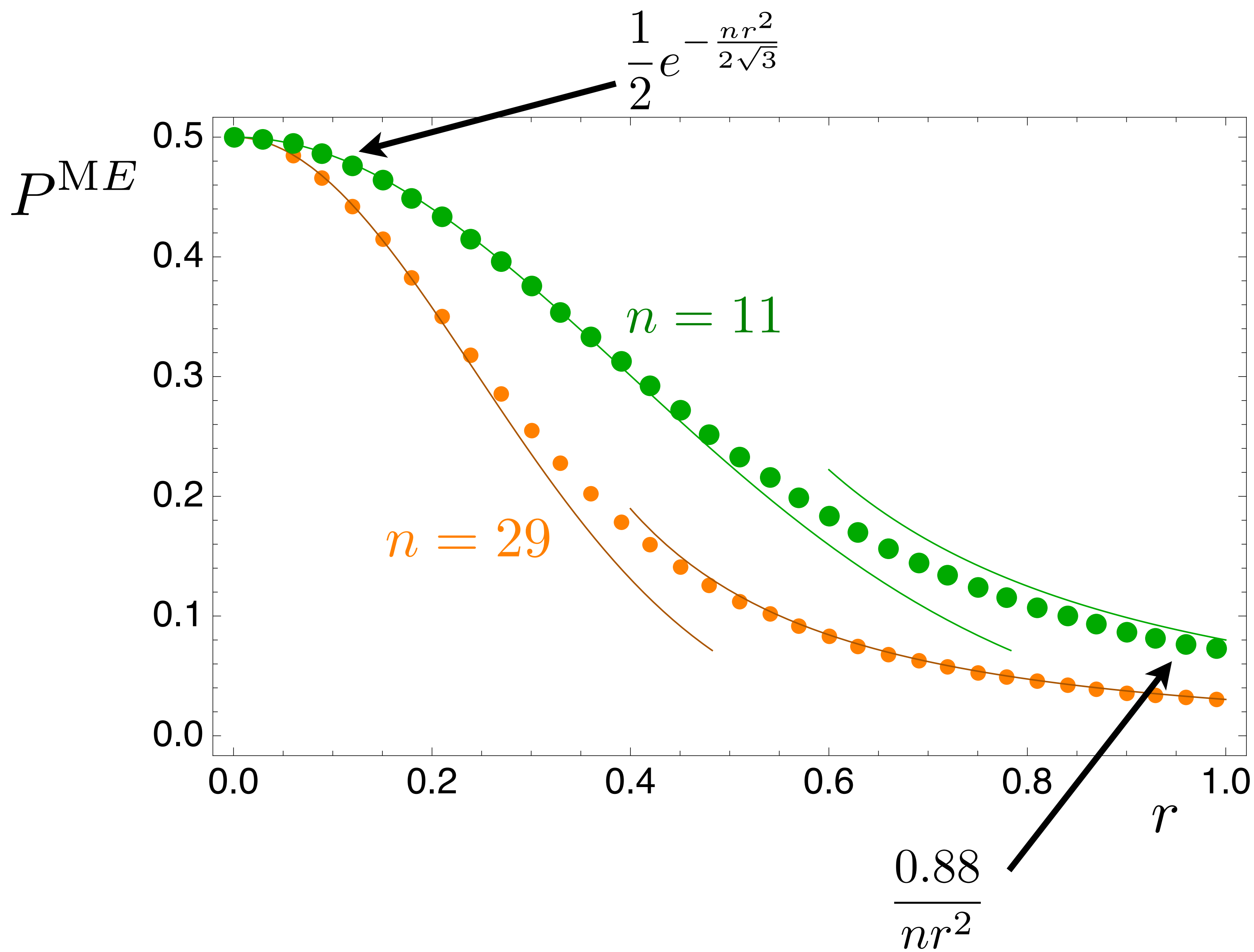
Exact asymptotic expression:

$$P^{\mathrm{ME}} = \frac{1}{2} - \frac{r}{3} + \frac{1}{3rn}$$

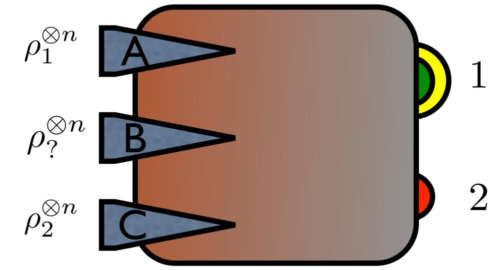
$n \to \infty$: Average Minimum Error

•n×n×n


$\rho_1^{\otimes n}$
A
$\rho_?^{\otimes n}$
B
$\rho_2^{\otimes n}$
C
1
2


$\frac{1}{2}e^{-\frac{nr^2}{2\sqrt{3}}}$
P^{ME}
$n = 11$
$n = 29$
r
$\frac{0.88}{nr^2}$
0.0
0.1
0.2
0.3
0.4
0.5
0.0
0.2
0.4
0.6
0.8
1.0

(Trully) Universal Discriminator

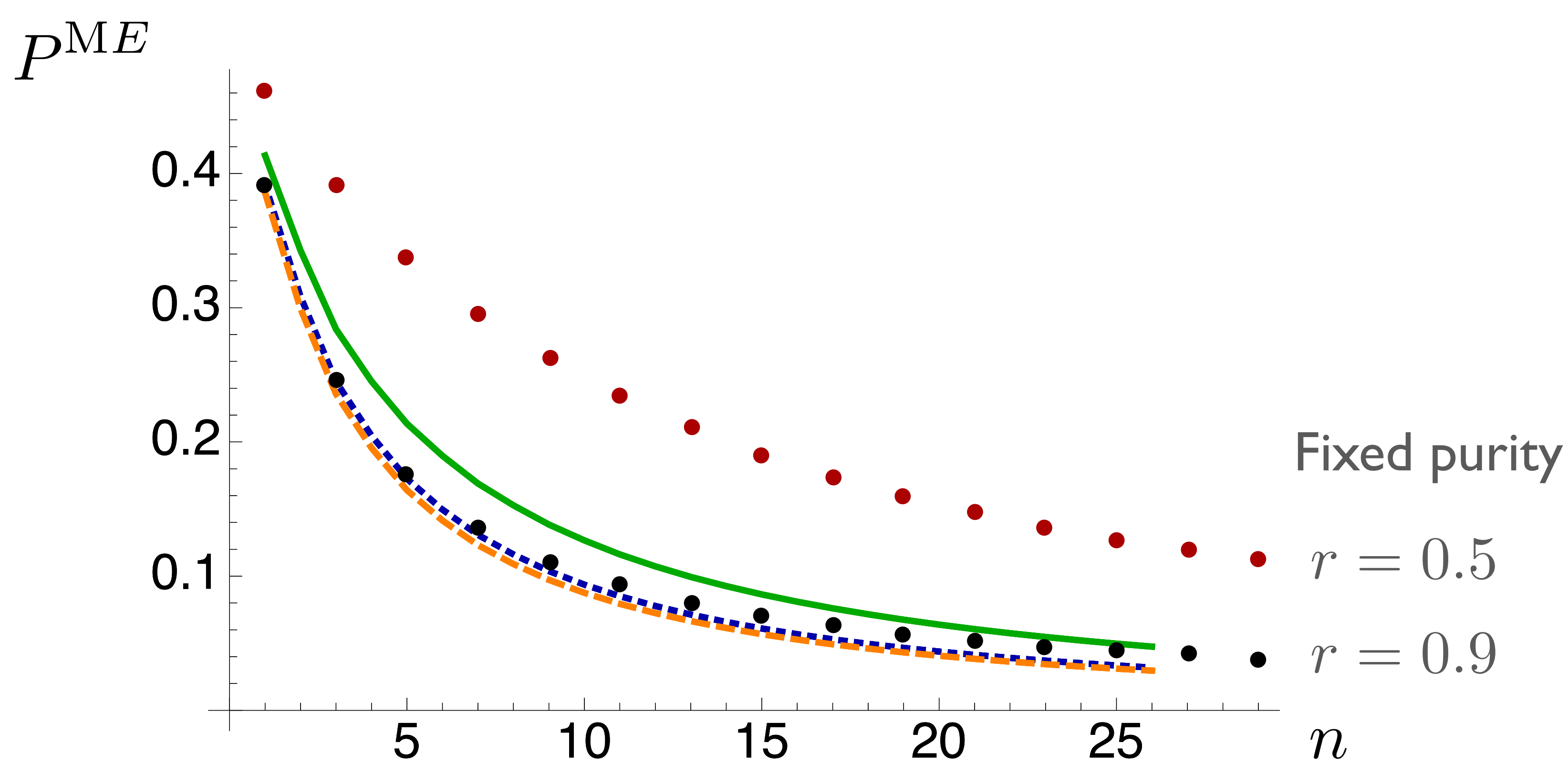


- Nothing is known about the states, not even its purity
- Average over purity prior distribution
- No unique (reasonable) prior distribution

Hard sphere $$d\rho^{\mathrm{HS}} = 3r^2 dr \frac{d\Omega}{4\pi}$$

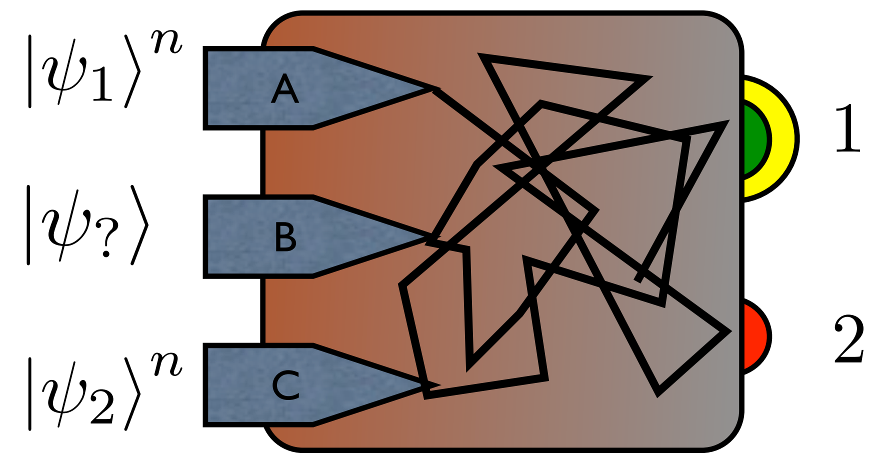
Bures $$d\rho^{\mathrm{Bu}} = \frac{4}{\pi} \frac{r^2}{\sqrt{1-r^2}} dr \frac{d\Omega}{4\pi}$$

Chernoff $$d\rho^{\mathrm{Ch}} = \frac{1}{\pi - 2} \frac{\left(\sqrt{1+r} - \sqrt{1-r}\right)^2}{\sqrt{1-r^2}} dr \frac{d\Omega}{4\pi}$$



MAD machines

Do programmable machines require the use of full quantum correlations?



- Measure→obtain guesses
- Discriminate as if the guesses were correct
- Average over all input states

(in general average first, here is the same)

- *n×1×n*

$$\tilde{P}^{\mathrm{ME}} = \frac{1}{6} + \frac{2}{3n}$$

$$P^{\mathrm{ME}} = \frac{1}{6} + \frac{1}{3n}$$

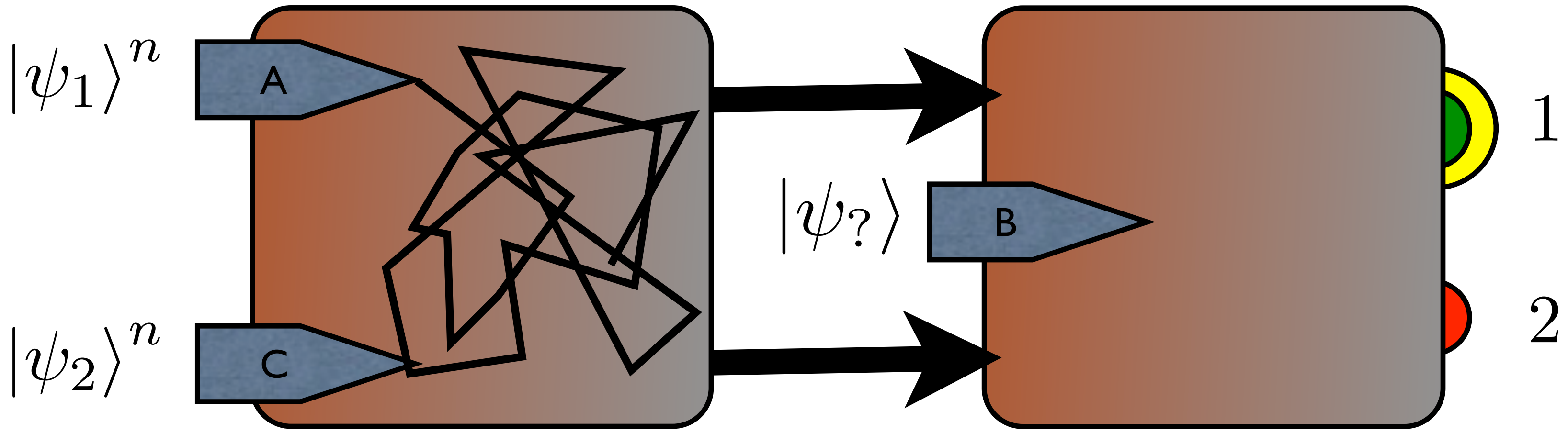
- Leading term coincides.
- Subleading term is a factor TWO larger (pure).

Quantum correlations seem to play a crucial role

MAD machines II

- Measure at the program ports with generalized measurement
- Discriminate at the data port

☛ No quantum memory would be required



- Measurement that prepares most orthogonal states

☛ *nx1xn* EXACTLY the optimal error probability

$$P^{\mathrm{ME}} = \frac{1}{6} + \frac{1}{3n}$$

Summary

- General expressions for unambiguous and minimum error probabilities (pure and mixed)
- State discrimination and state comparison is recovered
- Considered the universal discrimination machine
- Local MAD machines, a factor of two worse
- General MAD machines can match the optimal error probability.

G.Sentis, E. Bagan, J. Calsamiglia, RMT, PRA 82, 042312 (2010)
G.Sentis, E. Bagan, J. Calsamiglia, RMT, to appear