Introduction to topological insulators

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Plan of the talk

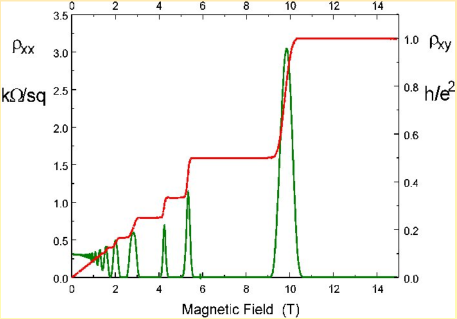
Electrons and atoms in the quantum world form many different states of matter - crystals, magnets, superconductors, etc

Classification of all these quantum states is through principle of spontaneous symmetry breaking - *e.g.*, crystal breaks translational symmetry, ferromagnet breaks spin symmetry, superconductivity breaks gauge symmetry, etc

Recent years, new way of classifying phases depending on topological quantum numbers

Quantum Hall effect

Quantum Hall effect discovered in eighties Hall conductance quantised $\sigma_{xy} = ne^2/h$



Problem is of electrons moving on a 2-dim surface in the presence of a magnetic field in the perpendicular direction - no electron-electron interactions

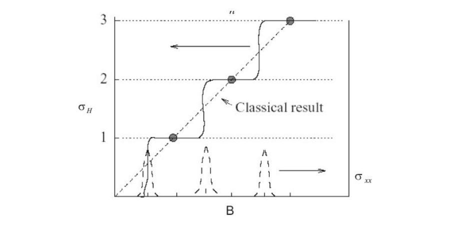
Solved in quantum mechanics course -

$$H = \sum_i \frac{(\mathbf{p}_i - e\mathbf{A}_i)^2}{2m}$$

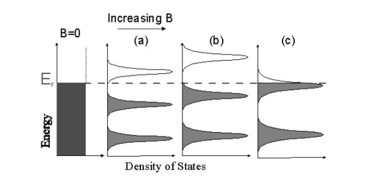
leads to degenerate single particle Landau levels - degeneracy is finite for a given area

When the degenerate states at a given Landau level are filled, gap to next level

Actually, the solution of the equations only give the conductance at the points where the bands are filled



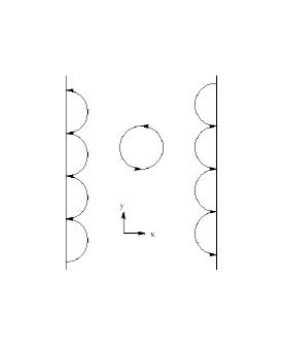
To understand the plateaux, one needs to understand how the levels are broadened by disorder



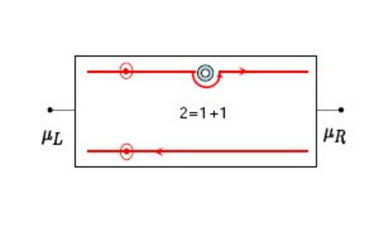
The extra-ordinary accuracy in the presence of disorder is a surprise!

What does this mean for conduction of electrons through the sample?

Semi-classical picture - closed orbits in the interior of the sample and hopping orbits at the edge of the sample



Uni-directional flow dictated by the sign of the magnetic field
Upper edge supports forward movers and lower edge backward movers



Because of spatial separation of forward and backward movers, no possibility of back-scattering due to impurities Explains robustness and accuracy of the quantisation of the Hall current - impervious to disorder or impurity scattering

Well-understood phenomenon by now

Yet another way of understanding quantisation of current - Physically measured current is related to a ‘topological invariant’

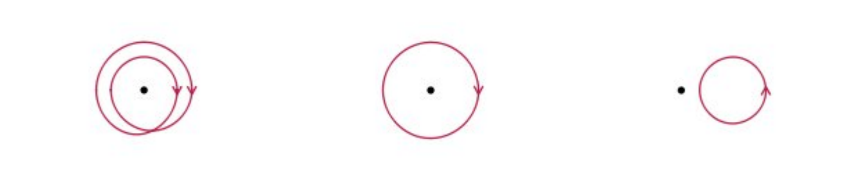
Topological invariant = quantity that does not change under continuous deformation (change of some parameter)

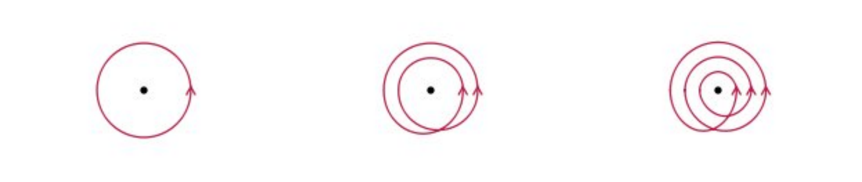
Small aside on topological quantum numbers

Simplest case - winding numbers

Winding number is a topological invariant
Depends only on the winding around the point and not on details such as how the winding is done

Topological quantum numbers





Explains why the quantisation of Hall current is so accurate, even in the presence of disorder
Can be related to a topological invariant

Small aside on Berry phase

Consider Hamiltonian that depends on time through parameter $\mathbf{R}(t)$

$$H = H(\mathbf{R}), \mathbf{R} = \mathbf{R}(t)$$

Interested in adiabatic evolution of the system as $\mathbf{R}(t)$ moves slowly along a path C in parameter space

$$H(\mathbf{R})|n(\mathbf{R}) >= \epsilon_n(\mathbf{R})|n(\mathbf{R}) >$$

where $|n(\mathbf{R}) >$ = basis function

State at time t is given by

$$\psi_n(t) = e^{i\gamma_n(t)} \;\; exp[\frac{i}{\hbar}\int_0^t dt' \epsilon_n(\mathbf{R}(t'))] \;\; |n(\mathbf{R}(t)) >$$

Second exponential is the dynamical phase factor

But the eigen value equation allows arbitrary R dependent phase of $|n(\mathbf{R}) >$ given by $e^{i\gamma_n(t)}$

But $\gamma_n(t)$ is also important
Using

$$i\hbar\frac{\partial}{\partial t}|\psi_n(t)> = H(\mathbf{R})|\psi_n(t)>$$

and multiplying on left by $< n(\mathbf{R}(t))|$, find

$$\gamma_n = \int_C d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R})$$

where

$$\mathbf{A}_n(\mathbf{R}) = i < n(\mathbf{R})|\frac{\partial}{\partial\mathbf{R}}|n(\mathbf{R})>$$

$\mathbf{A}_n(\mathbf{R})$ = Berry connection or Berry vector potential

Can also define Berry curvature

$$\mathbf{B}_n(\mathbf{R}) = \nabla_{\mathbf{R}} \times \mathbf{A}_n(\mathbf{R})$$

and define the Berry phase as

$$\gamma_n = \int_S d\mathbf{S} \cdot \mathbf{B}_n(\mathbf{R})$$

For closed paths C , Berry phase becomes gauge invariant physical quantity
Berry curvature is itself gauge invariant

Berry curvature integrated over closed manifold = topological invariant = $2\pi n$ = first Chern number

IQHE quantisation related to integral over Berry curvature

For electrons in periodic solid, electron momentum provides natural parameter space

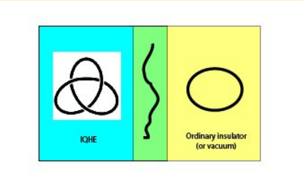
Berry connection and Berry curvature defined as

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}), \qquad \mathbf{A} = < u_{\mathbf{k}}| - i\nabla_k |u_{\mathbf{k}} >, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\sigma_H = \frac{e^2}{\hbar} \times \int_{FB} \frac{d^2k}{(2\pi)^2} \mathbf{B} = \frac{e^2}{h} n$$

where FB is a filled Landau band, n = integer

Thus, each plateau was related to a topological invariant



Can argue that where the system evolves from IQHE to ordinary insulator, the system cannot remain insulating
Else, the topological invariant cannot change
Hence, it implies conducting edge states

Hence, understand how edge states and topology are related

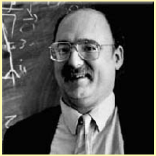
Started the idea of classifying phases of matter through topology

Two dimensional topological insulators

Natural question

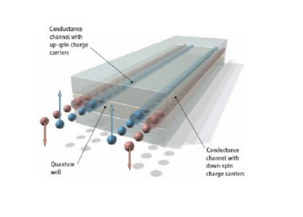
Can one achieve separation of chiral modes (left and right moving modes) or equivalently, can one achieve non-trivial topological phases without magnetic fields, or without time reversal symmetry breaking?

Haldane (1988) had a toy model which had quantum Hall physics without magnetic fields, due to non-trivial topology of Brillouin zone



Generalise the edge states to have two species at each edge
One going forwards and one backward, but with different spins

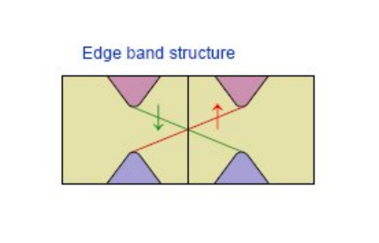
Overall time-reversal symmetry not broken, but states of unique chirality for each spin



Spatial separation of the edge states implies no back-scattering unless spin can change

Kane and Mele, 2005, Zhang and Bernevig 2006

First predicted for graphene with spin-orbit coupling (to make it an insulator), but gap is very small and hence, requires very low temperatures



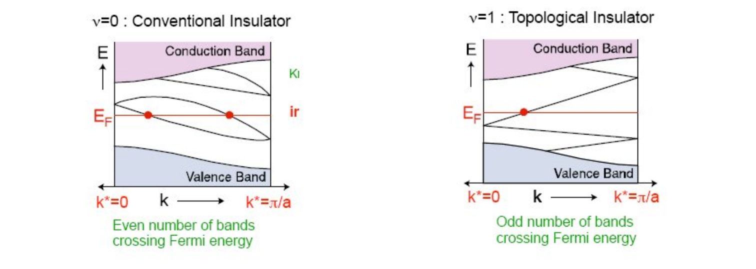
Kane and Mele

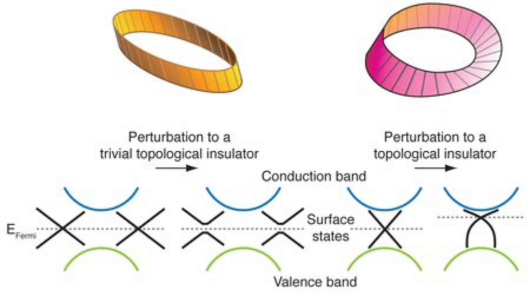
Hence, in absence of spin-flip scattering, expect the same physics as for quantum Hall systems
called quantum spin Hall insulator

But no strong magnetic fields here - time-reversal invariant
Hence, achieved edge states without strong magnetic fields

Zhang et al studied time-reversal invariant systems with half-integer spin,
Kramer's theorem implies all states are doubly degenerate
Realised that a single Kramers pair is topologically protected

Kramers degeneracy means that at time-reversal invariant points $k = 0, \pi$, the spectrum should be double degenerate





Realised that all time-reversal invariant insulators can be classified into 2 classes, depending on whether they have even or odd number of edge states

Protected against back-scattering even when impurity can change spin (spin-orbit coupling) as long as time-reversal symmetry unbroken

Another way of understanding
No mass term (or gap term) can be added without breaking time-reveral symmetry for one pair of edge states, whereas it can be added for even number of pairs

Hence all time-reversal invariant insulators can be classified into 2 classes, depending on whether they have even or odd number of Kramers pairs of edge states

Bernevig, Zhang and Hughes

Relation to topology

Oddness or evenness of edge states cannot be removed under any continuous deformation of band structure, so long as time reversal symmetry exists

Insulators with odd number of (pairs of) edge states belong to different topological class than those of ordinary insulators

But doubling number of edge states implies back-scattering allowed and edge states no longer topologically protected to be massless

Unlike quantum Hall effect, where topological quantum number was integer, for topological insulators, topological quantum number is Chern parity - it is only a Z_2 invariant

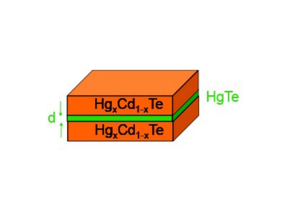
To find real materials that are topological insulators
Need to look at its band structure

If its band structure has odd number of edge states in the gap, it is a topological insulator

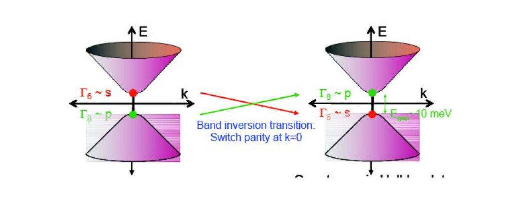
Zhang et al argued that materials where ordering of conduction and valence bands get inverted by spin-orbit coupling, will be topological insulators

Technically, described by model with negative Dirac mass. Hence guessed that at the edges between positive and negative masses, there needs to exist a domain wall, and hence they would be in a different topological class

Zhang, Bernevig and Hughes

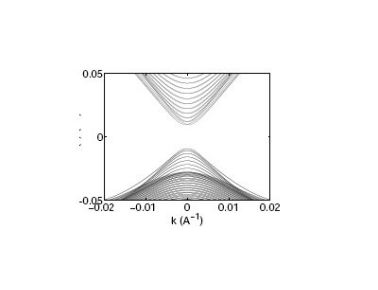


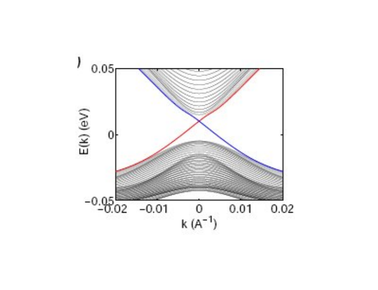
$Hg_xCd_{1-x}Te$
$Hg_xCd_{1-x}Te$
HgTe
d



E
k
$\Gamma_6 \sim s$
$\Gamma_8 \sim p$
Band inversion transition:
Switch parity at k=0
E
k
$\Gamma_8 \sim p$
$\Gamma_6 \sim s$
E_{gap} ~10 meV

By explicitly solving for the band structure of Mercury Telluride *HgTe* quantum well, they showed that for thickness greater than some $d_c (= 6.3nm)$, the bands are inverted and *HgTe* is a topological insulator





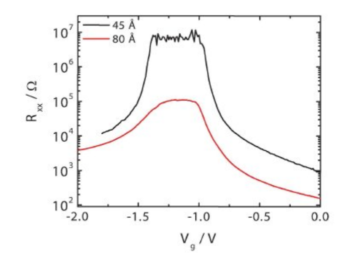
Amazingly, HgTe was first predicted and then found to be a topological insulator

Finding new topological class of matter or new phase of matter is like finding new particles in HEP

Just like in HEP, symmetries of standard model predicted top quark which was then found, here, using symmetry and topological consideration, new phase of matter was predicted and then found

Very unusual in condensed matter physics

No magnetic field required. Opposite spin excitations move in opposite directions
No Hall current, but net ordinary two-terminal current, because one spin species at each edge contributes
Experimentally measured $2e^2/h$ Hall plateau in zero magnetic field



Konig *et al*, Science, 2007

Three dimensional topological insulators

Idea of Z_2 topological invariant generalised to 3 dimensions

4 independent Z_2 invariants can be defined
3 of them η_i are just the generalisations of the 2D Z_2 to the 3 surfaces in 3D
The 4th one η_{3D} is a new Z_2 index

Weak topological insulator, one or more $\eta_i = -1$, $\eta_{3D} = +1$

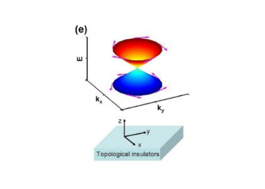
Strong topological insulator, $\eta_{3D} = -1$

Once again, materials (Bi_xSb_{1-x}, Bi_2Se_3, Bi_2Te_3, Sb_2Te_3) first predicted and then found to be topological insulators

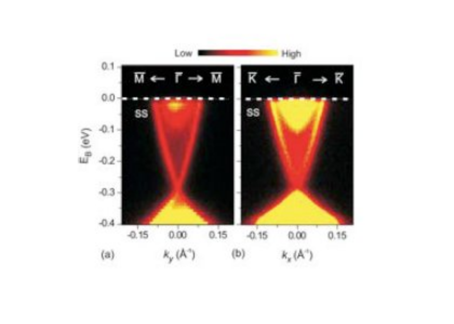
Fu and Kane, R. Roy, Moore and Balents

Bulk states are fully gapped, but topologically protected gapless surface states

**Surface states consist of single massless Dirac fermions (helical, with spin perpendicular to momentum) and dispersion forms Dirac cone
similar to Dirac electrons in graphene, but without the valley and spin degeneracies**



Using ARPES measurements, surface states were actually seen in Bi_2Se_3



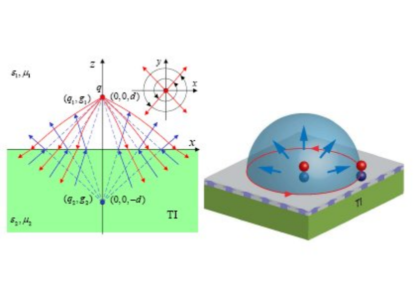
Dirac cone seen in what would have been a gap for a normal insulator

Hasan group, 2008

Electromagnetic properties

Break time reversal symmetry on the surface and not in the bulk
Leads to topological magneto-electric effect or axion electrodynamics

Unlike usual polarisation of a dielectric leading to image charge, here one has image magnetic charge as well



Charge-monopole composites or dyons behave like anyons - i.e., have fractional statistics

Connection to topological field theories

Quantum Hall effect described by the effective Chern-Simons field theory $S_{eff} = \frac{c_1}{4\pi} \int d^2x \int dt \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau$

Generalise to 4+1 dimensions

$$S_{eff} = \frac{c_2}{24\pi^2} \int d^4x \int dt \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau$$

Claim that this is the fundamental TFT from which effective theory for topological insulators in 3+1 dimensions and 2+1 dimensions can be derived

Bernevig,Zhang, Hughes

Quasi-particle excitations in topological insulators

Topological insulators in the proximity of a superconductor have surface excitations which are like Majorana particles (particles which are their own anti-particles) bound to vortices

Fu and Kane

Majorana particles expected to have 'non-abelian' statistics under exchange

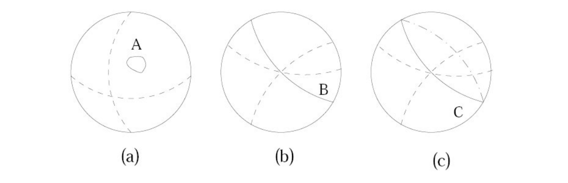
Short course on anyons and fractional statistics

Anyons are particles with 'any' statistics intermediate between bosons and fermions

Consider statistics of 2 indistinguishable particles $(\mathbf{r}_1, \mathbf{r}_2) = (\mathbf{r}_2, \mathbf{r}_1)$ and assume $\mathbf{r}_1 \neq \mathbf{r}_2$

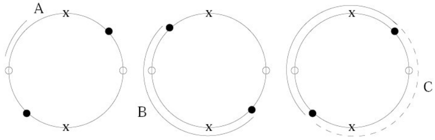
In 3 dimensions, represent relative space of 2 particles as a sphere as $(R_3 - origin)/Z_2$

Consider possible phases of the wave-function picked up when the particles are exchanged



Only possible paths are single exchange or no exchange
Hence, phase $\eta^2 = 1$, implies $\eta = -1$ denoting fermions or $\eta = +1$ denoting bosons

But in 2 dimensions, relative space is a circle with a point removed



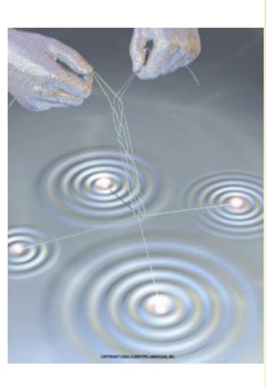
(a) (b) (c)

Can have multiple windings which are not deformable to the trivial winding
Implies one can have any statistics -i.e. anyons

Non-abelian anyons

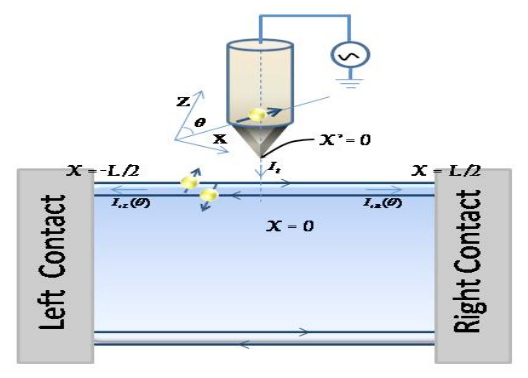
When the particles are exchanged, instead of a phase, one has a phase matrix
Implies multi-dimensional representations of the braid group

Point that is important here that this implies degenerate ground states
Important for topological quantum computation because this degeneracy depends on topology and is very robust



Currently, a race is on to find Majorana particles in condensed matter systems experimentally
These excitations expected to obey non-abelian statistics and expected to be relevant for quantum computation

Computation of charge and spin fractionalisation in helical Luttinger liquids
Proposed three terminal geometry



Sourin Das and S.R

Choose spin quantisation axis of electrons in edge state to be $\hat{z}$ axis. Spin of electrons in polarized STM tip chosen in $\hat{x}$-$\hat{z}$ plane, forming an angle θ with $\hat{z}$ axis as shown. $\hat{y}$-axis points out of screen

If spin of STM tip in tune with spin of edge electron, it implies uni-directional injection locally

In presence of electron-electron interactions, due to scattering, both right and left-movers
But asymmetry survives, can be measured

Uni-directional injection of electrons possible and hence left-right asymmetry of charge and spin currents
Leads to spin and charge fractionalisation and even a spin amplification effect

If polarisation of STM tip at angle θ with spin projection of edge electrons, specific computable θ dependence

But with electron-electron interactions, also interaction dependence parametrised by K

$$
\begin{aligned}
< I_{tR} > &= \frac{(1 + K\cos\theta)}{2} I_0 \\
< I_{tL} > &= \frac{(1 - K\cos\theta)}{2} I_0
\end{aligned}
$$

$$I_0 = \frac{2e^2}{h}|t^2|\frac{(T/\Lambda)^\nu}{(\hbar v_F)^2\Gamma(\nu+1)} \times V$$

ν is the Luttinger tunneling exponent given by $\nu = -1 + (K + K^{-1})/2$, K = interaction parameter, $T \gg T_L, T_V$

Expectation value of $\dot{S}_Y$ is zero, as expected

Spin current (vector) at right and left leads point in different direction than injected current

$$\mathbf{S}_{R/L}(\theta) = \left[\frac{K \mp \cos\theta}{2K\sin\theta}\, \widehat{Z} \pm \frac{1}{2K}\, \widehat{X} \right] I_{tR/L}(\theta)$$

Non-linear function of K

Total spin current in direction of injected spin

$$\langle \frac{d\mathbf{S}}{dt} \rangle = (\, \widehat{Z}\cos\theta + \widehat{X}\sin\theta \,)\, \frac{I_0}{2}$$

As if charge excitations with charge $(1+K)/2$ and $(1-K)/2$ moving to the left and right

Spin excitation with spin $(1 \pm K)/2K$ moving to left and right, spin amplification at one end

But interpretation of fractional charge is different from that of the $e/3$ charge of FQHE with filling fraction $1/3$
Gapped system and also, charge is quantised in that case
Here, charges of $(1 \pm K)/2e$ interpreted as property of electron injection

Importance of the work - chiral injection which was difficult with normal Luttinger liquids is easy here, because of the spin polarisation

New feature - spin current and spin fractionalisation
Need more work to understand spin fractionalisation

New paradigm of classifying phases of matter in terms of topological quantum numbers
Led to the discovery of new materials with interesting properties

Described the relation between topology and gapless edge states and consequently between topology and quantised conductances

Ended with a discussion on fractional statistics and in particular non-abelian statistics and why they are expected to be relevant in quantum computation