Geometric Discord: Some Analytic Results

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Outline

- **Geometric discord as a correlation measure**
  - What and why geometric discord
  - Some basic features of GD
  - Monogamy of GD
- **Analytic formulae**
  - Two-qubits
  - Arbitrary states∗
- **Interrelations between GD and Entanglement**
  - The conjecture $\mathscr{D} \geq \mathscr{N}^2$
  - Why the conjecture is interesting (though apparently GD ≥ entanglement is trivial)?
  - Counterexamples
- **Maximally discordant separable states of two-qubits**
  - The problem
  - Its solution among X states
  - Some characterizations for solution to the general problem
- **Conclusion**

Geometric discord (GD) and its properties

Geometric Discord (GD) [Dakić *et al.*, PRL 2010]

$$\mathscr{D} = \mathscr{D}_A(\rho_{AB}) = \frac{m}{m-1} \min_{\chi \in \Omega_0} \|\rho - \chi\|^2, \tag{1}$$

where

- $\Omega_0 = \{\chi = \sum_k p_k |k\rangle_A \langle k| \otimes \rho_k^B\}$ is the set of all *zero-discord*, or *classical-quantum* (CQ) states, with $\{k\rangle_A\}$ being an orthonormal basis of the Hilbert space for A.
- the norm is the usual Hilbert-Schmidt norm given by

$$\|X\|^2 = \langle X, X\rangle = \mathrm{Tr}\left(X^\dagger X\right) = \sum_{i,j} |X_{ij}|^2,$$

- the factor $m/(m-1)$ is for normalizing $\mathscr{D}$, so that $\mathscr{D} \in [0,1]$.

Properties

☺

- Non-negative: $\mathscr{D}(\rho) \geq 0$ for all states ρ
- Faithfulness: $\mathscr{D}(\rho) = 0$ iff ρ is a CQ state. (Hence $\delta_A = 0 \Leftrightarrow \mathscr{D} = 0$)
- LU Invariance: $\mathscr{D}(U \otimes V \rho U^\dagger \otimes V^\dagger) = \mathscr{D}(\rho)$
- Reaches maximum for Bell states ($\mathscr{D}_{\max} = 1$)

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$$\begin{cases} |0\rangle_A \to |0\rangle_A \\ |1\rangle_A \to |+\rangle_A := \frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right) \end{cases}$$

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⚠ Separable state may have non-zero Discord: a true *post-entanglement* correlation!

Pure states [Luo and Fu, PRL 2011]

- $\mathscr{D}(|\psi\rangle)$ is given by the *linear entropy* :

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|ii\rangle_{AB}, \quad \sum_i \lambda_i = 1 \tag{2a}$$

$$\text{then, unnormalized } \mathscr{D}(|\psi\rangle) = S_L(\rho^A) := 1 - \mathrm{Tr}(\rho^A)^2 = 1 - \sum_{i=1}^{m} \lambda_i^2. \tag{2b}$$

- $\mathscr{D}$ is maximum for MES $|\psi\rangle = (\sum|ii\rangle)/\sqrt{m}$. Hence the normalization factor of $m/(m-1)$.
- Normalized $\mathscr{D}(\rho) = 1$ iff ρ is MES.
- For $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^n$, $\mathscr{D}(|\psi\rangle) = 4\det\rho^A$.

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  - For $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^n$, $\mathscr{D}(|\psi\rangle) = 4\det\rho^A$.
- On the other hand, δ_A is given by the *entropy of entanglement*:

$$\delta(|\psi\rangle) = E(|\psi\rangle) := S(\rho^A) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{3}$$

Analytic expression of GD

2-qubit states [Dakić *et al.*, PRL 2010]

✡ Bloch Form: $$\rho_{AB} = \frac{1}{4}\left[I \otimes I + \mathbf{x}.\sigma \otimes I + I \otimes \mathbf{y}.\sigma + \sum T_{ij}\sigma_i \otimes \sigma_j\right]$$
$$:= (\mathbf{x}, \mathbf{y}, T)$$

✪ Discord: $$\mathscr{D}(\rho) = \frac{1}{2}\left[\|x\|^2 + \|T\|^2 - \lambda_{max}(\mathbf{x}\mathbf{x}^t + TT^t)\right]$$

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- Parametrization of $\chi \in \mathbb{C}^2 \otimes \mathbb{C}^2 \cap CQ$:

$$\chi = \sum_{i=1}^{2} p_i |\psi_i\rangle\langle\psi_i| \otimes \rho_i^B, \quad \langle\psi_i|\psi_j\rangle = \delta_{ij}, \quad p_1, p_2 \geq 0, \quad p_1 + p_2 = 1 \tag{4a}$$
$$= (q\mathbf{e}, \mathbf{s}_+, \mathbf{e}\mathbf{s}_-^t), \tag{4b}$$

where the parameters are given by

$$q = p_1 - p_2, \quad \mathbf{e} = \langle\psi_1|\sigma|\psi_1\rangle, \quad \mathbf{s}_\pm = \mathrm{Tr}[(p_1\rho_1 \pm p_2\rho_2)\sigma], \tag{5}$$

with the restrictions $q \in [-1, 1]$, $\|\mathbf{e}\| = 1$, $\|\mathbf{s}_\pm\| \leq 1$.

- The distance, in terms of these new parameters:

$$\|\rho-\chi\|^2 = \frac{1}{4}\left(1+\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+\|T\|^2\right) - \frac{1}{2}\left(1+q\mathbf{x}^t\mathbf{e}+\mathbf{y}^t\mathbf{s}_+ +\mathbf{e}^t T\mathbf{s}_-\right) + \frac{1}{4}\left(1+q^2+\|\mathbf{s}_+\|^2+\|\mathbf{s}_-\|^2\right). \tag{6}$$

- The distance as well as the constraints are convex. Hence it has a global minimum guaranteed by the positivity of the Hessian. So, as usual, differentiating Eq. (6) and equating to zero, leads to the analytic formula

$$\mathscr{D}(\rho) = \frac{1}{2}\left[\|\mathbf{x}\|^2+\|T\|^2-\lambda_{\max}(\mathbf{x}\mathbf{x}^t+TT^t)\right], \tag{7}$$

together with the optimal CQ state $\chi^\star$ being a state with the following Bloch components

$$\chi^\star = \left(\mathbf{e}^t\mathbf{x}\mathbf{e}, \mathbf{y}, \mathbf{e}\mathbf{e}^t T\right), \tag{8}$$

where $\mathbf{e}$ is the normalized eigenvector of $\mathbf{x}\mathbf{x}^t + TT^t$ corresponding to the maximum eigenvalue $\lambda_{\max}$.

Alternative form of $\mathscr{D}$: Minimization over measurements

- The convexity condition was satisfied due to the fact that $(I+\mathbf{x}.\sigma)/2$ is a qubit iff

$$|\mathbf{x}| \leq 1.$$

- But conditions for a vector $\mathbf{v} \in \mathbb{R}^{d^2-1}$ to represent the Bloch vector of a qudit are not known for $d \geq 3$. So this problem can not be solved analytically in general.

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Two equivalent expressions [Luo and Fu, PRA 2010]

$$\min_{\chi \in \Omega_0} \|\rho - \chi\|^2 = \min_{\Pi^A} \|\rho - \Pi^A(\rho)\|^2,$$

$\Pi^A(\rho)$: state after a measurement Π^A on $\mathscr{H}^A$, i.e.,

$$\Pi^A = \{|k\rangle_A\langle k|\} \Longrightarrow \Pi^A(\rho) := \sum_k (|k\rangle\langle k| \otimes I^B)\rho(|k\rangle\langle k| \otimes I^B).$$

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Not necessarily true in other (e.g. *trace*) norm.

GD as matrix optimization problem

$$\mathscr{D}(\rho) = \mathrm{Tr}(CC^t) - \max_A \mathrm{Tr}(ACC^tA^t), \tag{9}$$

- $C = (c_{ij})$ is an $m^2 \times n^2$ matrix given by the expansion

$$\rho = \sum c_{ij} X_i \otimes Y_j \tag{10}$$

in terms of orthonormal operators $X_i \in \mathscr{L}(\mathscr{H}^A)$, $Y_j \in \mathscr{L}(\mathscr{H}^B)$,

- $A = (a_{ki})$ is an $m \times m^2$ matrix given by

$$a_{ki} = \mathrm{Tr}\left(|k\rangle\langle k|X_i\right) = \langle k|X_i|k\rangle \tag{11}$$

for any orthonormal basis $\{|k\rangle\}$ of $\mathscr{H}^A$.

- The problem of determination of $\mathscr{D}(\rho)$ reduces to finding the maximum of

$$f(A) := \mathrm{Tr}\left(ACC^tA^t\right), \tag{12}$$

subject to the restriction in Eq. (11).

Sharpest bound on GD [Rana and Parashar, PRA 2012]

• Set the generators of $SU(m)$ (together with I_m) as the basis of $\mathscr{L}(\mathscr{H}^A)$ and similarly for $\mathscr{L}(\mathscr{H}^B)$ (the usual Bloch form). This gives

$$C = \frac{1}{\sqrt{mn}} \begin{pmatrix} 1 & \sqrt{\frac{2}{n}}\mathbf{y}^t \\ \sqrt{\frac{2}{m}}\mathbf{x} & \frac{2}{\sqrt{mn}} T \end{pmatrix}. \tag{13}$$

• The restriction on A in Eq. (11) basically gives the following three restrictions on A:

$$\mathbf{e} := (a_{k1})_{k=1}^{m} = (\langle k|X_1|k\rangle)_{k=1}^{m} = \frac{1}{\sqrt{m}}(1,1,\ldots,1)^t, \tag{14a}$$

$$\sum_{k=1}^{m} \mathbf{a_k}^t := \sum_{k=1}^{m} (a_{ki})_{i=2}^{m^2} = \left(\sum_{k=1}^{m} a_{ki}\right)_{i=2}^{m^2} = (\mathrm{Tr} X_i)_{i=2}^{m^2} = \mathbf{0}, \tag{14b}$$

$$\text{the } \textit{isometry condition} \quad AA^t = I_m, \tag{14c}$$

and in addition, $|k\rangle\langle k|$ should be a legitimate pure state. (14d)

- Writing $A = (\mathbf{e} \quad B)$, where B is an unknown $m \times (m^2 - 1)$ matrix, the objective function of Eq. (12) becomes

$$\begin{aligned} f(A) &= \frac{1}{mn}\left[1 + \frac{2}{n}\|\mathbf{y}\|^2 + 2\mathrm{Tr}\left\{B\left(\sqrt{\frac{2}{m}}\mathbf{x} + \frac{2\sqrt{2}}{n\sqrt{m}}T\mathbf{y}\right)\mathbf{e}^t\right\}\right. \\ &\left. + \mathrm{Tr}\left\{B\left(\frac{2}{m}\mathbf{x}\mathbf{x}^t + \frac{4}{mn}TT^t\right)B^t\right\}\right]. \end{aligned} \tag{15}$$

- For any vector $\mathbf{x}$, $\mathbf{x}\mathbf{e}^t = \frac{1}{\sqrt{m}}(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \Rightarrow \mathrm{Tr}(B\mathbf{x}\mathbf{e}^t) = \sum_{k=1}^m \mathbf{a}_\mathbf{k}.\mathbf{x} = 0$, by (14b). $T\mathbf{y}$ is vector $\Rightarrow \mathrm{Tr}(BT\mathbf{y}\mathbf{e}^t) = 0$. Thus, the optimization reduces to maximizing $g(B) := \mathrm{Tr}(BGB^t)$, where

$$G := \left(\frac{2}{m}\mathbf{x}\mathbf{x}^t + \frac{4}{mn}TT^t\right). \tag{16}$$

- $AA^t = I_m \Rightarrow BB^t = I_m - \mathbf{e}\mathbf{e}^t \xrightarrow{SVD} B = U\Sigma V^t, \quad \Sigma = \mathrm{diag}\{1, 1, \ldots, 1_{m-1}, 0\}.$

$$\begin{aligned} g(B) &= \mathrm{Tr}\left[BGB^t\right] = \mathrm{Tr}\left[U\Sigma V^t G V \Sigma^t U^t\right] \\ &= \mathrm{Tr}\left[\Sigma^t U^t U \Sigma V^t G V\right] = \mathrm{Tr}\left[\Delta V^t G V\right], \end{aligned} \tag{17}$$

$$\therefore \qquad \max g(B) = \sum_{k=1}^{m-1} \lambda_k^{\downarrow}(G) \tag{18}$$

Arbitrary bipartite $(m \otimes n)$ states, $m \leq n$ [**Rana and Parashar**, PRA 2012]

✡ Bloch Form: $\rho_{AB} = \frac{1}{mn}\left[I \otimes I + \mathbf{x}.\mu \otimes I + I \otimes \mathbf{y}.\nu + \sum_{i,j} T_{ij}\sigma_i \otimes \sigma_j \right]$

✪ Discord: $\mathscr{D}(\rho) \geq \frac{2}{m(m-1)n}\left[\|x\|^2 + \frac{2}{n}\|T\|^2 - \sum_{k=1}^{m-1} \lambda_k^{\downarrow}(\mathbf{x}\mathbf{x}^t + \frac{2}{n}TT^t) \right]$

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- All $2 \otimes n$ states saturate this bound. So, monogamy of GD could be checked for $2 \otimes d_2 \otimes d_3 \cdots \otimes d_N$ systems.
- Upper bound on MIN in a single shot:

$$\mathscr{M}(\rho) \leq \frac{4}{m(m-1)n^2} \sum_{k=1}^{m^2-m} \lambda_k^{\downarrow}(TT^t).$$

- Non-unique U, V in SVD $\Rightarrow$ non-unique optimal measurements!

Monogamy of GD

Monogamy

A correlation measure Q is monogamous iff for any N-partite (typically $N=3$) state $\rho^{12\cdots N}$,

$$Q(\rho^{12})+Q(\rho^{13})+\cdots+Q(\rho^{1N})\leq Q(\rho^{1|23\cdots N})$$

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Examples:

- ☺ C^2, $E_{\overleftarrow{D}}$, E_{sq}, Reny entropy, Distillable key etc.
- ☹ E_f, E_C, distributed entanglement etc.

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Monogamy of discord

- Zurek's discord δ may or may not be monogamous for 3-qubit $GGHZ$ but always strictly non-monogamous for GW states!
- GD is monogamous for all 3-qubit pure states, all N-qubit $GGHZ$ and GW sates; but not necessarily for 3-qubit mixed states, and pure states beyond 3 qubits.

$$|\psi\rangle = \sqrt{p}|00\cdots0_N\rangle + \sqrt{1-p}|+1\cdots1_N\rangle, \quad N \geq 3.$$

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- Clearly, all

$$p \in \left(\frac{2}{N+1}, \frac{N-1}{N+1}\right)$$

violate Eq. (19). Thus GD is not necessarily monogamous for all pure states beyond 3-qubits.

Partial Transposition (PT): Entanglement detector

Partial transposition (PT)

PT of $\rho \in \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ w. r. t. A is defined as $\rho^{T_A} = (T \otimes I)\rho$

PPT Criteria for Separability

ρ is separable $\Rightarrow \rho^{T_A} \geq 0$.

- So, $\rho^{T_A} \ngeq 0 \Rightarrow \rho$ is entangled (NPT=Entangled)
- The converse is true for $\dim(\mathcal{H}) \leq 6$
- $\rho^{T_A} \geq 0 \;\Rightarrow \rho$ is undistillable

States known to satisfy PPT criteria

- Pure states
- $2 \otimes 2$, $2 \otimes 3$, $3 \otimes 2$
- Werner and isotropic states
- $\rho = \rho^{T_A}$

Negativity ($\mathcal{N}$)

Definition [Vidal and Werner, PRA 2002]

For $m \otimes n$ $(m \leq n)$ state ρ,

$$\mathcal{N} = \mathcal{N}(\rho) = \frac{1}{m-1}\left(\|\rho^{T_A}\|_{\mathrm{tr}} - 1\right) = \frac{2}{m-1}\sum_{\lambda_i<0} |\lambda_i(\rho^{T_A})|$$

Facts/Properties

- Easily computable and so the stand alone measure for mixed states. However, fails to detect PPT entanglement.
- Does not reduce to entropy of entanglement for pure states.
- $\mathcal{N}(\sum \alpha_i |ii\rangle) = \frac{1}{m-1}[(\sum \alpha_i)^2 - 1]^2$.
- Both $\mathcal{N}$ and $\mathcal{N}^2$ are convex and monotone under LOCC, hence are legitimate entanglement measures.
- The logarithmic version is additive, not asymptotically continuous and gives upper bound to distillable entanglement!!

The conjecture $\mathcal{D} \geq \mathcal{N}^2$ [Girolami and Adesso, PRA 2011]

- In a sequence of papers, they have tried to develop an interrelation between discord and entanglement.
- The common belief is that, discord being *weaker* correlations than entanglement, the conjecture must hold.
- Interesting: $\mathcal{D}$ is geometric distance but $\mathcal{N}$ is not!

States satisfying the conjecture

- PPT states (from definition)

√ Pure states (from the explicit known values)

√ Two-qubit states (complicated arguments)

√ Werner and isotropic states (from the explicit known values)

?? $2 \otimes 3$ states (extensive numerical evidence: $\sim 10^5$ random states)

Violation of the conjecture [Rana and Parashar, PRA(R) 2012]

We need to extend a $2 \otimes 2$ result from [Sanpera *et al.*, PRA 1998]

PT of any $2 \otimes n$ state can not have more than $(n-1)$ negative eigenvalues

Proof:

- Any hyperplane (indeed, subspace) of dimension n in $\mathbb{C}^2 \otimes \mathbb{C}^n$ must contain at least one product vector. [Kraus *et al.*, PRA 2000]
- If possible, let ρ^{T_A} has n negative eigenvalues λ_i with corresponding eigenvectors $|\psi_i\rangle$.
- Expand the product vector $|e,f\rangle = \sum c_i |\psi_i\rangle \Rightarrow \langle e,f|\rho^{T_A}|e,f\rangle = \sum \lambda_i |c_i|^2 < 0$.
- But this would imply $\langle e^*, f|\rho|e^*, f\rangle < 0$ which is impossible as $\rho \geq 0$.

$\blacksquare$

Violation of the conjecture in $2 \otimes n$, for any $n > 2$

There are $2 \otimes 3$ states violating $\mathscr{D} \geq \mathscr{N}^2$

Proof:

- The optimal classical-quantum state χ satisfies $\mathrm{Tr}[\chi^2] = \mathrm{Tr}[\rho\chi]$.
- Hilbert-Schmidt norm is invariant under PT. So, $\mathscr{D} = 2\|\rho - \chi\|^2 = 2\,\mathrm{Tr}[\rho^2 - \chi^2] = 2\,\mathrm{Tr}[(\rho^{T_A})^2 - \chi^2]$
- Let the eigenvalues of ρ^{T_A} and χ be given by

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0 \geq \lambda_5 \geq \lambda_6 \tag{20a}$$

$$\text{and} \quad \xi_1 \geq \xi_2 \geq \ldots \xi_6 \geq 0 \tag{20b}$$

Then the Hoffman-Wielandt theorem gives $\|\rho^{T_A} - \chi\|^2 \geq \sum_{i=1}^{6} (\lambda_i - \xi_i)^2$ and hence we have

$$\sum_{i=1}^{6} \xi_i^2 \leq \sum_{i=1}^{6} \lambda_i \xi_i \tag{21}$$

- We also have the following constraints for a given (fixed) negativity $\mathcal{N}$,

$$|\lambda_5| + |\lambda_6| = \frac{\mathcal{N}}{2} \tag{22a}$$

$$\sum_{i=1}^{4} \lambda_i = 1 + \frac{\mathcal{N}}{2} \tag{22b}$$

$$\sum_{i=1}^{6} \xi_i = 1 \tag{22c}$$

- Now setting

$$f(\lambda, \xi) := \sum_{i=1}^{6} (\lambda_i^2 - \xi_i^2) - \frac{\mathcal{N}^2}{2} = \frac{\mathcal{D} - \mathcal{N}^2}{2} \tag{23}$$

  and using Lagrange's multiplier method repeatedly, we have

$$f \geq \frac{\mathcal{N}}{16}(2 - 5\mathcal{N}) \tag{24}$$

  So, whenever $\mathcal{N} \geq 2/5$, $\mathcal{D} \ngeq \mathcal{N}^2$.

- **However, $f \geq 0$ for the Bell States having $\mathcal{N} = 1$. The reason is that the conditions we used, are only a subset of the necessary conditions. But (24) indicates the possibility of existence of counterexamples. So, we must resort to numerical techniques.**

- And now comes the dilemma: they have already given numerical evidence with 5×10^5 random states and we are to do the same thing. But fortunately, we are able to find (at most one) counterexample with 6×10^5 random states. The proof is completed with this example.

$\blacksquare$

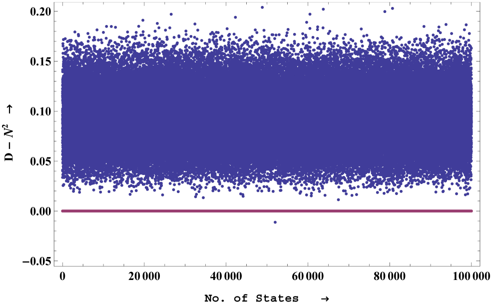

Figure: Only one counterexample in 6×10^5 random $2 \otimes 3$ states

- However, it is very easy to see that the arguments of $2 \otimes 3$ case also apply to $2 \otimes n$ states with $n > 3$. It also indicates that the violation $(\mathcal{D} - \mathcal{N}^2)$ increases with n. Indeed, the number of states violating this conjecture increases very rapidly with n.

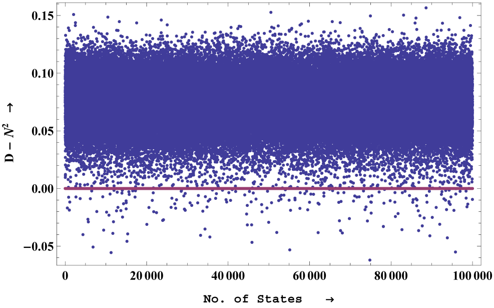

Figure: $\mathcal{D} - \mathcal{N}^2$ (dimensionless) for 10^5 random $2 \otimes 4$ states

Analytic example: $4 \otimes 4$ Werner states

- The $m \otimes m$ Werner state is given by

$$\rho_W = \frac{m-z}{m^3-m}\mathbf{I} + \frac{mz-1}{m^3-m}F, \quad z \in [-1,1]$$

where $F = \sum |k\rangle\langle l| \otimes |l\rangle\langle k|$. It is well known that

$$\mathcal{D}(\rho_W) = \left(\frac{mz-1}{m^2-1}\right)^2$$

- In general $2 \otimes 3 \neq 3 \otimes 2$: Separable or entangled? [K.-C. Ha, PRA 2010]

$$\rho = \frac{1}{6}\begin{pmatrix} 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- When seen as $2 \otimes 8$ states, $\mathcal{D}(\rho_W)$ does not change from the $4 \otimes 4$ case, but $\mathcal{N}(\rho_W)$ changes a lot.

$$\mathcal{N}_{2\otimes 8} = \begin{cases} \frac{1}{10}(-2-7z), & \text{if } z \in [-1, -\frac{2}{7}) \\ 0, & \text{if } z \in [-\frac{2}{7}, \frac{2}{3}] \\ \frac{1}{10}(-2+3z), & \text{if } z \in (\frac{2}{3}, 1] \end{cases}$$

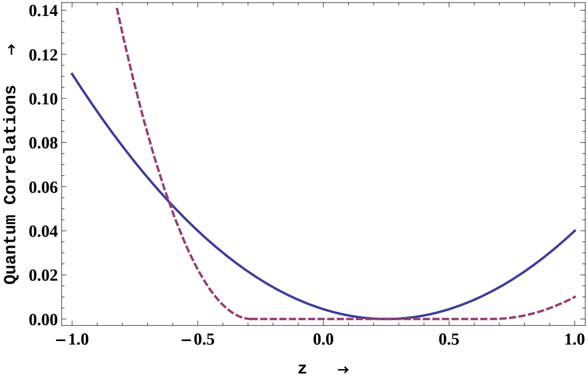

Figure: $\mathcal{D} < \mathcal{N}^2 \ \forall \ z \in [-1, -8/13)$

Discussion on Hilbert-Schmidt norm

Criteria for geometric measure of entanglement [Vedral and Plenio, PRA 1998]

If a *distance function d* satisfies

i. Positivity: $d(\rho,\sigma) \geq 0 \quad \forall\ \rho,\sigma$, with equality iff $\rho = \sigma$,

ii. Monotonicity: $d(\mathscr{E}(\rho),\mathscr{E}(\sigma)) \leq d(\rho,\sigma)$ for all CPTP map $\mathscr{E}$,

then an entanglement measure can be defined through this distance as

$$E(\rho) = \inf_{\sigma\in\{\text{Separable states}\}} d(\rho,\sigma).$$

✓ REE :

$$d(x,y) = S(x\|y) := \begin{cases} \mathrm{Tr}(x\log x - x\log y), & \text{if support } x \subseteq \text{ support } y \\ +\infty, & \text{otherwise} \end{cases}$$

✓ Bures metric: $d(x,y) = 2-2\sqrt{F(x,y)}$, where $F(x,y) := [\mathrm{Tr}\{\sqrt{x}y\sqrt{x}\}^{1/2}]^2$

✓ Trace distance: $d(x,y) = \|x-y\|_1 := \mathrm{Tr}\,|x-y|$

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✖ Hilbert-Schmidt distance $d(x,y) = \|x-y\|_2$ is *non-monotonic* [Ozawa, PLA 2000].

MDSS is a unique rank two state?

The problem

What is the maximum value of geometric discord among separable two-qubit states?

Conjecture [Gharibian, PRA 2012]: Unique MDSS of rank two with GD $1/4$.

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- *Interesting* similar inequalities:

$$\|T\|_1 := \sum_{i=1}^{3} \sqrt{\lambda_i^{\downarrow}(TT^t)} \leq 1 \tag{26a}$$

$$M(\rho) := \sum_{i=1}^{2} \lambda_i^{\downarrow}(TT^t) \leq 1. \tag{26b}$$

Unique MDS X state of two qubits [arXiv: 1311.1671]

The maximum of $\mathscr{D}$ among two qubit separable X-states is $1/4$.

Moreover, the maximal state is unique and has rank 2.

$$\rho_X = \begin{pmatrix} a & 0 & 0 & p \\ 0 & b & q & 0 \\ 0 & q & c & 0 \\ p & 0 & 0 & d \end{pmatrix},$$

- WLOG, all entries ≥ 0, as the LU transformation

$$|0\rangle_k \to \exp\left(i \frac{-\theta_p + (-1)^k \theta_q}{2} \right) |0\rangle_k$$

will drive out the phases of p, q, and neither $\mathscr{D}$ nor rank changes under LU.

- $\rho_X \geq 0$ iff $p^2 \leq ad$ and $q^2 \leq bc$
- PPT $\Leftrightarrow$ separability $\Leftrightarrow$ $p \leftrightarrow q$

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$$\Downarrow$$

- ρ_X **is a separable state iff** $\max\{p, q\} \leq \min\{\sqrt{bc}, \sqrt{ad}\}$**.**

• $\mathbf{x} = (0, 0, a+b-c-d)$ and $G = \mathrm{diag}\{4(p+q)^2, 4(p-q)^2, 2(a-c)^2 + 2(b-d)^2\}$. Therefore,

$$\sum_{i=1}^{2} \lambda_i^{\uparrow}(G) \leq 8(p^2+q^2) \tag{27a}$$

$$\leq 16 \min\{ad, bc\}, \tag{27b}$$

where equality occurs in Eq. (27a) iff

$$4(p+q)^2 \leq 2(a-c)^2 + 2(b-d)^2 \tag{28}$$

and equality occurs in Eq. (27b) iff

$$p = q = \min\{\sqrt{ad}, \sqrt{bc}\} \tag{29}$$

• As we are seeking for maximum, it follows from Eq. (27b) that the maximum occurs iff

$$ad = bc, \tag{30}$$

and the maximum value in Eq. (27) becomes $\max\{16ad\}$ subject to

$$ad = bc = \frac{1}{8}\left[(a-c)^2 + (b-d)^2\right] \tag{31a}$$

$$a+b+c+d = 1. \tag{31b}$$

- This maximum occurs at $a = b = (2 \pm \sqrt{2})/8, c = d = 1/(32a)$ and hence maximum possible value of $\mathscr{D}$ is $1/4$.

- The conditions (29) and (30) were necessary to achieve this maximum. Thus, it is necessary that the state has rank 2 and up to LU, the unique separable X-state having the maximum $\mathscr{D}$ is given by

$$\rho = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{2}+1 & 0 & 0 & 1 \\ 0 & \sqrt{2}+1 & 1 & 0 \\ 0 & 1 & \sqrt{2}-1 & 0 \\ 1 & 0 & 0 & \sqrt{2}-1 \end{pmatrix} \qquad (32)$$

$\blacksquare$

- This state is LU to

$$\sigma = \frac{1}{2}\left(|00\rangle\langle 00| + |+1\rangle\langle +1|\right) = \frac{1}{4} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \qquad (33)$$

No two qubit separable state with x = 0, or $TT^t = \lambda^2 I$ could be MD.

Clearly, an MDSS must have $\mathscr{D}(\rho) \geq 1/4$.
Now, a necessary condition for separability is $\sum \sigma_i(T) \leq 1$. So, assuming the singular values of T as $a, b, c \geq 0$, we must have

$$\begin{aligned} a+b+c &\leq 1 \\ a^2+b^2+c^2-\max\{a^2,b^2,c^2\} &\geq \frac{1}{2} \end{aligned}$$

which is clearly impossible, as the maximum of $a^2+b^2+c^2-\max\{a^2,b^2,c^2\}$ subject to the constraints $a+b+c \leq 1$ and non-negative a, b, c is $2/9 < 1/2$.

The second assertion follows by noticing that the eigenvalues of G then become $\{\|x\|^2+\lambda^2, \lambda^2, \lambda^2\}$.

Remark: The separability condition can not be ignored

❶ Werner state:

$$\rho_W = p|\Psi\rangle\langle\Psi| + \frac{(1-p)}{4} I$$

where $|\Psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, has $\mathbf{x} = \mathbf{0}$ and $\mathscr{D} = p^2$ thereby $\mathscr{D} > 1/2$ whenever $p > 1/\sqrt{2}$. Thus, separable Bell-diagonal states are never MD!

❷ The rank two (entangled) state

$$\rho_\epsilon = \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\epsilon}{3}}\right)|\Psi\rangle\langle\Psi| + \left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\epsilon}{3}}\right)|00\rangle\langle 00| \qquad (34)$$

has $D(\rho_\epsilon) = 1 - \epsilon$.
$\therefore$ The function $\mathscr{D}$ has no maximum among rank 2 two qubit states!

Conclusion

- An axiomatic measure should have the advantage of analytic expression, or at the least, it must be calculable. However, it looks there might be pay off for this advantage.
- The choice of norm (distance, metric) plays the most important role while constructing a geometric measure. Proper care has to be taken in establishing interrelations, otherwise there will certainly be many pitfalls.
- The conjecture on MDSS of two qubits is still open!
- For now, the only innocent geometric measure of discord is through the trace norm.

Thank You!