Power-law Random Banded Unitary Matrices: A new random matrix ensemble

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Plan of the talk

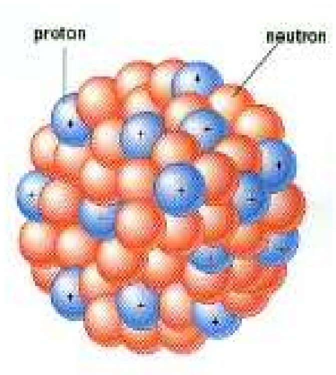
- What is Random Matrix Theory (RMT)?
- Why Random Matrices?
- Developments of RMT in last century (1950-2000)
  - RMT and Nuclear spectra
  - RMT and quantum chaotic systems
- Recent developments in RMT: Mainly its widespread applications
  - Entanglement in quantum chaotic systems
  - Spectra of complex network
  - Applications of RMT in QIP
  - Other applications
- RMT model for critical systems (eg. Anderson transition)
- Time-dependent system with critical behavior
- PRBUM ensemble
- Conclusion

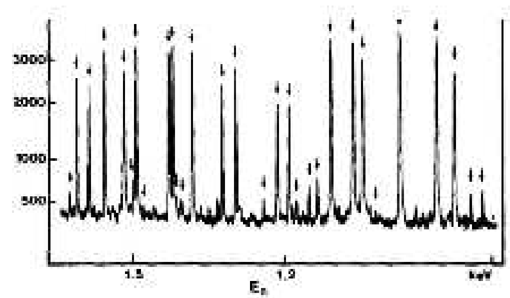
What is RMT?

- Definition: Matrices with entries drawn randomly from various probability distributions
- Three classical ensembles of random matrices: GOE, GUE, and GSE
  - GOE: Ensemble of real symmetric matrices (O = Orthogonal)
  - GUE: Ensemble of complex Hermitian matrices (U = Unitary)
  - GSE: Complex self-adjoint quaternion matrices (S = Symplectic)
- Elements of these matrices are independent, normally distributed, mean zero and the variances are adjusted to ensure the invariance of its joint probability distribution under similarity transformations (Orthogonal, Unitary, Symplectic)
- Ensemble of unitary matrices having the same invariance properties form another important classes of RM ensemble: COE, CUE, and CSE (C = Circular).
- Eigenvalues of circular ensembles are restricted on the unit circle in the complex plane

Why RMT?

- Origin of RMT could be traced back to work of Wishart (in 1928) in the field of statistics
- He constructed an ensemble of random matrices of the form $W = A^T A$, where the elements of A are sample data points. The eigenvectors of W are the Principal Components (correlated variables into a set of values of linearly uncorrelated variables) of the data set
- In Physics, RMT got its importance into the study of Excitation spectrum of heavy nuclei (eg. U_{238}. This data was available in plenty during 1950's from the neutron scattering experiment





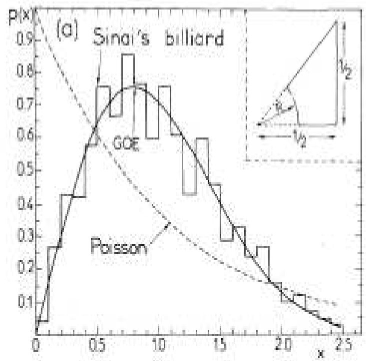
- Protons and neutrons in the nucleus of U_{238} interact with each other in a complicated way. The Hamiltonian is too complex. Its spectrum is difficult to compute either theoretically or by simulation.

Why RMT?

- Wigner suggested that the fluctuations in the excitation spectrum can be described in terms of statistical properties of the eigenspectrum of very large real symmetric random matrices.
- RMT is a ‘new’ kind of statistical theory. In statistical mechanics, we do not ask the question about the exact state of the system. In RMT, we renounce the knowledge about the nature of the system itself.
- Its basic assumption is that, for complete lack knowledge about the system (Hamiltonian), it is wise to treat the system as a ‘black box’ and adopt a kind of statistical description.
- The input for this theory should be very general properties of underlying generic Hamiltonians, eg. Hermiticity, time-reversal symmetry, any other symmetries, etc.

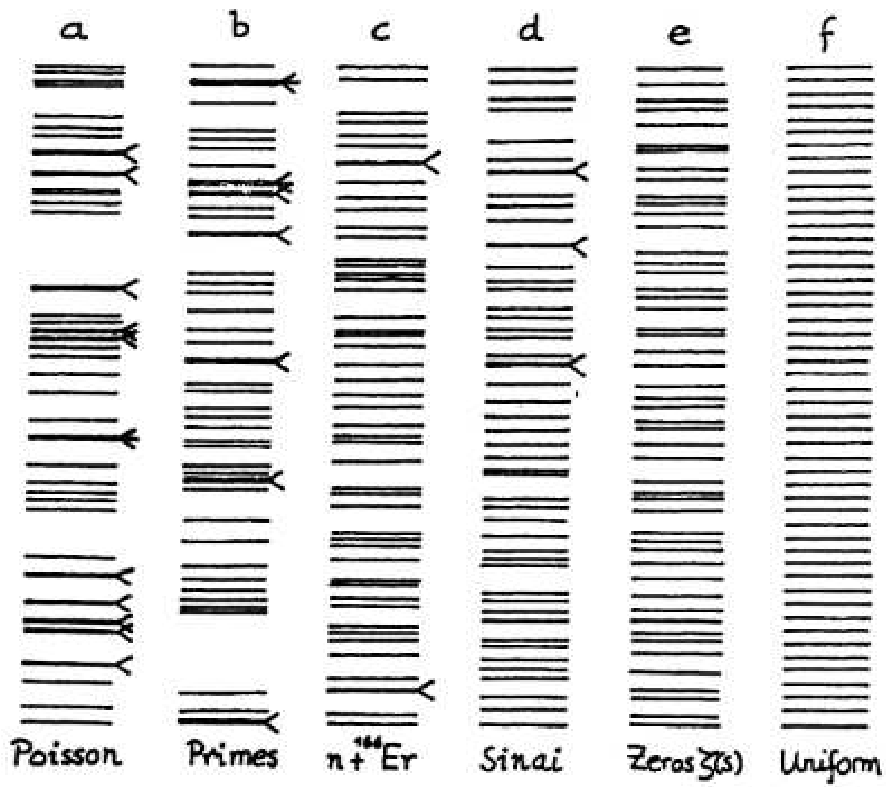
Developments of RMT (1950-2000)

- Study revealed an important feature of nuclear energy levels, that is “level repulsion" - absence of two very close energy levels
- Eigenvalues of Hermitian random matrices whose elements are Gaussian distributed random numbers also show the similar “level repulsion"
- In 1985, Bohigas et al identified the similar “level repulsion" property in quantum chaotic spectrum



- The level repulsion property reflects in the fluctuations of the energy levels which is observed in the distribution of the nearest neighbor spacing s between two energy levels:
  - $P(s) = As^{\beta} \exp(-Bs^2)$ with $P(0) = 0$ (level repulsion)
- In case of non-chaotic or regular system, the level repulsion is absent and the spacing distribution is Poisson:
  - $P(s) = \exp(-s)$ with $P(0) = 1.0$ (level clustering)

RMT in different scenarios



Recent developments in RMT (its Applications)

....and “my encounter” with it

- Entanglement in quantum chaotic systems
- Spectra of complex network
- Spectra of critical systems

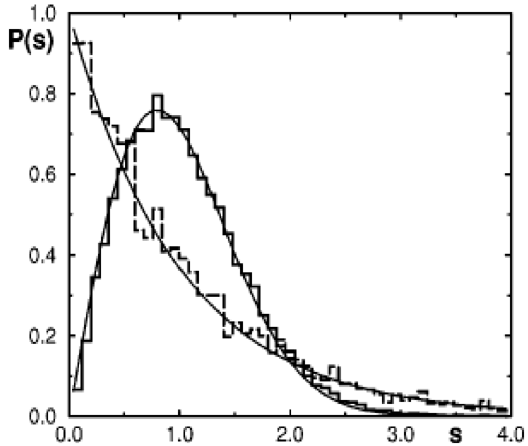
Entanglement in Quantum Chaotic Systems : Motivation

- Quantum Computer (QC) : a collection of many particles (qubits)
- The energy between the two states of the qubits may fluctuate from one qubit to another
- An interaction between the qubits is necessary to implement two-qubit gates
- This interaction is switched on and off to use the gates, but this cannot be done with perfect accuracy and there will be residual random couplings that act permanently on the system
- Georgeot and Shepelyansky modelled a quantum computer with residual random coupling by

$$H = \sum_i \Gamma_i \sigma_i + \sum_{i<j} J_{ij} \sigma_i \sigma_j$$

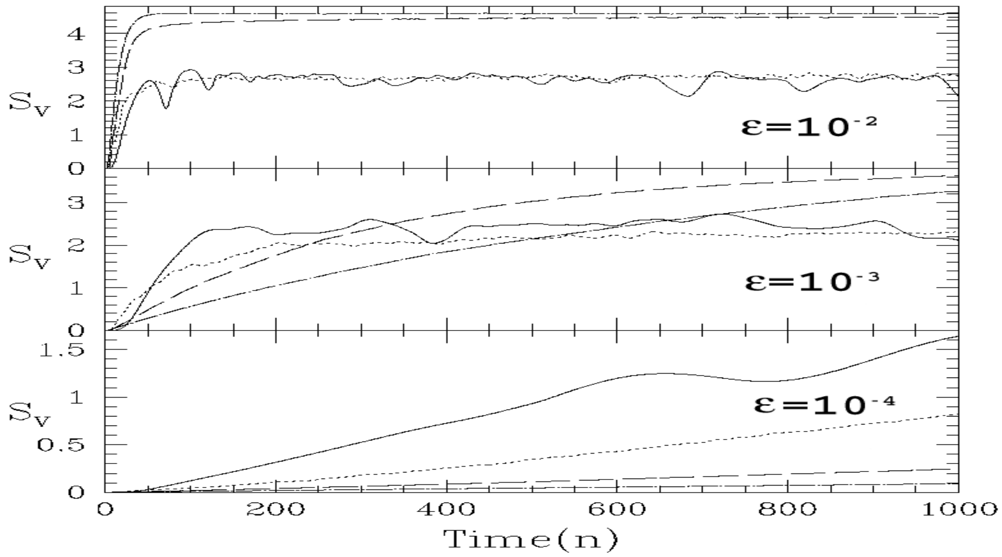
- The nearest neighbor interaction strength J_{ij} are random uniform in $[-J, J]$; Γ_i is random $[\Delta - \delta/2, \Delta + \delta/2]$, where Δ is the average energy difference between two states of one qubit

- Georgeot and Shepelyansky showed : There exists a critical value of the interaction strength $J = J_c$, where $J_c \sim \delta/n_q$, the quantum computer model shows quantum chaos



- Entanglement is probably the most crucial property of quantum computer
- Georgeot and Shepelyansky showed the presence of quantum chaos in quantum computer
- The above facts link "Entanglement" and "Quantum Chaos"

Entanglement production in coupled chaotic system



$k = 1$ (solid), $k = 2$ (dotted), $k = 3$ (dashed), $k = 6$ (dashed-dot)

Entanglement measure : von-Neumann entropy

$$S_V = -\text{Tr}\rho \ln \rho = -\sum_{m=1}^{N} \lambda_m \ln \lambda_m$$

- Important results :
  - In general, presence of chaos in a system produces more entanglement
  - There exists a typical upper bound on entanglement which is a function Hilbert space dimensions of the systems [von-Neumann entropy : $S_V = \ln(0.6N)$]
  - This bound is universal : shown by many different quantum chaotic systems

- We pointed out, from RMT, the eigenvalue distribution of Reduced Density matrix (when the total system is bipartite and its overall state is pure):

$$f(\lambda) = \frac{1}{2\pi}\sqrt{\frac{4-\lambda}{\lambda}}$$

- The RDM for this case is identical to the Wishart matrix, i.e., $\rho = A^T A$ or $\rho = A^\dagger A$.
- Using this distribution, we estimated the von Neumann entropy

$$S_V = \ln N - \frac{1}{2} = \ln(\gamma N) \text{ with } \gamma = 1/\sqrt{e} = 0.6065$$

References:

JNB and A. Lakshminarayan, Phys. Rev. Lett. 89, 060402 (2002)

JNB and A. Lakshminarayan, Phys. Rev. E 69, 016201 (2004)

Spectra of complex network

- Many complex systems can be viewed as a distributed system of many interrelated parts, and the system can be represented in terms of (complex) networks
- Complex network consists of two things:
  - Nodes = A collection of entities which have properties that are somehow related to each other (e.g., people, rivers, proteins, webpages,...)
  - Links = Connections between nodes
    - may be real and fixed (rivers)
    - real and dynamic (airline routes)
    - abstract with physical impact (hyperlinks)
    - Links may be directed or undirected (A can influence B, but not the vice versa)
    - Links may be binary or weighted

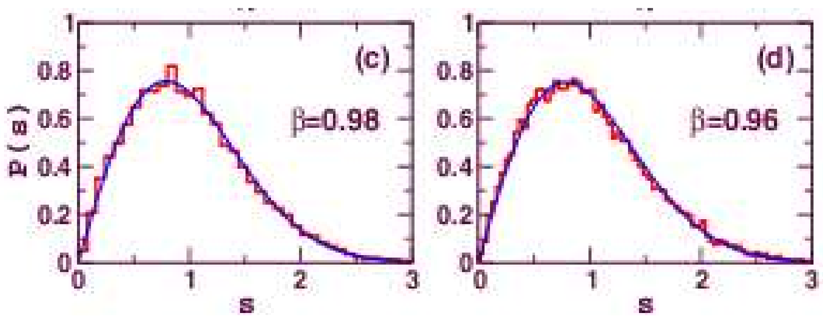
Binary: either connected or not connected

Weighted: how much strength of the connections

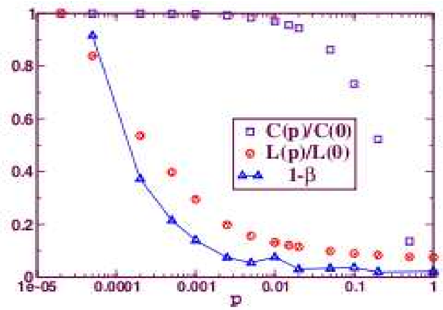
- Adjacency matrix: a graph or network is represented by a matrix A with link weight a_{ij} for nodes i and j in entry (i, j)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{Binary and Directed}$$

- We study the statistical properties of the eigenvalues of the adjacency matrix representing binary and undirected network: Artificially (computer) generated scale-free network and Real world network



- We studied Strogatz algorithm for the generation of Small-World network
<u>Small-world network</u>: most nodes are not neighbors of one another, but most nodes can be reached from every other by a small number of steps.
- For Strogatz algorithm, we observed that eigen-spectrum of Adjacency matrix makes transition from Poisson to GOE exactly at the point where SW transition takes place



References:

JNB and S. Jalan, Phys. Rev. E 76, 026109 (2007)

S. Jalan and JNB, Phys. Rev. E 76, 046107 (2007)

S. Jalan and JNB, EPL 87, 48010 (2009)

RMT and QIP

- Generation of pseudo-random numbers:
  - The fundamental role of random numbers in classical information theory, is played by random unitary operators
  - Emerson et al proposed a method to generate pseudo-random unitary operators that can reproduce those statistical properties of random unitary operators most relevant to quantum information tasks.
  - This method requires exponentially fewer resources, and hence enables the practical application of random unitary operators in quantum communication and information processing protocols
  - Random unitary operators also allow the construction of more efficient data-hiding schemes and provide a means to reduce the key length required for the (approximate) encryption of quantum states
- Very recently, random matrices are used for quantum data locking which is a uniquely quantum protocol that allows for a small secret key to lock an exponentially longer message by encoding it into a quantum state

Ref: J. Emerson et al, Science 302, 2098 (2003); C. Lupo, M. M. Wilde, S. Lloyd, arXiv:1311.5212v1 [quant-ph]

RMT and Quantum Critical system

- The standard Anderson model on a $3D$ simple cubic lattice : a tight-binding Hamiltonian with elements

$$H_{ij} = \epsilon_i \delta_{ij} + t_{ij} \quad : \quad i, j \text{ lattice sites}$$

  - Site energies $\{\epsilon_i\}$ are randomly distributed within an interval $-\frac{W}{2} < \epsilon_i < \frac{W}{2}$.
  - Off-diagonal elements $t_{ij} = 1$ for the nearest neighbors and $t_{ij} = 0$ for otherwise.
  - Depends on W, eigenstates of the Anderson Hamiltonian make delocalization-localization (metal-insulator) transition

- Power-law Random Banded Matrix (PRBM) model:
  - Ensemble of random Hermitian matrices : $\{H_{ij}\}$ are independently distributed Gaussian random numbers with $\langle H_{ij} \rangle = 1$ and the variance

$$\sigma^2(H_{ij}) = \left[1 + \left(\frac{|i-j|}{b} \right)^{2\alpha} \right]^{-1}$$

  - $\alpha = 1$ is the critical point (delocalized or extended state $\leftrightarrow$ localized state) and $0 < b < \infty$ characterizes the ensemble
  - Interpretation: it describes an $1D$ sample with random long-range hopping and the hopping amplitudes decay as $|i - j|^{-1}$

Time-dependent quantum critical system

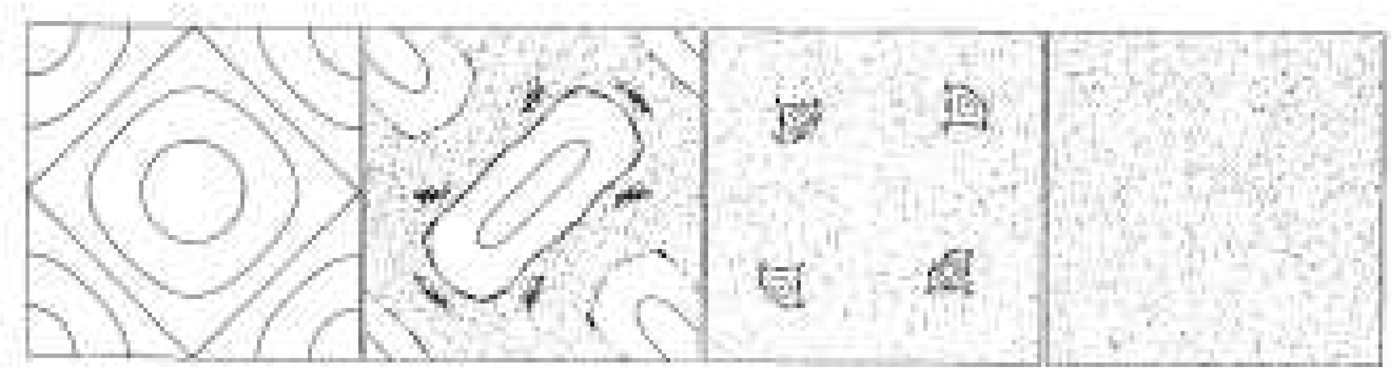
Kicked Harper model:

- Hamiltonian:

$$H(t) = L\cos p + K\cos x \sum_n \delta(t-n)$$

- Classical Map :

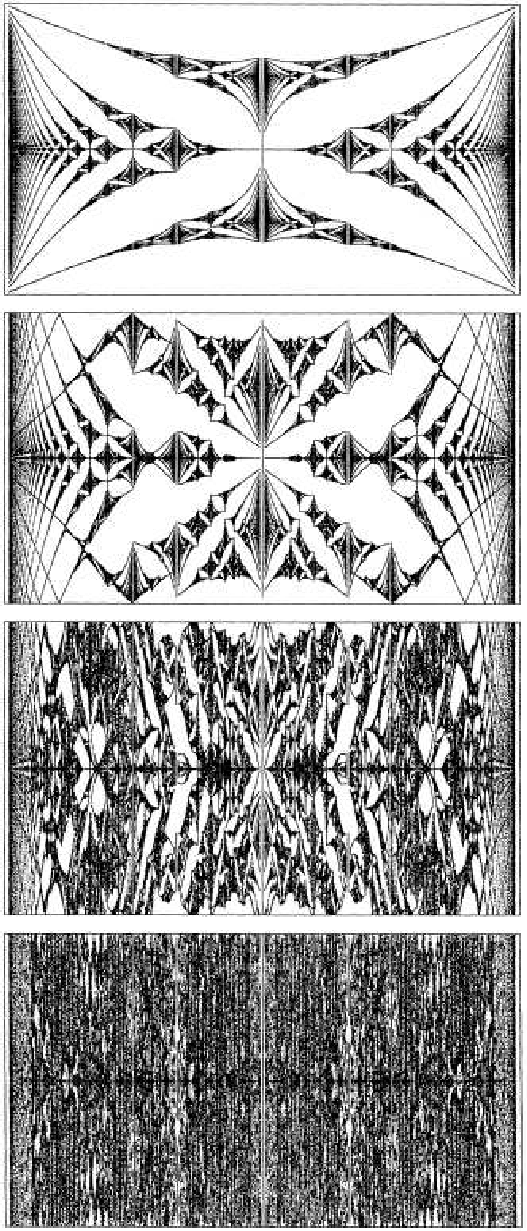
$$p_{n+1} = p_n + K\sin x_n, \quad x_{n+1} = x_n - L\sin p_n$$



Here $K = L = 0.01, 2.0, 3.5$, and 5.0

- Time-evolution operator :

$$U = \exp\left[-i\frac{L}{\hbar}\cos p\right]\exp\left[-i\frac{K}{\hbar}\cos x\right]$$

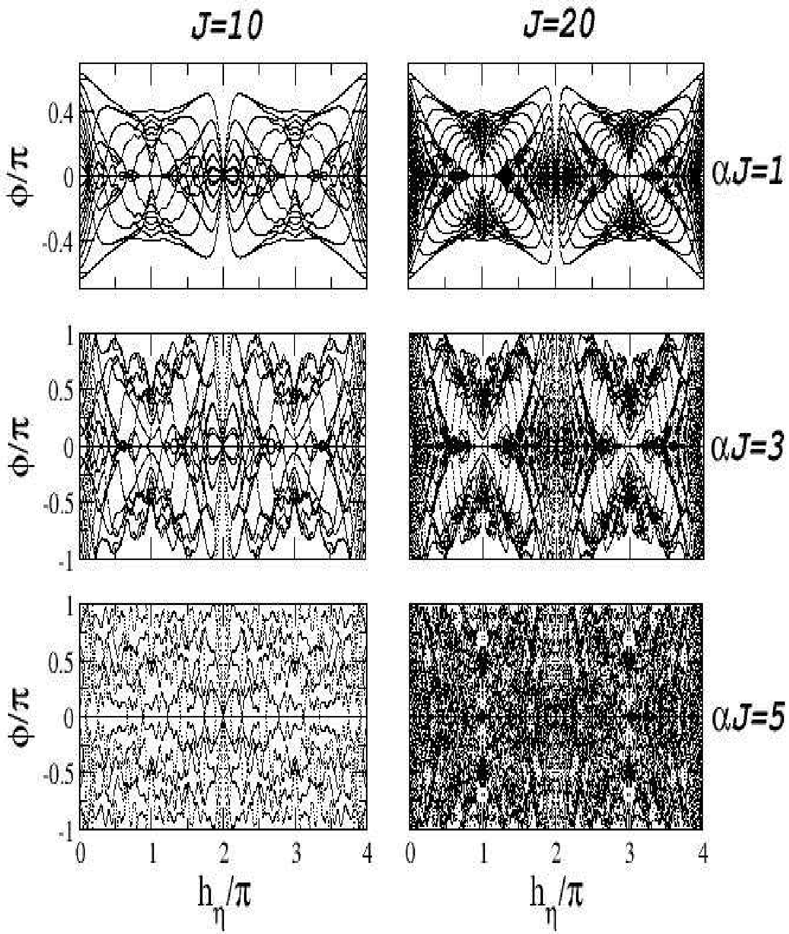


$\hbar$

- Quasienergy spectrum $[U|\phi\rangle = e^{i\phi}|\phi\rangle]$ of the KHM is (multi)fractal
- Figures are plotted for different $K = L = \kappa$
- Here underlying classical dynamics is chaotic, but the spectrum does not follow Bohigas-Giannoni-Schmit conjecture...interesting!!!

Another example

$$U_{DKRM} = \exp\left[-i\frac{(T-\eta)}{2\hbar}p^2\right]\exp\left[-i\frac{K_2}{\hbar}\cos q\right]\exp\left[-i\frac{\eta}{2\hbar}p^2\right]\exp\left[-i\frac{K_1}{\hbar}\cos q\right]$$



- A natural objective : to see how the criticality of the unitary operators different from the critical Hermitian operators

PRBUM: Algorithm

- Generation of random matrix ensemble for critical Hermitian matrices (PRBM) was very easy
- A major obstacle for the Unitary version of that ensemble is how to generate
- We use Mezzadri's algorithm which was proposed to generate random unitary matrices (both COE and CUE) from arbitrary complex random matrices
- Our scheme :
  - We start with an ensemble of PRBM
  - Then apply Mezzadri's algorithm to get random unitary matrices
  - Variance of the elements of these random matrices also roughly follows power-law : hence the name 'PRBUM'
  - Our scheme is successful : eigenstates of the ensemble of these unitary matrices indeed show multifractality
  - We can tune the parameters α and b of the initial PRBM to generate critical random unitary matrices with desired multifractal property

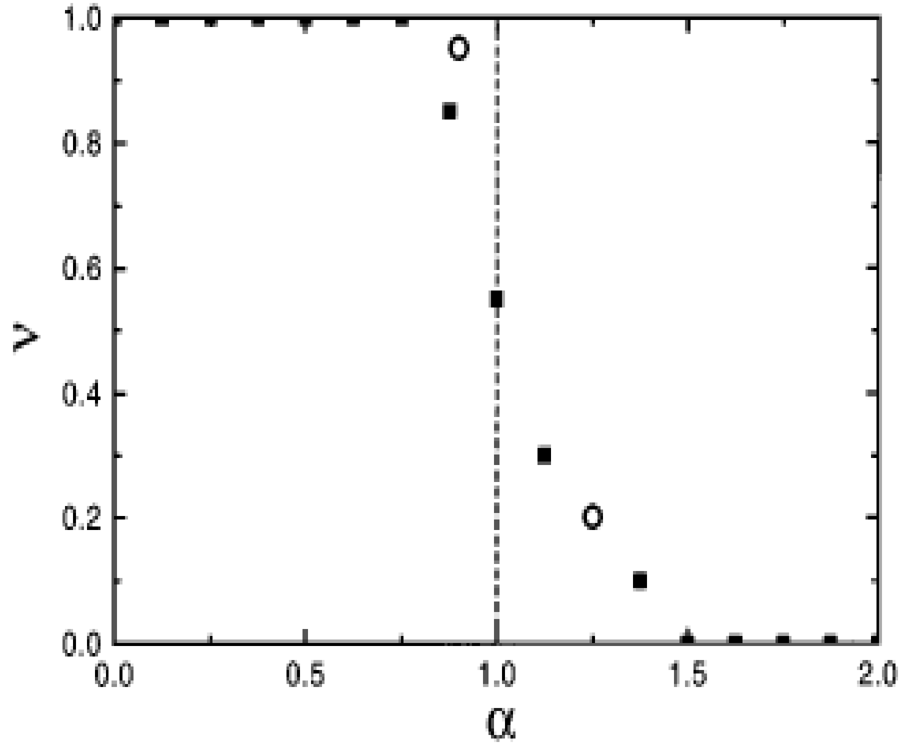
References:

JNB, J. Wang, J. B. Gong, Phys. Rev. E 81, 066212 (2010)

JNB and J. B. Gong, EPJB 85, 335 (2012)

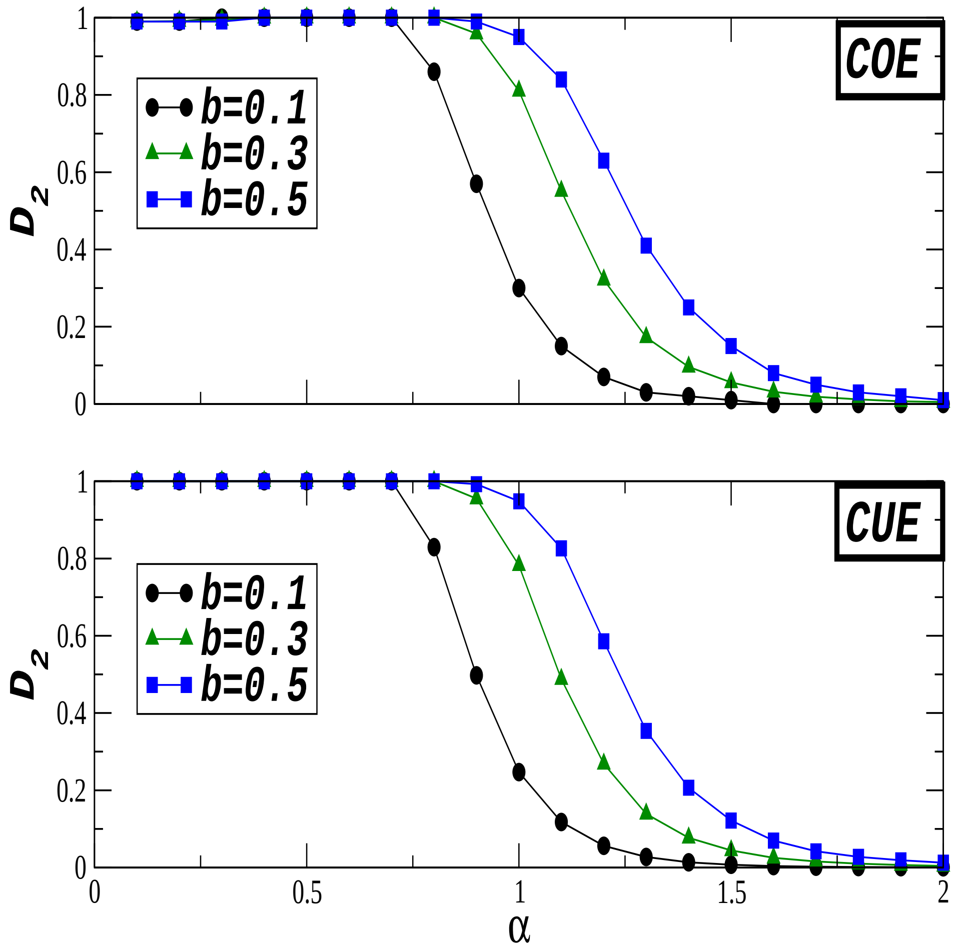
PRBUM vs PRBM

PRBM : α Vs. D_2



- Here, ν is D_2
- Transition point of Extended State $\leftrightarrow$ Localized State : $\alpha = 1.0$
- Ref. A. D. Mirlin et. al., PRE **54**, 3221 (1996).

PRBUM : α Vs. D_2



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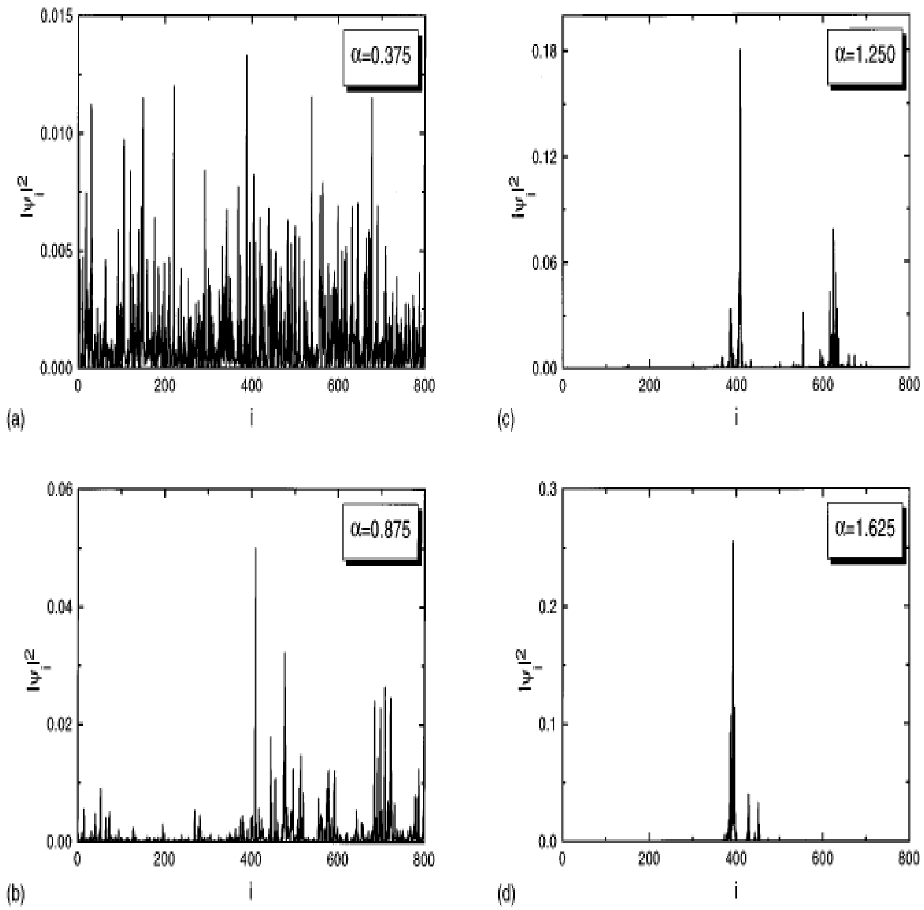
PRBM : Eigenstates



- $b = 1.0$
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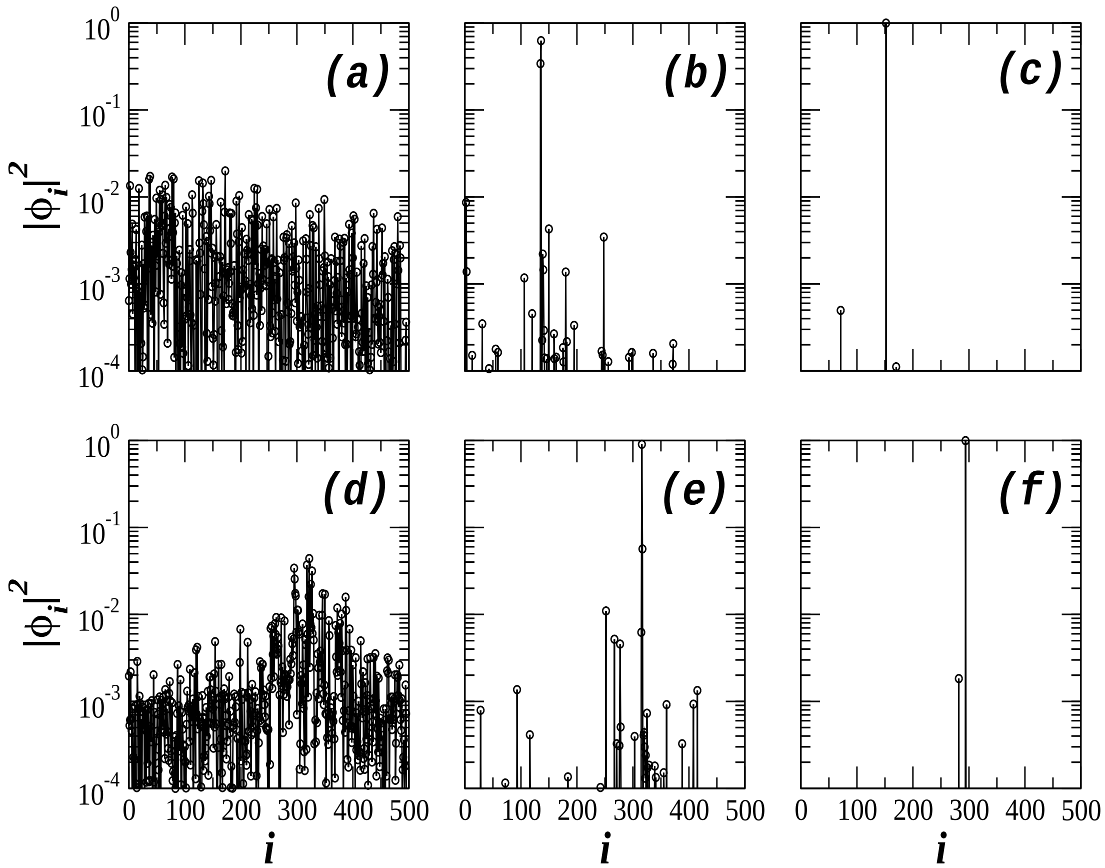
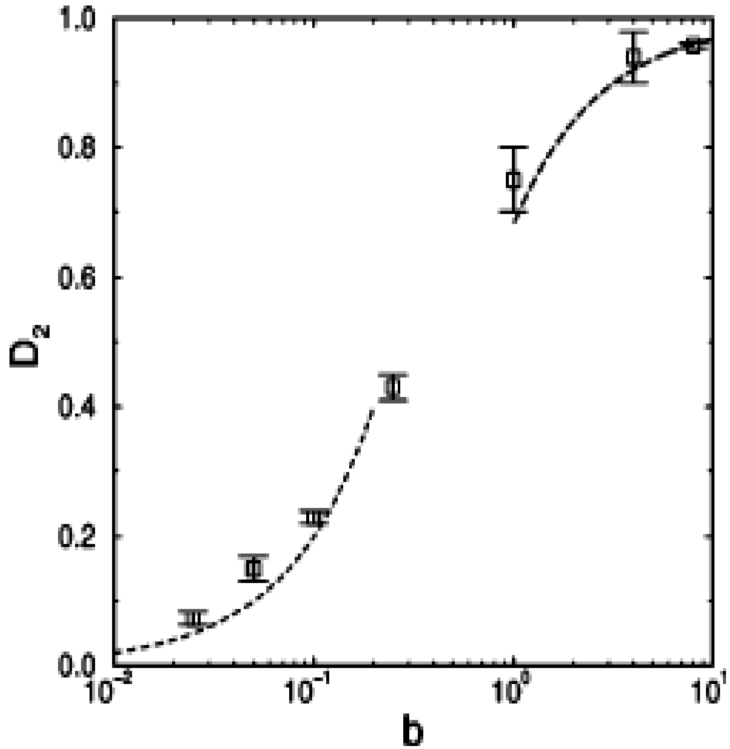
PRBUM : Eigenstates



- $b = 0.1$
- COE : (a)-(c) where a: $\alpha = 0.5$; b: $\alpha = 1.0$; c: $\alpha = 1.5$
- CUE : (d)-(f) where a: $\alpha = 0.5$; b: $\alpha = 1.0$; c: $\alpha = 1.5$

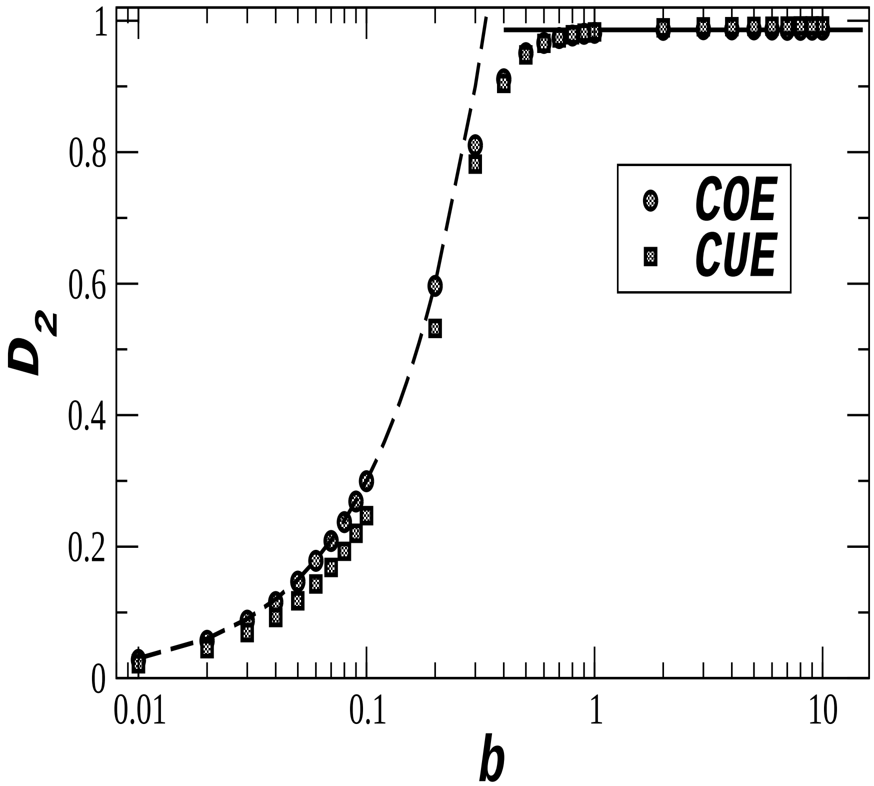
PRBM : D_2 Vs. b at the critical point $\alpha = 1.0$



From supersymmetry calculation :

- $b \ll 1 : D_2 = 2b$
- $b \gg 1 : D_2 = 1 - \frac{1}{\pi b}$
- Ref. A. D. Mirlin & F. Evers, PRB **62**, 7920 (2000).

PRBUM : D_2 **Vs.** b **at** $\alpha = 1.0$

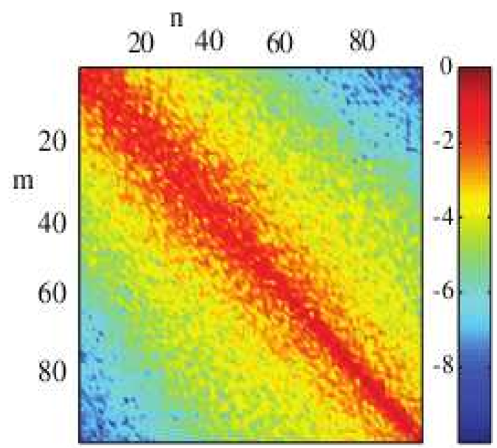


From numerics :

- $b \ll 1 : D_2 \simeq 3b$
- $b \geq 1 : D_2 \simeq 1$

Application of PRBUM

- Sound propagation in ocean can be formulated as a wave guide with a weakly random medium generating multiple scattering
- About 20 years ago, this was recognized as a quantum chaos problem, and yet RMT has never been introduced into the subject
- The modes of the wave guide provide a representation for the propagation, which in the parabolic approximation is unitary
- Scattering induced by the ocean's internal waves leads to a power-law random banded unitary matrix ensemble for long-range deep ocean acoustic propagation



- Many people are considering to extend the application of this ensemble to shallower water, higher frequencies and surface/bottom scattering

Ref: K. C. Hegewisch and S. Tomsovic, EPL 97, 34002 (2012)

Conclusion

- We have given a short introduction of RMT
- We have presented developments of RMT in last few decades
- We have discussed about its recent various applications including quantum chaos, complex networks, QIP, etc.
- We then discuss about the random matrix model for quantum critical systems
- We then discuss about time-dependent quantum critical system and introduced a new random matrix ensemble that is PRBUM
- In case of PRBUM ensemble, our numerics suggest strongly that $\alpha = 1.0$ is the transition point. However, we do not have any analytical understanding of this fact.
- For PRBM, this ensemble has been mapped with nonlinear σ model, and from this mapping, the critical point was analytically calculated as $\alpha = 1.0$

Topics which we could not discuss...

- Applications of RMT in data analysis:
  - stock-market data
  - medical data (eg. ECG, EEG, etc,)
  - atmospheric data
- Applications in Quantum graph/network
- Applications in Quantum Chromodynamics
- Applications in 2D Quantum Gravity
- Relation between the spectrum of random matrices and $1/f$ noise

  Ref. M. S. Santhanam and JNB, Phys. Rev. Lett. 95, 114101 (2005)

  M. S. Santhanam, JNB, and D. Angom, Phys. Rev. E 73, 015201(R) (2006)

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  - stock-market data
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  - atmospheric data
- Applications in Quantum graph/network
- Applications in Quantum Chromodynamics
- Applications in 2D Quantum Gravity
- Spectrum of random matrices can also be analyzed from time series analysis point of view

  Ref. M. S. Santhanam and JNB, Phys. Rev. Lett. 95, 114101 (2005)

  M. S. Santhanam, JNB, and D. Angom, Phys. Rev. E 73, 015201(R) (2006)

Thank you!