Polaron dynamics and decoherence in an interacting two-spin system coupled to optical phonon environment

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Outline

- Introduction:
  charge qubit from electron in tunnel-coupled double
    quantum dot (DQD)
  oxides good candidates for miniaturization and coherence
- Two spin interacting system locally coupled to optical phonons
- Decoherence analysis using non-Markovian master equation
- Markovian dynamics
- Quantum control of decoherence using dynamic decoupling
- Summary and conclusions

- Quantum superposition essential for quantum computation
- System-environment interaction produces decoherence or destruction of quantum superposition.
- Quantum robustness strategy can be passive [e.g., decoherence free subspaces (DFS)] or active [e.g., quantum Zeno effect (QZE)].
- Long coherence times in GaAs and Si based DQDs using QZE
- Propose oxide based DQD with small decoherence

Double quantum dot

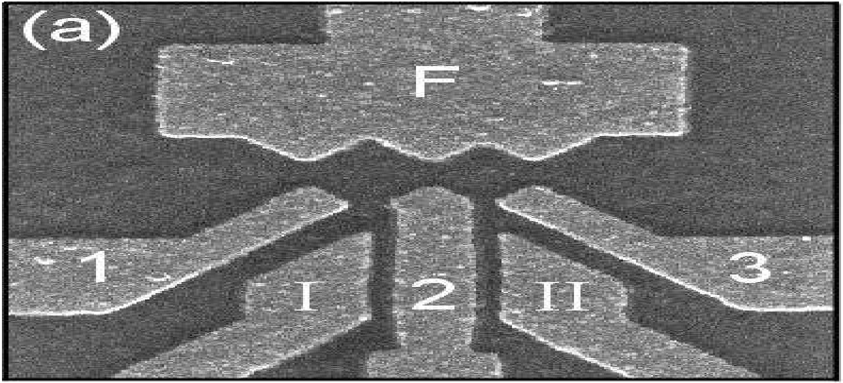

Figure: (a) SEM micrograph of a double quantum dot defined by metallic gates (light gray areas) from Kouwenhoven et al. RMP **75**, 1 (2003).

- Miniaturization demands replacement of silicon technology.
- Oxides a promising alternative due to small extent of wavefunction.
- Oxide modeling and fabrication more challenging.
- Goal to exploit tunability, rich physics, coupling between various degrees of freedom, and develop control to produce new functionalities.
- Substantial experimental evidence for strong electron-phonon interaction (EPI) in manganites (EXAFS).
- Significant progress made in understanding bulk doped oxides.
- Quantum dots from oxides a new area of research.
- Novel phenomena, with no counter part in bulk samples, emerge from quantum-dot/nano-structure physics.
- Need to technologically exploit new physics and develop new devices to meet future challenges such as miniaturization, decoherence-free and dissipationless operations, etc.

Interacting two spin system

Anisotropic Heisenberg interaction:

$$H_s = J_{\parallel} S_1^z S_2^z + \frac{J_{\perp}}{2}(S_1^+ S_2^- + S_2^+ S_1^-).$$

Local phonon Hamiltonian:

$$H_{env} = \sum_{k;i=1,2} \omega_k a_{k,i}^{\dagger} a_{k,i}.$$

Spin-phonon local interaction (strong coupling $g > 1$):

$$H_I = \sum_{k;i=1,2} g_k \omega_k S_i^z (a_{k,i} + a_{k,i}^{\dagger}).$$

Initially consider only one k-mode. Lang-Firsov transformation:

$$H_{LF} = e^S H e^{-S}$$

$S = -\sum_i g S_i^z (a_i - a_i^{\dagger}) \rightarrow$ Transformation generator.

$$H_s^{LF} = J_{\parallel} S_1^z S_2^z + \frac{J_{\perp} e^{-g^2}}{2}(S_1^+ S_2^- + S_2^+ S_1^-)$$

$\Rightarrow$ spins coupled to the mean phonon field and with reduced hopping amplitude due to formation of polaron. Harmonic oscillators are displaced.

$$H_{env}^{LF} = \sum_{i=1,2} \omega a_i^{\dagger} a_i .$$

$$H_I^{LF} = \frac{1}{2}[J_{\perp}^{+} S_1^{+} S_2^{-} + J_{\perp}^{-} S_2^{+} S_1^{-}]$$

$\Rightarrow$ Spins coupled to local phonon fluctuations around mean field. This contains uncontrolled degrees of freedom.

$$J_{\perp}^{\pm} = J_{\perp} e^{\pm g[(a_2 - a_2^{\dagger}) - (a_1 - a_1^{\dagger})]} - J_{\perp} e^{-g^2}$$

$$\langle J_{\perp} e^{\pm g[(a_2 - a_2^{\dagger}) - (a_1 - a_1^{\dagger})]} \rangle_{T=0} = J_{\perp} e^{-g^2}$$

For fixed $S_T^z (= 0)$ only the spin flipping part in the Hamiltonian H_s contributes to the excitation gap.
The two $S_T^z = 0$ eigenstates are :
$|\varepsilon_t\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$
and
$|\varepsilon_s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

So, the energy gap is much smaller than the phonon energy in the strong coupling ($g^2 \gg 1$) and non-adiabatic ($\frac{J_\perp}{\omega} \leq 1$) limit:
$J_\perp e^{-g^2} \ll \omega$.

Decoherence analysis using non-Markovian master equation

Using time convolutionless (TCL) projection operator technique the non-Markovian master equation up to second order in perturbation is given by

$$\frac{d\tilde{\rho}_s(t)}{dt} = -\int_0^t d\tau \, Tr_R[\tilde{H}_I(t), [\tilde{H}_I(\tau), \tilde{\rho}_s(t) \otimes R_0]].$$

where $\tilde{\rho}_s(t) \equiv Tr_R[\tilde{\rho_T}(t)]$
Assume initially $\rho_T(0) = \rho_s(0) \otimes R_0$.
Initial Bath state: $R_0 = \sum_n \frac{e^{-\beta\omega_n}}{Z} |n\rangle_{ph} \, {}_{ph}\langle n|$.

Interaction picture: $\tilde{H}_I(t) = e^{iH_o t} H_I e^{-iH_o t}$ and
$\tilde{\rho}_T(t) = e^{iH_o t} \rho_T(t) e^{-iH_o t}$
where $H_0 = H_s + H_{env}$

Preparation of separable intial state $\rho_T(0) = \rho_s(0) \otimes R_0$

Start with gate voltage set to $J_\perp = 0$ and introduce an electron in one of the quantum dots to obtain the state $|10\rangle \propto |\varepsilon_s\rangle + |\varepsilon_t\rangle$.

Introduce small tunneling $J_\perp/\omega \ll 1$ (say 10^{-3}) rapidly and let the system evolve.

For small $J_\perp/\omega$, $|\varepsilon_s\rangle$ and $|\varepsilon_t\rangle$ are approximate eigenstates (of the Hamiltonian in the LF frame) with probability larger than $1 - J_\perp^2/(g^4\omega^2)$ (i.e.,> 0.999999).

The evolved state is a general separable initial state (in the dressed basis) given by:

$$|\psi(t)\rangle \propto [e^{i\phi} \cos(J_\perp e^{-g^2} t/2)|10\rangle - i \sin(J_\perp e^{-g^2} t/2)|01\rangle] \otimes |0,0\rangle_{ph}$$

where $e^{i\phi}$ is Aharnov-Bohm phase factor due to a magnetic flux.

Change the gate voltage and the magnetic flux rapidly to get the desired value of tunneling $J_\perp$.

The master equation simplifies as:

$$
\begin{aligned}
\frac{d\langle\varepsilon_s|\tilde{\rho}_s(t)|\varepsilon_t\rangle}{dt} &= -K\Big[\sum_{m_1,m_2}{}^{\prime} C_{m_1,m_2}\langle\varepsilon_s|\tilde{\rho}_s(t)|\varepsilon_t\rangle\frac{sin(\omega_{m_1,m_2}t)}{\omega_{m_1,m_2}} \\
&+ \sum_{|m_1-m_2|=odd} C_{m_1,m_2}\langle\varepsilon_t|\tilde{\rho}_s(t)|\varepsilon_s\rangle\frac{sin(\omega_{m_1,m_2}t)}{\omega_{m_1,m_2}} \\
&+ \sum_{|m_1-m_2|=even,0}{}^{\prime} C_{m_1,m_2}\langle\varepsilon_s|\tilde{\rho}_s(t)|\varepsilon_t\rangle\frac{sin(\omega_{m_1,m_2}t)}{\omega_{m_1,m_2}}\Big],
\end{aligned}
$$

$\sum'_{m_1,m_2} \rightarrow$ sum over all (m_1, m_2) values excluding $m_1 = m_2 = 0$.
$\omega_{m_1,m_2} = \omega_{m_1} + \omega_{m_2} = \omega(m_1 + m_2)$, $K = \frac{J_\perp^2}{2}e^{-2g^2}$ and
$C_{m_1,m_2} = \frac{g^{2(m_1+m_2)}}{m_1!m_2!}$.

For large coupling strength ($g^2 \gg 1$), the long time values of the matrix elements are estimated as:

$$
\begin{aligned}
|\langle \varepsilon_s|\rho_s(t)|\varepsilon_t\rangle|\Big|_{t\to\infty} &= |\langle \varepsilon_s|\rho_s(0)|\varepsilon_t\rangle|\; exp\Big[-\frac{1}{4g^2}\Big(\frac{J_\perp}{g\omega}\Big)^2\Big] \\
\langle \varepsilon_s|\rho_s(t)|\varepsilon_s\rangle\Big|_{t\to\infty} &= \frac{1}{2}\langle \varepsilon_s|\rho_s(0)|\varepsilon_s\rangle\Big\{1 + exp\Big[-\frac{1}{8g^2}\Big(\frac{J_\perp}{g\omega}\Big)^2\Big]\Big\} \\
&\quad +\frac{1}{2}\langle \varepsilon_t|\rho_s(0)|\varepsilon_t\rangle\Big\{1 - exp\Big[-\frac{1}{8g^2}\Big(\frac{J_\perp}{g\omega}\Big)^2\Big]\Big\}
\end{aligned}
$$

Here, the initial density matrix $\rho_s(0)$ is considered to be real.

The coherence factor:

$$D(t) = \frac{|\langle \varepsilon_s|\rho_s(t)|\varepsilon_t\rangle|}{|\langle \varepsilon_s|\rho_s(0)|\varepsilon_t\rangle|}$$
$$= exp[-2K \sum_{m_1,m_2}' C_{m_1,m_2} \frac{(1-cos(\omega_{m_1,m_2} t))}{\omega^2_{m_1,m_2}}]$$

Inelastic factor (indicating dissipation) or population difference:

$$P(t) = \frac{\langle \varepsilon_s|\rho_s(t)|\varepsilon_s\rangle - \langle \varepsilon_t|\rho_s(t)|\varepsilon_t\rangle}{\langle \varepsilon_s|\rho_s(0)|\varepsilon_s\rangle - \langle \varepsilon_t|\rho_s(0)|\varepsilon_t\rangle}$$
$$= exp[-2K \sum_{|m_1-m_2|=odd} C_{m_1,m_2} \frac{(1-cos(\omega_{m_1,m_2} t))}{\omega^2_{m_1,m_2}}]$$

Plots:

coherence factor:

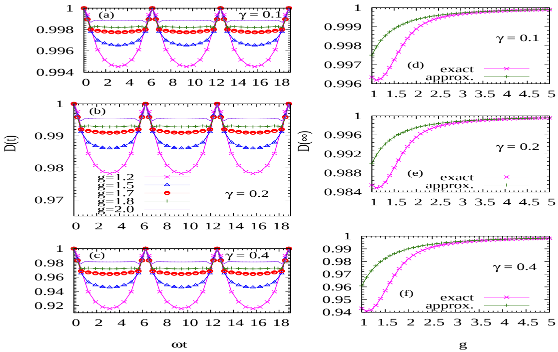

Figure: $\gamma = \frac{J_{\perp}}{g\omega}$. For large values of g, $D(\infty)$ match with the D(t) values between $2n\pi$ and $2(n+1)\pi$ values of ωt.

Including the effect of small $J_\perp e^{-g^2}/\omega$:

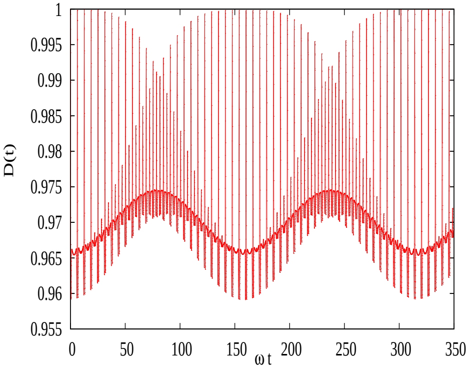

Figure: $J_\perp e^{-g^2}/\omega = 0.02$ and $g = 2$.

Including large number of bath modes ($0.9\omega_c \leq \omega_k \leq \omega_c$):

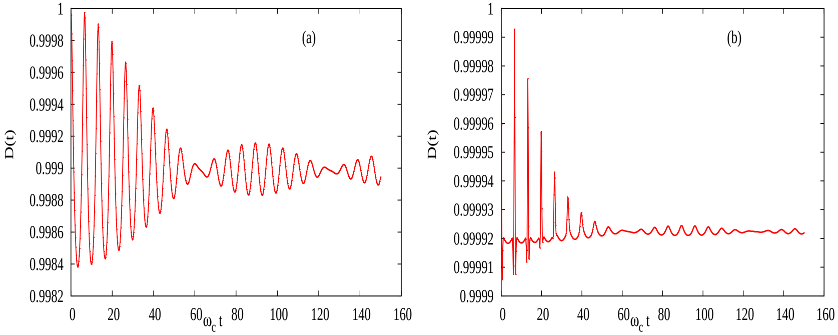

Figure: (a) $J_\perp/\omega_c = 0.05$ and $\frac{\sum_k g_k^2}{N} = 1$; (b) $J_\perp/\omega_c = 0.05$ and $\frac{\sum_k g_k^2}{N} = 4$.

Inelasticity (dissipation):

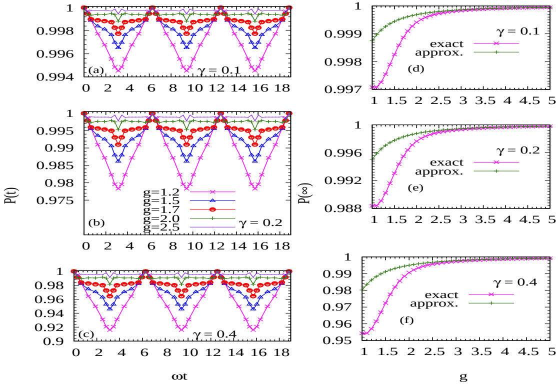

Figure: $\gamma = \frac{J_\perp}{g\omega}$. For large values of g, $P(\infty)$ match with P(t) at ωt values between two consecutive multiples of π .

Decoherence and dissipation are less for smaller γ and larger g values.

Markovian dynamics

As $J_\perp e^{-g^2} \ll \omega$, we can assume that the time scale over which the system considerably changes is much larger than the correlation time for the environmental fluctuations, $\tau_s \gg \tau_c$. In the TCL, raise the upper limit of integration to ∞.

$$\frac{d\tilde{\rho}_s(t)}{dt} = -\int_0^\infty d\tau\, Tr_R[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_s(t) \otimes R_0]].$$

$\Rightarrow$ does not keep memory as the environmental dynamics is not resolved in system time scale.
By following the same analysis as for non-Markovian case, we get

$$\frac{d\tilde{\rho}_s(t)}{dt} = i \sum_n \left[\frac{|_{ph}\langle 0|H_I|n\rangle_{ph}|^2}{\omega_n} \tilde{\rho}_s(t) - \tilde{\rho}_s(t) \frac{|_{ph}\langle 0|H_I|n\rangle_{ph}|^2}{\omega_n} \right].$$

Solution:

$$\begin{aligned}
\langle \varepsilon_s|\rho_s(t)|\varepsilon_t\rangle &= \langle \varepsilon_s|\rho_s(0)|\varepsilon_t\rangle e^{-i(\varepsilon_s-\varepsilon_t)t}, \\
\langle \varepsilon_t|\rho_s(t)|\varepsilon_s\rangle &= \langle \varepsilon_t|\rho_s(0)|\varepsilon_s\rangle e^{-i(\varepsilon_t-\varepsilon_s)t}, \\
\langle \varepsilon_s|\rho_s(t)|\varepsilon_s\rangle &= \langle \varepsilon_s|\rho_s(0)|\varepsilon_s\rangle, \\
\langle \varepsilon_t|\rho_s(t)|\varepsilon_t\rangle &= \langle \varepsilon_t|\rho_s(0)|\varepsilon_t\rangle.
\end{aligned}$$

No decoherence is observed.

Up to second order in perturbation, the dominant process for the effective system is twice the adjacent spin flipping simultaneously. The energy scale for this is $\frac{J_\perp^2}{g^2\omega}$. So, the condition $\gamma^2 = (\frac{J_\perp^2}{g^2\omega})/\omega \ll 1$ leads to the condition $\tau_s \gg \tau_c$ for Markovian dynamics. It is evident from the figures that, with decreasing γ, decoherence becomes less.

Quantum control for non-Markovian case

Due to the system-environmental coupling, the environment can distinguish among different states of the system. Thus different states acquire random relative phases and the reduced system faces decoherence.

Strategy for protection:

The system is perturbed much faster than the environment response time; the environment can not follow the change of system states anymore. So, the system is effectively decoupled from the environmental fluctuations.

Driving pulse: $P_\pi = S_1^+ S_2^- + S_2^+ S_1^- \Rightarrow$ flips both the spins simultaneously.

Composite evolution operator:
$U_1 = \tilde{U}(t + 2\delta t, t + \delta t) P_\pi \tilde{U}(t + \delta t, t) P_\pi$
where

$$\tilde{U}(t, t^{'}) = \tilde{U}(t, 0)\tilde{U}^{\dagger}(t^{'}, 0) = e^{iH_0 t} e^{-iHt} e^{iHt^{'}} e^{-iH_0 t^{'}}$$
$$= e^{iH_0 t} e^{-iH(t-t^{'})} e^{-iH_0 t^{'}}.$$

Now,

$$P_\pi e^{-i(H_0 + H_I)\delta t} = e^{-i(H_0 - H_I)\delta t} P_\pi$$

$\Rightarrow P_\pi$ pulse changes the sign of interaction. Thus applying P_π rapidly, produces decoupling from the environment.

$$\begin{aligned} U_1 &= \tilde{U}(t+2\delta t, t+\delta t)P_\pi \tilde{U}(t+\delta t, t)P_\pi \\ &= I + O(\delta t^2) \end{aligned}$$

If δt is small enough, evolution is almost unitary.
N equispaced pulses (i.e., $\delta t = \frac{t}{N}$) yields:

$$\begin{aligned} \tilde{\rho}_T(t) &= \left[I + O[(\delta t^2)]\right]^{\frac{N}{2}} \rho_T(0) \left[I + O[(\delta t^2)]\right]^{\frac{N}{2}} \\ \lim_{N\to\infty} \rho_s(t) &= e^{-iH_s t}\rho_s(0)e^{iH_s t}. \end{aligned}$$

Error: $O[N(\delta t^2)] \sim O[(\delta t)]$ very small for $\lim_{N\to\infty}$.

Decoupling up to second order in δt:

$$e^{-i(H_0+H_I)\delta t} \approx e^{-iH_0\delta t} e^{-iH_I\delta t} e^{\frac{1}{2}[H_0,H_I]\delta t^2} + O(\delta t^3)$$

$$\begin{aligned} U_2 &= U_1 P_\pi U_1 P_\pi \\ &= \tilde{U}(t+4\delta t, t+3\delta t) P_\pi \tilde{U}(t+3\delta t, t+2\delta t) \tilde{U}(t+2\delta t, t+\delta t) P_\pi \tilde{U}(t+\delta t, t) \\ &= I + O[\delta t^3] \end{aligned}$$

$\Rightarrow$ Composite operator with unequally spaced pulses at δt and $3\delta t$.

Actually, ignoring terms of order δt^4 and higher:

$$\begin{aligned} \langle \varepsilon_s | \rho_s(t) | \varepsilon_t \rangle &= [1 - i(J_\perp g^2 \omega^2 t \delta t^2)] \langle \varepsilon_s | \rho_s(0) | \varepsilon_t \rangle e^{-i(\varepsilon_s - \varepsilon_t)t}, \\ \langle \varepsilon_s | \rho_s(t) | \varepsilon_s \rangle &= \langle \varepsilon_s | \rho_s(0) | \varepsilon_s \rangle. \end{aligned}$$

Summary and conclusions

- Using GaAs DQDs, Petta et al. [PRL **86**, 246804 (2010)], Ritchie et al. [Nano Letters **10**, 2789 (2010)] obtain decoherence times ~ 10 ns.
- In oxide materials, dominant interaction is with optical phonons. Analyzing optical phonon environment, we get a small decoherence even for local noise [arXiv:1309.5824].
- For Markov processes, we do not have any decoherence.
- For Heisenberg interaction, dynamical decoupling is possible for the spin states.
- Qubits, based on oxide DQDs, hold promise in terms of coherence and miniaturization.

Thank you

Including large number of bath modes ($0.9\omega_c \leq \omega_k \leq \omega_c$):

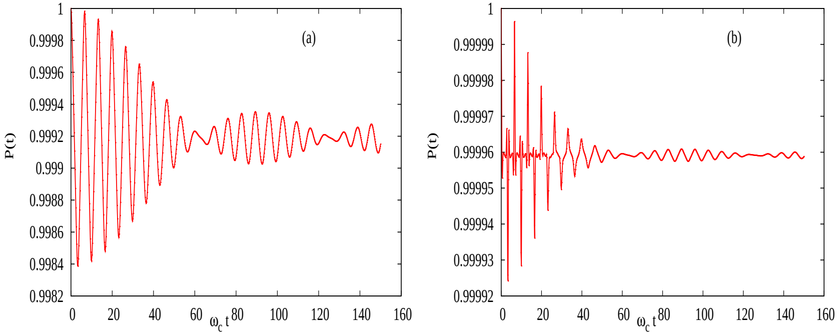

Figure: (a) $J_\perp/\omega_c = 0.05$ and $\frac{\sum_k g_k^2}{N} = 1$; (b) $J_\perp/\omega_c = 0.05$ and $\frac{\sum_k g_k^2}{N} = 4$.

Including the effect of small $J_\perp e^{-g^2}/\omega$:

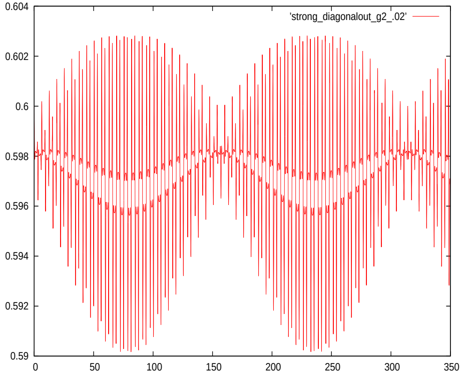

Figure: $J_\perp e^{-g^2}/\omega = 0.02$ and $g = 2$.

Expanding the double commutator and using the complete set $\sum_n |m\rangle_{ph\,ph}\langle m| = I$

$$\begin{aligned}\frac{d\tilde{\rho}_s(t)}{dt} &= -\frac{1}{z}\sum_n \int_0^t d\tau \Big[{}_{ph}\langle n|\tilde{H}_I(t)|m\rangle_{ph}\ {}_{ph}\langle m|\tilde{H}_I(\tau)|n\rangle_{ph}\tilde{\rho}_s(t)e^{-\beta\omega_n} \\ &\quad - {}_{ph}\langle n|\tilde{H}_I(t)|m\rangle_{ph}\tilde{\rho}_s(t)_{ph}\langle m|\tilde{H}_I(\tau)|n\rangle_{ph}e^{-\beta\omega_m} \\ &\quad - {}_{ph}\langle n|\tilde{H}_I(\tau)|m\rangle_{ph}\tilde{\rho}_s(t)_{ph}\langle m|\tilde{H}_I(t)|n\rangle_{ph}e^{-\beta\omega_m} \\ &\quad + \tilde{\rho}_s(t)_{ph}\langle n|\tilde{H}_I(\tau)|m\rangle_{ph}\ {}_{ph}\langle m|\tilde{H}_I(t)|n\rangle_{ph}e^{-\beta\omega_n}\Big]\end{aligned}$$

$T=0K$ analysis:

First term:

$$\begin{aligned}&{}_{ph}\langle 0|\tilde{H}_I(t)|m\rangle_{ph}\ {}_{ph}\langle m|\tilde{H}_I(\tau)|0\rangle_{ph}\tilde{\rho}_s(t) \\ &= \sum_{\varepsilon,\varepsilon',\varepsilon''}\Big[|\varepsilon\rangle\langle\varepsilon|\ {}_{ph}\langle 0|H_I|m\rangle_{ph}|\varepsilon'\rangle\langle\varepsilon'|\ {}_{ph}\langle m|H_I|0\rangle_{ph}|\varepsilon''\rangle\langle\varepsilon''|\Big] \\ &\quad \times e^{i[(\varepsilon-\varepsilon')t+(\varepsilon'-\varepsilon'')\tau]}\tilde{\rho}_s(t)e^{-i\omega_m(t-\tau)}\end{aligned}$$

For fixed $S_T^z(=0)$ only the spin flipping part in the Hamiltonian H_s contributes to the excitation gap. The two $S_T^z = 0$ eigenstates are $|\varepsilon_t\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $|\varepsilon_s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. So, the energy gap in the strong coupling ($g^2 \gg 1$) and non-adiabatic ($\frac{J_\perp}{\omega} \leq 1$) limit $J_\perp e^{-g^2} \ll \omega$. Using this one can write the first term as

$$
\begin{aligned}
{}_{ph}\langle 0|\tilde{H}_I(t)|m\rangle_{ph}\ {}_{ph}\langle m|\tilde{H}_I(\tau)|0\rangle_{ph}\tilde{\rho}_s(t) &= {}_{ph}\langle 0|H_I|m\rangle_{ph}\ {}_{ph}\langle m|H_I|0\rangle_{ph} \\
&\quad \times \tilde{\rho}_s(t) e^{i(\omega_n - \omega_m)(t-\tau)}
\end{aligned}
$$

Using the same procedure for all four terms the master equation at $T = 0K$

$$
\begin{aligned}
\frac{d\tilde{\rho}_s(t)}{dt} &= -\int_0^t d\tau \Big[\sum_m [|{}_{ph}\langle 0|H_I|m\rangle_{ph}|^2\ \tilde{\rho}_s(t) e^{-i\omega_m(t-\tau)} \\
&\quad + \tilde{\rho}_s(t)\ |{}_{ph}\langle 0|H_I|m\rangle_{ph}|^2 e^{i\omega_m(t-\tau)}] \\
&\quad - \sum_n [{}_{ph}\langle n|H_I|0\rangle_{ph}\tilde{\rho}_s(t)\,{}_{ph}\langle 0|H_I|n\rangle_{ph} e^{i\omega_n(t-\tau)} \\
&\quad + {}_{ph}\langle n|H_I|0\rangle_{ph}\tilde{\rho}_s(t)\,{}_{ph}\langle 0|H_I|n\rangle_{ph} e^{-i\omega_n(t-\tau)}] \Big].
\end{aligned}
$$

Phonon states $|m\rangle_{ph} \equiv |m_1, m_2\rangle_{ph}$

$$
{}_{ph}\langle 0,0|H_I|m_1,m_2\rangle_{ph} = \frac{J_\perp}{2} e^{-g^2} \frac{g^{m_1+m_2}}{\sqrt{m_1! m_2!}} (-1)^{m_1} \times (S_1^+ S_2^- + (-1)^{m_2-m_1} S_2^+ S_1^-),
$$

where m_1 and m_2 are not zero simultaneously.

$$
{}_{ph}\langle 0,0|H_I|0,0\rangle_{ph} = 0.
$$

Off-diagonal matrix elements of the reduced density matrix:

$$\begin{aligned}
\langle\varepsilon_s|\rho_s(t)|\varepsilon_t\rangle e^{i(\varepsilon_s-\varepsilon_t)t} &= \frac{1}{2}\Bigg[\langle\varepsilon_s|\rho_s(0)|\varepsilon_t\rangle\Big\{exp[-2K\sum_{m_1,m_2}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}] \\
&\qquad +exp[-2K\sum_{|m_1-m_2|=even,0}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\} \\
&\quad +\langle\varepsilon_t|\rho_s(0)|\varepsilon_s\rangle\Big\{exp[-2K\sum_{m_1,m_2}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}] \\
&\qquad -exp[-2K\sum_{|m_1-m_2|=even,0}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\}\Bigg], \\
\langle\varepsilon_t|\rho_s(t)|\varepsilon_s\rangle e^{i(\varepsilon_t-\varepsilon_s)t} &= \frac{1}{2}\Bigg[\langle\varepsilon_t|\rho_s(0)|\varepsilon_s\rangle\Big\{exp[-2K\sum_{m_1,m_2}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}] \\
&\qquad +exp[-2K\sum_{|m_1-m_2|=even,0}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\} \\
&\quad +\langle\varepsilon_s|\rho_s(0)|\varepsilon_t\rangle\Big\{exp[-2K\sum_{m_1,m_2}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}] \\
&\qquad -exp[-2K\sum_{|m_1-m_2|=even,0}{}' C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\}\Bigg].
\end{aligned}$$

These show some amount of decoherence.

Diagonal matrix elements of the reduced density matrix:

$$
\begin{aligned}
\langle \varepsilon_s|\rho_s(t)|\varepsilon_s\rangle &= \frac{1}{2}\Bigg[\langle \varepsilon_s|\rho_s(0)|\varepsilon_s\rangle \Big\{1 + exp[-2K \sum_{|m_1-m_2|=odd} C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\} \\
&\quad + \langle \varepsilon_t|\rho_s(0)|\varepsilon_t\rangle \Big\{1 - exp[-2K \sum_{|m_1-m_2|=odd} C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\}\Bigg], \\
\langle \varepsilon_t|\rho_s(t)|\varepsilon_t\rangle &= \frac{1}{2}\Bigg[\langle \varepsilon_t|\rho_s(0)|\varepsilon_t\rangle \Big\{1 + exp[-2K \sum_{|m_1-m_2|=odd} C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\} \\
&\quad + \langle \varepsilon_s|\rho_s(0)|\varepsilon_s\rangle \Big\{1 - exp[-2K \sum_{|m_1-m_2|=odd} C_{m_1,m_2}\frac{(1-cos(\omega_{m_1,m_2}t))}{\omega^2_{m_1,m_2}}]\Big\}\Bigg].
\end{aligned}
$$

Indicates change of probabilities.

$$
\begin{aligned}
\int_0^\infty dt\,\frac{sin(\omega_{m_1,m_2}t)}{\omega_{m_1,m_2}} &= \int_0^\infty dt\,\frac{e^{i\omega_{m_1,m_2}t} - e^{-i\omega_{m_1,m_2}t}}{2i\omega_{m_1,m_2}} \\
&= \frac{1}{2i\omega_{m_1,m_2}}\Big[\int_0^\infty dt\; e^{i(\omega_{m_1,m_2}+i\eta)t} - \int_0^\infty dt\; e^{-i(\omega_{m_1,m_2}-i\eta)t}\Big] \\
&= \frac{1}{\omega^2_{m_1,m_2}}
\end{aligned}
$$