Shared Purity of Multipartite Quantum States

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Outline of the talk

1. Motivation
2. Shared purity: definition & properties
3. Bipartite and multipartite mixed states: some examples
4. Monogamy properties
5. Application: detecting criticality in quantum spin models
6. Summary

Quantum Information protocols: some examples

- Quantum teleportation
- Superdense coding
- Entanglement based quantum crytography
- Quantum nonlocality without entanglement
- Deterministic quantum computation with one qubit
- Secure deterministic communication without entanglement

Quantum information protocols can achieve higher efficiencies than their classical counterparts, if they exist.

Why is the efficiency higher?

Quantum mechanical resource

- Quantum entanglement
- Some of them do not employ entanglement

What then is the resource?

- Quantum discord seems to be the answer
- However, intriguing questions remain

- Quantum entanglement $\longrightarrow$ entanglement-separability
- Quantum discord $\longrightarrow$ information-theoretic

Quantum Correlation measures

Entanglement-separability paradigm

1. Concurrence
2. Logarithmic negativity
3. Geometic measure, etc.

Information-theoretic paradigm

1. Quantum discord
2. Quantum work deficit
3. Symmetric discord, etc.

Shared purity $\longrightarrow$ not a measure of quantum correlation

Outline of the talk

Fidelity

Pure quantum states are privileged

- Maximum advantage $\longrightarrow$ for pure shared states
- Pure quantum states $\longrightarrow$ vanishing entropy $\Longrightarrow$ full information of the system is available

- Fidelity $\longrightarrow$ distance of a state from another state

  Pure (another)

- Shared purity defined for an arbitrary quantum state of shared systems of an arbitrary number of parties in arbitrary dimensions

Global fidelity

Definition

The "global fidelity" of an N-party arbitrary (pure or mixed) quantum state, $\rho_{1\ldots N}$, on $\mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N}$,

$$F_G = \max_{\{|\phi\rangle_{1\ldots N} \in \mathcal{H}\}} {}_{1\ldots N}\langle\phi|\rho_{1\ldots N}|\phi\rangle_{1\ldots N},$$

where the maximization is performed over all elements (pure states) of $\mathcal{H}$.

- Measures the lack of disorder present in the system
- Unity for all pure states in arbitrary dimensions

Proposition

For an arbitrary mixed state $\rho_{1\ldots N}$, F_G is the largest eigenvalue in the spectrum of the state.

Proof.

- $\rho_{1\ldots N} = \sum_i \lambda_i |e_i\rangle\langle e_i|$, $\{|e_i\rangle\} \longrightarrow$ (orthonormal basis spanning $\mathcal{H}$)
- $|\phi\rangle_{1\ldots N} = \sum_i a_i |e_i\rangle$, $\sum_i |a_i|^2 = 1$
- $F_G = \max_{a_i} \sum_i |a_i|^2 \lambda_i$
- $\lambda_r \longrightarrow$ largest eigenvalue
- $a_r = 1$ and $a_{i,i\neq r} = 0$ (assumption) $\Longrightarrow F_G \geq \lambda_r$
- $F_G \leq \max_{a_i} \sum_i |a_i|^2 \lambda_r = \lambda_r$, since $\lambda_i \leq \lambda_r \quad \forall i$
- $F_G = \lambda_r$

□

Local fidelity

Definition

The "local fidelity", of the N-party quantum state $\rho_{1\ldots N}$

$$F_L = \max_{\{|\phi\rangle_{1\ldots N} \in S\}} {}_{1\ldots N}\langle\phi|\rho_{1\ldots N}|\phi\rangle_{1\ldots N},$$

where the maximization is carried out over a certain set S, of pure product states.

Hierarchy of local fidelities depending on S
Two extreme cases:

- fully separable states
- genuinely multiparty entangled states

Proposition

For an arbitrary pure N-party state $|\psi\rangle_{1\ldots N}$, F_L^{n-gen} is the square of the maximal Schmidt coefficient among all bipartitions. (optimization over states that are not genuinely multiparty entangled)

Proof.

- $F_L = \max_{\{|\phi\rangle_{1\ldots N} \in S_{n-gen}\}} |\langle\phi|\psi\rangle|^2 = 1 - \mathcal{E}$, $\mathcal{E} \longrightarrow$ generalized geometric measure
- $\mathcal{E}(|\psi\rangle) = 1 - \max\{\lambda^2_{\mathcal{A}:\mathcal{B}} | \mathcal{A} \cup \mathcal{B} = \{1, \ldots, N\}, \mathcal{A} \cap \mathcal{B} = \emptyset\}$, where $\lambda_{\mathcal{A}:\mathcal{B}}$ is the maximal Schmidt coefficient in the $\mathcal{A} : \mathcal{B}$ bipartition

□

Theorem

For an arbitrary bipartite (pure or mixed) state, on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the minimum value attained by F_L is λ_r/d, where $d = \min\{d_1, d_2\}$, and λ_r is the largest eigenvalue in the spectrum of ρ.

Proof.

We have

$$\begin{aligned} F_L(\rho) &= \max_{\{|\phi\rangle \in S_L\}} \langle\phi|\rho|\phi\rangle \\ &= \max_{\{|\phi\rangle \in S_L\}} \sum_i \lambda_i |\langle\phi|e_i\rangle|^2, \end{aligned}$$

where $\sum_i p_i |e_i\rangle\langle e_i|$ is a spectral decomposition of the bipartite quantum state ρ. Therefore,

$$F_L(\rho) \geq \max_{\{|\phi\rangle \in S_L\}} \lambda_r |\langle\phi|e_r\rangle|^2.$$

The property follows from the fact that $F_L \geq \frac{1}{d}$ for any pure state in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. ■

Corollary 3.1. For an arbitrary bipartite (pure or mixed) state, on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the maximum value attained by S_P is $\lambda_r(1 - 1/d)$, where $d = \min\{d_1, d_2\}$, and λ_r is the largest eigenvalue in spectrum of ρ. □

Properties

Shared purity: $S_P = F_G - F_L$

- The shared purity vanishes for pure product states of the form $|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle$.
- For an arbitrary N-party pure state $|\psi\rangle_{1\ldots N}$ in arbitrary dimensions, the shared purity is a geometric measure of entanglement.
- The shared purity is invariant under local unitary operations.
- For classically correlated states, the global and local fidelities are equal.
- For a state of the form $\rho_1 \otimes \ldots \otimes \rho_N$ on $\mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N}$, shared purity vanishes.

Outline of the talk

Bipartite mixed states

Admixtures of a Bell state with a pure product state

$$\rho_{ent} = p|00\rangle\langle 00| + (1-p)|\psi^-\rangle\langle\psi^-|$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \text{ and } 0 \leq p \leq 1$$

The state is entangled for any value of $p < 1$.

$$\begin{aligned} S_P &= \frac{(1-p)(1-2p)}{2-3p}, & 0 \leq p < \frac{1}{2}, \\ &= 0, & \frac{1}{2} \leq p \leq 1. \end{aligned}$$

- Shared purity $\longrightarrow$ zero for this mixed entangled states
- Distillable entanglement $\longrightarrow$ zero for some entangled states

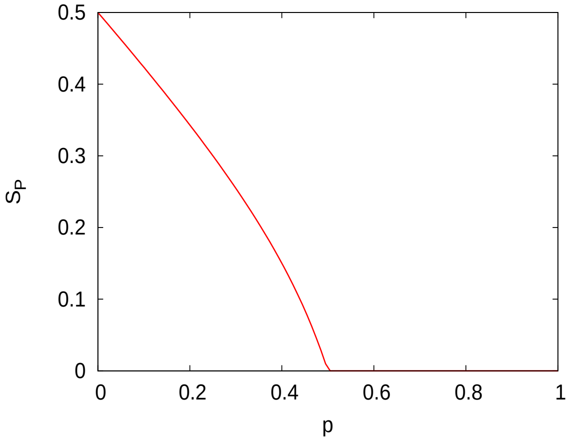


Figure : The most interesting region is $\frac{1}{2} \leq p < 1$, where the shared purity vanishes, although the state has a nonzero entanglement there.

Admixtures of pure states with noise

- $\rho_{gen} = p|\psi\rangle\langle\psi| + \frac{(1-p)}{4} I \otimes I$,
- $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ with $0 \leq \theta \leq \pi/4$, $0 \leq p \leq 1$
- $S_P = p\sin^2\theta$
- $\rho_{gen} \longrightarrow$ Werner state when $\theta = \frac{\pi}{4}$, $S_P = \frac{p}{2}$
- Werner state is entangled for $p > 1/3$

- Quantum discord and quantum work-deficit are non-vanishing for $p > 0$
- Shared purity can be positive for separable states

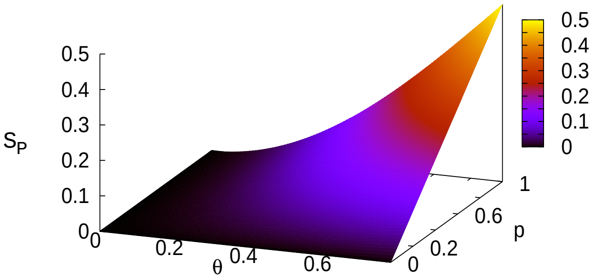


Figure : S_P is always nonzero for $p > 0$ & $\theta > 0$.

Bell mixtures

$$\rho_{Bell} = p|\psi^-\rangle\langle\psi^-| + (1-p)|\psi^+\rangle\langle\psi^+|,$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \text{ and } 0 \leq p \leq 1.$$

Note that the state is entangled for all values of p except $p = 1/2$.

$$\begin{aligned} S_P &= p - \frac{1}{2} \quad \text{for} \quad p \geq \frac{1}{2}, \\ &= \frac{1}{2} - p \quad \text{for} \quad p < \frac{1}{2}. \end{aligned}$$

Just like any quantum correlation measure, shared purity, in this case, is also a mirror reflection with respect to the $p = 1/2$ line. This is a result of the local unitary invariance of shared purity.

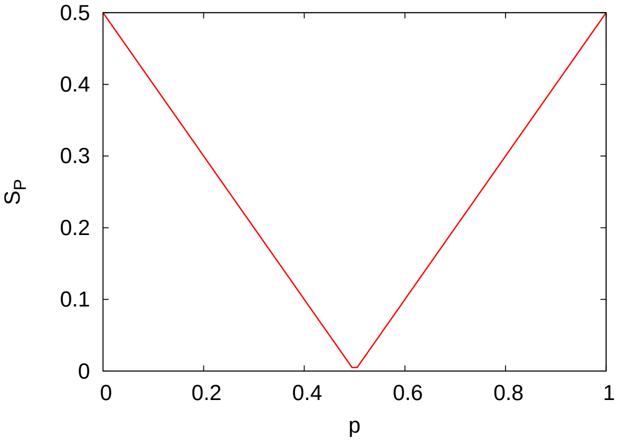


Figure : S_P is vanishing only at $p = 1/2$.

Multipartite mixed state

N-party Greenberger-Horne-Zeilinger state, mixed with white noise, in $\left(\mathbb{C}^d\right)^{\otimes N}$.

$$\rho_{GHZ_N} = p|\psi\rangle\langle\psi| + (1-p)\left(\frac{1}{d}I_d \otimes \ldots \otimes \frac{1}{d}I_d\right),$$

$$|\psi\rangle = \frac{1}{\sqrt{d}}(|0_1 \ldots 0_N\rangle + \ldots + |(d-1)_1 \ldots (d-1)_N\rangle) \text{ and } 0 \leq p \leq 1.$$

I_d denotes the identity operator on $\mathbb{C}^d$, and $\{|i_j\rangle\}_{i=0}^{d-1}$ for $j = 1, \ldots, N$ forms an orthonormal basis in the Hilbert space of the jth particle.

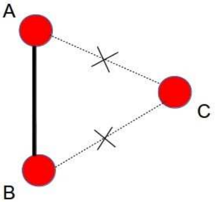
$$S_P = p\left(1 - \frac{1}{d}\right).$$

Note that the shared purity never vanishes except at $p = 0$.

Outline of the talk

Monogamy

- Multiparty quantum system in a state $\longrightarrow \rho_{1\ldots N}$
- $\mathcal{Q} \longrightarrow$ two-party physical quantity
- $\mathcal{Q}$ will be monogamous if
  a high amount of $\mathcal{Q}(\rho_{12})$ implies that neither the party 1 nor the party 2 will be able to share a substantial amount of $\mathcal{Q}$ with any other party
- Sharing of classical correlations of a multiparty quantum system does not have any such restriction



Shared purity: Quantum property of shared systems

Theorem

If two quantum systems, irrespective of their dimensions, have the maximal amount of shared purity, they cannot share any purity with any third quantum system.

Proof.

- Three-party system ρ_{123} in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- Assumption: $S_P(\rho_{12}) = (1 - 1/d) \longrightarrow$ (maximal), where $d = \min\{d_1, d_2\}$
- Now, $S_P(\rho_{12}) = 1 - 1/d$, only for a pure state ρ_{12}
- Therefore, ρ_{123} must be of the form $\rho_{12} \otimes \rho_3$
- Therefore, $S_P(\rho_{13}) = S_P(\rho_{23}) = 0$
- Shared purity is qualitatively monogamous

□

Monogamy: quantitative investigation

The monogamy condition

$$S_P(\rho_{12}) + S_P(\rho_{13}) \leq S_P^{1:23}(\rho_{123})$$

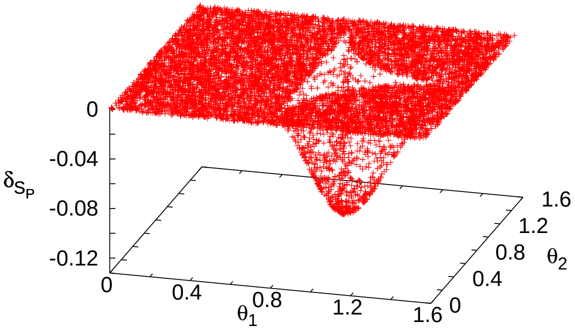
- The generalized GHZ states always satisfy the monogamy condition
- The generalized W states always violate it
- However, the above statements are not true for the GHZ class and W class states

Generalized GHZ state

- $|\psi\rangle_{GHZ}^{G} = \cos\theta|000\rangle + e^{i\phi}\sin\theta|111\rangle$, $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$
- $\rho_{1j}^{G} = \cos^2\theta|00\rangle\langle 00|_{1j} + \sin^2\theta|11\rangle\langle 11|_{1j}$, $j \in \{2,3\}$
- Since the state is classically correlated, $S_P(\rho_{1j}^{G}) = 0 \ \forall j$
- Since $|\psi\rangle_{GHZ}^{G}$ is a pure state, $F_G^{1:23} = 1$
- $F_L^{1:23} = \max\{\cos^2\theta, \sin^2\theta\}$ (calculated)
- $S_P^{1:23}(|\psi\rangle_{GHZ}^{G}) = 1 - \max\{\cos^2\theta, \sin^2\theta\}$
- Monogamy condition is satisfied

Generalized W state

- $|\psi\rangle_W^G = \sin\theta_1 \cos\theta_2 |001\rangle + \sin\theta_1 \sin\theta_2 e^{i\phi_1} |010\rangle + \cos\theta_1 e^{i\phi_2} |100\rangle$
- $\delta_{S_P} = S_P(\rho_{1:23}) - (S_P(\rho_{12}) + S_P(\rho_{13})) \longrightarrow$ shared purity monogamy score



Outline of the talk

Anisotropic quantum XY spin chain in one dimension

- $H_{XY} = \frac{J}{2}\left(\sum_{i=1}^{N}(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y\right) + h\sum_{i=1}^{N}\sigma_i^z$
- $J \longrightarrow$ coupling constant for the nearest neighbor interaction
- $\gamma \in (0,1] \longrightarrow$ anisotropy parameter
- σ's $\longrightarrow$ the Pauli spin matrices
- $h \longrightarrow$ the external transverse magnetic field
- Periodic boundary condition is assumed

- H_{XY} can be diagonalized by applying Jordan-Wigner, Fourier, and Bogoliubov transformations successively
- At zero temperature, the system undergoes a quantum phase transition driven by the external transverse magnetic field
- Concurrence, geometric measures, quantum discord $\longrightarrow$ detects this transition

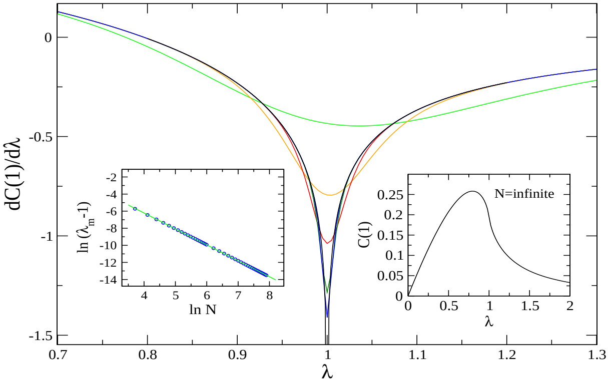


Figure : Concurrence detects quantum phase transition.

A. Osterloh, L. Amico, G. Falci and R. Fazio, Nature **416**, 608, (2002) We investigate the behavior of the shared purity of the nearest neighbor density matrix of the ground state near the known quantum critical point at $\lambda = \frac{h}{J} = 1$

- $F_G \longrightarrow$ maximum eigenvalue of the density matrix ρ_{AB}
- $F_L \longrightarrow$ obtained by numerical maximization of the density matrix ρ_{AB} with respect to the product states in $\mathbb{C}^2 \otimes \mathbb{C}^2$

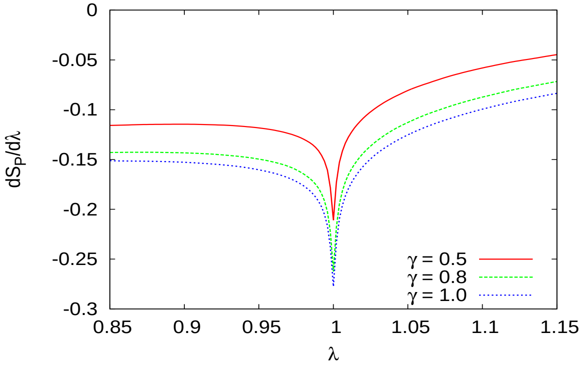


Figure : Shared purity detects quantum phase transition.

Scaling analysis

- Scaling analysis $\longrightarrow$ helps us to understand the viability of detecting the critical point in finite-sized systems
- the point of divergence approaches $\lambda = \lambda_c$ as $N^{-1.40}$, *i.e.*,

$$\lambda = \lambda_c + kN^{-1.40}$$

- For concurrence, $\lambda \longrightarrow \lambda_c$ as $N^{-1.87}$
- For quantum discord, $\lambda \longrightarrow \lambda_c$ as $N^{-1.28}$

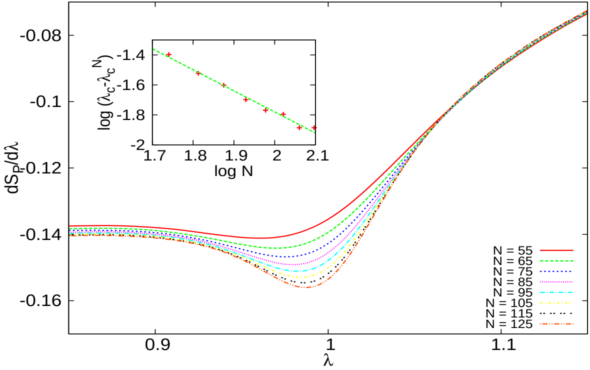


Figure : Finite-size scaling analysis for shared purity. $(\gamma = 0.8)$

Outline of the talk

Summary

- New property of shared quantum systems
- Different from quantum correlations
  - can be nonzero for unentangled states
  - can be zero for entangled states
- Quantum property (monogamous)
- Scaling in transverse Ising model different from both entanglement & discord

A. B., A. Sen(De), U. Sen, arXiv:1306.0669

Thank you.