State Dependent Operators and the Interior of a Black Hole

Suvrat Raju

International Centre for Theoretical Sciences

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Outline

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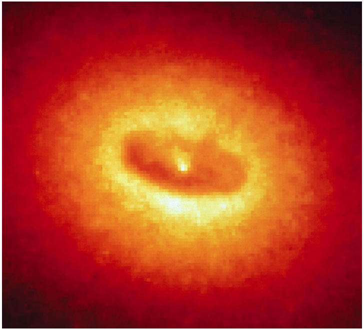
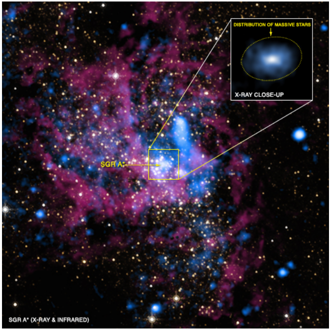


- This is a super-massive black-hole, 1.2 billion times as massive as the sun and 50 million light years away.
- This image was taken by the Hubble Space Telescope in 1992 and shows the black-hole eating a surrounding disk of dust.

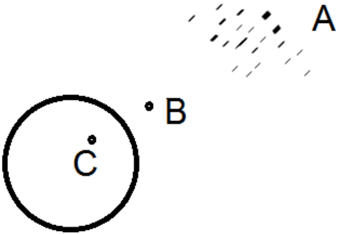


- This is an X-ray image, taken by the Chandra X-Ray Observatory of the super-massive black-hole in the center of the Milky Way.

Black Holes

- Black holes are real!
- We believe that the horizon of big black holes like the one on the previous slide is very smooth; indistinguishable from flat space.
- But black holes also evaporate by Hawking radiation.
- The question is whether the unitarity of this process is in conflict with the smoothness of the horizon.

Three Subsystems

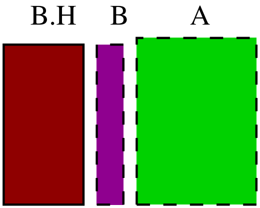


They key point is to think of three subsystems

1. The radiation emitted long ago – A
2. The Hawking quanta just being emitted – B
3. Its partner falling into the BH – C

A Pure Black Hole

- Imagine that the original black hole was prepared in a pure state.
- Then the Hawking radiation process is dividing this pure state into three entangled parts 1)"$B.H$", 2)B, 3)A.
- Statistically, once size(A) > size(B.H), the system B is generically largely "entangled" with A, and not with the "B.H"

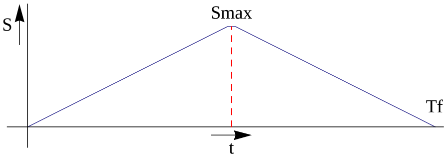


Entropy of A

- We can repeat this argument more precisely. Say the Black Hole is formed by the collapse of a pure state.
- Consider the entropy of system A

$$S_A = -\text{Tr}\rho_A \ln \rho_A$$

- Very general arguments due to Page tell us this must eventually start decreasing.



Strong Subadditivity contradiction?

- Now, consider an old black hole, beyond its "Page time" where S_A is decreasing. We must have

$$S_{AB} < S_A$$

since B is purifying A.

- Second, the pair B, C is related to the Bogoliubov transform of the vacuum of the infalling observer, we have

$$S_{BC} = 0$$

- Finally, both B and C are thermal, so

$$S_B = S_C > 0$$

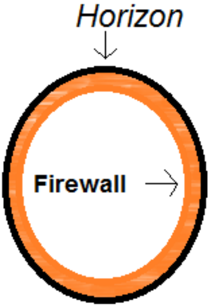
- However, a very general theorem tells us that for any three distinct systems A, B, C, we have

$$S_A + S_C < S_{AB} + S_{BC}$$

The Firewall Proposal

- The firewall proposal is the suggestion that we should drop the idea that B (the particle emitted outside) and C (its partner inside) are entangled.

- Once we do this, it is very hard to prevent the infalling observer from burning up at the horizon — a firewall.



Outline

A Spin Chain Toy Model

- The essential Quantum Information features of this model can be well captured by a simple spin-chain toy model.
- Consider a system of N spin-(1/2) spins. This has 2^N states. We can label these states by numbers and read off the individual spins using the binary expansion of the number.

$$
\begin{aligned}
|000.\ldots\ldots00\rangle &\equiv |0\rangle \\
|000.\ldots\ldots01\rangle &\equiv |1\rangle \\
|000.\ldots\ldots10\rangle &\equiv |2\rangle \\
|000.\ldots\ldots11\rangle &\equiv |3\rangle \\
&\ldots
\end{aligned}
$$

Pure States and Hawking Evaporation

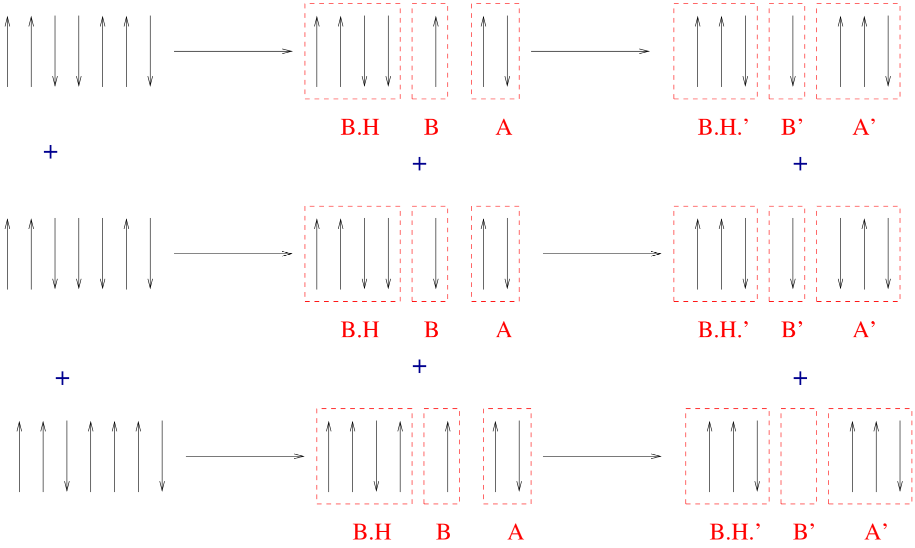
- Consider a generic pure state in this spin-model

$$|\Psi\rangle = \sum_{i=0}^{2^N-1} a_i |i\rangle$$

  where the a_i are chosen to some random complex numbers, satisfying $\sum |a_i|^2 = 1$.

- Our model of Hawking evaporation is simply to break off the spins one by one.

Spin Chain Model of Hawking Evaporation



Similarities with the CFT

- From an information theoretic perspective, this toy-model is not so different from the real CFT.
- It is clear how to realize A, and B in this model, and these objects have the right properties.

Properties of B: Thermality of Hawking Radiation

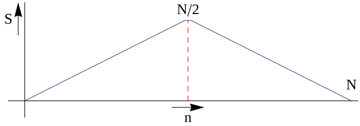
- Note that if we consider any particular emitted spin, it is not in a pure state.
- For a generic state $|\Psi\rangle = \sum_{i=0}^{2^N-1} a_i|i\rangle$, the density matrix of each emitted spin is very close to the identity.
- $$\rho_B = \frac{1}{2}\begin{pmatrix} 1+O\left(2^{\frac{-N}{2}}\right) & O\left(2^{\frac{-N}{2}}\right) \\ O\left(2^{\frac{-N}{2}}\right) & 1+O\left(2^{\frac{-N}{2}}\right) \end{pmatrix}$$

Unitarity of Hawking Evaporation

- For n emitted qubits, we can estimate the von Neumann entropy of the system A, which consists of these qubits.

$$S_n = -\mathrm{Tr}(\rho_n \ln \rho_n) = \left[n\theta\left(\frac{N}{2} - n\right) + (N-n)\theta\left(n - \frac{N}{2}\right)\right] + \mathrm{O}\left(2^{-\frac{N}{2}}\right)$$

- This has precisely the expected behaviour



- Once we cross the “Page Time” (i.e. once half the spin-chain evaporates), each B purifies the old A.

Where are Infalling Quanta?

Where are the Infalling Quanta in the spin chain?

Expected Properties of the Infalling Quanta

- We want a set of operators that satisfy a $SU(2)$ algebra, and represent the C qubit

$$[\widetilde{\mathbf{s}}^i_a, \widetilde{\mathbf{s}}^j_b] \doteq 2i\epsilon^{abc}\delta^{ij}\widetilde{\mathbf{s}}^i_c,$$

- On the state of the theory, the C-qubit is perfectly anti-correlated with the B qubit

$$\langle\Psi|\widetilde{\mathbf{s}}^i_a \mathbf{s}^j_b|\Psi\rangle = -\delta^{ij}\delta_{ab}$$

- The fact that this is an effectively independent degree of freedom is represented by

$$[\widetilde{\mathbf{s}}^i_a, \mathbf{s}^j_b] \doteq 0,$$

Mimicking a set of Bell Pairs

- There is another simple way to understand these criterion.
- In a sense we will make precise, from the point of view of the $\mathbf{s}^i_a$ and $\widetilde{\mathbf{s}}^i_a$, the pure state mimics a product state of Bell pairs.

$$|\Psi\rangle \sim (|01\rangle + |10\rangle)^N$$

An Important Approximation

- We don't need the equations on the previous slide to hold exactly as operator equations.

- If we do not have infinite accuracy, and N is large then we can measure correlators
$$\langle \Psi | \mathbf{s}_{a_1}^{i_1} \ldots \mathbf{s}_{a_K}^{i_K} | \Psi \rangle$$
where $K \ll N$.

- So, we need
$$\langle \Psi | \mathbf{s}_{a_1}^{i_1} \ldots \mathbf{s}_{a_K}^{i_K} \widetilde{\mathbf{s}}_a^i \mathbf{s}_b^j | \Psi \rangle = -\delta^{ij} \delta_{ab} \langle \Psi | \mathbf{s}_{a_1}^{i_1} \ldots \mathbf{s}_{a_K}^{i_K} | \Psi \rangle$$

- And
$$\langle \Psi | \mathbf{s}_{a_1}^{i_1} \ldots [\mathbf{s}_{a_l}^{i_l}, \widetilde{\mathbf{s}}_{a_{l+1}}^{i_{l+1}}] \mathbf{s}_{a_K}^{i_K} | \Psi \rangle = 0$$

Defining the Mirror Operators

- We now describe a remarkably simple definition of the mirror operators using a set of linear equations.

- First,

$$\widetilde{\mathbf{s}}^i_a |\Psi\rangle = -\mathbf{s}^i_a |\Psi\rangle$$

- Second,

$$\widetilde{\mathbf{s}}^i_a \prod_{j=1}^{p} \mathbf{s}^{i_1}_{a_1} \ldots \mathbf{s}^{i_p}_{a_p} |\Psi\rangle = \left(\prod_{j=1}^{p} \mathbf{s}^{i_1}_{a_1} \ldots \mathbf{s}^{i_p}_{a_p} \right) \widetilde{\mathbf{s}}^i_a |\Psi\rangle.$$

- These two rules can recursively be used to specify the action of $\widetilde{\mathbf{s}}^i_a$ on $|\Psi\rangle$ and on any descendant of $|\Psi\rangle$ produced by acting with up to K ordinary operators.

Consistency of the Definition

- Note that $\widetilde{\mathbf{s}}^i_a$ is a $2^N \times 2^N$ matrix.
- The rules above specify the action of this matrix on a set of D_A vectors
$$D_A = \sum_{j=0}^{K} \binom{N}{j} 3^j$$
- Moreover, these vectors are linearly independent for all except for a measure zero set of states.
- So, as long as $D_A < 2^N$, the linear equations produced by these two rules can be consistently solved to define $\widetilde{\mathbf{s}}^i_a$.

Properties of the Definition

- The definition satisfies all the properties that we need.

- Clearly,

$$\langle \Psi | \mathbf{s}^j_a \widetilde{\mathbf{s}}^i_b | \Psi \rangle = -\langle \Psi | \mathbf{s}^j_a \mathbf{s}^i_b | \Psi \rangle = \delta^{ij}_{ab}$$

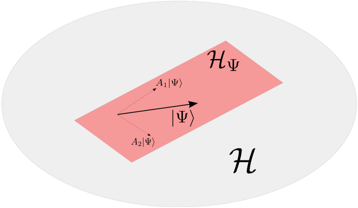
- Also,

$$\begin{aligned} &\langle \Psi | \mathbf{s}^{i_1}_{a_1} \ldots [\mathbf{s}^{i_l}_{a_l}, \widetilde{\mathbf{s}}^{i_{l+1}}_{a_{l+1}}] \mathbf{s}^{i_K}_{a_K} | \Psi \rangle \\ &= \langle \Psi | \mathbf{s}^{i_1}_{a_1} \ldots \mathbf{s}^{i_l}_{a_l} \mathbf{s}^{i_K}_{a_K} \widetilde{\mathbf{s}}^{i_{l+1}}_{a_{l+1}} | \Psi \rangle - \langle \Psi | \mathbf{s}^{i_1}_{a_1} \ldots \mathbf{s}^{i_l}_{a_l} \mathbf{s}^{i_K}_{a_K} \widetilde{\mathbf{s}}^{i_{l+1}}_{a_{l+1}} | \Psi \\ &= 0 \end{aligned}$$

- Both these properties hold on the state $|\Psi\rangle$ and on its descendants produced by acting with up to K ordinary sigma matrices.

State Dependence of the Mirror Operators

- The mirror operators representing C that we have defined are slightly unusual: they depend on the state of the system.



- Eg. think of the density matrix $\rho = |\Psi\rangle\langle\Psi|$. This is always a good operator, but has a good physical interpretation only in a given state.

Resolving Paradoxes using State Dependent Operators

These operators can be used to resolve ALL the recent paradoxes regarding the interior of the black hole and the information paradox

Resolving the Strong Subadditivity Paradox

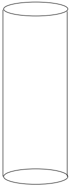
- The resolution to the strong subadditivity paradox is straightforward in this model.
- *A* (the old radiation) and *C* (the particle inside the B.H.) are not independent!
- Q: How can this be, given that the commutator $[\widetilde{\mathbf{s}}^i_a, \mathbf{s}^j_b] \doteq 0$, where *j* could index a spin that is in *A*?
- Ans: The commutator vanishes only effectively. The $\widetilde{\mathbf{s}}^j_b$ matrices secretly act on all degrees of freedom, and only appear to be independent.

Numerical Demonstration of this construction in the spin chain.

Outline

Repeating the Construction in Quantum Gravity

- We can repeat this construction in a theory of quantum gravity using the AdS/CFT conjecture.
- Anti-de Sitter space is a space with a negative cosmological constant. It is like a box.



- The AdS/CFT duality relates quantum gravity in this box to a lower-dimensional non-gravitational quantum field theory.

Condition for a smooth horizon

- The boundary theory has operators $\mathcal{O}^i_{\omega_n,m}$, where ω_n is the energy.
- These operators can be used to describe excitations outside the B.H in a reasonably precise manner.

$$\phi^i_{\text{CFT}}(t,\Omega,z) = \sum_{m,n} \left[\mathcal{O}^i_{\omega_n,m} f_{\omega_n,m}(t,\Omega,z) + \text{h.c.}\right],$$

- These operators describe B and A.

Rephrasing the firewall paradox

All aspects of the recent debate around the information paradox can be phrased in terms of whether the CFT contains operators that describe C

Properties of the Mirror Operators

- More precisely, the condition for smoothness of the horizon is that there should exist new operators $\widetilde{\mathcal{O}}^{i}_{\omega,m}$ satisfying

$$\langle\Psi|\mathcal{O}^{i_1}_{\omega_1,m_1}\ldots\widetilde{\mathcal{O}}^{j_1}_{\omega'_1,m'_1}\ldots\widetilde{\mathcal{O}}^{j_l}_{\omega'_l,m'_l}\ldots\mathcal{O}^{i_n}_{\omega_n,m_n}|\Psi\rangle$$
$$= e^{-\frac{\beta}{2}(\omega'_1+\ldots\omega'_l)}\langle\Psi|\mathcal{O}^{i_1}_{\omega_1,m_1}\ldots\mathcal{O}^{i_n}_{\omega_n,m_n}(\mathcal{O}^{j_l}_{\omega'_l,m'_l})^{\dagger}\ldots(\mathcal{O}^{j_1}_{\omega'_1,m'_1})^{\dagger}|\Psi\rangle.$$

- This equation looks simple, but this is a little deceptive.
- Note that on the RHS, the tilde-operators have been moved to the right and reversed.
- Apart from the $e^{-\beta\omega_l/2}$ factors, this is very similar to our demands in the spin-chain.

Defining the Mirrors in the CFT

- Consider any polynomial in the CFT operators

$$A_\alpha = \sum_N \alpha(N) (\mathcal{O}^i_{\omega_n,m})^{N(i,n,m)},$$

- With an appropriate bound on energy and number of insertions the number of such polynomials is smaller than the dim of the Hilbert space. Except for a measure-zero set of states, we have

$$A_\alpha |\Psi\rangle \neq 0$$

- Now, we define

$$\widetilde{\mathcal{O}}^i_{\omega_n,m} A_\alpha |\Psi\rangle = A_\alpha e^{-\frac{\beta\omega_n}{2}} (\mathcal{O}^i_{\omega_n,m})^\dagger |\Psi\rangle.$$

Resolving all Paradoxes

- Briefly speaking, Mathur, and AMPS have set forward four paradoxes
  1. The strong subadditivity paradox.
  2. The large "commutator" as an counter-argument to complementarity.
  3. The lack of a left-inverse paradox.
  4. the $N_a \neq 0$ paradox.
- This construction neatly resolves all the paradoxes.
- We already described the resolution to (1) and (2) above, and now we will briefly describe the other solutions.

Lack of a Left-Inverse Paradox

- The lack of a left-inverse paradox is that, on the one hand

$$[\widetilde{c}_{\omega}, \widetilde{c}_{\omega}^{\dagger}] A_{\alpha} |\Psi\rangle = A_{\alpha} |\Psi\rangle$$

  and so

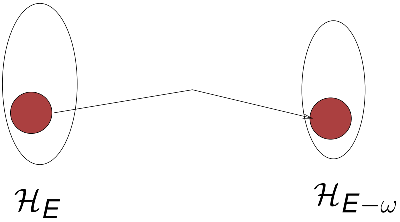
$$\left(\frac{1}{1 + \widetilde{c}_{\omega}^{\dagger} \widetilde{c}_{\omega}} \right) \widetilde{c}^{\dagger} = 1?$$

- But

$$[H_{\text{cft}}, \widetilde{c}^{\dagger}] = -\omega \widetilde{c}_{\omega}^{\dagger}$$

- Since the growth of number of states with energy in the CFT is monotonic, $\widetilde{c}_{\omega}^{\dagger}$ cannot have a left inverse?

State Dependent Operators can be Sparse



- The action of $\widetilde{c}_\omega, \widetilde{c}_\omega^\dagger$ is correct only on $|\Psi\rangle$ and its descendants produced by excitations with bounded energy and insertions.
- In the full Hilbert space, the action of $\widetilde{c}_\omega^\dagger$ can be sparse!
- No contradiction with Linear Algebra!

$N_a \neq 0$ Paradox

- Marolf and Polchinski pointed out that if take some fixed operator $\widetilde{\mathcal{O}}_{\omega,m}$ then the condition

$$\widetilde{\mathcal{O}}_{\omega,m}|\Psi\rangle = e^{\frac{-\beta\omega}{2}}(O_{\omega,m})^{\dagger}|\Psi\rangle$$

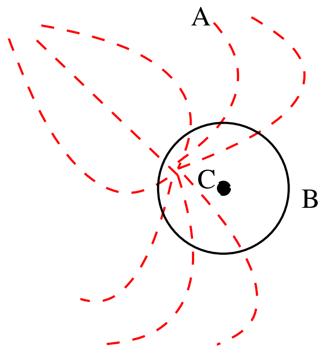
  is rather special.

- This is the condition for a smooth horizon: if one state in the CFT satisfies it, another will not.

- However, our state-dependent operators are defined to satisfy this condition! So, with the use of these operators, the infalling observer measures no particles at the horizon.

Outline

Embedding the Exterior in the Interior



- Construction leads to the funny picture shown above where the interior of the black hole is not independent of the exterior.

- However to detect this unusual picture of spacetime, we need to measure a very high point correlator — precise version of B.H. complementarity.

Are State-Dependent Operators Okay?

- Our construction clearly resolves all recent paradoxes associated with the black-hole interior.

- But it uses state-dependent operators

$$\widetilde{\mathbf{s}}_a^i \prod_{j=1}^{p} \mathbf{s}_{a_1}^{i_1} \ldots \mathbf{s}_{a_p}^{i_p} |\Psi\rangle = \left(\prod_{j=1}^{p} \mathbf{s}_{a_1}^{i_1} \ldots \mathbf{s}_{a_p}^{i_p} \right) \mathbf{s}_a^i |\Psi\rangle.$$

- So, which operator on the Hilbert space that the infalling observer calls the bulk field ϕ, depends on the state.

- This is a little like allowing the observer to measure the "density matrix". Which operator is measured depends on the state!

- This facet of the black-hole interior deserves to be understood better.