Non-classicality and its Macro-limit for the Schrödinger Coherent State of Harmonic Oscillator using Leggett-Garg Inequality

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SOME FUNDAMENTAL QUESTIONS CONCERNING THE NATURE OF QUANTUM REALITY

1. What is the relationship between the weird microscopic world described by quantum physics and the everyday macroscopic world of human experience? To what extent it is possible to test quantum mechanics in the macro-limit? **How the macroscopic emerges from the microscopic, and what concepts are needed to understand this transition** remains one of the most intriguing issues in modern physics.

2. Are *realist models* possible that can account for the observable quantum phenomena by ascribing definite values to *all* observables at any given instant for a *complete specification* of the state of an individual system? By the term '*complete specification of state*' one means a theory which supplements the standard wave function description by some suitable *additional variables* (the so-called 'hidden variables'). **To what extent the possibility of such a realist model of quantum phenomena can be restricted that can be experimentally tested?**

TESTABLE FEATURES OF QUANTUM MECHANICS WITH FUNDAMENTAL CONCEPTUAL IMPLICATIONS

The key Goal of such studies is to provide feasible tests that demonstrate quantum behaviour having implications for realist models of quantum phenomena that ascribe definite values to all observables for a complete specification of the relevant state.

Bell's inequality and its variants $\rightarrow$ Testable algebraic consequence of the combination of the notions of *realism* and *locality*.

Contextuality inequalities $\rightarrow$ Testable algebraic consequence of the combination of the notions of *realism* and *noncontextuality* (i.e., assuming that the individual outcome of measuring an observable does not depend on which other *commuting observable* is measured along with it).

Bell-type inequalities involve correlations between results of joint measurements on *spatially separated systems*, whereas Contextuality inequalities involve correlations between results of joint measurements of commuting pairs of observables for a *single system*.

On the other hand, *Leggett-Garg inequality* (LGI) is concerned with measurements of an observable at different instants, thereby providing another fundamental signature of non-classicality of Quantum Mechanics(QM) through QM violation of LGI.

Motivations underlying LGI:

(i) To show non-classicality by probing *QM violation* of the *'realist' notion* of a definite value ascribed to any observable at *any* instant t, independent of testing the notion of locality or noncontextuality.

(ii) To formulate an effective scheme for probing the *Macroscopic limit of QM* or the *Quantum-Classical Transition*.

In our work, we use LGI in the context of QM treatment of a *system with classical analogue*, like a harmonic oscillator, so that by varying the value of mass one can study the extent up to which the system displays quantum behaviour and then approaches classicality.

HISTORICAL DEVELOPMENT

Some key breakthroughs

EPR paper (1935)$\rightarrow$ Concept of a '*realist model*' underlying quantum phenomena, based on '*complete specification*' of a *state*; 'Non-locality' of QM.

Schrödinger's paper (1935) $\rightarrow$ *Entanglement, Quantum Steerability, Quantum Measurement Problem, Macro-limit of QM*.

Bohm's explicit construction of a realist model accounting for quantum phenomena (1952).

Bell's inequality (BI) showing testability of local realist models (1964).

Expt. tests of BI (1974, 1981 - 83, ...).

Leggett-Garg inequality (LGI) (1985).

Application of BI for security check in Quantum Cryptography (1991).

Application of Entanglement for more efficient information transfer - Dense Coding (1992).

HISTORICAL DEVELOPMENT

Use of Entanglement to transfer Quantum State-Quantum Teleportation (1993). Development of the research area of Quantum Information (.... 2013).

Expt. test of BI achieved for 18 km separation between detectors (2008).

Quantum Teleportation achieved over a distance of 143 km (2012).

Quantum Interference of C_{60} molecules (size $\sim$ 1 nm) with mass $=$ 720 a.m.u (1999).

Quantum Interference of bigger biomolecules of size $\sim$2 nm and mass $\sim 2 \times 10^3$ a.m.u (2003).

Quantum Interference of Large Organic Molecules, molecules of size $\sim 60\AA$ *with 430 atoms and masses up to* ~ 7000 *a.m.u (2011).*

Expt. tests of LGI (2000, 2002,...) for SQUID system involving superposition of micro-amperes current flowing along clockwise and anticlockwise directions.

Expt. tests of LGI for systems, ranging from *solid-state qubit systems* to *nuclear spins* precessing in an external magnetic field (2006 - 2012).

- Leggett-Garg inequality (LGI)[1] [2] is a *temporal analogue* of Bell's inequality (BI) in terms of *time-separated correlation functions* corresponding to successive measurement outcomes for a system whose state evolves in time.
- Notion of *realism* is invoked in deriving LGI by assuming that a system, during its time evolution, is at *any* given time in a definite one of the available states.
- *Noninvasive measurability* (NIM) is assumed which means that, it is possible to determine which of the states the system is in, *without affecting* the state itself or the system's subsequent dynamics.

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[1] A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857 (1985).
[2] A. J. Leggett, J. Phys. Condens. Matter **14**, R415 (2002).

- NIM can be satisfied classically for *'negative result' type measurements*. QM violation of LGI in suitable examples would, therefore, signify repudiation of the notion of realism that includes the assumption of NIM.

- Thus, in furnishing a signature of distinctly quantum behaviour, LGI can be regarded as complementing BI in providing insight into the nature of physical reality, while enabling probing of the applicability of QM for *mesoscopic/macroscopic systems*, or, can be used for studying the Quantum-Classical transition.
- Hence it has been of considerable interest to investigate the extent to which LGI is violated by QM for various types of systems.

- The original motivation that led to LGI was to use it for probing the possible limits of quantum mechanics in the macroscopic regime; e.g., experiments involving the rf-SQUID device.[3]
- In recent years, LGI has been extensively studied for various types of microsystems and for probing the Quantum-Classical transition. [4] [5] [6] [7] [8]

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[3] C. H. van der Wal *et al.*, Science **290**, 773 (2000); J. R. Friedman et al. Nature **406**, 43 (2002).; A. Palacios *et al.* Nature Phys. **6**, 442 (2010).

[4] J. Kofler and C. Brukner, Phys. Rev. Lett. **101**, 090403 (2008); **99**, 180403 (2007).

[5] R. Ruskov et al., Phys. Rev. Lett. **96**, 200404 (2006).

[6] M. E. Gossin *et al.*, Proc. Natl Acad. Sci. **108**, 1256 (2011).

[7] J. Dressel et al., Phys. Rev. Lett. **106**, 040402 (2011).

[8] G. Waldherr et al., Phys. Rev. Lett. **107**, 090401 (2011); V. Athalye et al., Phys. Rev. Lett. **107**, 130402 (2011); G. C. Knee et al., Nature Communications **3**, 606 (2012); D. Home et al., Phys. Rev. A **88**, 022115 (2013).

DERIVATION OF THE LEGGETT-GARG INEQUALITY

- We focus on a two-state system whose temporal evolution consists of oscillations between the states, say, 1 and 2.
- Let Q(t) be an observable quantity such that, whenever measured, it is found to take a value $+1(-1)$ depending on whether the system is in the state 1(2).
- Next, consider a collection of runs starting from the identical initial state such that on the first series of runs Q is measured at times t_1 and t_2, on the second at t_2 and t_3, on the third at t_3 and t_4, and on the fourth at t_1 and t_4 (here $t_1 < t_2 < t_3 < t_4$).

- It is possible to adapt in this context, the standard argument leading to a Bell-type inequality with the times t_i playing the role of apparatus settings.
- One can then use the following consequence of the assumptions of realism and NIM that were mentioned earlier. For any set of runs corresponding to the *same initial state*, any individual $Q(t_i)$ has the *same* definite value, irrespective of the pair $Q(t_i)Q(t_j)$ in which it occurs; i.e., the value of $Q(t_i)$ in any pair does *not* depend on whether any prior measurement has been made on the system.

- Consequently, the combination $[Q(t_1)Q(t_2) + Q(t_2)Q(t_3) + Q(t_3)Q(t_4) - Q(t_1)Q(t_4)]$ is always $+2$ or -2. If all the individual product terms in this expression are replaced by their averages over the entire ensemble of such sets of runs, the following form of LGI is then obtained

$$C \equiv |C_{12} + C_{23} + C_{34} - C_{14}| \leq 2 \quad (1)$$

where the temporal correlation $C_{ij} \equiv \langle Q(t_i)Q(t_j)\rangle$.

- The above is, thus, an inequality imposing realist constraints on the temporal correlations pertaining to oscillations in any two-state system.

THE IDEA OF NEGATIVE RESULT MEASUREMENT JUSTIFYING NIM AS A CONSEQUENCE OF REALISM

- Let us arrange our measuring apparatus so that if $Q(t)$ is, say, $+1$, it is triggered, while if $Q(t) = -1$ it is *not*. We then do a series of runs in which Q is measured at some instant, say, t_0. One then uses the results of those runs on which $Q(t_0) = -1$, and the rest are discarded.
  We then invert the measuring setup so that a value of $Q(t) = -1$ triggers it, while for $Q(t) = +1$, it does *not*. For this case, one then uses the results of those runs on which $Q(t_0) = +1$, and the rest are discarded.
  In this way, one can evaluate the temporal correlations which occur in the argument leading to LGI. Note that it is only the first measurement of any pair which needs to be noninvasive.

LEGGETT-GARG INEQUALITY FROM JOINT PROBABILITY DISTRIBUTION

If the assumptions used in deriving LGI are true then one would be able to define an overall joint probability distribution $\rho(Q_1, Q_2, Q_3, Q_4)$ for an ensemble of systems prepared in an identical state where *any* Q_i has a definite value (+1 or -1) assigned to the observable Q at *any* time $t_i, i = 1, 2, 3, 4$ $(t_1 < t_2 < t_3 < t_4)$. Then the pair-wise joint probability distributions like $P(Q_1, Q_2)$,$P(Q_2, Q_3)$,$P(Q_3, Q_4)$,$P(Q_1, Q_4)$ can be obtained by appropriate marginalizations. For example,

$$P(Q_1, Q_2) = \sum_{Q_3=\pm 1, Q_4=\pm 1} \rho(Q_1, Q_2, Q_3, Q_4)$$

$$= \rho(Q_1, Q_2, +, +) + \rho(Q_1, Q_2, -, +) + \rho(Q_1, Q_2, +, -) + \rho(Q_1, Q_2, -, -).$$

LEGGETT-GARG INEQUALITY FROM JOINT PROBABILITY DISTRIBUTION

Hence the correlation function like C_{12} is given by

$$C_{12} = \langle Q_1 Q_2 \rangle = P(Q_1+, Q_2+) + P(Q_1-, Q_2-) - P(Q_1-, Q_2+) - P(Q_1+, Q_2-)$$

$$= \sum_{Q_3=\pm 1, Q_4=\pm 1} [\rho(+,+,Q_3,Q_4) + \rho(-,-,Q_3,Q_4) - \rho(-,+,Q_3,Q_4) - \rho(+,-,Q_3,Q_4)]$$

Similarly, one can write the relevent expressions for C_{23}, C_{34}, C_{14}. It can then be seen that

$$|C_{12} + C_{23} + C_{34} - C_{14}| \leq 2 \tag{2}$$

which can be obtained by using $\sum \rho(Q_1, Q_2, Q_3, Q_4) = 1$ where the summation is over all possible outcomes $Q_1, Q_2, Q_3, Q_4 = \pm 1$.

A SIMPLE EXAMPLE SHOWING QM VIOLATION OF LGI

Consider a system oscillating between two states $|A\rangle$ and $|B\rangle$ which are degenerate eigenstates of the Hamiltonian H_0

$H_0|A\rangle = E_0|A\rangle$, $H_0|B\rangle = E_0|B\rangle$ and $\langle A|H_0|A\rangle = \langle B|H_0|A\rangle = 0$

Oscillatory transitions between $|A\rangle$ and $|B\rangle$ are induced by $H^{'}$ where $\langle A|H^{'}|B\rangle = \langle B|H^{'}|A\rangle = \Delta E$.

If we take at $t = 0$, $|\psi(0)\rangle = |A\rangle$, then from $(H_0 + H^{'}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$ we obtain

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar}(cos(\Delta Et/\hbar)|A\rangle - isin(\Delta Et/\hbar)|B\rangle) \quad (3)$$

whence

$$|(A|\psi(t)|^2 = \frac{1}{2}(1 + cos\omega t) \quad (4)$$

$$|(B|\psi(t)|^2 = \frac{1}{2}(1 - cos\omega t) \quad (5)$$

where $\omega = \frac{2\Delta E}{\hbar}$

A SIMPLE EXAMPLE SHOWING QM VIOLATION OF LGI

If we measure an observable $Q = |A\rangle\langle A| - |B\rangle\langle B|$ at two different distants 0 and t, then the correlation function $C_{12} = \langle Q(t=0)Q(t=t)\rangle$ will be given by

$$C_{12} = P(++) + P(--) - P(-+) - P(+-) = cos\omega t \tag{6}$$

Choosing equal time intervals $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t$,

$$C_{12} + C_{23} + C_{34} - C_{14} = 3cos(\omega\Delta t) - cos(3\omega\Delta t)$$

If $\omega\Delta t = \pi/4$, the above quantity maximally violates LGI.

EXAMPLE USING SPIN SYSTEM

Consider an ensemble of spin-$\frac{1}{2}$ particles where an external magnetic field is applied along x-direction. The Hamiltonian of the system is, $H = \frac{1}{2}\omega\sigma_x$, where ω is the angular precession frequency (taking $\hbar = 1$). Let the initial state be a mixture of two eigenstates of σ_x with equal weightage. Then the relevant density operator is $\frac{\mathcal{I}}{2}$. Now, if we measure the dichotomic observable σ_z (spin along $z =$ direction) at different instants with the time intervals Δt, then the correlation function will be (taking the initial time $t = 0$)

$$C_{12} = \langle Q_1 Q_2 \rangle = P(Q_1+, Q_2+) + P(Q_1-, Q_2-) - P(Q_1-, Q_2+) - P(Q_1+, Q_2-)$$

$$= \frac{1}{2}cos^2(\frac{\omega\Delta t}{2}) + \frac{1}{2}cos^2(\frac{\omega\Delta t}{2}) - \frac{1}{2}sin^2(\frac{\omega\Delta t}{2}) - \frac{1}{2}sin^2(\frac{\omega\Delta t}{2}) = cos(\omega\Delta t)$$

Choosing equal time interval $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t$, one obtains

$$C_{12} + C_{23} + C_{34} - C_{14} = 3cos(\omega\Delta t) - cos(3\omega\Delta t).$$

Taking $\omega\Delta t = \frac{\pi}{4}$, the above expression has the value $2\sqrt{2}$, thereby maximally violating LGI.

SCHEMATIC DESCRIPTION OF OUR WORK

Linear Harmonic Oscillator

- Initial wavepacket is

$$\psi(x,0) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_0}} \exp\left(-\frac{x^2}{4\sigma_0^2} + \frac{ip_0 x}{\hbar}\right) \tag{7}$$

- Here one considers measuring localization of the particle. If the particle is found in the region between $x \to -\infty$ and $x = 0$, then the measurement outcome is denoted by $+1$. If the particle is found in the region between $x = 0$ and $x \to \infty$, then the outcome is denoted by -1.
- The above mentioned condition is satisfied by defining the following measurement operator

$$\hat{O} = \int_{-\infty}^{0} |x\rangle\langle x| dx - \int_{0}^{\infty} |x\rangle\langle x| dx \tag{8}$$

PROPERTIES OF THE OBSERVABLE $\widehat{O}$

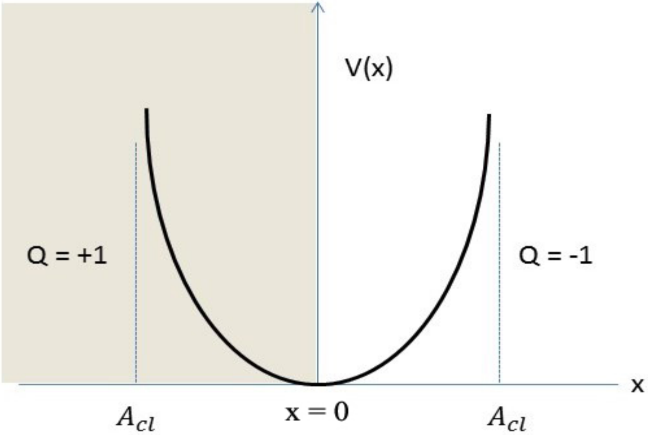
- The observable $\widehat{O}$ has two eigenstates having eigenvalues $+1$ and -1 respectively. For the eigenvalue $+1$, we have the corresponding eigenstate defined by

$$\hat{O}\int_{-\infty}^{0}\langle x|\psi\rangle|x\rangle dx = +1\int_{-\infty}^{0}\langle x|\psi\rangle|x\rangle dx \tag{5}$$

- For the eigenvalue -1 we have the corresponding eigenstate defined by

$$\hat{O}\int_{0}^{\infty}\langle x|\psi\rangle|x\rangle dx = -1\int_{0}^{\infty}\langle x|\psi\rangle|x\rangle dx \tag{6}$$

OUR SETUP



TIME EVOLUTION

- Initial wave packet is evolved by the following propagator

$$K(x', t'; x, t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t-t')}} \tag{9}$$
$$\exp\left(\frac{im\omega}{2\hbar\sin\omega(t-t')}\left((x'^2+x^2)\cos\omega(t-t')-2xx'\right)\right)$$

- Wave packet at the instant t is given by

$$\psi(x,t) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_t}}\exp\left(-\frac{\alpha(t)+\beta x+\gamma(t)x^2}{4\hbar\sigma_0\sigma_t}\right) \tag{10}$$

TIME EVOLUTION

- $\alpha(t), \beta, \gamma(t)$ are given by

$$\begin{aligned} \alpha(t) &= 2i\sigma_0^2\langle x(t)\rangle \\ \beta &= 2ip_0\sigma_0^2 \\ \gamma(t) &= \hbar\cos\omega t + 2im\omega\sigma_0^2\sin\omega t \\ \langle x(t)\rangle &= \frac{p_0}{m\omega}\sin\omega t \\ \sigma_t &= \frac{i\hbar\sin\omega t + 2m\omega\sigma_0^2\cos\omega t}{2m\omega\sigma_0.} \end{aligned} \tag{11}$$

- Note that $p_0/m\omega = A_{Cl}$ is the amplitude of the corresponding classical oscillation.

MEASUREMENT RESULTS AT TIME t

- Probability at time t of finding the particle in the region between $x \to -\infty$ and $x = 0$ is given by

$$P_{+}(t) = \int_{-\infty}^{0} |\psi(x,t)|^2 dx = \frac{1}{2}\left(1 - Erf(\frac{\langle x(t)\rangle}{\sqrt{2}|\sigma_t|})\right) \qquad (12)$$

- Probability at time t of finding the particle in the region between $x = 0$ and $x \to \infty$ is given by

$$P_{-}(t) = \int_{0}^{\infty} |\psi(x,t)|^2 dx = \frac{1}{2}\left(1 + Erf(\frac{\langle x(t)\rangle}{\sqrt{2}|\sigma_t|})\right) \qquad (13)$$

ERROR FUNCTION

- Error function is defined as

$$Erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-z^2) dz \tag{14}$$

- Few properties of error function are

$$Erf(\infty) = 1 \tag{15}$$

$$Erf(-t) = -Erf(t) \tag{16}$$

POST-MEASUREMENT STATE AT TIME t

- When the particle is found at the instant t in the region between $x \to -\infty$ and $x = 0$, the post-measurement state is given by

$$|\psi_{+}^{PM}(t)\rangle = \int_{-\infty}^{0} \psi(x', t)|x'\rangle dx' \tag{17}$$

- When the particle is found at the instant t in the region between $x = 0$ and $x \to \infty$, the post-measurement state is given by

$$|\psi_{-}^{PM}(t)\rangle = \int_{0}^{\infty} \psi(x', t)|x'\rangle dx' \tag{18}$$

FURTHER EVOLUTION OF THE STATE AFTER 1st MEASUREMENT

- If $+1$ result is obtained at, say, $t = t_1$, then the post-measurement state under the harmonic oscillator potential evolves into the following state at the instant $t = t_2$

$$|\psi_+^{PM}(t_2)\rangle = \int_{-\infty}^{\infty} K(x', t_1; x, t_2)\psi(x', t_1)_+^{PM}|x'\rangle dx' \tag{19}$$

- If -1 result is obtained at, say, $t = t_1$, then the post-measurement state under the harmonic oscillator potential evolves into the following state at the instant $t = t_2$

$$|\psi_-^{PM}(t_2)\rangle = \int_{-\infty}^{\infty} K(x', t_1; x, t_2)\psi(x', t_1)_-^{PM}|x'\rangle dx' \tag{20}$$

JOINT PROBABILITIES AFTER THE 2nd MEASUREMENT

- Conditional Probability of finding the particle in the region between $x \to -\infty$ and $x = 0$ at the instant t_2 when $\pm$ result for the measurement of the localization operator $\widehat{0}$ has been obtained at the instant t_1 is given by

$$P_{\pm/+}(t_1, t_2) = \int_{-\infty}^{0} |\psi(x, t_2)_{\pm}^{PM}|^2 dx \tag{21}$$

- Similarly, the Conditional Probability of finding the particle in the region between $x = 0$ and $x \to \infty$ at the instant t_2 when $\pm$ result for the measurement of the localization operator $\widehat{0}$ has been obtained at the instant t_1 is given by

$$P_{\pm/-}(t_1, t_2) = \int_{0}^{\infty} |\psi(x, t_2)_{\pm}^{PM}|^2 dx \tag{22}$$

TEMPORAL CORRELATION FUNCTIONS

- The *temporal correlation function*, say, C_{12} occurring in the Leggett-Garg inequality is given by

$$C_{12} = P_{++}(t_1, t_2) - P_{+-}(t_1, t_2) + P_{--}(t_1, t_2) - P_{-+}(t_1, t_2) \tag{23}$$

- where $P_{++}(t_1, t_2)$ is the *joint probability* of finding the measurement outcomes $+1$, $+1$ at the respective times t_1 and t_2; similarly, $P_{+-}(t_1, t_2)$, $P_{--}(t_1, t_2)$, and $P_{-+}(t_1, t_2)$ denote the corresponding *joint probabilities*. Thus, by evaluating these *joint probabilities*, one can calculate the quantity C_{12}. In a similar way, the other temporal correlation functions C_{23}, C_{34}, C_{14} occurring in LGI can also be calculated, thereby checking whether $|C_{12} + C_{23} + C_{34} - C_{14}| \leq 2$

SCHRÖDINGER COHERENT STATE

- Initial wave packet is given by

$$\psi(x,0) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_0}} \exp\left(-\frac{x^2}{4\sigma_0^2} + \frac{ip_0 x}{\hbar}\right) \tag{24}$$

- At time t it evolves into (Here $\sigma_0 = \sqrt{\frac{\hbar}{2m\omega}}$)

$$\psi(x,t) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_t}} \exp\left(-\sqrt{m\omega}\frac{\alpha(t) + \beta x + \gamma(t)x^2}{4\hbar^{3/2}\sigma_t}\right) \tag{25}$$

- Here $\alpha(t), \beta, \gamma(t), \sigma_t$ are defined as earlier.

SCHRÖDINGER COHERENT STATE

- Probability density of this state is given by

$$|\psi(x,t)|^2 = \sqrt{\frac{m\omega}{\hbar\pi}} \exp\left(-m\omega \frac{(x - \frac{p_0}{m\omega}\sin\omega t)^2}{\hbar}\right) \qquad (26)$$

- Thus, the probability density of this wave packet oscillates *without spreading* or *changing shape* with its *peak* following classical motion and $\Delta x \Delta p = \hbar/2$. Hence coherent state is regarded as the "best possible" quasi-classical quantum description of the motion of a classical harmonic oscillator.

SALIENT FEATURES OF CALCULATIONAL RESULTS

In our setup, the key parameters are p_0, ω where p_0 is the initial peak momentum (expectation value of momentum corresponding to the initial wave packet) and ω is the angular frequency of the corresponding classical oscillation. Suitably choosing p_0, ω and by appropriate tuning of $t, \Delta t$, the QM violation of LGI for a given mass (m) may be shown.

In the calculational results we present, p_0 and ω are throughout chosen such that the corresponding classical amplitude of oscillation ($A_{Cl} = p_0/m\omega$) ranges from that of the *order of centimetres* to *thousands of nanometer*, and the time period (T) of oscillation remains of the order of seconds.

SALIENT FEATURES OF CALCULATIONAL RESULTS

The results are given for the following cases:

(a) The QM violation of LGI with *changing mass* is studied by keeping p_0, ω *fixed*.

(b) The QM violation of LGI is studied by changing the corresponding classical amplitude of oscillation for a given mass. This is done by *changing* ω, *keeping* p_0 *fixed* (A_{Cl} then varies from *centimetres* to a *metre*).

(c) For *different masses* up to, say, 10^{15} *amu* (this range can be extended), by appropriately choosing p_0, ω, $t_1, \Delta t$ for a given mass, the QM violation of LGI can be shown by keeping the classical amplitude of oscillation ranging from *centimetres* to *thousands and hundreds of nanometre*, and the time period of oscillation within of the order of *seconds*, up to a *minute*.

The action quantity relevant to our setup is given by $p_0 A_{Cl} (= p_0^2/m\omega)$ and the QM violation of LGI is exhibited under the conditions when $(p_0^2/m\omega) \gg \hbar$; or $(S/\hbar) \gg 1$ where $S = p_0^2/m\omega$

QUANTUM VIOLATION OF LGI WITH CHANGING MASS

- Here the key parameters p_0, ω are kept fixed and $t_1, \Delta t$ are appropriately chosen to maximize quantum violation of LGI.

MASS(amu)	$S/\hbar$	$A_{Cl}(m)$	C
10	1.66×10^5	10^{-2}	2.72
100	1.66×10^4	10^{-3}	2.701
500	3.32×10^3	10^{-3}	2.70
1000	1.66×10^3	10^{-4}	2.69
10000	1.66×10^2	10^{-5}	2.36
20000	83	10^{-5}	2.09
30000	55	10^{-5}	1.85

- $\omega = 0.4Hz, p_0 = 3.32 \times 10^{-28} kgm/s, T = 15s, A_{Cl} = p_0/m\omega$, $t_1 = 7.5s, \Delta t = 12s$.

QUANTUM VIOLATION OF LGI WITH CHANGING MASS

- Here the key parameters p_0, ω are kept fixed and $t_1, \Delta t$ are appropriately chosen to maximize quantum violation of LGI.

MASS(amu)	$S/\hbar$	$A_{Cl}(m)$	C
10	3.32×10^5	10^{-1}	2.78
100	3.32×10^4	10^{-2}	2.73
500	6.64×10^3	10^{-3}	2.71
1000	3.32×10^3	10^{-3}	2.7
10000	3.32×10^2	10^{-4}	2.54
20000	1.66×10^2	10^{-5}	2.35
30000	1.1×10^2	10^{-5}	2.2
50000	66	10^{-5}	1.97

- $\omega = 0.2Hz, p_0 = 3.32 \times 10^{-28} kgm/s, T = 31s, A_{Cl} = p_0/m\omega,$ $t_1 = 15s, \Delta t = 24s.$

QUANTUM VIOLATION OF LGI FOR CHANGING CLASSICAL AMPLITUDE

- For a given mass(here $100amu$), keeping ω fixed, classical amplitude is changing with p_0.

$p_0(Kgm/s)$	$A_{Cl}(m)$	C
3.32×10^{-28}	10^{-2}	2.81
1.66×10^{-27}	0.05	2.69
3.32×10^{-27}	0.01	2.65
1.66×10^{-26}	0.5	2.51
3.32×10^{-26}	1	2.30

- $\omega = 0.2Hz, t_1 = 15s, \Delta t = 24s, T = 31s.$
- Quantum violation of LGI decreases with increasing classical amplitude.

QUANTUM VIOLATION OF LGI FOR LARGER MASSES

- Tuning p_0 and ω, QM violation of LGI is computed for larger masses

MASS (amu)	$p_0(Kgm/s)$	$\omega(Hz)$	$A_{Cl}(m)$	T (s)	C
3000	3.3×10^{-27}	.2	3×10^{-3}	31	2.7
5000	3.3×10^{-27}	.2	2×10^{-3}	31	2.68
10000	3.3×10^{-27}	.1	2×10^{-3}	63	2.7
50000	3.3×10^{-26}	.1	4×10^{-3}	63	2.77
10^5	3.3×10^{-26}	.1	2×10^{-3}	63	2.74
10^6	3.3×10^{-26}	.1	2×10^{-4}	63	2.66
10^7	3.3×10^{-25}	.1	2×10^{-4}	63	2.48
10^8	3.3×10^{-25}	.1	2×10^{-5}	63	2.73
10^9	3.3×10^{-25}	.1	2×10^{-6}	63	2.7
10^{10}	3.3×10^{-24}	.1	2×10^{-6}	63	2.75
10^{12}	3.3×10^{-23}	.1	2×10^{-7}	63	2.78
10^{15}	3.3×10^{-21}	.1	2×10^{-8}	63	2.67

- For $T = 31s$, $t_1 = 15s$, $\Delta t = 24s$, and when $T = 63s$, $t_1 = 30s$, $\Delta t = 48s$.

FURTHER DIRECTIONS OF STUDY

(a) To study the effect of decoherence due to coupling with dissipative environment and probe the QM violation of LGI for the Schrödinger Coherent State for a wider range of variation of the relevant parameters.

(b) To compare in detail the QM violation of LGI for different cases by varying the choice of the initial wave packet (say, comparing the cases of non-spreading and spreading wave packets, or minimum uncertainty vis-a-vis non-minimum uncertainty wave packets).

(c) To explore in the context of such an example the feasibility of relevant experimental studies.

CONCLUDING REMARKS

In our treatment, we've *dichotomised* the outcomes of relevant measurements for the time evolution of a system involving continuous variables.

This enables us to show QM violation of LGI for a quasi-classical quantum state of linear harmonic oscillator, viz. for the Schrödinger Coherent State, even for masses quite *large* compared to the typical microscopic masses.

Such QM violation of LGI, if experimentally detected, would thus enable to *repudiate* the notion of *realism*, even for large masses undergoing linear harmonic oscillation; also, would enable empirical study of precisely how the quantum-classical transition occurs for such a system.

Einstein had once remarked, "*I like to think that the moon is there even if I don't look at it.*" LGI is a powerful tool for providing scripts for *quantitative tests* of such an assertion for *different* types of *systems* corresponding to various degrees of *macroscopicness*. Besides, like what happened in the case of BI, LGI might offer similar surprises of its own as regards possible applications in quantum communication.

ENERGY ESTIMATION

- $E = \frac{1}{2} m\omega^2 A_{Cl}^2 = \frac{p_0^2}{2m}$

MASS(amu)	ω(Hz)	p_0(kgm/s)	T(s)	A_{Cl}(m)	E(eV)
100	0.2	3.3×10^{-28}	31	10^{-2}	2×10^{-12}
10000	0.2	3.3×10^{-28}	31	10^{-4}	2×10^{-14}
5×10^4	0.2	3.3×10^{-28}	31	10^{-5}	4×10^{-15}
10^6	0.1	3.3×10^{-26}	63	10^{-4}	2×10^{-12}
10^7	0.1	3.3×10^{-25}	63	10^{-4}	2×10^{-11}
10^9	0.1	3.3×10^{-25}	63	10^{-6}	2×10^{-13}
10^{10}	0.1	3.3×10^{-24}	63	10^{-6}	2×10^{-12}

RELATION OF ERROR FUNCTION WITH GAMMA FUNCTION

- Error function can be expressed as

$$Erf(z) = \frac{1}{\sqrt{\pi}}\gamma(1/2, z^2), Erfc(z) = \frac{1}{\sqrt{\pi}}\Gamma(1/2, z^2) \qquad (27)$$

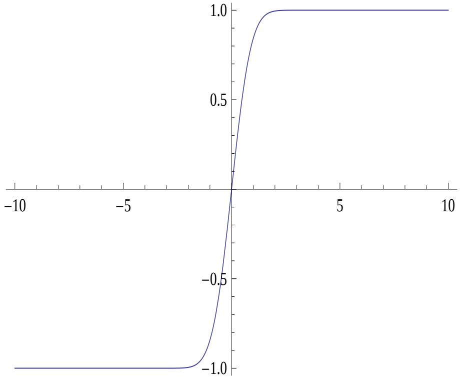
$$\gamma(a, x) = \int_0^x \exp^{-t} t^{a-1} dt, \Gamma(a, x) = \int_x^\infty \exp^{-t} t^{a-1} dt \quad (28)$$

- Normalization of these functions are

$$\Gamma(a, x) + \gamma(a, x) = \gamma(a), Erf(z) + Erfc(z) = 1 \qquad (29)$$

PICTORIAL REPRESENTATION OF ERROR FUNCTION

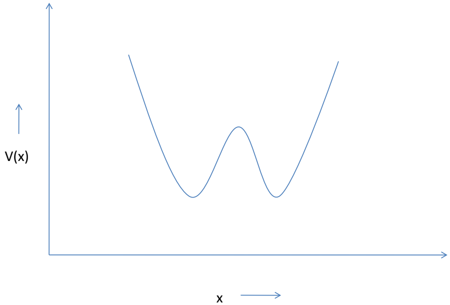
- Error function with real argument.



- Error function is plotted along vertical axis while argument is along horizontal axis..

DOUBLE WELL POTENTIAL

- In double well potential there is two distinct possibilities whether the particle or flux will be in 1st or 2nd well. Superposition of these two possibilities arise as well.



THANK YOU