Majorana modes and non-abelian statistics

Sumathi Rao

Harish-Chandra Research Institute, Allahabad, India

December 2013

Plan of the talk

Introduction

What are Majorana fermions or Majorana modes?

Named after Ettore Majorana

But refer to different kinds of excitations in high energy physics and condensed matter physics

Nomenclature

In particle physics, Majorana fermions are fermionic particles $\equiv$ anti-particles, $\psi = \psi^\dagger$, where $\psi, \psi^\dagger$ are annihilation and creation operators

In condensed matter physics, hermitian operators $\gamma = \gamma^\dagger$ satisfying fermionic algebra

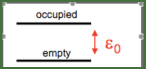
$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} \quad \gamma_a^2 = 1,$$

form zero energy modes of some system.
Not true fermions (sometimes called weird fermions!), but instead obey non-abelian statistics under exchange

Pairs of Majorana fermions can be combined into ordinary fermions

$$c = \frac{1}{2}(\gamma_1 + i\gamma_2), c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2),$$

form a single 2 level system



If the Majorana fermions are spatially separated, implies fermion state is delocalised,
Protected from local changes that affect only one of the Majorana fermions, hence protected from decoherence
Expected to be relevant for topological quantum computation

Majorana fermion expected to be equal superposition of particle and hole states
S-wave superconductors have Boguliobov quasi-particles which are superpositions of spin up and down fermions

$$b = uc_{\uparrow}^{\dagger} + vc_{\downarrow}$$

But Majorana fermions

$$\begin{aligned} \gamma_1 &= c_\sigma + c_\sigma^{\dagger}, \\ \gamma_2 &= -i(c_\sigma - c_\sigma^{\dagger}) \end{aligned}$$

made up of superpositions of fermions with equal spin
Need effectively spinless or *p*-wave superconductors

Kitaev Model

Kitaev Model

Kitaev's toy model for 1D spinless p wave superconductor:

$$H = -\mu \sum_x c_x^\dagger c_x - \frac{1}{2} \sum_i (t c_x^\dagger c_{x+1} + \Delta c_x c_{x+1} + h.c.)$$

$\Delta \geq 0$ **= p wave pairing amplitude**

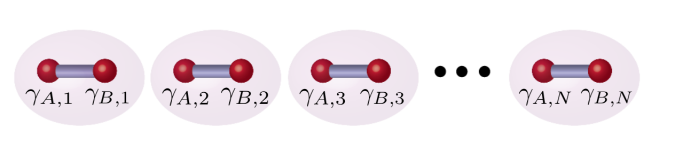
Can rewrite in terms of Majorana fermions

$$\begin{aligned} c_x &= \frac{1}{2}(\gamma_{x,1} + i\gamma_{x,2}), \\ c_x^\dagger &= \frac{1}{2}(\gamma_{x,1} - i\gamma_{x,2}) \end{aligned}$$

The two limits LIMIT 1 :

$\mu < 0, t = \Delta = 0$ (Topologically trivial phase)

$$H = -\frac{\mu}{2}\sum_{x=1}^{N}(1 + i\gamma_{B,x}\gamma_{A,x})$$

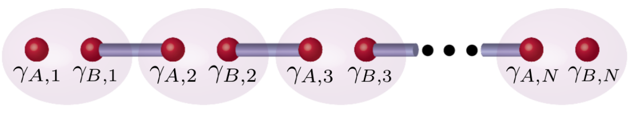


Here, fermion at single site is broken up into 2 Majoranas
The ground state is unique and the ends of the chain do not play any major role

The two limits LIMIT 2 :

$\mu = 0, t = \Delta \neq 0$ (Topological phase)

$$H = -i\frac{t}{2}\sum_{x=1}^{N-1} \gamma_{B,x}\gamma_{A,x+1}$$



Here, bond couples Majorana fermions at different sites

But unpaired *zero energy* Majorana modes $\gamma_1 \equiv \gamma_{A,1}$ and $\gamma_2 \equiv \gamma_{B,N}$ at the two ends of the chain
The Hamiltonian has no dependence on these 2 Majorana fermions - $\gamma_{A,1}$ or $\gamma_{B,N}$

We can form an ordinary but *nonlocal* fermion
$f = \frac{1}{2}(\gamma_1 + i\gamma_2)$
Energy is independent of whether or not this fermion state is occupied
- i.e., non unique ground state [$|0\rangle$ and $|1\rangle = f^\dagger |0\rangle$]

$\gamma_i^2 = 1$ implies no Pauli principle for Majorana particles and no notion of occupation number

Number operators only for fermions formed from Majorana particles

Depending on occupation or not of zero mode of fermion, odd or even number of fermions in ground state, odd or even parity state

To change parity, electrons have to be added or removed from the superconductor

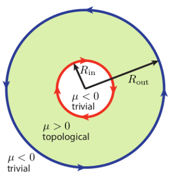
Non-abelian statistics

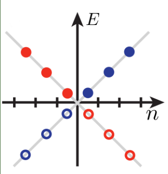
Non-abelian statistics

Consider 2 dimensional p-wave superconductor

$$H = \int d^2\mathbf{r}\left\{\psi^\dagger\left(-\frac{\nabla^2}{2m} - \mu\right)\psi + \frac{\Delta}{2}\left[e^{i\phi}\psi(\partial_x + i\partial_y)\psi + h.c.\right]\right\}$$

Assuming $\mu(r)$ is slowly varying, localized chiral Majorana modes at the "edges" of the topological phase ($\mu > 0$)



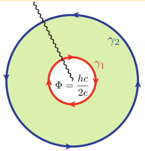


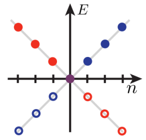
Majorana modes in p-wave superconductors

When a flux quantum $\Phi = \frac{hc}{2e}$ threads the central region

$$\Delta \longrightarrow \Delta e^{i\delta\theta} \Longleftrightarrow \psi \longrightarrow \psi e^{i\delta\theta/2}$$

$\delta\theta = 2\pi$ shift is irrelevant for the Cooper pairs, but to account for sign change of unpaired fermions at $\delta\theta = 2\pi$, need to introduce branch cuts





Branch cuts give rise to non-abelian statistics

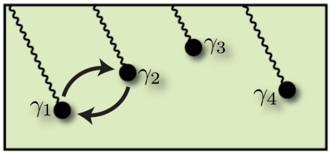
Suppose we have 2N Majorana zero modes $\gamma_1, \gamma_2, ..., \gamma_{2N}$
Ground state 2^N fold degenerate

Adiabatic exchange of Majorana particles can take the system from one ground state to another
Difficult to see using evolution of many body wave-function

Simplest picture requires 4 Majorana fermions, because with 2 Majorana fermions, cannot change the occupancy of the fermion formed by 2 Majorana particles, since parity (oddness or evenness of the number of fermions) preserved

Non-Abelian Statistics(in 2D)

Consider four well separated Majorana zero modes $\gamma_{1,2,3,4}$ with their associated branch cuts



For exchange shown above, one crossing of branch cut, $\gamma_1 \longrightarrow -\gamma_2$ and $\gamma_2 \longrightarrow \gamma_1$ and naturally, $\gamma_{3,4} = \gamma_{3,4}$

Implemented by $U_{j,j+1} = \frac{1}{\sqrt{2}}(1 + \gamma_j \gamma_{j+1})$ such that $U_{j,j+1} \gamma_{j/j+1} U^{\dagger}_{j,j+1} = \mp \gamma_{j+1/j}$

With this definition,

$$U_{12}\left|n_1, n_2\right\rangle = e^{i\frac{\pi}{4}(1-2n_1)}\left|n_1, n_2\right\rangle$$

$$U_{23}\left|n_1, n_2\right\rangle = \frac{1}{\sqrt{2}}\left[\left|n_1, n_2\right\rangle + i(-1)^{n_1}\left|1-n_1, 1-n_2\right\rangle\right]$$

$$U_{34}\left|n_1, n_2\right\rangle = e^{i\frac{\pi}{4}(1-2n_2)}\left|n_1, n_2\right\rangle$$

Exchanging Majorana fermions at the same site gives rise to phase, but exchanging Majoranas from different sites leads to a different state in the degenerate ground state

$U_{23} \Longrightarrow$ nontrivial rotation in the ground state manifold

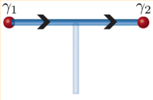
For sequential exchanges, final state depends on order of operations $\longrightarrow$ non-Abelian statistics of the Majoranas

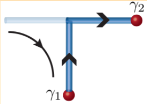
Demonstrated by $[U_{j-1,j}, U_{j,j+1}] = \gamma_{j-1}\gamma_{j+1}$

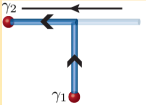
Non-Abelian Statistics(in 1D)

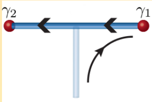
Hard to implement exchange of two particles in one dimension without having particles go through one another
Hence, exchange statistics not well defined in 1d

Need to construct networks of 1D wires to meaningfully exchange modes





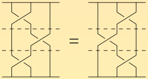




Can be shown that non-Abelian statistics persists in one dimension

An aside: Abelian versus non-abelian anyons

Abelian anyons acquire phase under exchange $\psi(x_1, x_2) = e^{i\alpha}\psi(x_2, x_1)$, obey braid group statistics, but order of exchange unimportant



Ground state is unique, and after exchange, state returns to the ground state

Hard to detect, because statistical phase factors difficult to isolate from other phase factors like the Aharanov-Bohm phase factor, dynamical phase, etc and has not been achieved so far

Signatures expected to be cleaner for particles obeying non-abelian statistics

Upon exchange, non-abelian particles may end up in a different state that does not interfere with the original state
Hence, expect suppression of interference
Easier to detect than precise phase information needed for abelian statistics

Non-abelian statistics perhaps seen in $\nu = 5/2$ FQHE state?

R.L. Willet, Pfeiffer and West, PRB 82, 205301 (2010)

Importance for quantum computing

2 separated Majorana particles = 1 fermion
Occupied or unoccupied give rise to 2 degenerate states = 1 qubit

2N separated Majorana particles = N qubits

Quantum information is stored non-locally, hence immune from local decoherence

Braiding performs unitary operations in the degenerate space of N qubits

Braiding particles with non-abelian statistics entangles them, because with non-abelian statistics, the probability amplitude depends on the order in which particles are braided

Change of state by braiding particle 2 and 3 as seen earlier
Entanglement possible even if none of the particles are near each other, because result only depends on topological properties of the system
Hence fault tolerant quantum computer

Majorana fermions (also called Ising anyons), not sufficient for universal quantum computation

Has not stopped the enormous interest in the field of Majorana fermions in condensed matter physics

Has started interest in looking for non-abelian quasi-particles beyond Majorana particles

Hosting and detecting Majorana particles

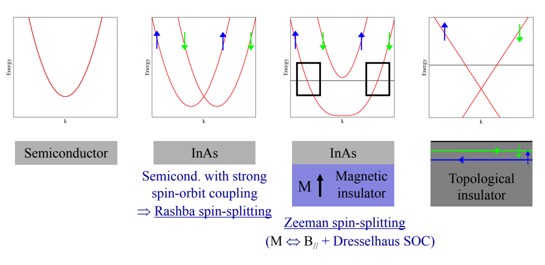
Majorana particles obey non-abelian statistics

Majorana fermions are zero energy states that do not conserve particle number, but conserve parity or particle number modulo2 so need
(a) superconductivity (b) p-wave
Candidate Sr_2RuO_4

Read and Green, Ivanov,Kitaev, 2001

Can be engineered by standard superconductor in proximity with topological insulator

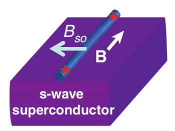
Fu and Kane, 2008



Can be engineered with s-wave superconductors, strong spin-orbit coupling and a magnetic field
Candidates *InAs*, *InSb* (Sau et al, Alicea, Oreg et al, 2010)

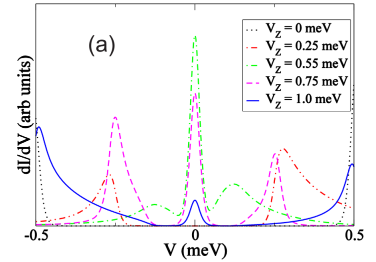
Predictions

Several detection scenarios have been proposed



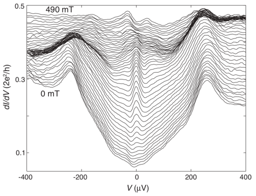
For small values of Δ, superconducting state topological and associated with Majorana edge states, as long as $|B| > \mu$ and B not so large that it aligns all spins in a band and makes superconductivity hard to induce
Actual condition $|B| > \sqrt{(\Delta^2 + \mu^2)}$ for topological superconductivity

Theoretical prediction $\longrightarrow$ tunneling spectroscopy into the end of the wire should reveal a state with zero energy



Experimental results

In 2012, observation of Majorana modes was first reported by two groups working in Netherlands and in Weizmann



For magnetic field ranging from 100mT to 400mT, zero bias peak clearly observed
Almost all other possible explanations eliminated

V. Mourik et al, A.Das et al

Conclusion and future directions

Other routes to detection of Majorana fermions

Anomalous Josephson effect

Signature of Majorana in interferometry experiments - Fabry-Perot, Mach-Zehnder and Hanbury-Brown-Twiss

No experimental results yet

Conclusion

Have discussed properties of Majorana fermions including their non-abelian statistics
Have also discussed where they can be found (and perhaps have been found!) in condensed matter systems

Other properties including the braiding and non-abelian statistics under exchange are yet to be verified

Far from the final aim, which is to control and manipulate Majorana fermions for use in quantum computation

Even more ambitious goal is to look for other non-abelian excitations beyond the Ising anyons (Majorana fermions)