Quantum Criticality, dynamics and Loschmidt echo

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S. Sharma, V. Mukherjee and A. Dutta, EPJ B **85** 143 (2012)
V. Mukherjee, S. Sharma and A. Dutta, PRB **86**, 020301 (R) (2012).
T. Nag, U. Divakaran and A. Dutta, PRB **86**, 020401(R) (2012).

QIPA,
HRI,
4tth December, 2013.

Outline

- Generic Definition
- Central Spin model and decoherence of the qubit
- Loschmidt echo close to a quantum critical point

  Equilibrium situation
- Non-Equilibrium situation

a: Non-equilibrium initial state

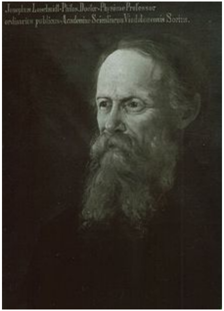
b: Dynamics of the decoherence of the qubit:

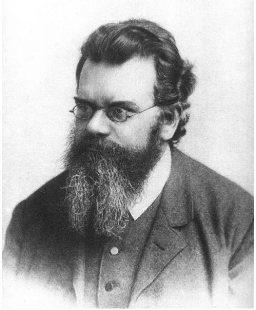
  universal scaling of decoherence factor

Generic Definition

Fight between J. Loschmidt and L. Boltmann

Second Law and time reversal invariance

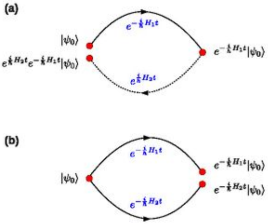




- J. Loschmidt
- L.Boltzmann

Source:Wikipedia

Loschmidt Echo



The generic definition:

$$\mathcal{L}(t) = |\langle\psi_0|e^{iH_1 t}e^{-iH_2 t}|\psi_0\rangle|^2 = |\langle\psi_0|e^{-iH_2 t}|\psi_0\rangle|^2$$

(If $|\psi_0\rangle$ Eigenstate of H_1)

• Overlap between two states evolving from the same initial state with different Hamiltonians.

• Sensitivity of the quantum evolution to external perturbation due to coupling to the environment.

Acknowledgement: Scholarpedia

Generic properties of the Loschmidt echo

- Characterized by a short-time decay.
- Partial revivals
- Asymptotic saturation

How does the proximity to a Q critical point influence the LE?

Static Counterpart Fidelity: $|\langle\psi_0(\lambda)|\psi_0(\lambda+\delta)\rangle|^2$

decays exponentially with δ for a many-body system.

Finite system: Sharp dip at the QCP.

Fidelity susceptibility and fidelity in the thermodynamic limit:

Interesting scaling relations.

Quantum Phase Transitions

Phase transitions are driven by fluctuations

• Zero temperature transition due to non-commuting terms in the Hamiltonian

• Driven by quantum fluctuations

simplest example: one dimensional Ising model in a transverse field

$$H = -\sum_{<ij>} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

• $h > 1$, $\langle \sigma^z \rangle = 0$; paramagnetic phase

• $h < 1$, $\langle \sigma^z \rangle \neq 0$; ferromagnetic phase

• Quantum phase transition at $h = 1$.

Quantum Phase Transitions: Critical Exponents

Notion of Universality:

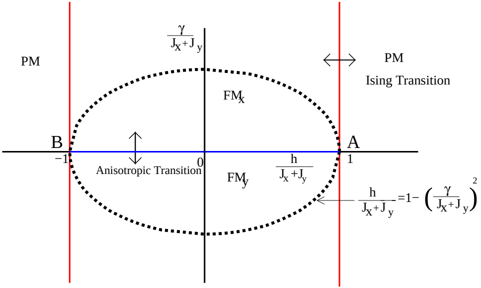
Symmetry, dimensionality and the nature of the fixed point

- $d \to (d+1)$
- Diverging length scale: $\xi \sim |\lambda|^{-\nu}$; $\lambda = h - 1$
- Diverging time scale: $\xi_\tau \sim |\lambda|^{-\nu z}$ Vanishing gap
- ν how one moves away from the critical point
- The dynamical exponent z associated with the critical point.

The model in consideration

Let us consider the Transverse XY spin chain

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$



- Critical Exponents for Ising transition $\nu = z = 1$
- Exponents with the Multicritical point $z_{mc} = 2$ and $\nu_{mc} = 1/2$.

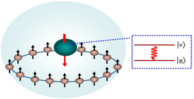
Two-Level System

Jordan-Wigner transformations: Spin-1/2's to Fermions

$$
\begin{aligned}
H &= \sum_{k>0} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \\
H_k &= 2 \begin{pmatrix} -(J_x + J_y)\cos(ka) - h & i(J_x - J_y)\sin(ka) \\ -i(J_x - J_y)\sin(ka) & (J_x + J_y)\cos(ka) + h \end{pmatrix},
\end{aligned}
$$

Decoupled two level systems

The central spin model and decoherence of a qubit



- A qubit coupled to a quantum critical many body system
- "Qubit" → a single Spin-1/2
- Environment → Quantum XY Spin chain
- A global coupling
- LE: Loss of phase information of the Qubit close to the QCP.

Does a QCP influence the Loschmidt echo?

The Central Spin model

- A central spin globally coupled to an environment.
- We choose the environment to be Transverse XY spin chain

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$

- and a global coupling $-\delta \sum_i \sigma_i^z \sigma_S^z$
- Qubit State: $|\phi_S(t=0)\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$
- The environment is in the ground state $|\phi_E(t=0)\rangle = |\phi_g\rangle$
- Composite initial wave function:

$$|\psi(t=0)\rangle = |\phi_S(t=0)\rangle \otimes |\phi_g\rangle$$

Quan *et al*, Phys. Rev. Lett. **96**, 140604 (2006).

Coupling and Evolution of the environmental spin chain

• At a later time t, the composite wave function is given by $|\psi(t)\rangle = c_1|\uparrow\rangle \otimes |\phi_+\rangle + c_2|\downarrow\rangle \otimes |\phi_-\rangle$.

$|\phi_\pm\rangle$ are the wavefunctions evolving with the environment Hamiltonian $H_E(h \pm \delta)$ given by the Schrödinger equation

$$i\partial/\partial t|\phi_\pm\rangle = \hat{H}[h \pm \delta]|\phi_\pm\rangle.$$

• The coupling δ essentially provides two channels of evolution of the environmental wave function with the transverse field $h + \delta$ and $h - \delta$.

What happens to the central spin?

The reduced density matrix:

$$\rho_S(t) = \begin{pmatrix} |c_1|^2 & c_1 c_2^* d^*(t) \\ c_1^* c_2 d(t) & |c_2|^2 \end{pmatrix}.$$

• The decoherence factor (Loschmidt Echo)

$$\mathcal{L}(t) = d^*(t)d(t) = |\langle \phi_+(t)|\phi_-(t)\rangle|^2$$

Overlap between two states evolved from the same initial state with different Hamiltonian

• $\mathcal{L}(t) = 1$, pure state. $\mathcal{L}(t) = 0$ Complete Mixing

• Coupling to the environment may lead to Complete loss of coherence

Loschmidt Echo and quantum criticality

We are interested in the small δ limit

Equilibrium Situation:

- No explicit time dependence in the Hamiltonian.

Non-Equilibrium Situation:

- $|\psi_0\rangle$ is an eigenstate of H_0; but there is a sudden quenching
- Explicit time dependence in the Hamiltonian

Equilibrium situation: the LE transverse Ising chain

$$\mathcal{L}(t) = |\langle\phi_0| \exp(iH_+ t) \exp(-iH_- t) |\phi_0\rangle|^2 = |\langle\phi_+(t)|\phi_-(t)|^2$$

$$H_k^{\pm}(t) = 2 \begin{pmatrix} h \pm \delta + \cos k & \sin k \\ \sin k & -(h \pm \delta + \cos k) \end{pmatrix}$$

Two sets of Bogoliubov transformations

$$\mathcal{L}(t) = \prod_{k>0} F_k = \left[1 - 2\sin^2(2\alpha_k)\sin^2(\epsilon_k(h_+)t)\right]$$

$2\alpha_k = (\theta_k(h_+) - \theta_k(h_-))$ and $\tan\theta_k(h_+) = \frac{\sin k}{h_+ + \cos k}$

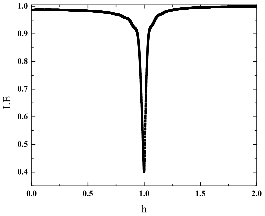
Equilibrium case: h independent of time

The decay of Loschmidt Echo close to QCP at a fixed t

Sum over modes close to the critical modes

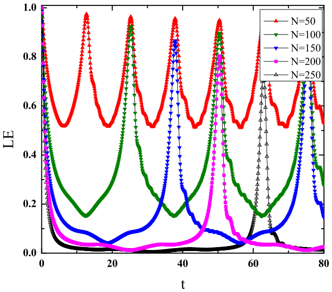
$$\mathcal{L}(t) = \exp(-\alpha t^2); \qquad \alpha \sim \frac{\delta^2}{(1-h)^2 N^2}$$

The scaling: $t \to t/p, N \to N/p$ or $\delta \to p\delta$



- Sharp dip at the quantum critical point
- Complete loss of coherence of the qubit

The collapse and revival at the QCP $h + \delta = 1$



Quasi-periodicity Scales with the system size L

Non-equilibrium initial state

- The initial state: not an eigenstate of uncoupled Hamiltonian H_F
- H_F is generated through a sudden quench $H_i(h_i) \to H_F(h_f)$.

$$H_F(\lambda) = H_0 + \lambda V_\lambda + g V_g$$

$$\mathcal{L}_q(\lambda, t, g) = |\langle G(\lambda, g=0)| e^{iH_F(\lambda,g)t} e^{-iH_F(\lambda+\delta,g)t} |G(\lambda, g=0)\rangle|^2$$

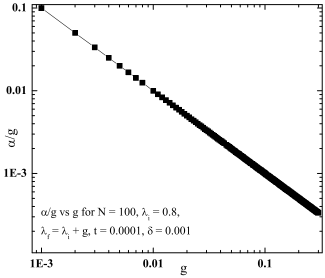
What happens to the temporal evolution of $\mathcal{L}$?

Early time decay $\mathcal{L}(t) \sim \exp(-\alpha t^2)$

$$\alpha = \frac{1}{2}\left[\langle\left(\frac{\partial H}{\partial\lambda}\right)^2\rangle - \langle\left(\frac{\partial H}{\partial\lambda}\right)\rangle^2\right]\delta^2 = \frac{1}{2}[\langle V_\lambda^2\rangle - \langle V_\lambda\rangle^2]\delta^2$$

- Independent of g

$\alpha \sim \delta^2 \lambda^{2\nu z-2} (\lambda \gg L^{-1/\nu})$ and $\alpha \sim \delta^2 L^{2/\nu - 2z} (\lambda \ll L^{-1/\nu})$



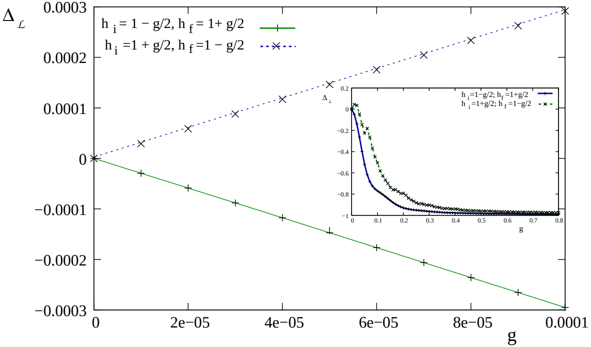
0.1
0.01
1E-3
α/g
1E-3
0.01
0.1
g
α/g vs g for N = 100, λi = 0.8,
λf = λi + g, t = 0.0001, δ = 0.001

When is the quenching relevant

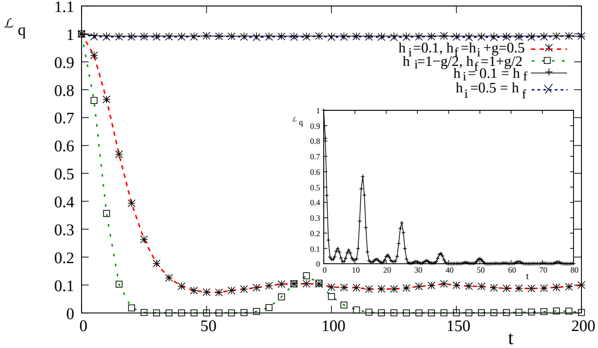
$$\mathcal{L}_q(\lambda, g, t) \approx \mathcal{L}(\lambda, 0, t) + g\frac{\partial \mathcal{L}_q(\lambda, g, t)}{\partial g}|_{g=0}$$

$\partial \mathcal{L}_q(\lambda, g, t)/\partial g|_{g=0} \sim g^{-1} \sim L^{1/\nu_g}$, where ν_g is the correlation length exponent.

$g \ll L^{-1/\nu_g}$, the correction due to quenching becomes irrelevant
For a fixed time $t = 20$ and $\delta = 0.025$



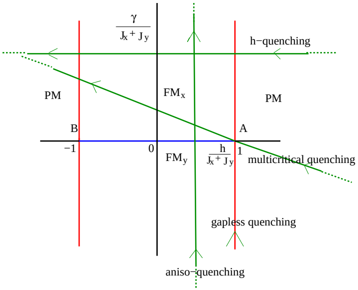
Variation with $\mathcal{L}$ with time: is there a revival



- faster decay in comparison to the equilibrium case.
- Partial revival when quenched to the QCP, $h_f + \delta = 1$.

What happens when the environmental spin chain is driven?

Different Quenching paths:



Assume $h(t) = 1 - t/\tau$, driven spin chain

$$H_k^{\pm}(t) = 2\begin{pmatrix} h(t) \pm \delta + \cos k & \gamma \sin k \\ \gamma \sin k & -(h(t) \pm \delta + \cos k) \end{pmatrix}.$$

B. Damski, Quan and Zurek, Phys. Rev. A **83**, 062104 (2011).

The decoherence factor $\mathcal{L}(t)$

$$|\phi^{\pm}(t)\rangle = \prod_k |\phi_k^{\pm}(t)\rangle = \prod_{k>0} \left[u_k^{\pm}(t)|0\rangle + v_k^{\pm}(t)|k,-k\rangle\right].$$

$$i\partial/\partial t \left(u_k^{\pm}(t), v_k^{\pm}(t)\right)^T = H_k^{\pm}(t) \left(u_k^{\pm}(t), v_k^{\pm}(t)\right)^T$$

with $\prod_k F_k(t) = \prod_k |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2$,

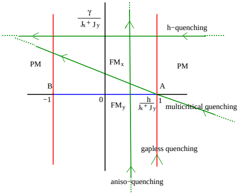
$$\mathcal{L}(t) = \exp\left[\frac{N}{2\pi}\int_0^{\pi} dk \;\; \ln F_k\right] \tag{1}$$

where F_k can be written in terms of $u_k^{\pm}$ and $v_k^{\pm}$.

Motivation: Kibble-Zurek Scaling

- Quenching through a QCP: Defect generation in the final state
- Universal scaling of the defect density: $n \sim 1/\tau^{\nu d/(\nu z+1})$

Different Quenching paths:



Critical point $h = t/\tau$; $n \sim \tau^{-1/2}$

Multicritical point

Quench $J_x = t/\tau$ with $h = 2J_y$; cross the MCP when $J_x = J_y$

We find Defect density: $n \sim \tau^{-1/6}$

Quenching through the gapless critical line $\gamma = t/\tau$:$n \sim \tau^{-1/3}$

A. Dutta, *et. al.*, arxiv:1012.0653

The question we address:

We assume $\delta \to 0$ and and work within the appropriate range of time;
λ is the driving parameter.

One finds

(i) $\ln \mathcal{L}(t) \sim (-t^2 f(\tau))$, if QCP is at $\lambda = 0$

(ii) $\ln \mathcal{L}(t) \sim \{-(t - \lambda_0 \tau)^2 f(\tau)\}$, if QCP is at λ_0

What is the scaling of this function $f(\tau)$?

- Is that identical to the scaling of the defect density?

Not necessarily! Even for this integrable system!

How to Calculate $\mathcal{L}(t)$?

Use the integrable two-level nature of the environmental Hamiltonian.

Far away from the QCP ($|h(t)| \gg 1$ $(t \to +\infty)$)

$$|\phi_k(h+\delta)\rangle = u_k|0\rangle + v_k e^{-i\Delta^+ t}|k,-k\rangle$$

$$|\phi_k(h-\delta)\rangle = u_k|0\rangle + e^{-i\Delta^- t} v_k|k,-k\rangle$$

$\Delta^+ = 4\sqrt{(h+\delta+1)^2 + \gamma^2 \sin k^2}$

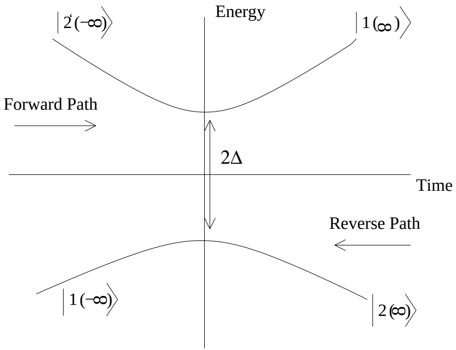
$\Delta^- = 4\sqrt{(h-\delta+1)^2 + \gamma^2 \sin k^2}$,

are the energy of two excitations in $|k,-k\rangle$ when the transverse field is equal to $h+\delta$ and $h-\delta$, respectively.

Excitations occur only in the vicinity of QCPs

F. Pollman *et al*, Phys. Rev. E **81** 020101 (**R**) (2010).

Landau-Zener formula



two approaching levels $\pm\sqrt{\epsilon^2 + \Delta^2}$ with $\epsilon = t/\tau$.

Probability of excitation $P = \exp(-\pi\Delta^2\tau)$

Gap protects from the excitation At the QCP , the gap for the critical mode vanishes; Gap is small for other modes close it.

Zener, Proc. R. Soc. London Ser A 137 (1932) 696; Landau and Lifshitz, Quantum Mechanics

How to Calculate $\mathcal{L}(t)$?...

How does one know u_k and v_k?

- Use the Landau-Zener transition formula:

$p_k = |u_k|^2 = \exp(-2\pi\tau\gamma^2 \sin^2 k)$

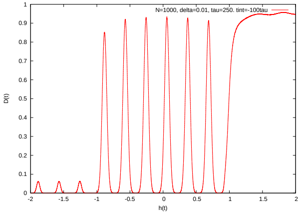
$$\begin{aligned} F_k(t) &= |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2 \\ &= \left||u_k|^2 + |v_k|^2 e^{-i(\Delta^+ - \Delta^-)t}\right|^2, \end{aligned} \tag{2}$$

In the vicinity of the quantum critical point at $h = 1$
$\Delta = (\Delta^+ - \Delta^-)/2$,

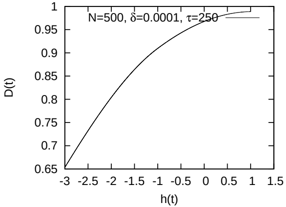
$$\begin{aligned} F_k(t) &= 1 - 4p_k(1-p_k)\sin^2(\Delta t) \\ &= 1 - 4\left[e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right]\sin^2(4\delta t) \end{aligned} \tag{3}$$

$\sin k$ has been expanded near the critical modes $k = \pi$, with $k' = \pi - k$ and we have taken the limit $\delta \to 0$.

Large δ and small δ



- Limit of large δ
- Oscillations



- Limit of small δ
- Gaussian Decay

How to calculate $\mathcal{L}(t)$?

Assume $\delta \to 0$

$$\begin{aligned}\mathcal{L}(t)(t) &= \exp \frac{N}{2\pi} \int_0^\infty dk \\ \ln &\left[1 - \left(e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right) 64\delta^2 t^2\right]\end{aligned}$$

Finally $\mathcal{L}(t)$ is given by

$$\mathcal{L}(t) \sim \exp\{-8(\sqrt{2}-1)N\delta^2 t^2/(\gamma\pi\sqrt{\tau})\}.$$

- $\ln \mathcal{L}(t) \sim \tau^{-1/2}$

The same scaling as the defect density

Non-linear Quenching

Non-linear Quenching: $h = 1 - \mathrm{sgn(t)}(\mathrm{t}/\tau)^{\alpha}$

The scaling form $p_k = |u_k|^2 = G(k^2 \tau^{2\alpha/(\alpha+1)})$

$$\mathcal{L}(t) = \exp(-CN\delta^2 t^2/\tau^{\alpha/(\alpha+1)})$$

- $\ln \mathcal{L}(t) \sim \tau^{-\alpha/(\alpha+1)}$

<u>Quenching through a MCP</u>

$$\ln \mathcal{L}(t) \sim (t - J_y\tau)^2/\tau^{1/6} \sim (J_x - J_y)\tau^{11/6}$$

- Quenching through Isolated critical points: $\ln \mathcal{L}(t)(\tau) \sim n$

Is this scenario true in general?

Quenching through a critical line

Change $\gamma = t/\tau$ with $h = 1$. Quenched through the MCP

Modified CSM with interaction:

$$H_{SE} = -(\delta/2)\sum_i (\sigma_i^x \sigma_{i+1}^x - \sigma_i^y \sigma_{i+1}^y)\sigma_S^z$$

The coupling δ provides two channels of the temporal evolution of the environmental ground state with anisotropy $\gamma + \delta$ and $\gamma - \delta$. The appropriate two-level Hamiltonain

$$H_k^{\pm}(t) = 2\begin{pmatrix} (\gamma \pm \delta)\sin k & h + \cos k \\ h + \cos k & -(\gamma \pm \delta)\sin k \end{pmatrix}.$$

- The defect density in the final state $n \sim \tau^{-1/3*}$

Does that mean $\ln \mathcal{L}(t) \sim \tau^{-1/3}$?

* U. Divakaran *et al*, Phys. Rev. B **78**, 144301 (2008).

A completely different Scaling

$$F_k = 1 - 4(e^{-\pi\tau k^3/2} - e^{-\pi\tau k^3})\sin^2(4\delta k t)$$

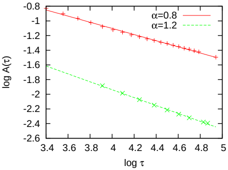
- An Gaussian decay:

$$\mathcal{L}(t) \sim \exp\{-2^{14/3} N\delta^2 t^2/(3\pi\tau)\}.$$

- Scaling of $\ln \mathcal{L}(t)(\sim \tau^{-1})$ is completely different!!

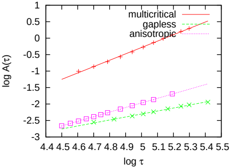
Numerical Justification

Non-linear Quenching



Slope $\simeq -\alpha/(\alpha+1)$

Different Qubit-environment interactions



Fairly good agreement with analytical predictions.

Integrability versus non-integrability

Ising model in a skewed field:

$$H = -\sum_{<ij>} \sigma_i^z \sigma_j^z - \sum_i \sigma_i^x - g(\sigma_i^x \cos\phi + \sigma_i^z \sin\phi)$$

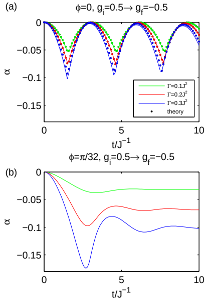
Integrable $\phi = 0, \phi$

- Start from the ground state of g_i ; Quench from $g_i \to g_f$
- The final state $|\psi(g_f, \tau)\rangle$

Look at the temporal evolution:

$$\mathcal{L}(t) = |\langle\psi(g_f, \tau)| \exp(-iH(g_f)t|\psi(g_f, \tau)\rangle|^2 = \exp(-\alpha(t)L)$$

Integrability versus non-integrability



Integrable Case

$$\alpha(t) = \frac{1}{2\pi}\int_0^{\infty} dk \log\left[1 + 4sin^2(\Delta_f t/2)P_k(1-P_k)\right]$$

F. Pollman *et al*, Phys. Rev. E **81** 020101 (**R**) (2010).

- Dynamical phase transitions

Heyl, Polkovnikov and Keherein, Phys. Rev. Lett (2013).

Concluding Comments:

- The LE shows interesting behavior close to a Quantum critical point: small δ; Universal Scaling?

- Non-equilbrium initial state Faster loss of coherence

- Scaling of the decoherence factor for a driven spin chain
- not necessarily identical to the scaling of the defect density.

- May be identical for quenching through isolated critical points.

- Clear deviation for quenching through critical lines.

- Dynamical Phase transitions

<u>Points to ponder</u>

- Integrable system reducible to two-level problems....

What happens beyond that?