Quantum Criticality, dynamics and Loschmidt echo

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• Generic Definition
• Central Spin model and decoherence of the qubit
• Loschmidt echo close to a quantum critical point
  Equilibrium situation
• Non-Equilibrium situation
  a: Non-equilibrium initial state
  b: Dynamics of the decoherence of the qubit:
    universal scaling of decoherence factor
Generic Definition

**Fight between J. Loschmidt and L. Boltmann**

**Second Law and time reversal invariance**

Loschmidt Echo

The generic definition:

$$\mathcal{L}(t) = |\langle \psi_0 | e^{iH_1 t} e^{-iH_2 t} | \psi_0 \rangle|^2 = |\langle \psi_0 | e^{-iH_2 t} | \psi_0 \rangle|^2$$

(If $| \psi_0 \rangle$ Eigenstate of $H_1$)

- Overlap between two states evolving from the same initial state with different Hamiltonians.
- Sensitivity of the quantum evolution to external perturbation due to coupling to the environment.

Acknowledgement: Scholarpedia
Generic properties of the Loschmidt echo

- Characterized by a **short-time** decay.
- **Partial revivals**
- **Asymptotic saturation**

How does the **proximity to a Q critical point** influence the LE?

**Static Counterpart Fidelity:** $|\langle \psi_0(\lambda) | \psi_0(\lambda + \delta) \rangle|^2$

decays exponentially with $\delta$ for a many-body system.

**Finite system:** Sharp dip at the QCP.

Fidelity susceptibility and fidelity in the thermodynamic limit:

Interesting **scaling** relations.
Phase transitions are driven by fluctuations

- Zero temperature transition due to non-commuting terms in the Hamiltonian
- Driven by quantum fluctuations

Simplest example: one dimensional Ising model in a transverse field

\[
H = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x
\]

- \( h > 1 \), \( \langle \sigma^z \rangle = 0 \); paramagnetic phase
- \( h < 1 \), \( \langle \sigma^z \rangle \neq 0 \); ferromagnetic phase
- Quantum phase transition at \( h = 1 \).
Notion of Universality:
Symmetry, dimensionality and the nature of the fixed point

- $d \rightarrow (d + 1)$
- Diverging length scale: $\xi \sim |\lambda|^{-\nu}; \lambda = h - 1$
- Diverging time scale: $\xi_\tau \sim |\lambda|^{-\nu z}$ Vanishing gap
- $\nu$ how one moves away from the critical point
- The dynamical exponent $z$ associated with the critical point.
The model in consideration

Let us consider the Transverse XY spin chain

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$

- Critical Exponents for Ising transition $\nu = z = 1$
- Exponents with the Multicritical point $z_{mc} = 2$ and $\nu_{mc} = 1/2$. 
Jordan-Wigner transformations: Spin-1/2's to Fermions

\[ H = \sum_{k > 0} \left( \begin{array}{cc} c_k^\dagger & c_{-k} \\ c_k & c_{-k}^\dagger \end{array} \right) H_k \left( \begin{array}{c} c_k \\ c_{-k}^\dagger \end{array} \right), \]

\[ H_k = 2 \begin{pmatrix} -(J_x + J_y) \cos(ka) - h & i(J_x - J_y) \sin(ka) \\ -i(J_x - J_y) \sin(ka) & (J_x + J_y) \cos(ka) + h \end{pmatrix}, \]

Decoupled two level systems
The central spin model and decoherence of a qubit

- A qubit coupled to a quantum critical many body system
- "Qubit" → a single Spin-1/2
- Environment → Quantum XY Spin chain
- A global coupling
- LE: Loss of phase information of the Qubit close to the QCP. Does a QCP influence the Loschmidt echo?
The Central Spin model

- A central spin globally coupled to an environment.
- We choose the environment to be Transverse XY spin chain

\[ H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z \]

- and a global coupling \(-\delta \sum_i \sigma_i^z \sigma_{S}^z\)

- Qubit State: \(|\phi_S(t = 0)\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle\)

- The environment is in the ground state \(|\phi_E(t = 0)\rangle = |\phi_g\rangle\)

- Composite initial wave function:

\[ |\psi(t = 0)\rangle = |\phi_S(t = 0)\rangle \otimes |\phi_g\rangle \]

At a later time $t$, the composite wave function is given by

$$|\psi(t)\rangle = c_1 |\uparrow\rangle \otimes |\phi_+\rangle + c_2 |\downarrow\rangle \otimes |\phi_-\rangle.$$  

$|\phi_\pm\rangle$ are the wavefunctions evolving with the environment Hamiltonian $H_E(h \pm \delta)$ given by the Schrödinger equation

$$i \frac{\partial}{\partial t} |\phi_\pm\rangle = \hat{H}[h \pm \delta]|\phi_\pm\rangle.$$  

The coupling $\delta$ essentially provides two channels of evolution of the environmental wave function with the transverse field $h + \delta$ and $h - \delta$. 
What happens to the central spin?

The reduced density matrix:

$$\rho_S(t) = \begin{pmatrix} |c_1|^2 & c_1 c_2^* d^*(t) \\ c_1^* c_2 d(t) & |c_2|^2 \end{pmatrix}.$$ 

- The decoherence factor (Loschmidt Echo)

$$\mathcal{L}(t) = d^*(t)d(t) = |\langle \phi_+(t) | \phi_-(t) \rangle|^2$$

Overlap between two states evolved from the same initial state with different Hamiltonian

- $\mathcal{L}(t) = 1$, pure state. $\mathcal{L}(t) = 0$ Complete Mixing
- Coupling to the environment may lead to Complete loss of coherence
We are interested in the small $\delta$ limit

**Equilibrium Situation:**
- No explicit time dependence in the Hamiltonian.

**Non-Equilibrium Situation:**
- $|\psi_0\rangle$ is an eigenstate of $H_0$; but there is a sudden quenching
- **Explicit** time dependence in the Hamiltonian
Equilibrium situation: the LE transverse Ising chain

\[ \mathcal{L}(t) = \left| \langle \phi_0 | \exp (iH_+ t) \exp (-iH_- t) | \phi_0 \rangle \right|^2 = \left| \langle \phi_+(t) | \phi_-(t) \rangle \right|^2 \]

\[ H_k^{\pm}(t) = 2 \begin{pmatrix} h \pm \delta + \cos k & \sin k \\ \sin k & -(h \pm \delta + \cos k) \end{pmatrix} \]

Two sets of Bogoliubov transformations

\[ \mathcal{L}(t) = \prod_{k>0} F_k = \left[ 1 - 2 \sin^2 (2\alpha_k) \sin^2 (\epsilon_k (h) t) \right] \]

\[ 2\alpha_k = (\theta_k (h_+) - \theta_k (h_-)) \text{ and } \tan \theta_k (h_+) = \frac{\sin k}{h_+ + \cos k} \]
Equilibrium case: $h$ independent of time

The decay of Loschmidt Echo close to QCP at a fixed $t$

Sum over modes close to the critical modes

$$\mathcal{L}(t) = \exp(-\alpha t^2); \quad \alpha \sim \frac{\delta^2}{(1 - h)^2 N^2}$$

The scaling: $t \to t/p$, $N \to N/p$ or $\delta \to p\delta$

- Sharp dip at the quantum critical point
- Complete loss of coherence of the qubit
The collapse and revival at the QCP $h + \delta = 1$

Quasi-periodicity Scales with the system size $L$
Non-equilibrium initial state

- The initial state: not an eigenstate of uncoupled Hamiltonian $H_F$
- $H_F$ is generated through a sudden quench $H_i(h_i) \rightarrow H_F(h_f)$.

$$H_F(\lambda) = H_0 + \lambda V_\lambda + g V_g$$

$$\mathcal{L}_q(\lambda, t, g) = |\langle G(\lambda, g = 0)|e^{iH_F(\lambda,g)t} e^{-iH_F(\lambda+\delta,g)t}|G(\lambda, g = 0)\rangle|^2$$

What happens to the temporal evolution of $\mathcal{L}$?

Early time decay $\mathcal{L}(t) \sim \exp(-\alpha t^2)$

$$\alpha = \frac{1}{2} \left[ \langle \left( \frac{\partial H}{\partial \lambda} \right)^2 \rangle - \langle \left( \frac{\partial H}{\partial \lambda} \right) \rangle^2 \right] \delta^2 = \frac{1}{2} [\langle V_\lambda^2 \rangle - \langle V_\lambda \rangle^2] \delta^2$$

- Independent of $g$

$$\alpha \sim \delta^2 \lambda^{2\nu z - 2}(\lambda \gg L^{-1/\nu}) \text{ and } \alpha \sim \delta^2 L^{2/\nu - 2z}(\lambda \ll L^{-1/\nu})$$
\( \alpha / g \) vs. \( g \) for \( N = 100, \lambda_i = 0.8 \),

\( \lambda_i = \lambda_i + g, t = 0.0001, \delta = 0.001 \)
When is the quenching relevant

\[ \mathcal{L}_q(\lambda, g, t) \approx \mathcal{L}(\lambda, 0, t) + g \frac{\partial \mathcal{L}_q(\lambda, g, t)}{\partial g} \bigg|_{g=0} \]

\[ \partial \mathcal{L}_q(\lambda, g, t)/\partial g \bigg|_{g=0} \sim g^{-1} \sim L^{1/\nu_g}, \] where \( \nu_g \) is the correlation length exponent.

\( g \ll L^{-1/\nu_g} \), the correction due to quenching becomes irrelevant.

For a fixed time \( t = 20 \) and \( \delta = 0.025 \)

![Graph showing \( \Delta \mathcal{L} \) vs. \( g \) for different values of \( h \)]
Variation with $\mathcal{L}$ with time: is there a revival

- **faster decay** in comparison to the equilibrium case.
- **Partial revival** when quenched to the QCP, $h_f + \delta = 1$. 
What happens when the environmental spin chain is driven?

Different Quenching paths:

Assume $h(t) = 1 - t/\tau$, driven spin chain

$$H_{k}^{\pm}(t) = 2 \begin{pmatrix} h(t) \pm \delta + \cos k & \gamma \sin k \\ \gamma \sin k & -(h(t) \pm \delta + \cos k) \end{pmatrix}.$$ 

The decoherence factor $\mathcal{L}(t)$

\[ |\phi^{\pm}(t)\rangle = \prod_k |\phi_k^{\pm}(t)\rangle = \prod_{k>0} \left[ u_k^{\pm}(t)|0\rangle + v_k^{\pm}(t)|k,-k\rangle \right]. \]

\[ i\partial/\partial t \begin{pmatrix} u_k^{\pm}(t) \\ v_k^{\pm}(t) \end{pmatrix}^T = H_k^{\pm}(t) \begin{pmatrix} u_k^{\pm}(t) \\ v_k^{\pm}(t) \end{pmatrix}^T \]

with $\prod_k F_k(t) = \prod_k |\langle \phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2$, 

\[ \mathcal{L}(t) = \exp \left[ \frac{N}{2\pi} \int_0^{\pi} dk \ln F_k \right] \] (1)

where $F_k$ can be written in terms of $u_k^{\pm}$ and $v_k^{\pm}$. 
Motivation: Kibble-Zurek Scaling

- Quenching through a QCP: **Defect generation in the final state**
- **Universal scaling** of the defect density: \( n \sim \frac{1}{\tau^\nu d/(\nu z+1)} \)

**Different Quenching paths:**

\[ \begin{align*}
\text{Critical point } h &= \frac{t}{\tau}; \quad n \sim \tau^{-1/2} \\
\text{Multicritical point} \\
\text{Quench } J_x = \frac{t}{\tau} \text{ with } h = 2J_y; \text{ cross the MCP when } J_x = J_y \\
\text{We find Defect density: } n \sim \tau^{-1/6} \\
\text{Quenching through the gapless critical line } \gamma = \frac{t}{\tau}; n \sim \tau^{-1/3} \\
\text{A. Dutta, et. al., arxiv:1012.0653} 
\end{align*} \]
The question we address:

We assume $\delta \to 0$ and work within the appropriate range of time; $\lambda$ is the driving parameter.

One finds

(i) $\ln L(t) \sim (-t^2 f(\tau))$, if QCP is at $\lambda = 0$
(ii) $\ln L(t) \sim \{- (t - \lambda_0 \tau)^2 f(\tau)\}$, if QCP is at $\lambda_0$

What is the scaling of this function $f(\tau)$?

• Is that identical to the scaling of the defect density?

Not necessarily! Even for this integrable system!
How to Calculate $\mathcal{L}(t)$?

Use the integrable two-level nature of the environmental Hamiltonian.

Far away from the QCP ($|h(t)| \gg 1$ ($t \to +\infty$))

$$|\phi_k(h + \delta)\rangle = u_k |0\rangle + v_k e^{-i\Delta^+ t} |k, -k\rangle$$

$$|\phi_k(h - \delta)\rangle = u_k |0\rangle + e^{-i\Delta^- t} v_k |k, -k\rangle$$

$$\Delta^+ = 4 \sqrt{(h + \delta + 1)^2 + \gamma^2 \sin k^2}$$

$$\Delta^- = 4 \sqrt{(h - \delta + 1)^2 + \gamma^2 \sin k^2},$$

are the energy of two excitations in $|k, -k\rangle$ when the transverse field is equal to $h + \delta$ and $h - \delta$, respectively. 

Excitations occur only in the vicinity of QCPs F. Pollman et al, Phys. Rev. E 81 020101 (R) (2010).
two approaching levels \( \pm \sqrt{\epsilon^2 + \Delta^2} \) with \( \epsilon = t/\tau \).

Probability of excitation \( P = \exp(-\pi \Delta^2 \tau) \)

Gap protects from the excitation At the QCP, the gap for the critical mode vanishes; Gap is small for other modes close it.

Zener, Proc. R. Soc. London Ser A 137 (1932) 696; Landau and Lifshitz, Quantum Mechanics
How to Calculate $\mathcal{L}(t)$?...

How does one know $u_k$ and $v_k$?

- Use the Landau-Zener transition formula:

\[
p_k = |u_k|^2 = \exp(-2\pi \tau \gamma^2 \sin^2 k)
\]

\[
F_k(t) = \left| \langle \phi_k(h(t) + \delta) | \phi_k(h(t) - \delta) \rangle \right|^2
\]

\[
= \left| |u_k|^2 + |v_k|^2 e^{-i(\Delta^+ - \Delta^-)t} \right|^2, \quad (2)
\]

In the vicinity of the quantum critical point at $h = 1$

\[
\Delta = (\Delta^+ - \Delta^-)/2,
\]

\[
F_k(t) = 1 - 4p_k(1 - p_k) \sin^2(\Delta t)
\]

\[
= 1 - 4 \left[ e^{-2\pi \tau \gamma^2 k'^2} - e^{-4\pi \tau \gamma^2 k'^2} \right] \sin^2(4\delta t), \quad (3)
\]

$\sin k$ has been expanded near the critical modes $k = \pi$, with $k' = \pi - k$ and we have taken the limit $\delta \to 0$. 
Large $\delta$ and small $\delta$

- Limit of large $\delta$
- Oscillations
- Limit of small $\delta$
- Gaussian Decay
How to calculate $\mathcal{L}(t)$?

Assume $\delta \rightarrow 0$

$$
\mathcal{L}(t)(t) = \exp \left( \frac{N}{2\pi} \int_{0}^{\infty} dk \ln \left[ 1 - \left( e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2} \right) 64\delta^2 t^2 \right] \right)
$$

Finally $\mathcal{L}(t)$ is given by

$$
\mathcal{L}(t) \sim \exp \left\{ -8(\sqrt{2} - 1)N\delta^2 t^2 / (\gamma\pi\sqrt{\tau}) \right\}.
$$

- $\ln \mathcal{L}(t) \sim \tau^{-1/2}$

The same scaling as the defect density.
Non-linear Quenching: $h = 1 - \text{sgn}(t)(t/\tau)^\alpha$

The scaling form $p_k = |u_k|^2 = G(k^2\tau^{2\alpha/\alpha + 1})$

$$\mathcal{L}(t) = \exp(-CN\delta^2 t^2/\tau^{\alpha/\alpha + 1})$$

- $\ln \mathcal{L}(t) \sim \tau^{-\alpha/(\alpha + 1)}$

Quenching through a MCP

$$\ln \mathcal{L}(t) \sim (t - J_y\tau)^2/\tau^{1/6} \sim (J_x - J_y)\tau^{11/6}$$

- Quenching through Isolated critical points: $\ln \mathcal{L}(t)(\tau) \sim n$

Is this scenario true in general?
Quenching through a critical line

Change $\gamma = t/\tau$ with $h = 1$. Quenched through the MCP

Modified CSM with interaction:

$$H_{SE} = -(\delta/2) \sum_{i}(\sigma_i^x \sigma_{i+1}^x - \sigma_i^y \sigma_{i+1}^y)\sigma_S^z$$

The coupling $\delta$ provides two channels of the temporal evolution of the environmental ground state with anisotropy $\gamma + \delta$ and $\gamma - \delta$. The appropriate two-level Hamiltonian

$$H_k^\pm(t) = 2 \begin{pmatrix} (\gamma \pm \delta) \sin k & h + \cos k \\ h + \cos k & -(\gamma \pm \delta) \sin k \end{pmatrix}.$$  

• The defect density in the final state $n \sim \tau^{-1/3}$

Does that mean $\ln \mathcal{L}(t) \sim \tau^{-1/3}$?

A completely different Scaling

\[ F_k = 1 - 4(e^{-\pi\tau k^3/2} - e^{-\pi\tau k^3})\sin^2(4\delta kt) \]

- An Gaussian decay:

\[ \mathcal{L}(t) \sim \exp\{-2^{14/3}N\delta^2 t^2/(3\pi\tau)\}. \]

- Scaling of \( \ln \mathcal{L}(t)(\sim \tau^{-1}) \) is completely different!!
Numerical Justification

Non-linear Quenching

Slope \simeq -\alpha/(\alpha + 1)

Different Qubit-environment interactions

Fairly good agreement with analytical predictions.
Integrability versus non-integrability

Ising model in a skewed field:

\[
H = - \sum_{<ij>} \sigma_i^z \sigma_j^z - \sum_i \sigma_i^x - g(\sigma_i^x \cos \phi + \sigma_i^z \sin \phi)
\]

Integrable \( \phi = 0, \phi \)

- Start from the ground state of \( g_i \); Quench from \( g_i \to g_f \)
- The final state \( |\psi(g_f, \tau)\rangle \)

Look at the temporal evolution:

\[
\mathcal{L}(t) = |\langle \psi(g_f, \tau) | \exp(-iH(g_f)t |\psi(g_f, \tau)\rangle|^2 = \exp(-\alpha(t)L)
\]
Integrability versus non-integrability

\[ \alpha(t) = \frac{1}{2\pi} \int_0^\infty dk \log \left[ 1 + 4\sin^2(\Delta f t/2)P_k(1 - P_k) \right] \]


- Dynamical phase transitions

Concluding Comments:

- The LE shows interesting behavior close to a Quantum critical point: small $\delta$; Universal Scaling?
- Non-equilibrium initial state Faster loss of coherence
- Scaling of the decoherence factor for a driven spin chain
- Not necessarily identical to the scaling of the defect density.
- May be identical for quenching through isolated critical points.
- Clear deviation for quenching through critical lines.
- Dynamical Phase transitions

Points to ponder

- Integrable system reducible to two-level problems....

What happens beyond that?