A universal framework for correlation detection

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Problem

- Problem:

  Given a subset of pure states (= pure state density matrices) of a quantum system characterize the convex hull of this set (= a specific subset of mixed states of the system)

- Examples:

  1. mixed separable states as the convex hull of pure separable states
  2. "bosonic" mixed separable states as the convex hull of pure symmetric separable states
  3. "fermionic" mixed separable states as the convex hull of simple Slater determinants (pure uncorrelated fermionic states)
  4. mixed coherent states (mixed states with positive P-representation)
  5. fermionic Gaussian states

- In all cases:
  - original set of pure states = uncorrelated pure states
  - its convex hull = uncorrelated mixed states

Notation

- $\mathcal{H}$ - a Hilbert space (finite- or infinite-dimensional) of a quantum system

- pure states = trace-one rank one density matrices = one-dimensional projections

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}, \quad |\psi\rangle \in \mathcal{H}$$

- $\mathcal{M}$ - a subset of the above - uncorrelated pure states

- $\mathrm{conv}(\mathcal{M})$ - uncorrelated mixed states

$$\mathrm{conv}(\mathcal{M}) \ni \rho = \sum_k p_k \rho_{\psi_k}, \quad \psi_k \in \mathcal{M}, \quad p_k > 0, \quad \sum_k p_k = 1$$

Characterization (strong)

- Ideally,
  - a function vanishing only on uncorrelated mixed states

$$f(\rho) = \begin{cases} 0 & \rho \text{ −uncorrelated} \\ \geq 0 & \text{otherwise} \end{cases}$$

  - given by the expectation values of an observable (a Hermitian operator on $\mathcal{H}$)

$$f(\rho) = \mathrm{Tr}(V\rho)$$

- Such f, in general, does not exist even for pure uncorrelated states $\mathcal{M}$.
  - set of vectors $|\psi\rangle \in \mathcal{H}$ such that $\rho_\psi \in \mathcal{M}$ - "small" subset of $\mathcal{H}$ in any reasonable sense (metric, topological), but...
  - usually spans $\mathcal{H}$
  - consequently, $f(\rho) = \mathrm{Tr}(V\rho) \equiv 0$

Characterization (mild)

- We have to content ourselves with
  - ρ - mixed uncorrelated $\Leftrightarrow f(\rho) \geq 0$ (as we demanded), but
  - f can be bounded by the mean value of an observable on multiple copies of ρ

$$f(\rho) \geq \mathrm{Tr}(V\rho^{\otimes n})$$

- For arbitrary $\mathcal{M}$ it is (probably) still difficult, but...
  - In all enumerated cases $\mathcal{M}$ is an **orbit** of some natural group of symmetry ('local symmetry') acting in $\mathcal{H}$, i.e. all uncorrelated pure states can be obtained from a single one by a symmetry group action
  - The orbit is a very special one, i.e. the state from which all other uncorrelated pure states (i.e. the set $\mathcal{M}$) can be obtained has particular properties determined by the concrete representation of the symmetry group in the underlying Hilbert space $\mathcal{H}$

Notation

- K - a (Lie) group of symmetries
- $\mathfrak{k}$ - its Lie algebra (the algebra of generators of the group K, characterized by commutation relations between generators)
- the group K acts on the Hilbert states $\mathcal{H}$ via its (irreducible) unitary representation, $K : \mathcal{H} \to \mathcal{H}$; if we choose a basis in $\mathcal{H}$ elements of K will be represented by unitary matrices
- the Lie algebra $\mathfrak{k}$ is also (irreducibly) represented on $\mathcal{H}$ via antihermitian matrices
- $\mathfrak{g}$ - the complexification of $\mathfrak{k}$,
  $$\mathfrak{g} \ni a = x + iy, \quad x, y \in \mathfrak{k}$$
  (represented in $\mathcal{H}$ by general complex matrices)
- the Lie algebra $\mathfrak{g}$ can be decomposed into three pieces
  $$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$
  in a chosen basis in $\mathcal{H}$ we can identify $\mathfrak{n}^-$ with strictly lower-triangular matrices, $\mathfrak{h}$ with diagonal matrices, and $\mathfrak{n}^+$ with strictly upper-triangular matrices
- every irreducible representation $\mathcal{H}^\mu$ of $\mathfrak{g}$ (and, consequently of $\mathfrak{k}$ and K) is uniquely characterized by a particular vector $|\mu\rangle$ in $\mathcal{H}$ (the highest-weight vector) which is a common eigenvector of all matrices in $\mathfrak{h}$ and annihilated by all matrices from $\mathfrak{n}^+$
  $$h|\mu\rangle = \lambda_h|\mu\rangle, \quad \eta|\mu\rangle = 0, \quad h \in \mathfrak{h}, \quad \eta \in \mathfrak{n}^+$$
  Example: angular momentum $K = SU(2)$, total spin j, $|\mu\rangle = |j, j\rangle$, $J_z|\mu\rangle = j|\mu\rangle$, $J_+|\mu\rangle = 0$
- the highest weight orbit - $\mathcal{O}_\mu = K.\mu$

Our examples

- L-partite separable states of (identical albeit distinguishable) N-dimensional systems ('quNits') with the single-particle Hilbert space $\mathcal{H}_N \simeq \mathbb{C}^N$

$$\mathcal{H} = \underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{L-\text{times}}, \quad K = \underbrace{SU(N) \times \cdots \times SU(N)}_{L-\text{times}}, \quad |\mu\rangle = |0, \ldots, 0\rangle$$

  highest weight orbit, $\mathcal{O}_\mu = K.\mu$, - pure separable states

- L bosons in N-dimensional space

$$\mathcal{H} = \underbrace{\mathbb{C}^N \vee \cdots \vee \mathbb{C}^N}_{L-\text{times}} =: \mathrm{Sym}^L\left(\mathbb{C}^N\right), \quad K = SU(N), \quad |\mu\rangle = |0, \ldots, 0\rangle$$

  highest weight orbit, $\mathcal{O}_\mu = K.\mu$, - pure uncorrelated bosonic states (Eckert et al., 2002)

- L fermions in N-dimensional space

$$\mathcal{H} = \underbrace{\mathbb{C}^N \wedge \cdots \wedge \mathbb{C}^N}_{L-\text{times}} =: \bigwedge^L\left(\mathbb{C}^N\right), \quad K = SU(N), \quad |\mu\rangle = |e_1\rangle \wedge \cdots \wedge |e_L\rangle$$

  highest weight orbit, $\mathcal{O}_\mu = K.\mu$, - pure uncorrelated fermionic states (Schliemann et al., 2001)

Our examples

- Coherent states: K - a Lie group, $\mathcal{H}$ - (irreducible) representation space. (Perelomov, 1972)

  Example: $K = SU(2)$, $\mathcal{H} = \mathbb{C}^{2j+1}$,
  $|\mu\rangle = |j, j\rangle$ - spin j coherent states ('atomic coherent states' in quantum optics.
  (Radcliffe, 1971)

- Gaussian fermionic states

$$\mathcal{H} = \mathcal{H}_{\text{Fock}} = \bigoplus_{L=0}^{L=d} \bigwedge^{L} \left(\mathbb{C}^d\right),$$

  canonical set of fermionic (anti-commuting) creation and annihilation operators

$$\left\{a_i,\, a_j^\dagger\right\} = \delta_{ij}\mathbb{I},\ \{a_i,\, a_j\} = 0\,.$$

  a class of non-interacting (quadratic) Hamiltonians

$$H = \theta_{ij} a_i^\dagger a_j^\dagger + h_{ij} a_i^\dagger a_j + \bar{\theta}_{ij} a_i a_j\,,$$

  pure fermionic Gaussian states = orbit of the group generated by such Hamiltonians

$$\mathcal{M} = \left\{e^{iH}|0\rangle\right\}$$

Bilinear characterization of pure uncorrelated states

- The highest weight orbit $\mathcal{O}_\mu$ (= pure uncorrelated states $\mathcal{M}$) can be identified using the following procedure (Liechtenstein theorem)
  - take the symmetrized two fold tensor product of the relevant representation and decompose it into irreducible parts
  $$\mathcal{H}^\mu \vee \mathcal{H}^\mu = \mathcal{H}^{2\mu} \oplus \cdots$$
  - $|\psi\rangle \in \mathcal{O}_\mu$ iff $\langle \psi \otimes \psi | I \otimes I - \mathbb{P}^{2\mu} | \psi \otimes \psi \rangle = 0$

    where $\mathbb{P}^{2\mu}$ - projection on $\mathcal{H}^{2\mu}$ in $\mathcal{H}^\mu \vee \mathcal{H}^\mu$
- $f(\rho_\psi) = \langle \psi \otimes \psi | A | \psi \otimes \psi \rangle$ with $A = I \otimes I - \mathbb{P}^{2\mu} | \psi \otimes \psi \rangle$ fulfills our demands for pure states - vanishes only on uncorrelated pure states, is positive otherwise (A is positive-definite)
- formally one can thus characterize the convex hull via the convex roof construction introducing
$$f(\rho) = \inf_{\sum_i p_i |\phi_i\rangle\langle\phi_i| = \rho} \left(\sum_i p_i f\left(\rho_{\phi_i}\right) \right)$$
- f vanishes only on uncorrelated mixed states, but the required optimization makes the procedure hardly effective

Estimates

- To estimate f we use the following theorem:

  Let $\mathcal{M}$ a subset of the set of pure states on $\mathcal{H}$. Assume three exist a Hermitian operator (an observable) on $\mathcal{H} \otimes \mathcal{H}$ such that for an arbitrary $|v\rangle \in \mathcal{M}$ and an arbitrary $|w\rangle \in \mathcal{H}$

  $$\langle v \otimes w | V | v \otimes w \rangle \leq 0$$

  then for an arbitrary $B \geq 0$ acting on $\mathcal{H}$ and for an arbitrary ρ from the convex hull of $\mathcal{M}$ (i.e. for an arbitrary uncorrelated mixed state)

  $$\mathrm{Tr}((\rho \otimes B)V) \leq 0$$

- at first sight a bit contrived, but $A - \mathbb{P}^{\mathrm{assym}}$, with $\mathbb{P}^{\mathrm{assym}}$ = the projection on the antisymmetric part $\bigwedge^2 \mathcal{H}$ of the two-fold tensor product, fulfills the conditions imposed on V
- Nonlinear correlation witness - if we find B such that $\mathrm{Tr}((\rho \otimes B)V) > 0$ - ρ - correlated
- We can choose $B = \rho$, one can show that $f(\rho) \geq \mathrm{Tr}((\rho \otimes \rho)V)$ (and this is what we were looking for!!!)
- Algorithm:
  1. find $\mathcal{H}^{2\mu}$ as the largest irreducible part of the symmetrized tensor product $\mathcal{H} \otimes \mathcal{H}$
  2. calculate A (the projection on the complement of the above)
  3. calculate V (by subtracting from A a projection on the antisymmetric part $\bigwedge^2 \mathcal{H}$

Applications

- For distinguishable particles - the Mintert-Buchleitner bound for the generalized N-partite concurrence
(F. Mintert and A. Buchleitner, Phys. Rev. Lett. **98**, 140505 (2007))

- Generalization to indistinguishable particles
(M. Oszmaniec, M. K., Phys. Rev. A **88**, 052328 (2013))

- Generalizations to infinite-dimensional Hilbert spaces for distinguishable and indistinguishable particles - a bit tricky (no Lie group structure at hand)
(*ibid.*)

- Estimation of the fraction of (un)correlated states among all density matrices with the same spectrum (*via* concentration of measure for transitive group actions)
(M. Oszmaniec, M. K. - in preparation)

- Identification of particular fermionic Gaussian states (see F. de Melo, P. Ćwikliński, B. M. Terhal, *The Power of Noisy Fermionic Quantum Computation*, New J. Phys. **15** 013015 (2013))
(*ibid.*)

Remarks, generalizations, outlook

- infinite dimensional case
- no (compact semisimple) group
- nevertheless some machinery works