Quantum Simulation
&
Quantum Primes

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Outline

- Quantum Simulations
- Quantum Counting of Prime Numbers

Validation of theory

- Exact Calculations

  Integrable models (from Hamiltonian), CFT (from symmetry),

  ADS/CFT (from conjecture)

- Approximate methods

  Perturbation theory, toy models, non-perturbative techniques,...

- Numerics

  Monte Carlo, Tensor Networks (MPS, PEPS, MERA,...)

other powerful instruments?

Analogue Simulation

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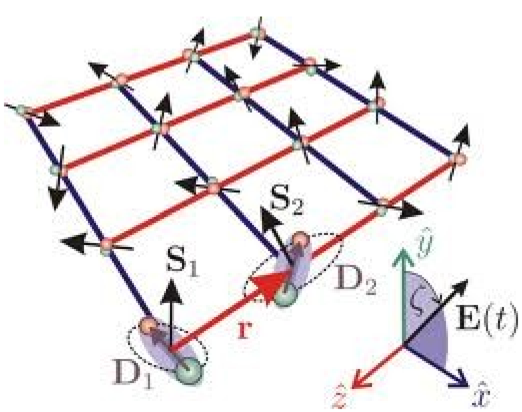

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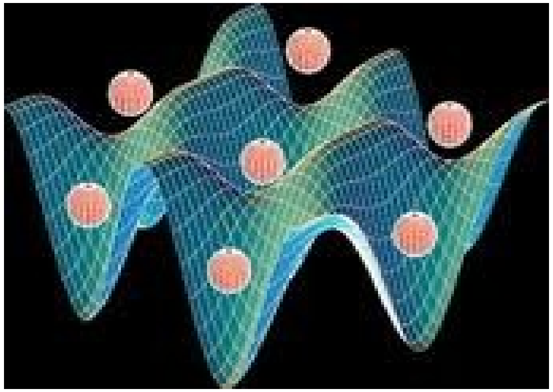


$$H \approx H'(\lambda_i)$$

controlled parameters

Analogies have to be analyzed very critically

Why?
Quantum Simulation is an intelligent window to QC

Quantum Computer

General purpose quantum computation
Shor's factorization algorithm
Oracle problems, NAND trees, …

Few problems, few algorithms!

Quantum Simulator

Efficient analysis of specific quantum problems
Explore new Physics

Experimentally achievable

Classical Computer

Tensor Networks: PEPS, MERA
Monte Carlo

Not sufficient

Quantum Simulation

Why → What?

- Q Simulation of models beyond classical simulation
- Q Simulation of criticality, frustration, topological order,...
- Q Simulation of non-abelian gauge theories
- Q Simulation of unphysical models, Klein paradox, Zitterbewegung,...
- **Q Simulation of gravity, geometry, topology**

Where?

- Ion traps
- Cold gases
- Molecules, solids, graphene, …

Quantum Simulation of gravitational backgrounds

Carriers in graphene are described by the Dirac equation

Graphene acts a quantum simulator for Dirac fermions

$$\left(\gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2\right)\psi = 0$$

Can we simulate the Dirac equation on optical lattices?

Can we simulate curved spaces?

O. Boada, A. Celi, M. Lewenstein, JIL (2011)

Dirac Hamiltonian in 2+1 dimensions

$$i\partial_t\psi=H\psi=-i\gamma_0\left(\gamma_1\partial_1+\gamma_2\partial_2\right)\psi$$

$$\gamma_0\gamma_1=-\gamma_2=\sigma_x \qquad \gamma_0\gamma_2=\gamma_1=\sigma_y \qquad \gamma_1\gamma_2=\gamma_0=i\sigma_z$$

$$H\psi=-i\left(\sigma_x\partial_x+\sigma_y\partial_y\right)\psi=0$$

$$\mathrm{H}=\int dx\,dy\,\psi^\dagger H\psi$$

Discretized Dirac Hamiltonian

$$\mathrm{H}=\frac{1}{2a}\sum_{m,n}\left(\psi^\dagger_{m+1,n}\left(i\sigma_x\right)\psi_{m,n}+\psi^\dagger_{m,n+1}\left(i\sigma_y\right)\psi_{m,n}\right)+h.c.$$

SU(2) Fermi Hubbard model

Dirac equation in curved space-time

$$D_\mu = \partial_\mu + \frac{1}{2} w_\mu^{ab} \gamma_{ab}$$

$$\gamma^\mu D_\mu \psi = 0$$

$$\gamma^\mu = e_a^\mu \gamma^a$$

$$\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$$

If there exists a timelike Killing vector
(time translation invariance in certain coordinates)

➡ there exists *H* conserved and well defined

Sufficient condition $\partial_t g_{\mu\nu} = 0$

$$H = -i\gamma_t \left(\gamma^i \partial_i + \frac{1}{4} \gamma^i w_i^{ab} \gamma_{ab} + \frac{1}{4} \gamma^t w_t^{ab} \gamma_{ab} \right)$$

$$\mathrm{H} = \int \sqrt{-g}\, dx\, dy\, \psi^\dagger \gamma_0 \gamma^t H \psi$$

Rindler space-time

$$ds^2 = -(Cx)^2 dt^2 + dx^2 + dy^2$$

$$e^0 = |Cx|\,dt \qquad e^1 = dx \qquad e^2 = dy$$

Steady Rindler observer is an accelerated Minkowski observer

acceleration $\frac{1}{Cx}$ → Unruh effect → **temperature** $\frac{1}{Cx}$

Rindler is the near horizon limit of Schwarzschild black hole

For any metric of the form

$$ds^2 = -e^{\Phi(x,y)} dt^2 + dx^2 + dy^2$$

The lattice version turns out to be

$$\mathrm{H} = \frac{1}{2a} \sum_{m,n} J_{mn} \left(\psi^{\dagger}_{m+1,n} \sigma_x \psi_{m,n} + \psi^{\dagger}_{m,n+1} \sigma_y \psi_{m,n} \right) + h.c.$$

$$J_{mn} = e^{\Phi(am, an)}$$

geometry = energy cost for jumping to a nearest neighbor

Site dependent couplings!

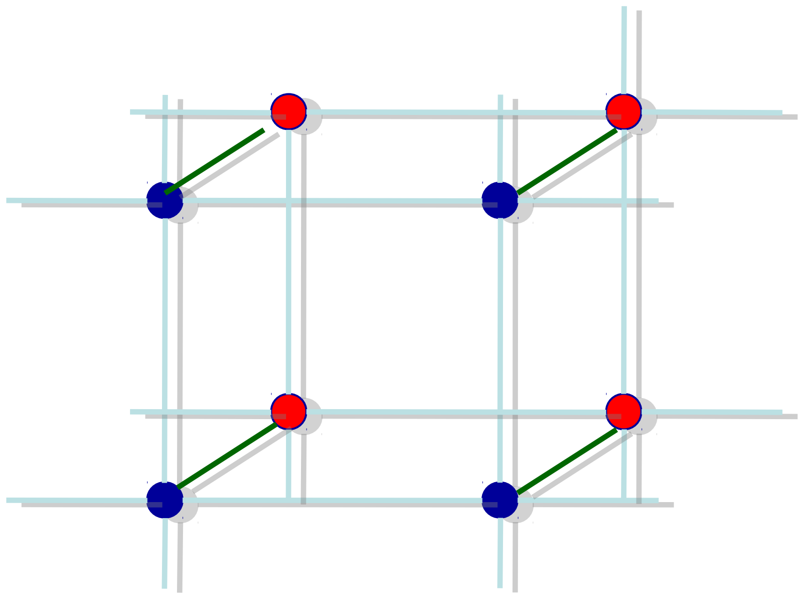
Discretized Dirac equation in a Rindler space

$$H = \frac{1}{2a} \sum_{m,n} c\, m \left(\psi^{\dagger}_{m+1,n} \sigma_x \psi_{m,n} + \psi^{\dagger}_{m,n+1} \sigma_y \psi_{m,n} \right) + h.c.$$

Experimental options

- **superlattice techniques**
- **laser waist**

Quantum Simulation of Extra Dimensions



dimensions = connectivity

D dimensions can be simulated in D-1 dimensions
by tuning appropriately the nearest neighbor couplings

O. Boada, A. Celi, Lewenstein, JIL (2012)

Dimension

$$H = -J \sum_{\vec{q}} \sum_{j=1}^{D+1} a^{\dagger}_{\vec{q}+\vec{u}_j} a_{\vec{q}} + h.c.$$

$$\vec{q} = (\vec{r}, \sigma)$$

D + 1

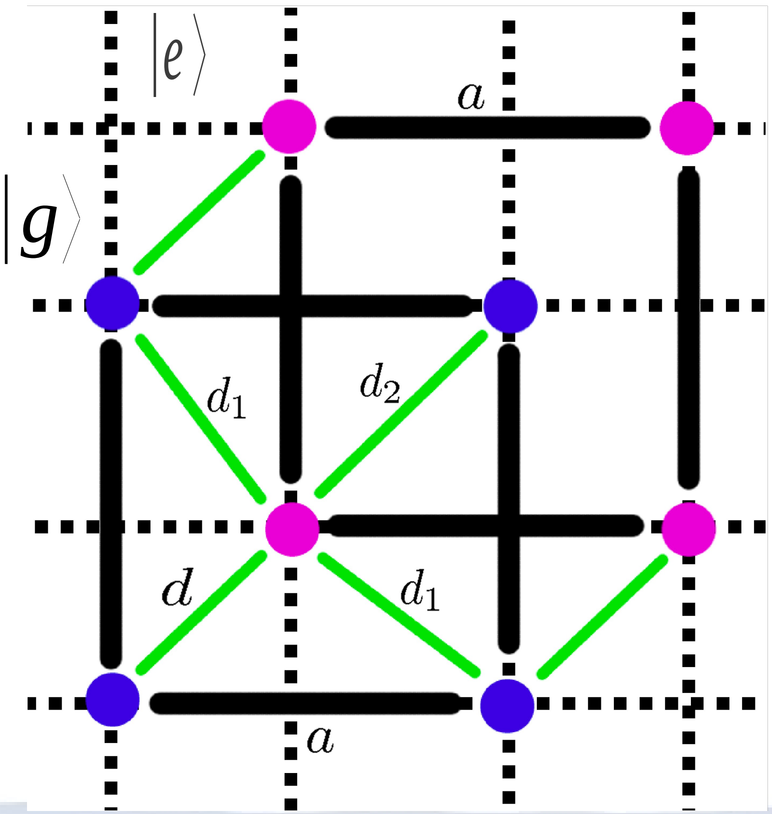
Species

$$H = -J \sum_{r,\sigma} \sum_{j=1}^{D} \left(a^{(\sigma)\dagger}_{\vec{r}+\vec{u}_j} a^{(\sigma)}_{\vec{r}} + a^{(\sigma+1)\dagger}_{\vec{r}} a^{(\sigma)}_{\vec{r}} \right) + h.c.$$

Bivolum

State dependent lattice / On site dressed lattice

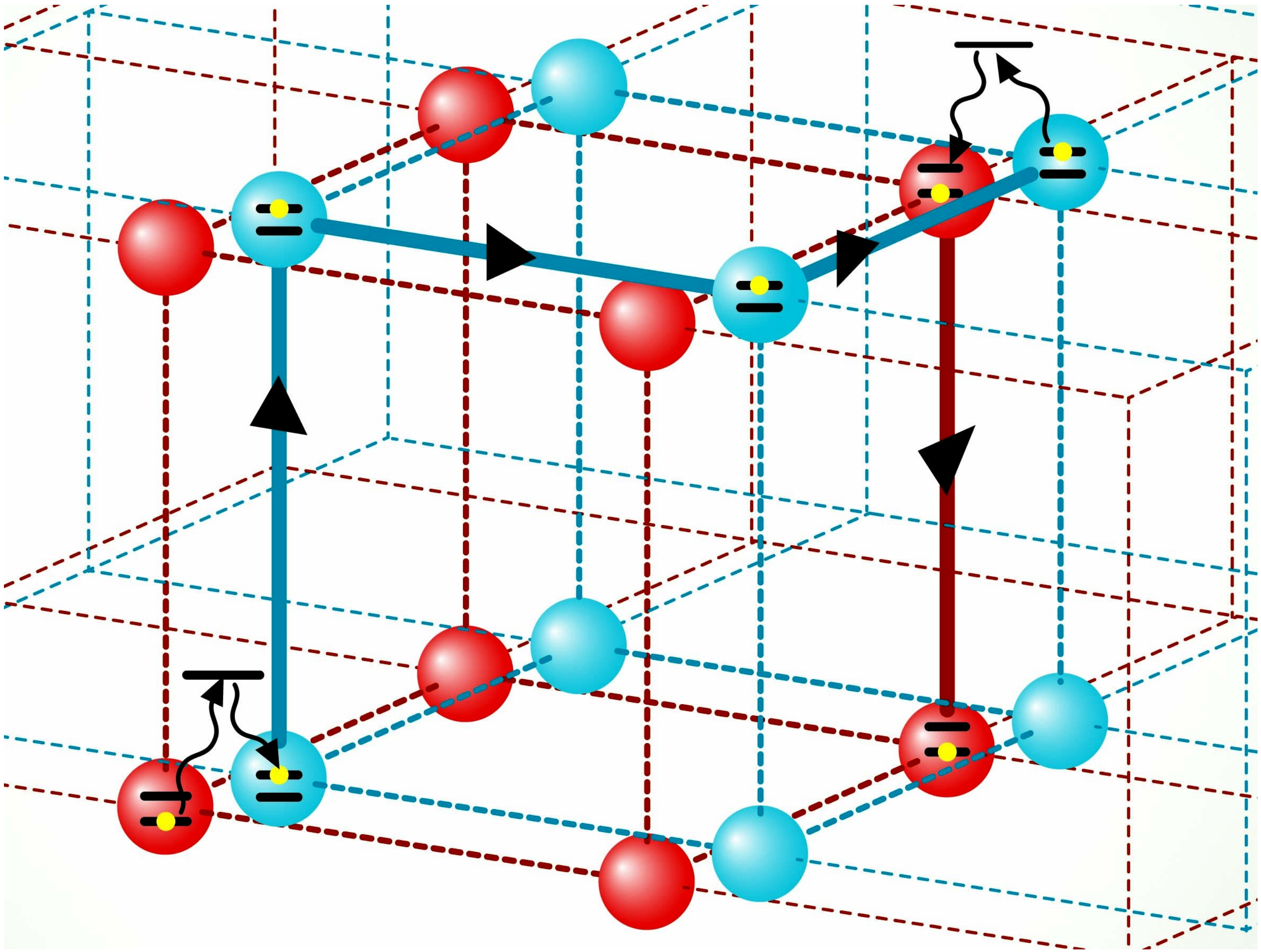
Two 3D sub-lattices are connected via Raman transitions



$$J_{bilayer} = \frac{\Omega}{2} \int d^{2x}\, w^*(\vec{x})\, w(\vec{x} - \vec{r})$$

Exponential decay of Wannier functions suppresses undesired transitions

Boada, Celi, Lewenstein, JIL PRL (2012)

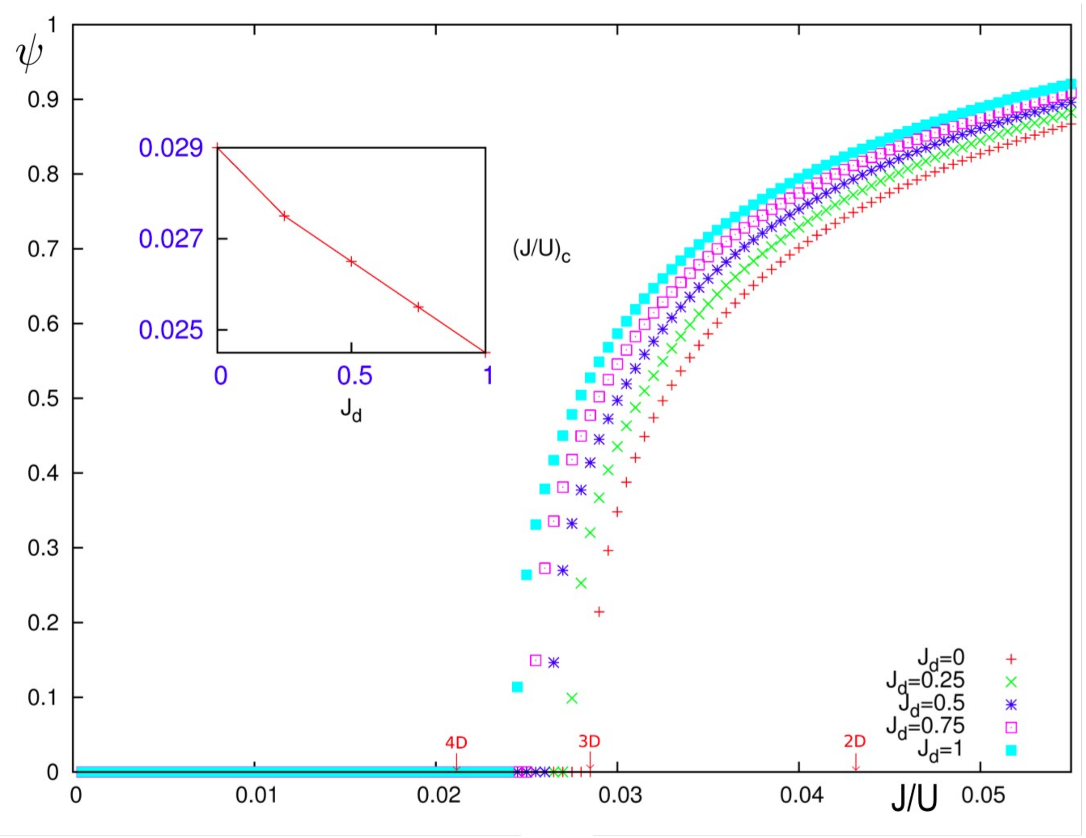


Single particle observable:

Kaluza-Klein modes
Single particle correlators take contributions from jumps back and forth to other dimensions in the form of exponential (KK) massive corrections

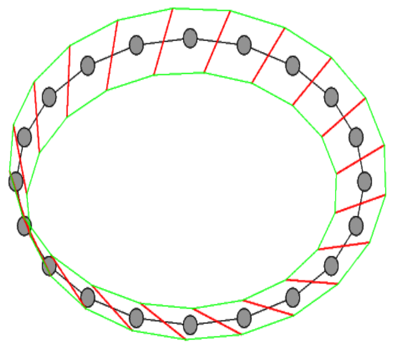
Many-body observables:

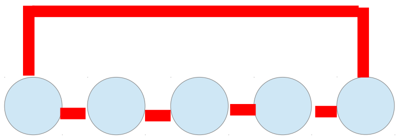
Shift of phase transition interpolates between dimensions

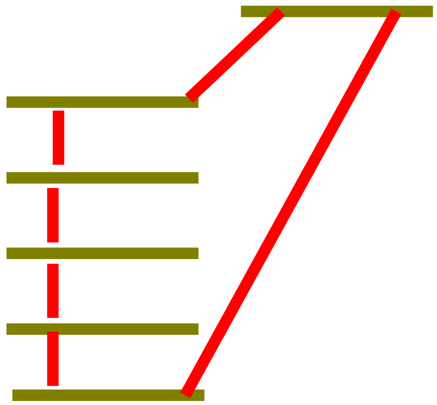


Quantum Simulation of boundary conditions
Quantum Simulation of Topology

Non-local interactions
can be artificially generated

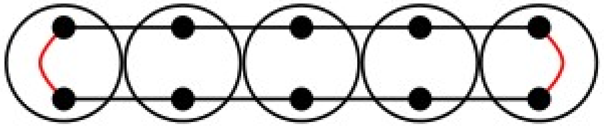






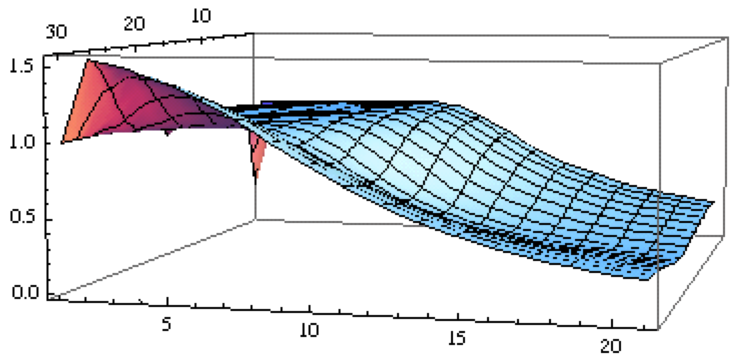
O. Boada, A. Celi, M. Lewenstein, J. Rodríguez-Laguna, JIL

1-D optical lattice with 2 species can be turned into a circle



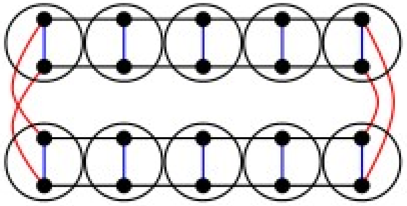
Frustration from boundary condition on a chain

$$H=\sum_1^n \sigma_i^x \sigma_n^x + \cos(\theta)\sigma_n^x \sigma_1^x + \lambda \sum_1^n \sigma_i^z$$

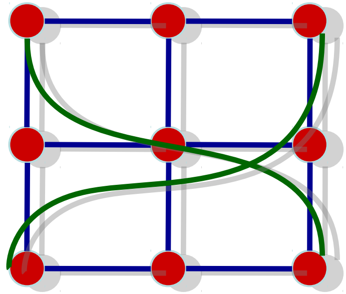


Entaglement entropy jumps

Ladders + Species = Moëbius band



Torus vs Klein bottle: Hubbard model



Quantum Counting
of
Prime Numbers

G. Sierra, JIL (Quant Comp. Comm. 2013)

What will be a Quantum Computer used for?

Build a Quantum Computer

↓

Break Classical Cryptography

↓

Use Quantum Cryptography

↓

Idle Quantum Computer?

Could we use a Quantum Computer for pure Maths?

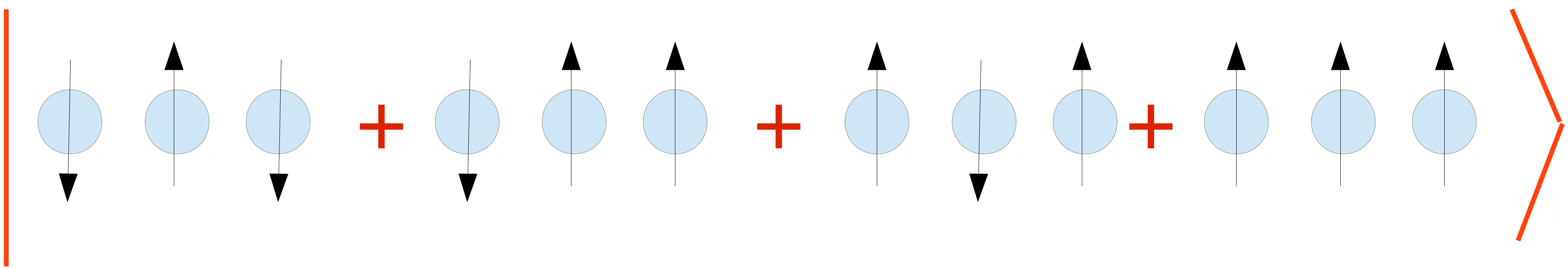
The Prime State

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p < 2^n \in Primes} |p\rangle$$

$\pi(2^n)$ is the prime counting function

Quantum Mechanics allows for the superposition of primes implemented as states of a computational basis

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p<2^n \in Primes} |p\rangle$$



$$|P(3)\rangle = \frac{1}{\sqrt{4}} \left(|2\rangle + |3\rangle + |5\rangle + |7\rangle \right)$$

Gauss, Legendre

Sieve of Eratosthenes

$$\pi(x) \approx \frac{x}{\ln x}$$

Prime Number Theorem

Gauss, Riemann

Hadamard, de la Vallée Poussin

Density of primes 1/log x

$$\pi(x) \approx Li(x)$$

$$Li(x) = \int_2^x \frac{dt}{\log t} \approx \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \ldots$$

$$\pi(10^{24}) = 18\,435\,599\,767\,349\,200\,867\,866$$

Platt (2012)

$$\pi(10^{24}) - \frac{10^{24}}{\ln(10^{24})} = 3.4\;10^{20}$$

$$Li(10^{24}) - \pi(10^{24}) = 1.7\;10^{9}$$

If the Riemann Conjecture is correct, fluctuations are bounded

$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$ If $\zeta(s)=0$ with $0\leq Real(s)\leq 1$ then $Real(s)=\frac{1}{2}$

$$|Li(x)-\pi(x)|<\frac{1}{8\pi}\sqrt{x}\ln x$$

The prime number function will oscillate around the Log Integral infinitely many times
Littlewood, Skewes

A first change of sign is expected for some $x<e^{727.9513468}$ ...

Could the Prime state be constructed?

Does it encode properties of prime numbers?

What are its entanglement properties?

Could it provide the means to explore Arithmetics?

Entanglement: single qubit reduced density matrices

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{i_{n-1},\dots,i_1,i_0=0,1} p_{i_{n-1}\dots i_1 i_0} |i_{n-1},\dots,i_1,i_0\rangle$$

$$p_{i_{n-1}\dots i_1 i_0} = \begin{cases} 1 & p = i_{n-1}2^{n-1}+\dots+i_0 = prime \\ 0 & otherwise \end{cases}$$

$$\rho^{(1)}_{ab} = \frac{1}{\pi(2^n)} \sum_{i_{n-1},\dots,i_2,i_0=0,1} p_{i_{n-1},\dots,i_2,a,i_0}\, p_{i_{n-1},\dots,i_2,b,i_0}$$

******10
******11
******01
~~******00~~

$$\rho^{(1)}_{00} = \frac{\pi_{4,1}(2^n)}{\pi(2^n)} \qquad \rho^{(1)}_{11} = \frac{1+\pi_{4,3}(2^n)}{\pi(2^n)} \qquad \rho^{(1)}_{01} = \frac{\pi_2^{(1)}(2^n)}{\pi(2^n)}$$

Each element counts sub-series of primes and twin primes

$\pi_{a,b}(x)$ counts the number of primes equal to *a* mod *b*

$\pi_2^{(1)}(x)$ counts twin primes equal to 1 mod 4

Using Euler totient function $S\left(\rho^{(i)}\right) \sim \log_2 2 = 1$

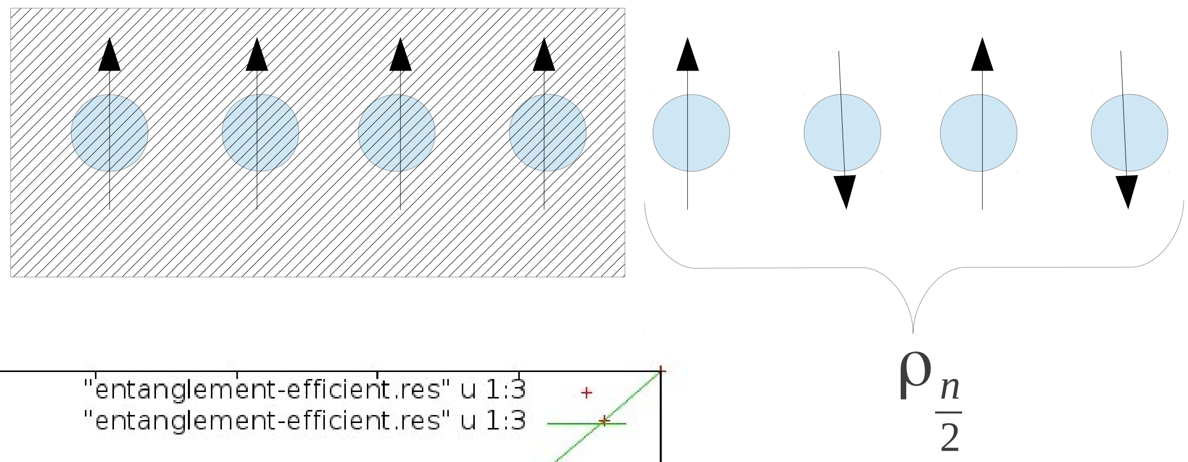
Twin primes 1 mod 4 $\left\langle \sigma_x^{(1)} \right\rangle = \frac{2\pi_2^{(1)}\left(2^n\right)}{\pi\left(2^n\right)}$

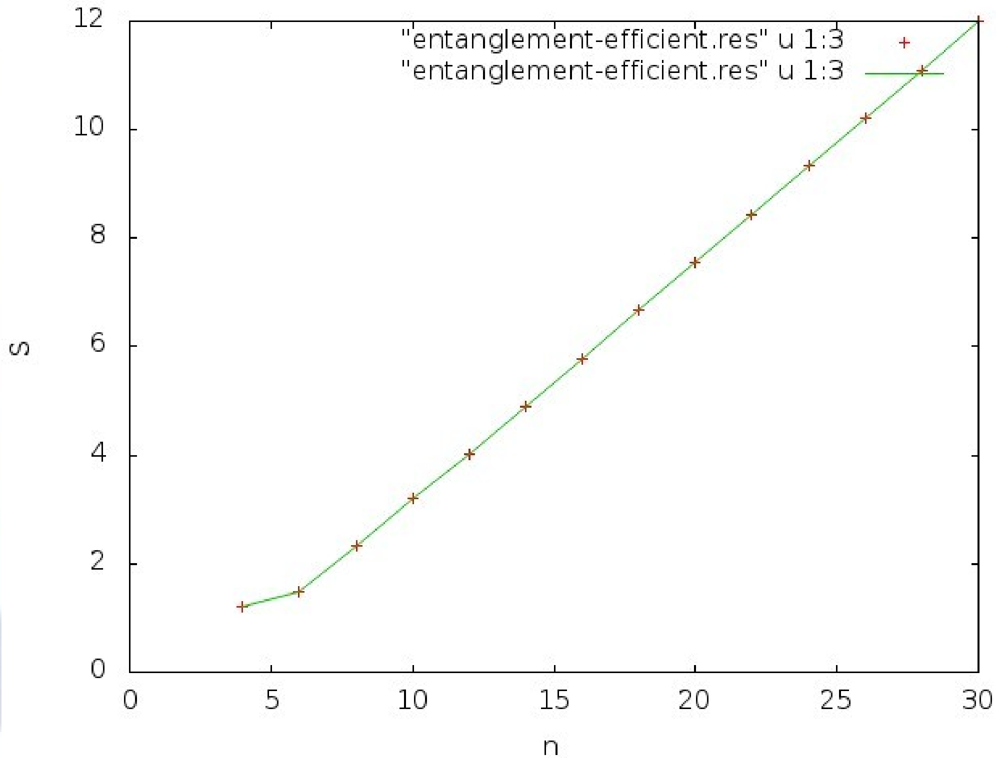
Chebyshev bias $\left\langle \sigma_z^{(1)} \right\rangle = \frac{-\Delta\left(2^n\right) - 1}{\pi\left(2^n\right)}$

$$\Delta(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$$

Sub-series of primes, twin primes, etc. are amenable to measurements

Entanglement entropy of the Prime state





"There is entanglement in the Primes"

Volume law scaling

$$S \sim .8858\, n + const$$

A. Monras, G. Sierra, JIL

Scaling of entanglement entropy

$$S \sim n - const$$

Random states

$$S \sim .8858\, n + const$$

Prime state

$$S \sim n^{\frac{d-1}{d}} + const$$

Area law in d-dimensions

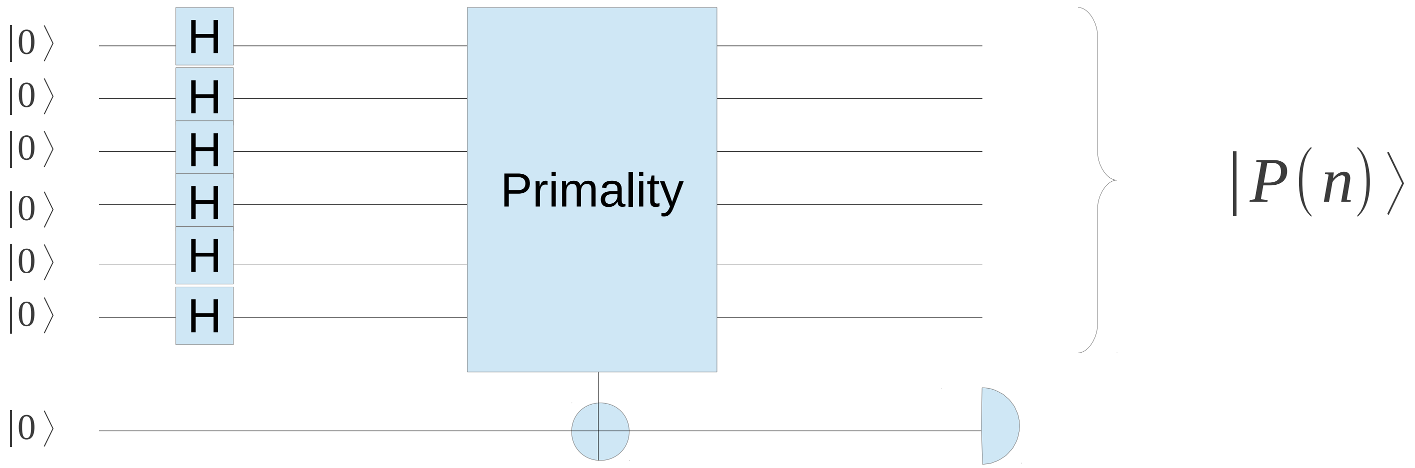
$$S \sim \frac{c}{3} \log n + const$$

Critical scaling in d=1
at quantum phase transitions

$$S \sim \log(\xi) = const$$

Finitely correlated states
away from criticality

Construction of the Prime state



$$U_{primality} \sum_x |x\rangle|0\rangle = |P(n)\rangle|0\rangle + \sum_{c \in composite} |c\rangle|1\rangle$$

$$Prob(|P(n)\rangle) = Prob(ancilla=0) = \frac{\pi(2^n)}{2^n} \approx \frac{1}{n \ln 2}$$

Efficient construction!

Construction of twin primes

$$U_{+2}|P(n)\rangle|0\rangle=\sum_{p\in primes}|p+2\rangle|0\rangle$$

$$U_{primality}U_{+2}|P(n)\rangle|0\rangle=\sum_{p,\,p+2\in primes}|p+2\rangle|0\rangle+\sum_{p+2\notin primes}|p+2\rangle|1\rangle$$

$$Prob(|twin\,primes\rangle)=\frac{\pi_2(2^n)}{\pi(2^n)}\approx\frac{1}{(n\ln 2)^2}$$

Efficient construction!

Grover construction of the Prime state

$$|\psi_0\rangle = \sum_{x<2^n} |x\rangle = \frac{1}{\pi(2^n)} \Big(\underbrace{\overbrace{\sum_{p\in primes} |p\rangle}^{M} + \sum_{c\in composites} |c\rangle}_{N} \Big)$$

calls to Grover

$$R(n) \leq \frac{\pi}{4}\sqrt{\frac{N}{M}} \leq \frac{\pi}{4}\sqrt{n \ln 2}$$

$$|\psi_f\rangle = |P(n)\rangle$$

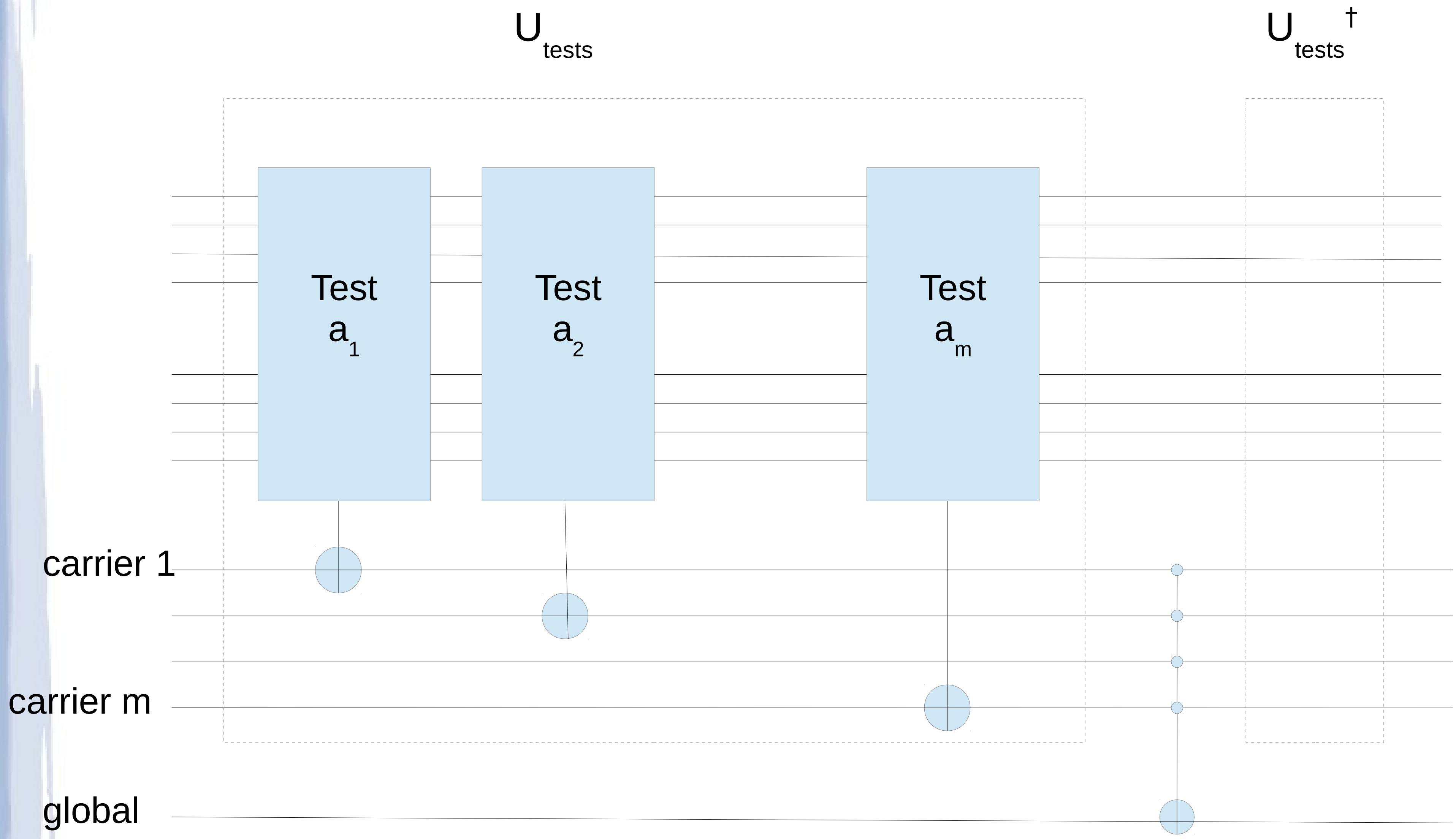
We need to construct an oracle!

Construction of a Quantum Primality oracle

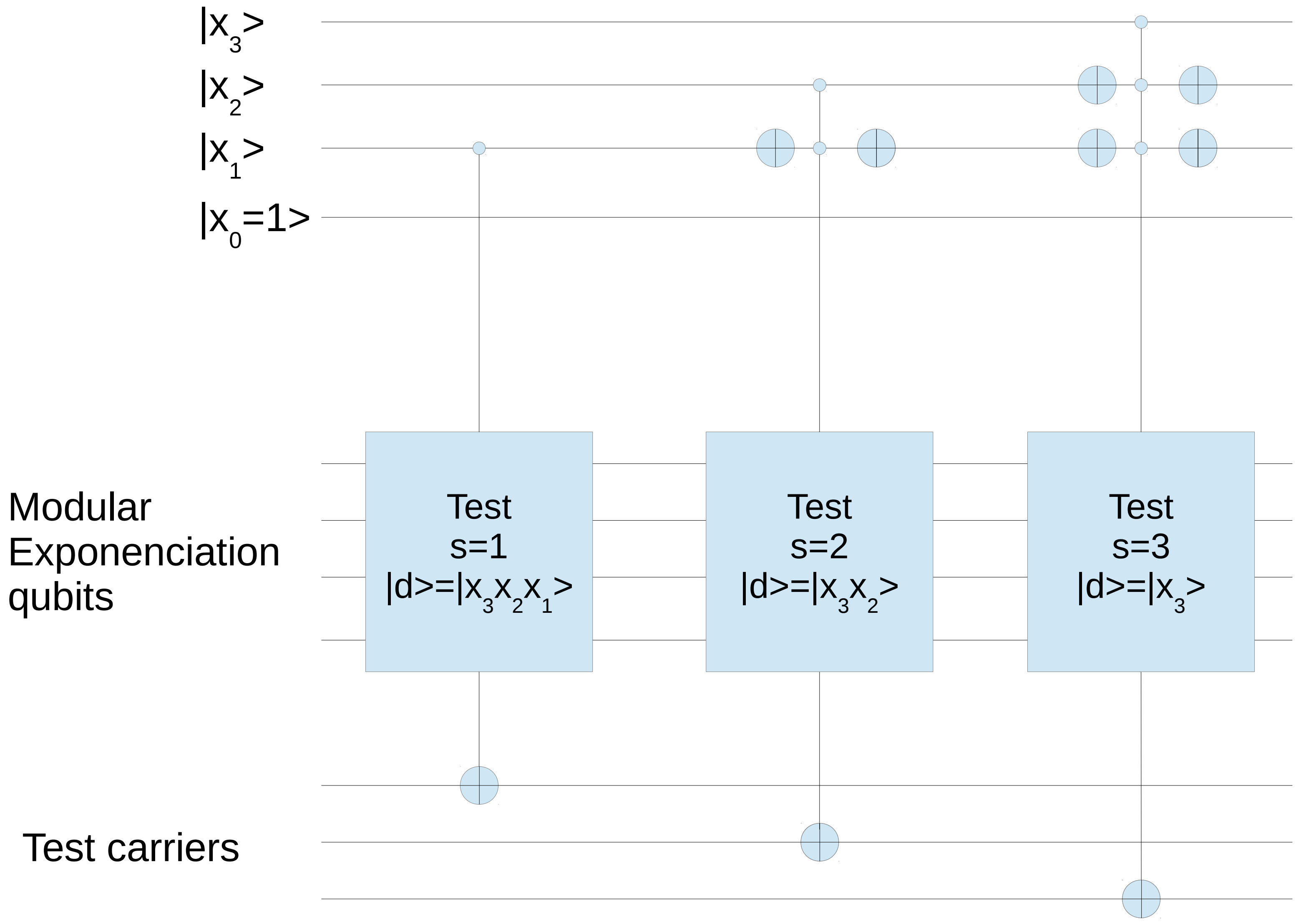
An efficient Quantum Oracle can be constructed following classical primality tests

Miller-Rabin primality test

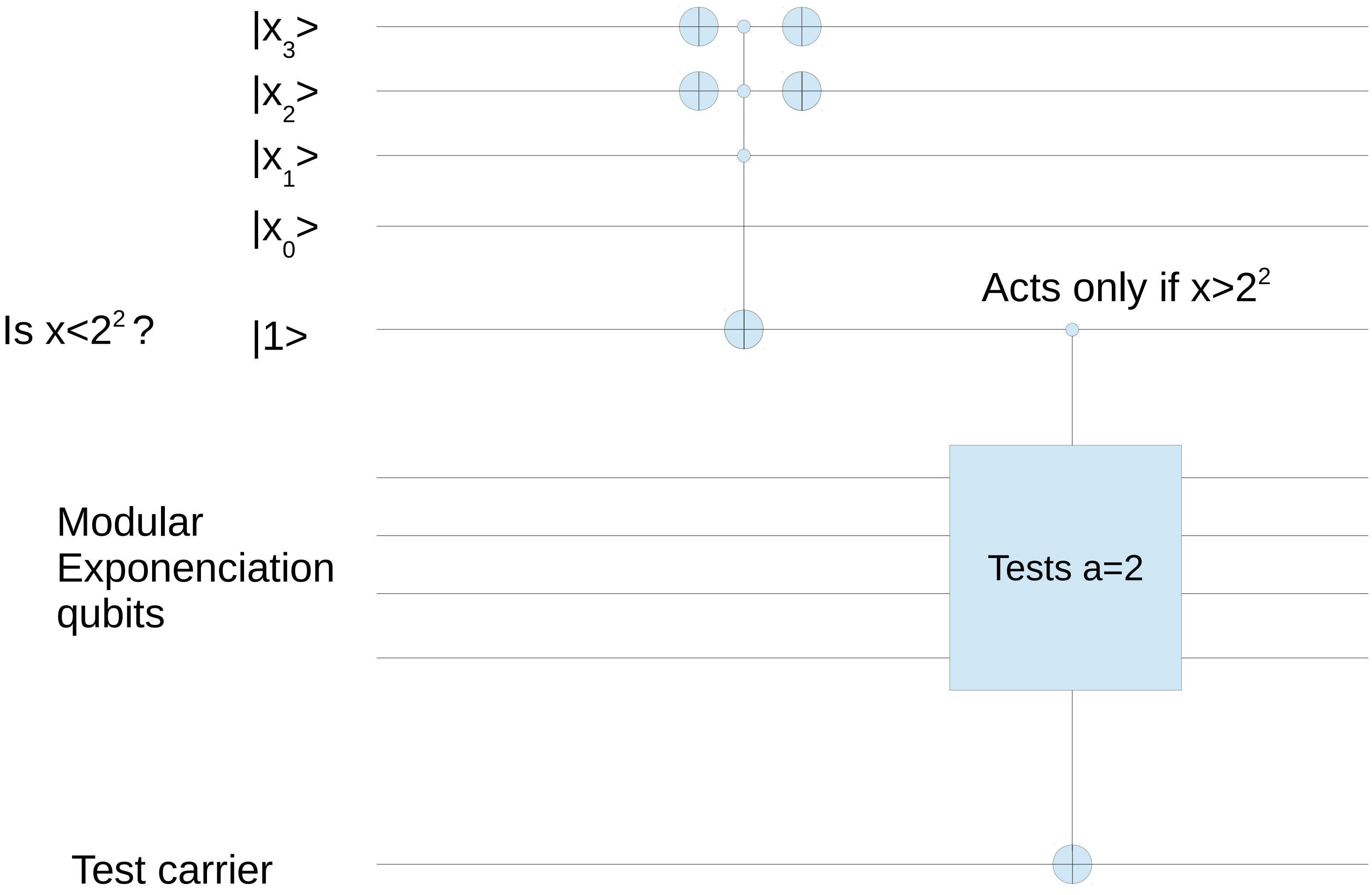
- Find $x \rightarrow x-1=2^s d$
- Choose "witness" $1 \leq a \leq x$
- If $a^d \neq 1 \, mod \, x$ then x is composite
  $a^{2^r d} \neq -1 \, mod \, x \quad 0 \leq r \leq s-1$
- If any test fails, x may be prime or composite: "liars"
- Eliminate strong liars checking less than $\ln(x)$ witnesses



Structure of the quantum primality oracle



Tests are conditioned to the actual value of *x*

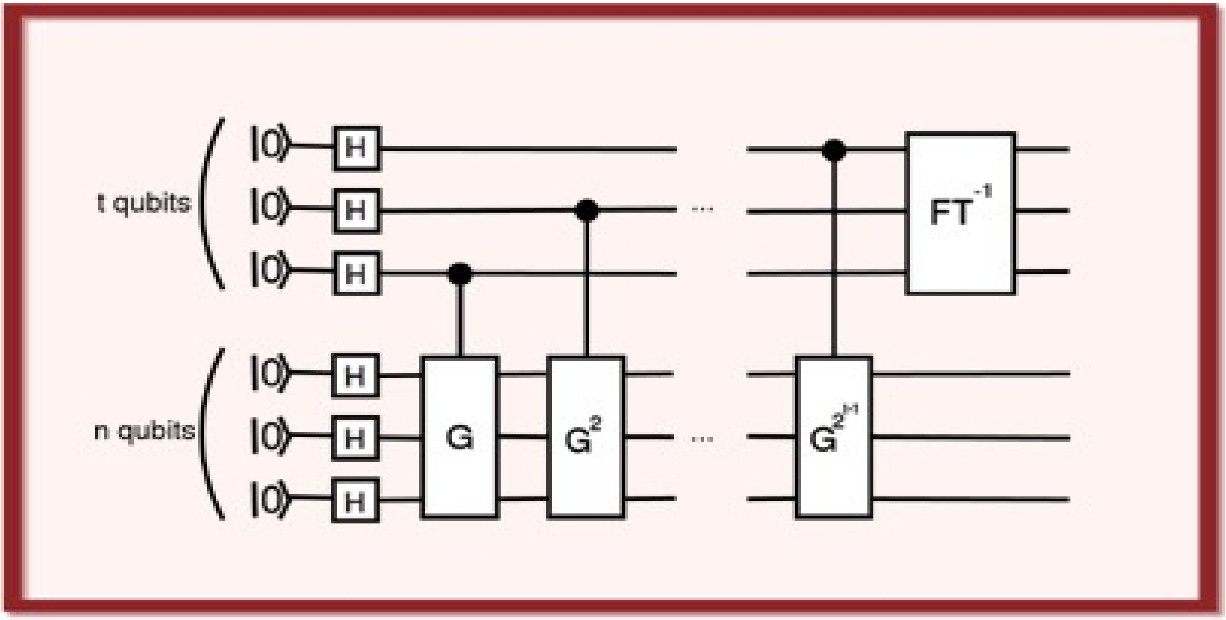


$|x_3>$
$|x_2>$
$|x_1>$
$|x_0>$
Acts only if $x>2^2$
Is $x<2^2$?
|1>
Modular Exponenciation qubits
Tests a=2
Test carrier

Quantum Counting of Prime numbers

quantum oracle + quantum Fourier transform
= quantum counting algorithm

Brassard, Hoyer, Tapp (1998)



Counts the number of solutions to the oracle

We want to count *M* solutions out of *N* possible states

We know an estimate $\tilde{M}$

$$|\tilde{M} - M| < \frac{2\pi}{c} M^{\frac{1}{2}} + \frac{\pi^2}{c^2}$$

Bounded error in quantum counting using $c\sqrt{N}$ calls

Brassard, Hoyer, Tapp (1998)

Bounded error in the quantum counting of primes

$$|Li(x) - \pi_{QM}(x)| < \frac{2\pi}{c} \frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}} x}$$

$$|Li(x) - \pi_{QM}(x)| < \frac{2\pi}{c} \frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}} x}$$

Error of counting is smaller than the bound for the fluctuations if Riemann conjecture is correct!

$$\frac{2\pi}{c} \frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}} x} < x^{\frac{1}{2}} \ln x$$

Best classical algorithm by Lagarias-Miller-Odlyzko (1987) implemented by Platt (2012)

$$T \sim x^{\frac{1}{2}} \qquad S \sim x^{\frac{1}{4}}$$

A Quantum Computer could calculate the size of fluctuations more efficiently than a classical computer

Beyond Prime numbers: the **q-functor**

$$f : S \subseteq X \rightarrow |S\rangle = \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$$

$$|S\rangle = \frac{1}{\sqrt{|S|}} \sum_{x \in X} \chi_S(x) |x\rangle \qquad \chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

Primes
Average (Cramér) primes
Dirichlet characters
...

Only needs the construction of a quantum oracle for $\chi_s(x)$

Conclusion

I) Quantum Simulation of basic concepts

Geometry ↔ Site-dependent coupling

Dimensions ↔ Connectivity

Boundary conditions ↔ Non-local coupling

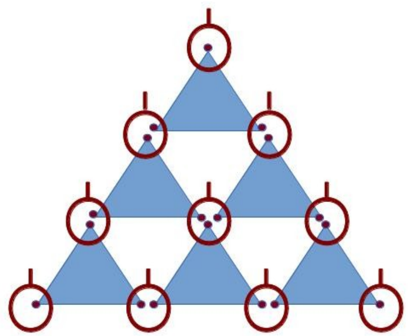
II) Quantum Simulation of Arithmetics

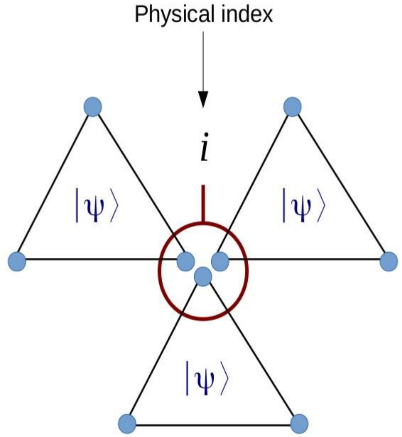
Superposition of series of numbers using appropriate q-oracles

Measurements of arithmetic functions

Other results

- Tensor Networks based on W-simplices span the ground state manifold frustrated systems

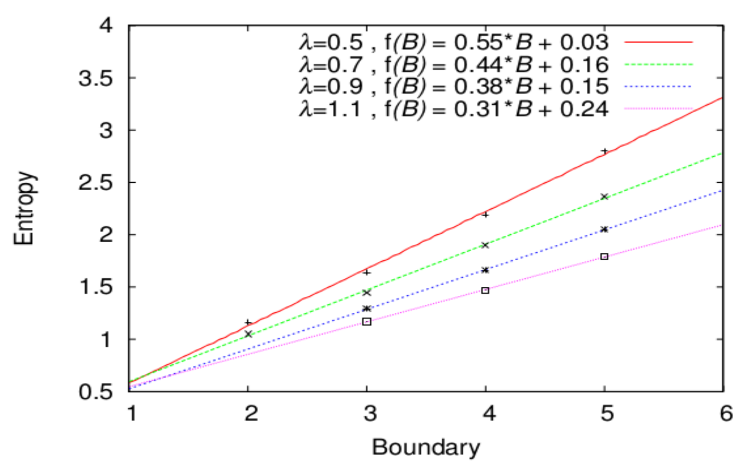




$$A^{i}_{\alpha\,\alpha'\alpha''}$$

- Area law coefficient is sensitive to frustration

$$S=\alpha L+ct$$



- Absolute Maximal Multipartite entangled states relates to Reed-Solomon codes, quantum sharing, multipartite teleportation,...