Probing the environment using system dynamics in open quantum evolution

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Open quantum dynamics

Isolated quantum systems

Schrödinger equation ⇔ Unitary evolution

$$i\frac{\partial\psi}{\partial t} = H\psi \quad ; \quad U(t) = \mathcal{T}\left[\exp\left\{-i\int_0^t dt' H(t')\right\}\right]$$

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Open quantum systems

Master equations $\rightleftharpoons$ Dynamical maps

$$\frac{\partial\rho}{\partial t} = \mathfrak{L}\rho \quad ; \quad \rho \to \Phi\rho$$

Dynamical maps take density matrices to density matrices

- Linear and trace preserving
- Preserves Hermiticity of ρ
- Maps positive matrices to positive matrices

The dynamical map

For finite dimensional systems

$$\rho_{rs} \longrightarrow A_{rs;r's'}\rho_{r's'} = (A\rho)_{rs}.$$

$$\begin{aligned} A_{sr;s'r'}(t) &= [A_{rs;r's'}(t)]^* && \text{(Hermiticity preserving)} \\ A_{rr;r's'} &= \delta_{r's'} && \text{(Trace preserving)} \\ x_r^* x_s A_{rs;r's'} y_{r'} y_{s'}^* &\geq 0 && \text{(Positivity)} \end{aligned}$$

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$$A_{rs;r's'}(t) \equiv B_{rr';ss'}(t)$$

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$$\begin{aligned} B_{ss';rr'}^*(t) &= B_{rr';ss'}(t) && \text{(Hermiticity)} \\ B_{nr';ns'} &= \delta_{r's'} && \text{(Trace preserving)} \\ x_r^* y_{r'} B_{rr';ss'} x_s y_{s'}^* &\geq 0 && \text{(Positivity)} \end{aligned}$$

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Operator sum form of B

Since B is hermitian, it can be written in terms of its eigenvalues and eigenvectors.

$$\rho_{rs} = \sum_{\alpha} \lambda_\alpha \zeta_{rr'}(\alpha) \rho_{r's'} \zeta^{\dagger}_{s's}(\alpha)$$

If all $\lambda_\alpha \geq 0$ then define $C(\alpha) \equiv \sqrt{\lambda_\alpha}\zeta(\alpha)$

$$\rho \longrightarrow \sum_{\alpha} C(\alpha)\rho C(\alpha)^{\dagger}$$

with

$$\sum_{\alpha} C(\alpha)^{\dagger} C(\alpha) = \mathbf{1}$$

B is then a **completely positive** map

The χ matrix representation

Starting from

$$\rho \longrightarrow \sum_{\alpha} C(\alpha)\rho C(\alpha)^{\dagger},$$

we can expand the $C(\alpha)$ in terms of a standard operator basis Σ_j:

$$C(\alpha) = \sum_j \alpha_j \Sigma_j$$

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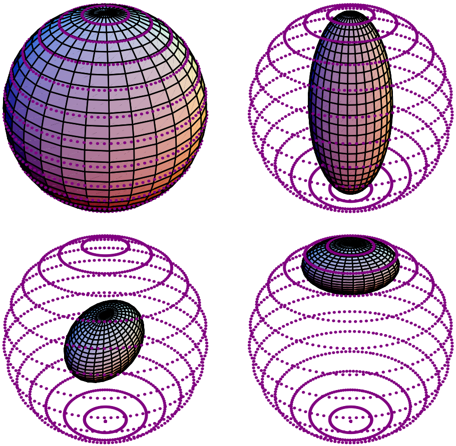
χ-matrix

$$\rho \longrightarrow \sum_{j,k} \chi_{jk} \Sigma_j \rho \Sigma_k^{\dagger}$$

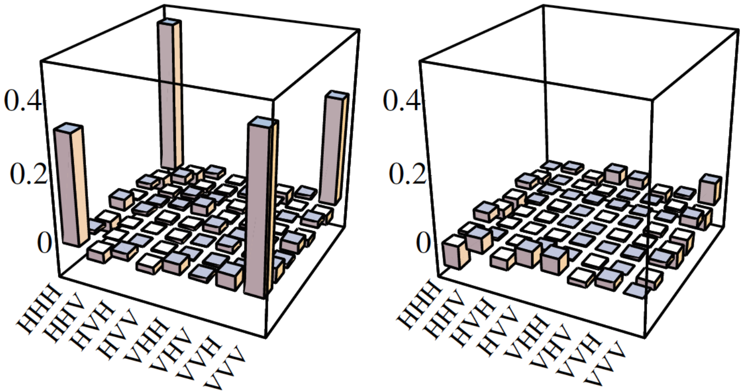
$$\chi_{jk} = \sum_{\alpha} \alpha_j \alpha_k.$$

Maps viewed on the Bloch sphere

The completely positive single qubit maps: Unitary rotation, pure dephasing, depolarizing map, and the pin map.

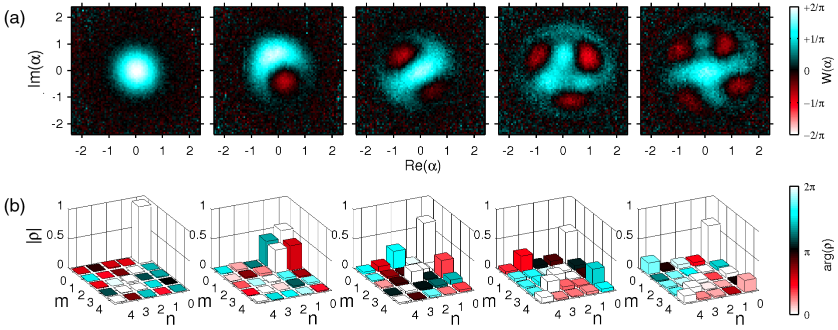


Quantum state tomography



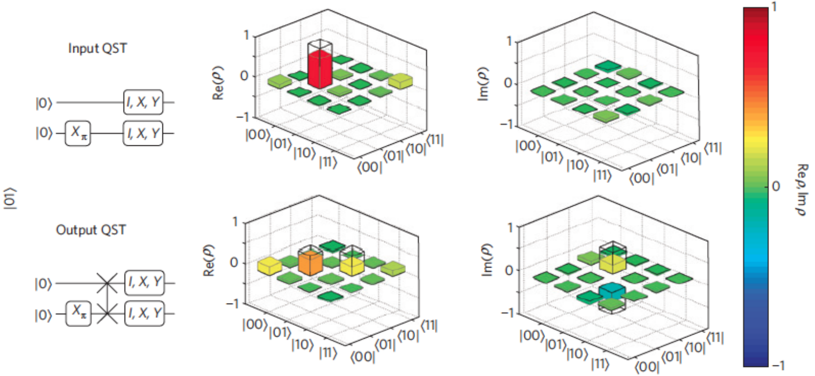
Quantum state tomography of an optical GHZ state - K. J. Resch, P. Walther and A. Zeilinger, PRL **94**, 07402 (2005)

Quantum state tomography



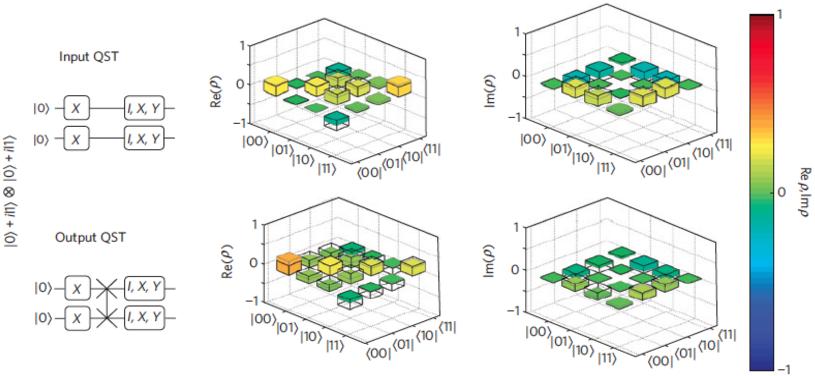
Wigner tomography of a superconducting anharmonic oscillator in a superposition of Fock states - Yoni Shalibo, Roy Resh, Ofer Fogel, David Shwa, Radoslaw Bialczak, John M. Martinis, and Nadav Katz, PRL **110**, 100404 (2013)

Quantum process tomography



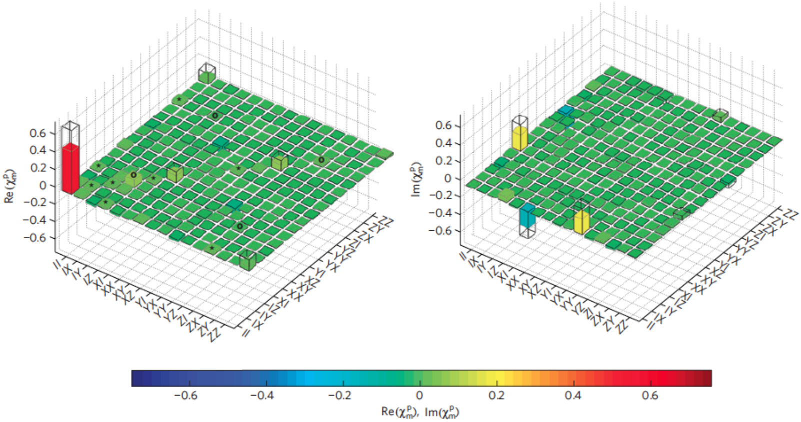
Quantum process tomography of a universal entangling gate on Josephson qubits (Input state 1) - Martinis group, Nature Physics (2010)

Quantum process tomography



Quantum process tomography of a universal entangling gate on Josephson qubits (Input state 2) - Martinis group, Nature Physics (2010)

Quantum process tomography



Quantum process matrix using 16 input states

- Martinis group, Nature Physics (2010)

Quantum process tomography

Realization of quantum process tomography in NMR

Andrew M. Childs,[1,2,3] Isaac L. Chuang,[1] and Debbie W. Leung[1,4,5]
[1] *IBM Almaden Research Center, San Jose, CA 95120*
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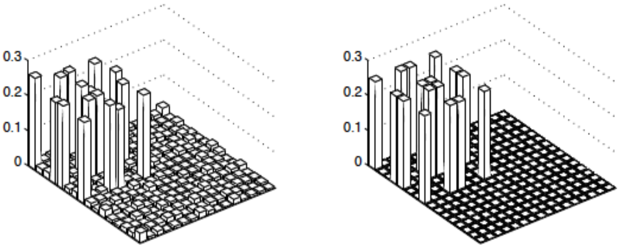
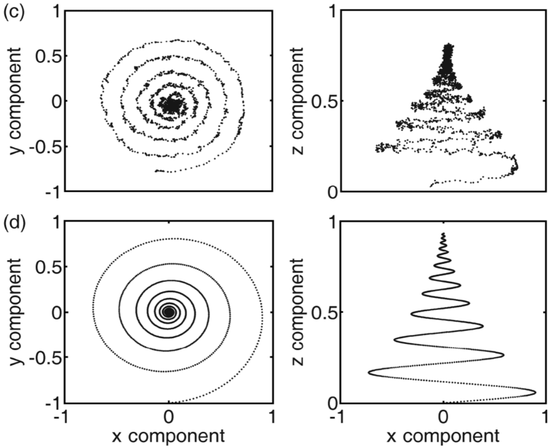
[3] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139*
[4] *Quantum Entanglement Project, ICORP, JST, Edward Ginzton Laboratory, Stanford University, Stanford, CA 94305*
[5] *IBM T. J. Watson Research Center, Yorktown Heights, NY 10598*
(6 December 2000)



Quantum process matrix for a controlled not gate

- arXiv:quant-ph/0012032v1

Quantum process tomography



Time evolution of different input states of a qubit

– Martinis group, PRL **97**, 050502 (2006)

Knowing the environment

The main question

Can the extensive data obtained about the state and evolution of a open quantum system through tomography be used to gain quantitative information about the environment of the system?

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Can the extensive data obtained about the state and evolution of a open quantum system through tomography be used to gain quantitative information about the environment of the system?

We find that this data can be put to good use assuming that the system is a qubit and that the environment is a generic N level quantum system.

A qubit with an N level environment: SU(2) $\times$ SU(N)

The state of the system of interest - a qubit (Σ-system) - is written in terms of the three Pauli matrices (SU(2) generators),

$$\vec{\Sigma} = \big(\Sigma_1,\, \Sigma_2,\, \Sigma_3\big).$$

$$[\Sigma_i,\, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k.$$

The environment is an N level quantum system with its state written in terms of the $N^2 - 1$ generators of SU(N) denoted by $\vec{\Lambda}$.

$$[\Lambda_i,\, \Lambda_j] = 2if_{ijk}\Lambda_k.$$

The parameters of the Hamiltonian

$$H = \frac{1}{2}\Big(\alpha_j \Sigma_j + \beta_k \Lambda_k + \sum_{j=1}^{3} \sum_{k=1}^{N^2-1} \gamma_{jk} \Sigma_j \Lambda_k\Big).$$

The parameters specifying the Hamiltonian are

$$\begin{aligned} \vec{\alpha} &= (\alpha_1,\, \alpha_2,\, \alpha_3), \\ \vec{\beta} &= (\beta_1,\, \ldots,\, \beta_{N^2-1}), \\ \overleftrightarrow{\gamma} &= \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N^2-1} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N^2-1} \\ \gamma_{31} & \gamma_{32} & \cdots & \gamma_{3N^2-1} \end{pmatrix} \equiv \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \end{pmatrix} \end{aligned}$$

$$H = \frac{1}{2}\big(\vec{\alpha} \cdot \vec{\Sigma} + \vec{\beta} \cdot \vec{\Lambda} + \vec{\Sigma} \cdot \overleftrightarrow{\gamma} \cdot \vec{\Lambda}\big).$$

We want to find $\vec{\alpha}$, $\vec{\beta}$ and $\overleftrightarrow{\gamma}$.

The experiment

The Σ-system is initialized in one of three preparations:

$$\rho_0^{(1)} = \frac{1}{2}(1+\Sigma_1), \qquad \rho_0^{(2)} = \frac{1}{2}(1+\Sigma_2), \qquad \text{and} \qquad \rho_0^{(3)} = \frac{1}{2}(1+\Sigma_3).$$

In the Schrödinger picture,

$$\rho_t^{(k)} = \frac{1}{2}\left(1 + a_1^{(k)}(t)\Sigma_1 + a_2^{(k)}(t)\Sigma_2 + a_3^{(k)}(t)\Sigma_3\right), \qquad k = 1, 2, 3.$$

Data collected

The nine functions $a_j^{(k)}(t)$ are obtained experimentally for some length of time t as part of a complete process tomography experiment.

Using the data

$$a_j^{(k)}(t) = \langle \Sigma_j \rangle_t^{(k)} = \mathrm{Tr}\big[\rho_t^{(k)} \Sigma_j\big].$$

In the Heisenberg picture

$$a_j^{(k)}(t) = \langle \Sigma_j(t) \rangle^{(k)} = \mathrm{Tr}\big[\rho_0^{(k)} \Sigma_j(t)\big].$$

Now consider the n^{th} time derivative of $a_j^{(k)}(t)$ in the Heisenberg picture,

$$\frac{d^n}{dt^n} a_j^{(k)}(t) = \left\langle \frac{d^n}{dt^n} \Sigma_j(t) \right\rangle^{(k)} = \mathrm{Tr}\left[\rho_0^{(k)} \frac{d^n}{dt^n} \Sigma_j(t)\right].$$

Time derivatives of the Heisenberg picture Pauli operators that appear on the right hand side are functions of the Hamiltonian parameters that we are trying to find.

The equation of motion for Σ_1

$$\frac{d}{dt}\Sigma_1(t) \otimes \mathbb{I}_e = i[H, \Sigma_1(t) \otimes \mathbb{I}_e].$$

$$\begin{aligned} \dot{\Sigma}_1(t) &= i[H,\, e^{iHt}\Sigma_1 e^{-iHt}] = ie^{iHt}[H,\, \Sigma_1]e^{-iHt} \\ &= \frac{i}{2}e^{iHt}[\vec{\alpha}\cdot\vec{\Sigma} + \vec{\beta}\cdot\vec{\Lambda} + \vec{\Sigma}\cdot \overleftrightarrow{\gamma} \cdot\vec{\Lambda},\, \Sigma_1]e^{-iHt} \\ &= \alpha_2\Sigma_3(t) - \alpha_3\Sigma_2(t) + \gamma_{2k}\Lambda_k\Sigma_3(t) - \gamma_{3k}\Lambda_k\Sigma_2(t), \end{aligned}$$

For small values of t so that $\Sigma_i(t) \simeq \Sigma_i(0) \equiv \Sigma_i$.

Finding α_2

$$\begin{aligned} \alpha_2 &= \frac{1}{2}\mathrm{Tr}\{\Sigma_3 i[H,\, \Sigma_1]\} = \frac{1}{2}\mathrm{Tr}\{(\mathbb{I} + \Sigma_3)i[H,\, \Sigma_1]\} \\ &= \mathrm{Tr}\left[\rho_0^{(3)}\dot{\Sigma}_1(0)\right] = \dot{a}_1^{(3)}(0). \end{aligned}$$

The first derivatives

Using the equations for $\dot{\Sigma}_2$ and $\dot{\Sigma}_3$ we get

$$\dot{a}(0) = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}.$$

Finding $\vec{\alpha}$

The first time derivatives of the nine functions $a_j^{(k)}$ form a real, anti-symmetric, 3×3 matrix whose three independent elements give us α_1, α_2 and α_3

This is not particularly surprising since α_j's are the coefficients of the part of the Hamiltonian that act only on the Σ-system.

Cross Products

Simplifying notation

$$[\vec{\alpha} \times \vec{\Sigma}]_i = \epsilon_{ijk} \alpha_j \Sigma_k$$

$$[\vec{\beta} \times \vec{\Lambda}]_i \equiv f_{ijk} \beta_j \Lambda_k .$$

Using this notation we can write the equation for the first time derivative as

$$\dot{\vec{\Sigma}} = i[H, \vec{\Sigma}] = \vec{\alpha} \times \vec{\Sigma} + \overleftrightarrow{\gamma} \cdot \vec{\Lambda} \times \vec{\Sigma}.$$

The second time derivative

$$\ddot{a}_j^{(k)}(0) = \mathrm{Tr}(\rho_0^{(k)}\ddot{\Sigma}_j) = \mathrm{Tr}\big(\rho_0^{(k)} i[H,\, i[H,\, \Sigma_j]]\big).$$

$$\begin{aligned} i[H,\, i[H,\, \vec{\Sigma}]] \;=\; & \vec{\alpha}\times\vec{\alpha}\times\vec{\Sigma} + \vec{\alpha}\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\Sigma} + \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\alpha}\times\vec{\Sigma} \\ & + \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\Sigma} + \overleftrightarrow{\gamma}\cdot(\vec{\beta}\times\vec{\Lambda})\times\vec{\Sigma} \\ & + \overleftrightarrow{\gamma}\cdot(\vec{\Sigma}\cdot\overleftrightarrow{\gamma}\times\vec{\Lambda})\times\vec{\Sigma}. \end{aligned}$$

$$\ddot{a}(0) = -\begin{pmatrix} \alpha_2^2+\alpha_3^2+|\tilde{\gamma}_2|^2+|\tilde{\gamma}_3|^2 & -\alpha_1\alpha_2-\tilde{\gamma}_1\cdot\tilde{\gamma}_2 & -\alpha_1\alpha_3-\tilde{\gamma}_1\cdot\tilde{\gamma}_3 \\ -\alpha_1\alpha_2-\tilde{\gamma}_1\cdot\tilde{\gamma}_2 & \alpha_1^2+\alpha_3^2+|\tilde{\gamma}_1|^2+|\tilde{\gamma}_3|^2 & -\alpha_2\alpha_3-\tilde{\gamma}_2\cdot\tilde{\gamma}_3 \\ -\alpha_1\alpha_3-\tilde{\gamma}_1\cdot\tilde{\gamma}_3 & -\alpha_2\alpha_3-\tilde{\gamma}_2\cdot\tilde{\gamma}_3 & \alpha_1^2+\alpha_2^2+\tilde{\gamma}_1^2+\tilde{\gamma}_2^2 \end{pmatrix}.$$

The six independent equations above can be used to find the lengths of the vectors $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ as well as the dot products $\tilde{\gamma}_1\cdot\tilde{\gamma}_2$, $\tilde{\gamma}_1\cdot\tilde{\gamma}_3$ and $\tilde{\gamma}_2\cdot\tilde{\gamma}_3$

The third time derivative

$$
\begin{aligned}
i[H,\, i[H,\, i[H,\, \vec{\Sigma}]]]_{\text{eff}} \;=\;& \vec{\alpha}\times\vec{\alpha}\times\vec{\alpha}\times\vec{\Sigma} + \vec{\alpha}\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\Sigma} \\
&+ \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\alpha}\times\vec{\Sigma} \\
&+ \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\alpha}\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\Sigma} \\
&+ 2\,\overleftrightarrow{\gamma}\cdot(\vec{\beta}\times\vec{\Lambda})\times \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times\vec{\Sigma} \\
&+ \overleftrightarrow{\gamma}\cdot\vec{\Lambda}\times \overleftrightarrow{\gamma}\cdot(\vec{\beta}\times\vec{\Lambda})\times\vec{\Sigma},
\end{aligned}
$$

$$
\dddot{a}_j^{(k)} = \epsilon_{jkl}\alpha_l\left(|\vec{\alpha}|^2+|\tilde{\gamma}_1|^2+|\tilde{\gamma}_2|^2+|\tilde{\gamma}_2|^2\right)+2\epsilon_{jkl}\alpha_m(\tilde{\gamma}_m\cdot\tilde{\gamma}_l)+3f_{plm}\beta_p\gamma_{jl}\gamma_{km}
$$

Number of equations

Putting it all together

The trace equations with the odd order commutators are antisymmetric and that with the even order commutators are symmetric. Hence we expect to get three independent equations each from the odd orders and six each from the even orders. Assuming an average of 4.5 parameters from each order, we can estimate the minimum order to which commutators are to be computed in order to have sufficient linearly independent equations so as to solve for the $3 + N^2 - 1 + 3(N^2 - 1) = 4N^2 - 1$ unknown parameters as $(4N^2 - 1)/4.5$.

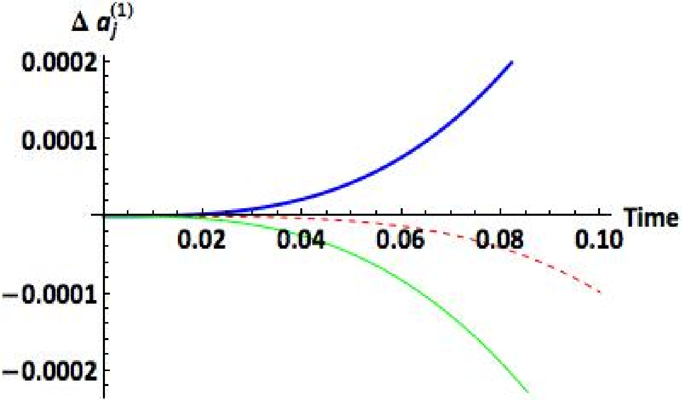
Qubit-Qutrit example

Parameters chosen

$\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ \beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 1,\ \beta_4 = 1,\ \beta_5 = 1,\ \beta_6 = 1,\ \beta_7 = 1,\ \beta_8 = 0.1,\ \gamma_{11} = 1,\ \gamma_{22} = 1,\ \gamma_{33} = 1.$ All the rest were set to zero

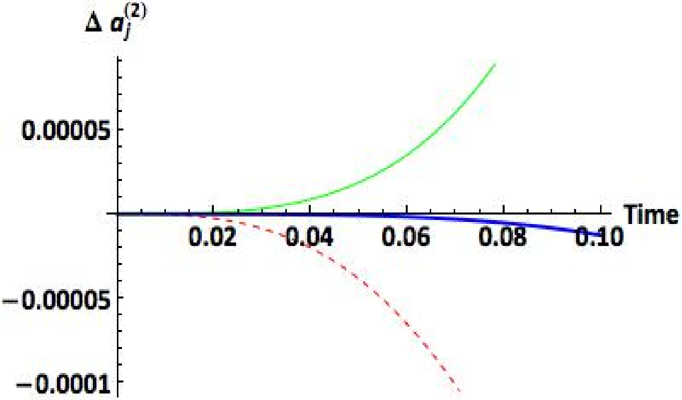
$$H = \begin{pmatrix} 2.52887 & 0.5 - i & 0.5 - 0.5i & 0.5 - i & 0 & 0 \\ 0.5 + i & 0.52868 & 0.5 - 0.5i & 1 & 0.5 - i & 0 \\ 0.5 + 0.5i & 0.5 + 0.5i & 1.44226 & 0 & 0 & 0.5 - i \\ 0.5 + i & 1 & 0 & -1.47113 & 0.5 - i & 0.5 - 0.5i \\ 0 & 0.5 + i & 0 & 0.5 + i & -1.47113 & 0.5 - 0.5i \\ 0 & 0 & 0.5 + i & 0.5 + 0.5i & 0.5 + 0.5i & -1.55774 \end{pmatrix}$$

Reconstruction



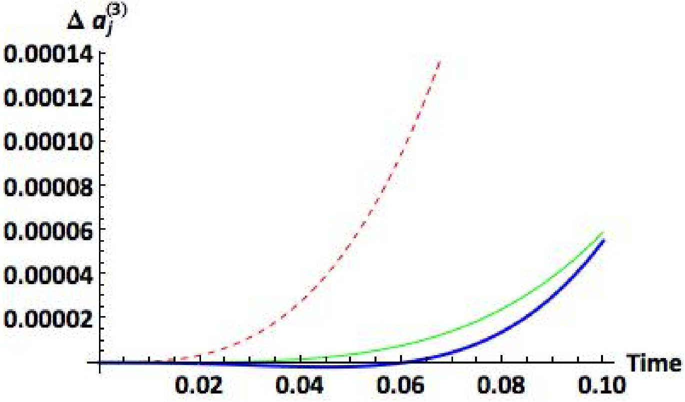
Difference between $a_j^{(1)}$(true) and $a_j^{(1)}$(reconstructed) versus time. j=1,2,3 are red, blue and green respectively

Reconstruction



Difference between $a_j^{(2)}$(true) and $a_j^{(2)}$(reconstructed) versus time. j=1,2,3 are red, blue and green respectively.

Reconstruction



Difference between $a_j^{(3)}$(true) and $a_j^{(3)}$(reconstructed) versus time. j=1,2,3 are red, blue and green respectively.

Conclusion

The parameters of the Hamiltonian of a qubit interacting with an N dimensional quantum system can be obtained, in principle, from the time dependance of the qubit alone.

The expressions for the quantities $a_j^{(k)}$ and their derivatives for varying N are shown to have a similar structure which is governed by the underlying SU(N) Lie algebra.

The Hamiltonian enables one to understand the details of the environment with which the qubit is interacting and this knowledge can help one to take necessary steps to minimize the decohering effects of the environment

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- Prof. E. C. G. Sudarshan
- Prof. Thomas F. Jordan

References

- Vinayak Jagdish and Anil Shaji, *SProbing the environment using system dynamics in open quantum evolution*, [to be submitted]
- T. F. Jordan, Anil Shaji and E. C. G. Sudarshan, *One qubit completely determines the dynamics of the other*, Phys. Rev. A. **76**, 012101 (2007)
- E. C. G. Sudarshan, P. M. Mathews and J. Rau, *Stochastic Dynamics of Quantum-Mechanical Systems*, Phys. Rev. **121**, 920-924 (1961)
- T. F. Jordan, Anil Shaji and E. C. G. Sudarshan, *Dynamics of initially entangled open quantum systems*, Phys. Rev. A. **70**, 042303 (2004)