Limit on Time-Energy Uncertainty with Multipartite Entanglement

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Outline

- *Introduction*
- *Geometric quantum uncertainty relation (GQUR)*
- *Multipartite entanglement: geometric*
- *GQUR and multiparty entanglement: applications*
- *Mixed states*
- *Conclusion*

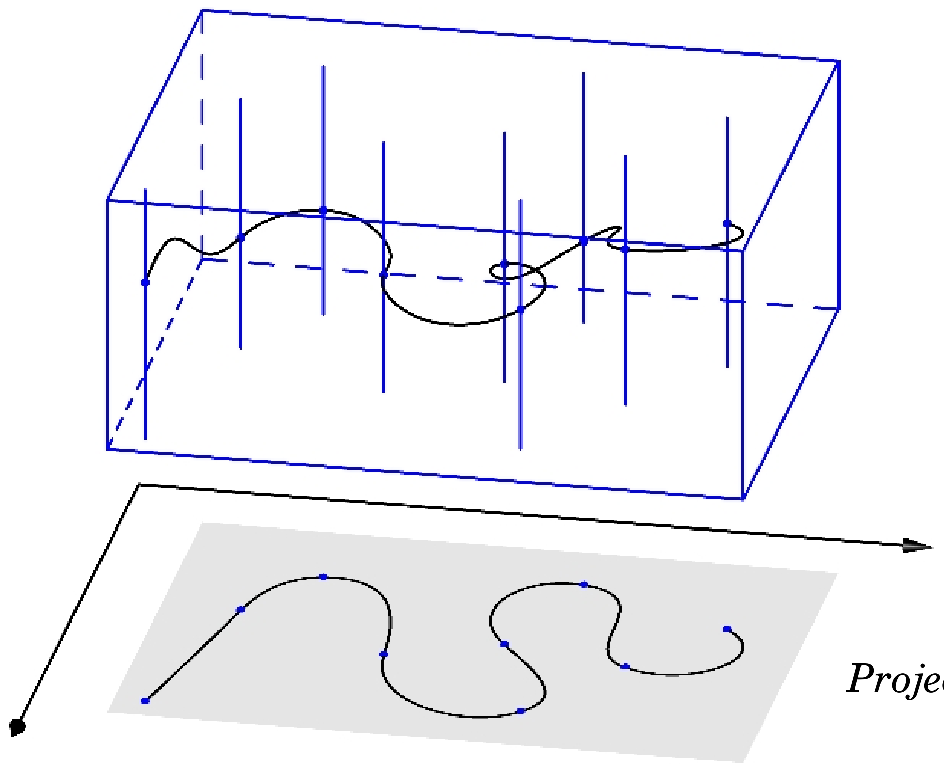
MNB, R. Prabhu, A K Pati, A Sen (De), U Sen, arXiv:1303.0706

Time-energy uncertainty relation

Quantum geometry

- *The projective Hilbert space*

$$\Pi : |\Psi\rangle \to |\Psi\rangle\langle\Psi|$$



Complex projective Hilbert space may be given a natural metric, the Fubini-Study metric.

Quantum geometry

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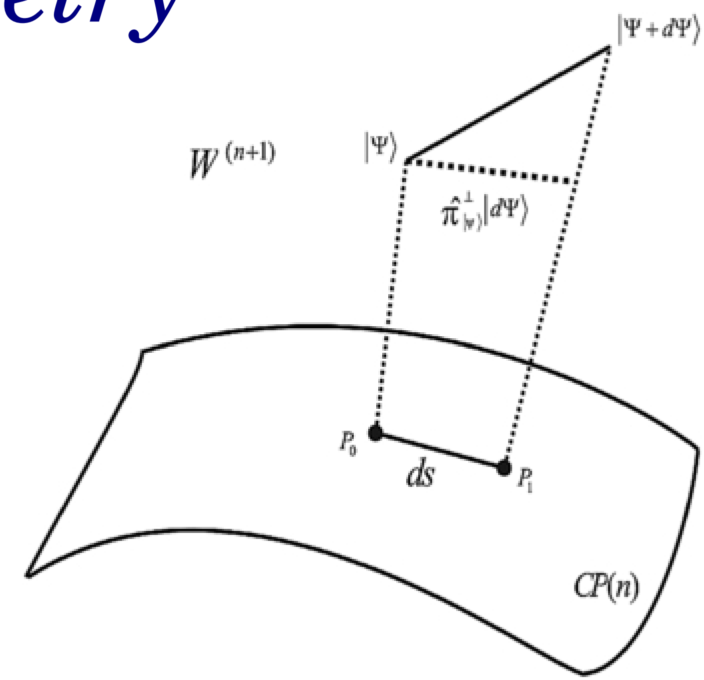

- *Fubini-Study (FS) metric:*

$$\begin{aligned} dS^2 &= 4\left(1 - |\langle\Psi(\bar{\lambda} + d\bar{\lambda})|\Psi(\bar{\lambda})\rangle|^2\right) \\ &= 4\left(\langle\partial_i\Psi|\partial_j\Psi\rangle - \langle\partial_i\Psi|\Psi\rangle\langle\Psi|\partial_j\Psi\rangle\right) d\lambda^i d\lambda^j \end{aligned}$$

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- *The distance, also called the Bargmann angle*

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \cos^2\left(\frac{\mathcal{S}}{2}\right)$$

Geometry of quantum evolution

Quantum dynamics

- *Quantum state evolves following the Schrodinger equation*

$$i\hbar|\dot{\Psi}(t)\rangle = H(t)|\Psi(t)\rangle$$

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- *After an infinitesimal time translation*

$$|\Psi(t)\rangle \longrightarrow |\Psi(t+dt)\rangle$$

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$$|\Psi(t)\rangle \longrightarrow |\Psi(t+dt)\rangle$$

- *Infinitesimal distance using FS metric*

$$dS^2 = 4\left(1 - |\langle\Psi(t+dt)|\Psi(t)\rangle|^2\right)$$

where

$$|\Psi(t+dt)\rangle = |\Psi(t)\rangle + |\dot{\Psi}(t)\rangle dt + \frac{1}{2}|\ddot{\Psi}(t)\rangle dt^2 + \ldots$$

Quantum dynamics: distance...

$$dS = \frac{2}{\hbar} \Delta H(t) dt$$

where $\Delta H(t)^2 = \langle \Psi(t) | H(t)^2 | \Psi(t) \rangle - \langle \Psi(t) | H(t) | \Psi(t) \rangle^2$

J. Anandan and Y. Aharonov (1990)

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Energy fluctuation of the state drives the quantum evolution !!
Distance traverse is directly proportional to the energy uncertainty present in the system.

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time-dep Hamiltonian

$$S = \frac{2}{\hbar} \int_0^\tau \Delta H(t) dt$$

time-indep Hamiltonian

$$S = \frac{2}{\hbar} \tau \, \Delta H$$

J. Anandan and Y. Aharonov (1990)

Quantum dynamics: speed...

$$dS = \frac{2}{\hbar} \Delta H(t) dt$$

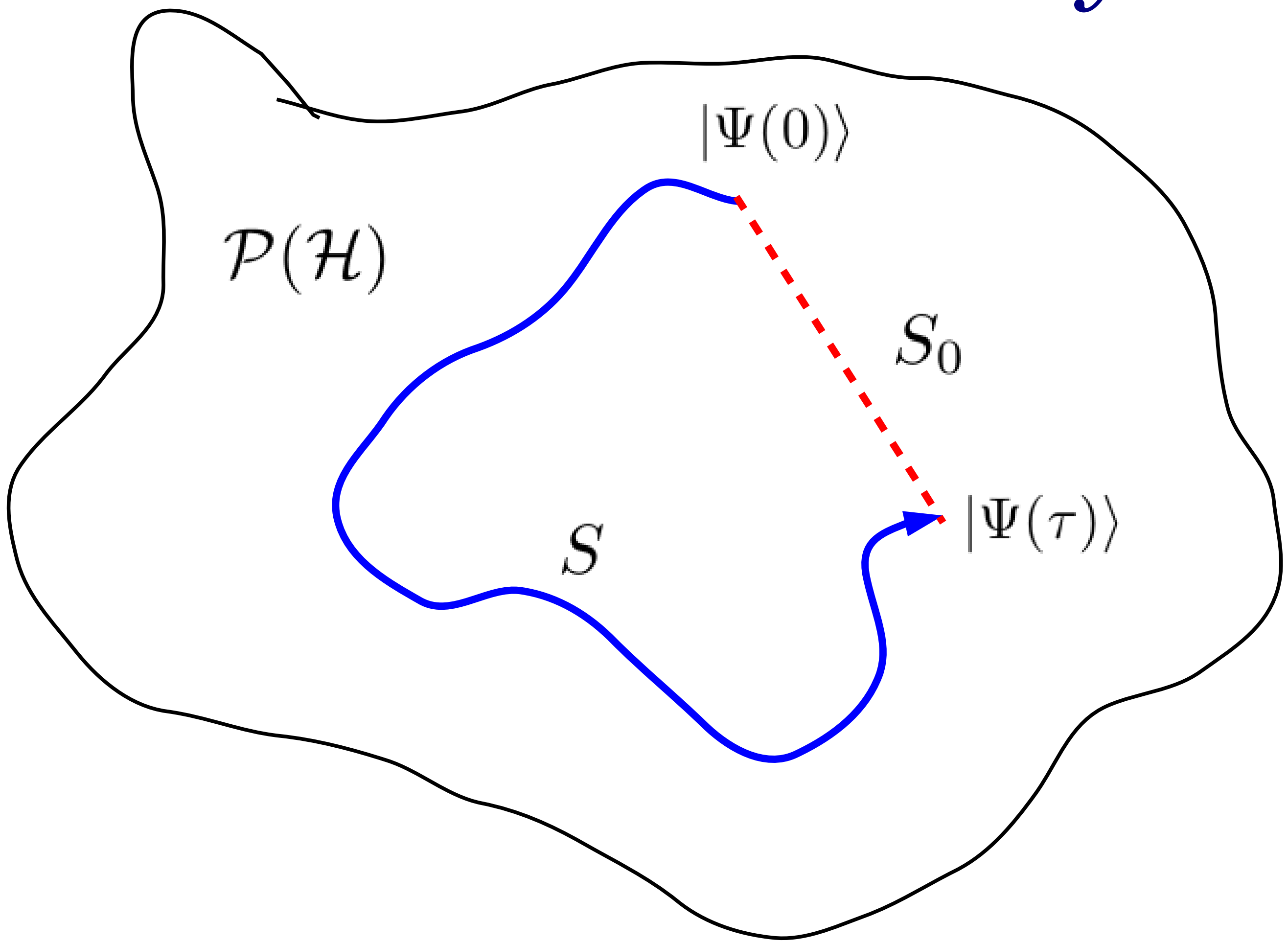
$$\Downarrow$$

$$\frac{dS}{dt} = \frac{2\Delta H(t)}{\hbar}$$

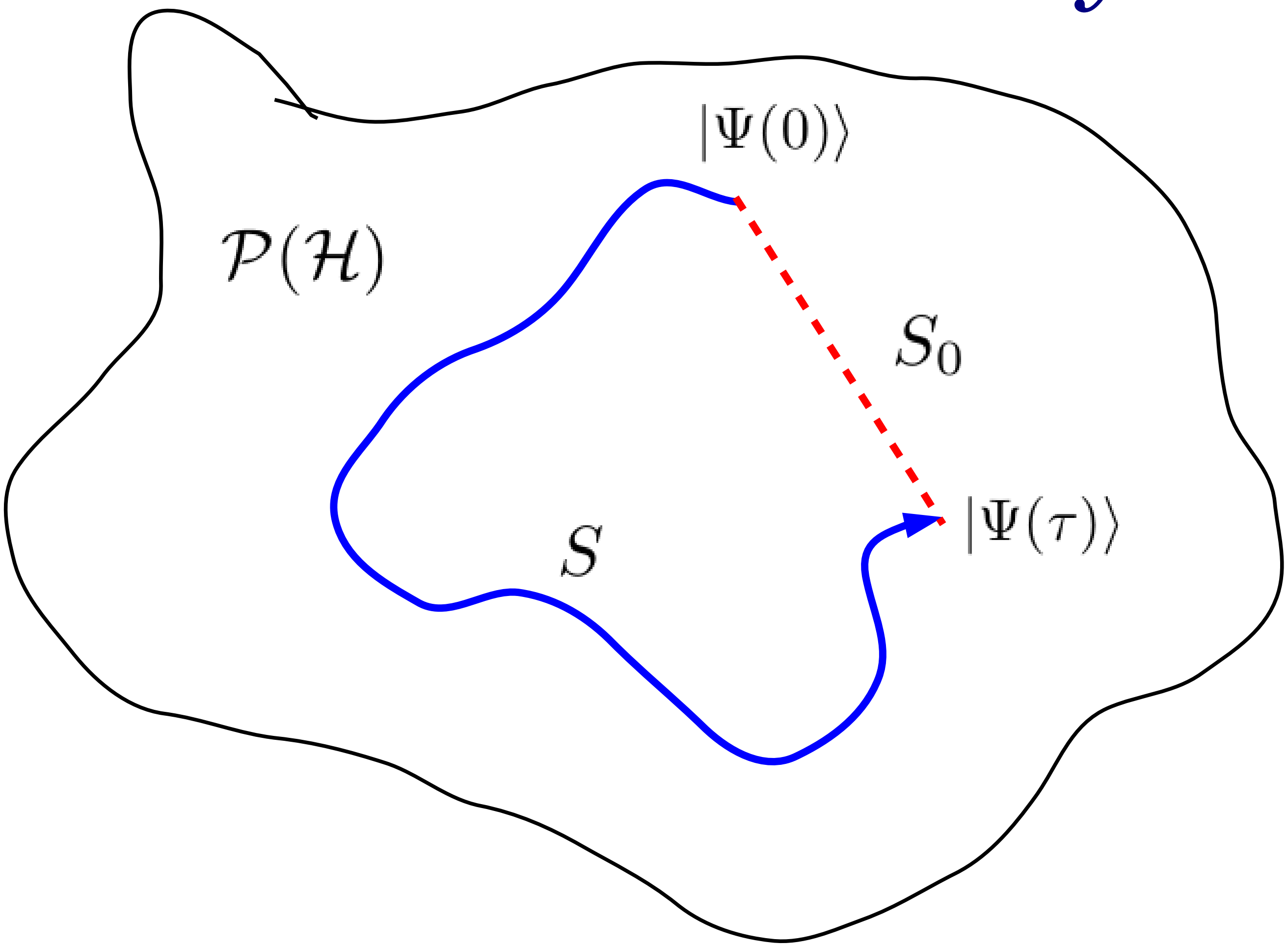
Infinitely many Hamiltonians can be used to transport the same initial state to the same final state.

J. Anandan and Y. Aharonov (1990)

Geometry ...

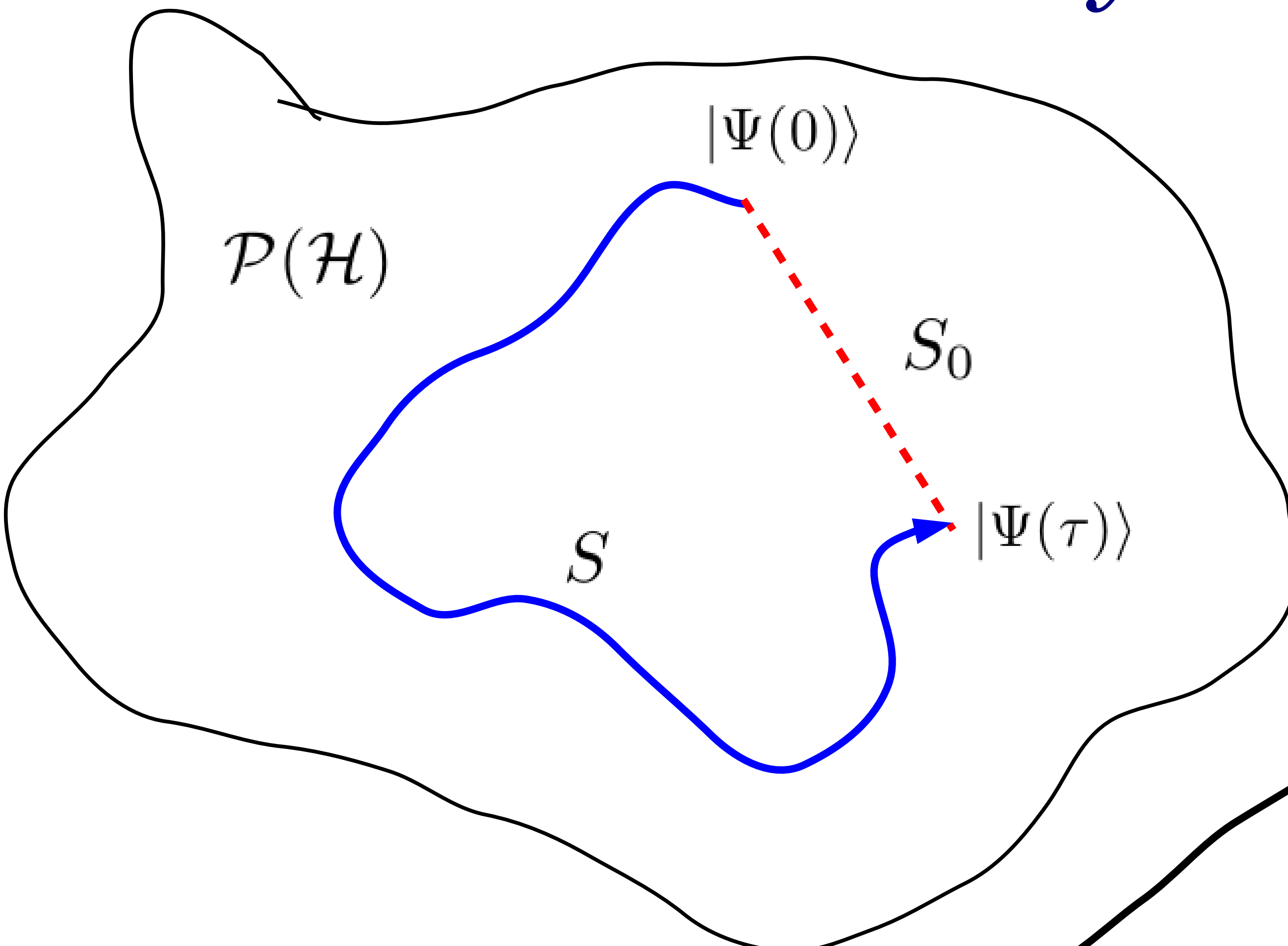


Geometry ...



$$S \geqslant S_0$$

Geometry ...

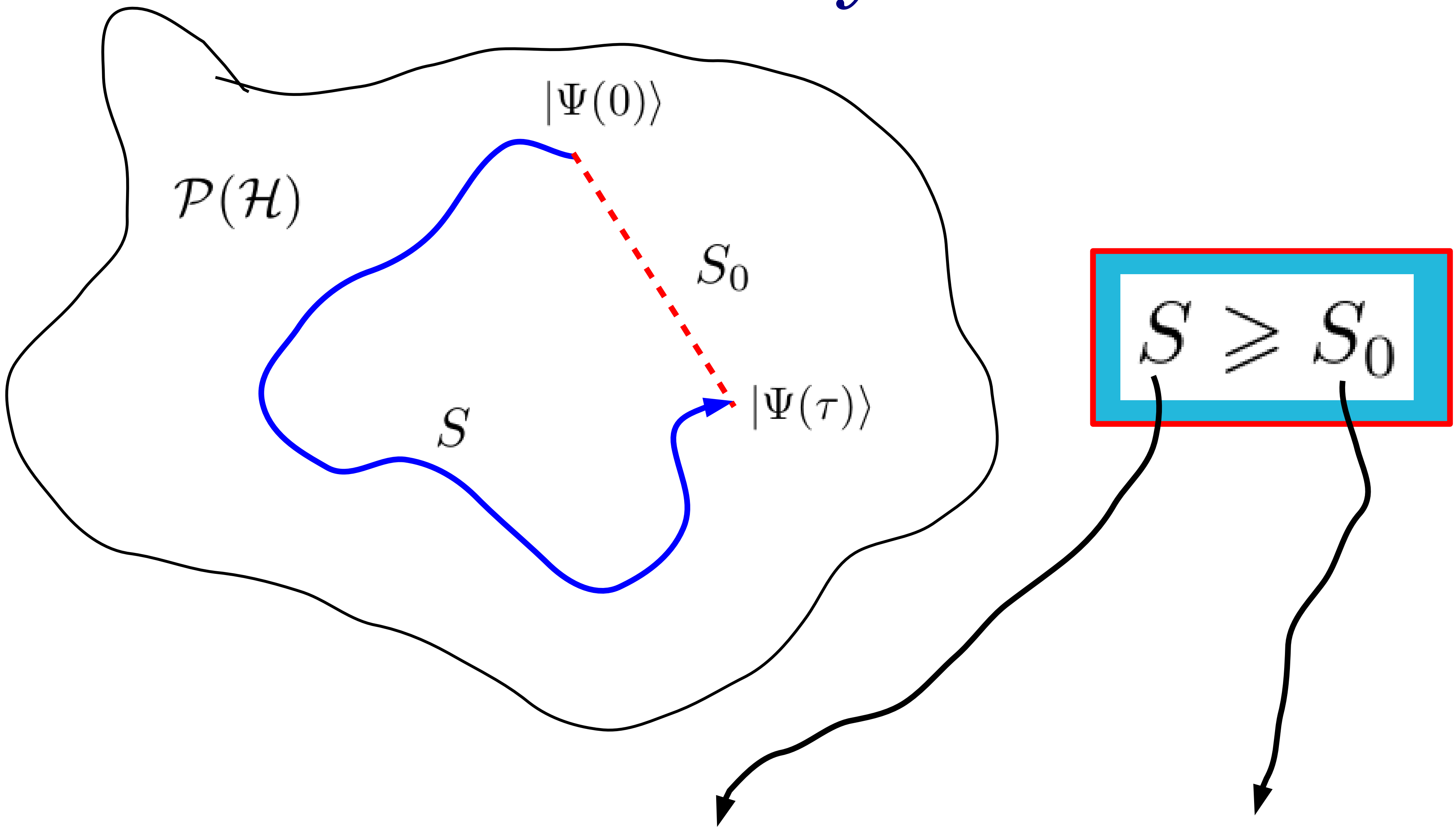


$$S \geqslant S_0$$

Again...

$$S = \frac{2}{\hbar}\int_0^{\tau} \Delta H(t)dt \quad \textit{and} \quad S_0 = 2\,\cos\left(\langle\psi(0)|\psi(t)\rangle\right)$$

Geometry ...



$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle \Psi(0)|\Psi(\tau)\rangle|)$$

Geometric Quantum Uncertainty Relation

$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

- *For a time-independent H*

$$\tau \Delta H \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

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- *For a time-independent H*

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- *When the initial and the final states are orthogonal to each other, the above raltion is the celebrated A-A time-energy uncertainty relation!!*

$$\tau\Delta H \geqslant \frac{h}{4}$$

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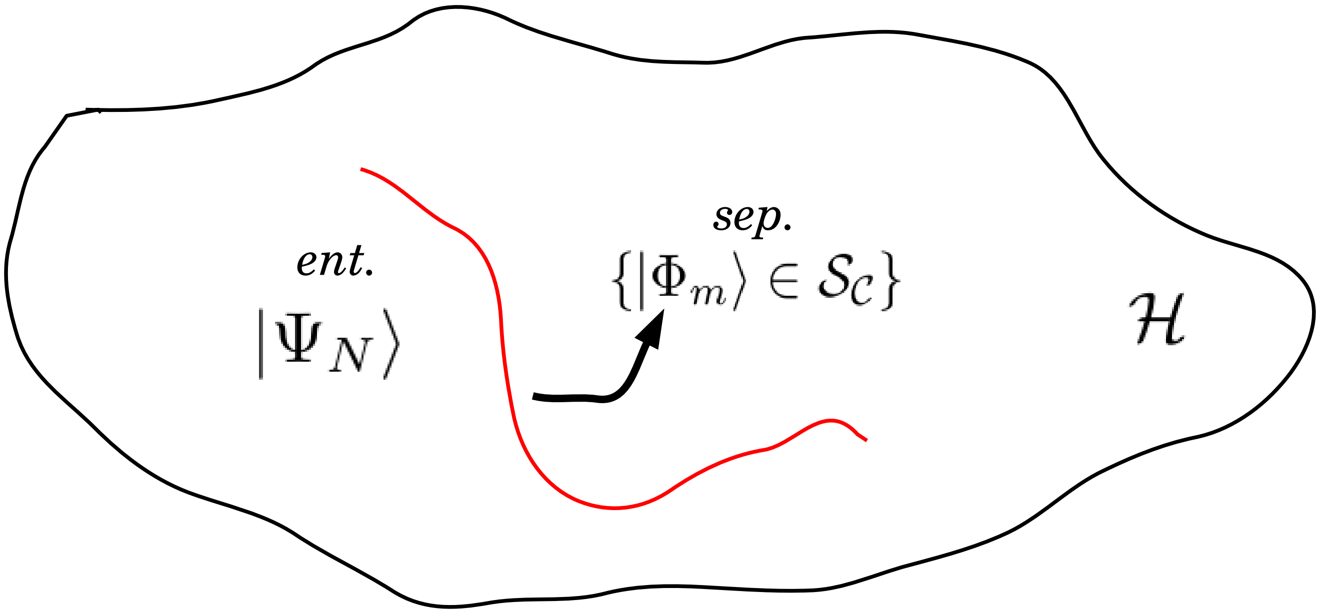
Multipartite entanglement

Multiparty entanglement: geometric



- *The Geometric measure of entanglement (GME)*

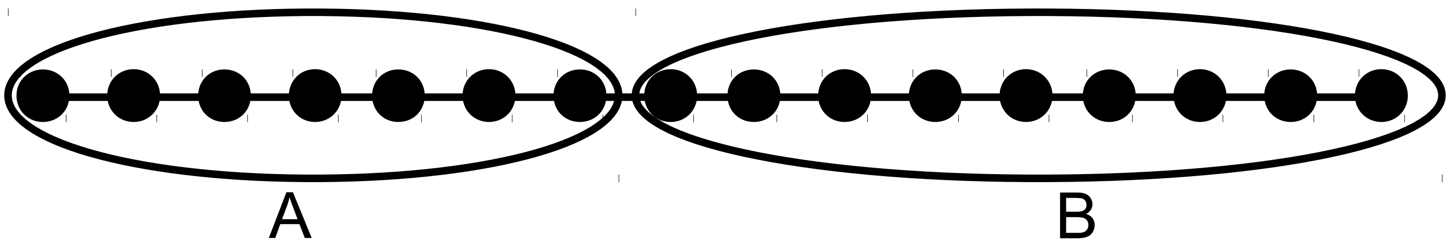
$$\mathcal{E}_\mathcal{C}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in S_\mathcal{C}\}} \left(1 - |\langle \Phi_m | \Psi_N \rangle|^2\right)$$

The distance is the measure of entanglement where the minimization is carried out over all pure states that are m-separable states.

Wei, Goldbart (2003)

Genuine multiparty entanglement (GGM)

- *A multiparty pure quantum state is said to be genuinely multiparty entangled if it is entangled across every bipartition of its constituent parties.*



A. Sen(De) and U. Sen (2010)

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- *GGM: generalized geometric measure*

$$\mathcal{E}_{\mathcal{G}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in \mathcal{S}_{\mathcal{B}}\}} \left(1 - |\langle\Phi_m|\Psi_N\rangle|^2\right)$$

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- *For pure states, it can be calculated, analytically, for arbitrary number of parties and dimensions.*

A. Sen(De) and U. Sen (2010)

Geometric entanglement & GQUR

From GQUR ⇨

$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

Geometric entanglement & GQUR

From GQUR ⇒

$$\int_0^{\tau} \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle \Psi(0)|\Psi(\tau)\rangle|)$$

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⇐ *From GME*

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⇦ *From GME*

- *By definition* $|\langle\Psi(0)|\Psi(\tau)\rangle|^2 \leqslant 1 - \mathcal{E}_{\mathcal{C}}\left(|\Psi(\tau)\rangle\right)$

Geometric entanglement & GQUR

From GQUR ⇨

$$\int_0^{\tau} \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

$$\mathcal{E}_{\mathcal{C}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in S_{\mathcal{C}}\}} \left(1 - |\langle\Phi_m|\Psi_N\rangle|^2\right)$$

⇦ *From GME*

- *By definition* $|\langle\Psi(0)|\Psi(\tau)\rangle|^2 \leqslant 1 - \mathcal{E}_{\mathcal{C}}\left(|\Psi(\tau)\rangle\right)$

- *We may write* $\cos^{-1}\left(|\langle\Psi(0)|\Psi(\tau)\rangle|\right) \geqslant \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right)$

where another entanglement measure

$$\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right) = \cos^{-1}\sqrt{1 - \mathcal{E}_{\mathcal{C}}}$$

Geometric entanglement & GQUR

From GQUR

$$\int_0^{\tau} \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

$$\mathcal{E}_{\mathcal{C}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in S_{\mathcal{C}}\}} \left(1 - |\langle\Phi_m|\Psi_N\rangle|^2\right)$$

From GME

- *Now we have...* $$\int_0^{\tau} \Delta H dt \geqslant \hbar\, \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right)$$

Geometric entanglement & GQUR

- *In terms of the time averaged energy fluctuation*

$$\overline{\Delta H}\tau \geqslant \hbar\, \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right)$$

where $\overline{\Delta H} = \dfrac{1}{\tau}\displaystyle\int_0^{\tau} \Delta H(t)dt$

Geometric entanglement & GQUR

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$$\overline{\Delta H}\tau \geqslant \hbar\, \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right)$$

where $$\overline{\Delta H} = \frac{1}{\tau}\int_0^{\tau} \Delta H(t)dt$$

For an arbitrary quantum evolution, the time interval multiplied by the time-averaged energy fluctuation is bounded below by the geometric measure of multipartite entanglement, provided the initial state is unentangled.

- *Applications:*

  Grover quantum serach

  Ising Hamiltonian

Grover quantum search

L. K. Grover (1997)

Grover quantum search

$$|\psi_0\rangle = 1/\sqrt{n}\sum_{i=0}^{n-1}|i\rangle$$

Grover operator

$$G = -I_0 H^{\otimes n} I_m H^{\otimes n}$$

$$I_0 = \mathbb{I} - 2|\psi_0\rangle\langle\psi_0|, \; I_m = \mathbb{I} - 2|m\rangle\langle m|$$

where $|m\rangle$ being the target state
H is the Hadamard transformation

L. K. Grover (1997)

Role of entanglement

After k iterations of the Grover operator the combined n-qubit state evolves to the state

$$|\psi_k\rangle = \frac{\cos\theta_k}{\sqrt{n-1}} \sum_{i \neq m} |i\rangle + \sin\theta_k |m\rangle$$

where $\theta_k = (2k+1)\theta_0$ and $\theta_0 = \sin^{-1}(1/\sqrt{n})$

S. L. Braunstein and A. K. Pati (2001).

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The state can be decomposed in the Schmidt basis

$$|\psi_k\rangle = \sqrt{\lambda_1(k)}|g'\rangle|e\rangle - \sqrt{\lambda_2(k)}|e'\rangle|g\rangle$$

where $\{|g\rangle, |e\rangle\}$ describes an orthonormal basis for i-th qubit and $\{|g'\rangle, |e'\rangle\}$ describes an orthonormal basis for other (n-1) qubits.

S. L. Braunstein and A. K. Pati (2001).

Role of entanglement

Where $\lambda_1(k)\lambda_2(k) = \frac{n(n-2)}{2(n-1)^2}\sin^2(\theta_k - \theta_0)\cos^2(\theta_k)$

The Grover search is complete when $\theta_k \to \pi/2$

The bound:

$$\delta = \frac{1}{\hbar}\int_{\theta_0}^{\theta_k} \Delta H_Q(\theta_k)d\theta_k - \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}(|\psi_{\theta_k}\rangle)\right)$$

Grover quantum serach...

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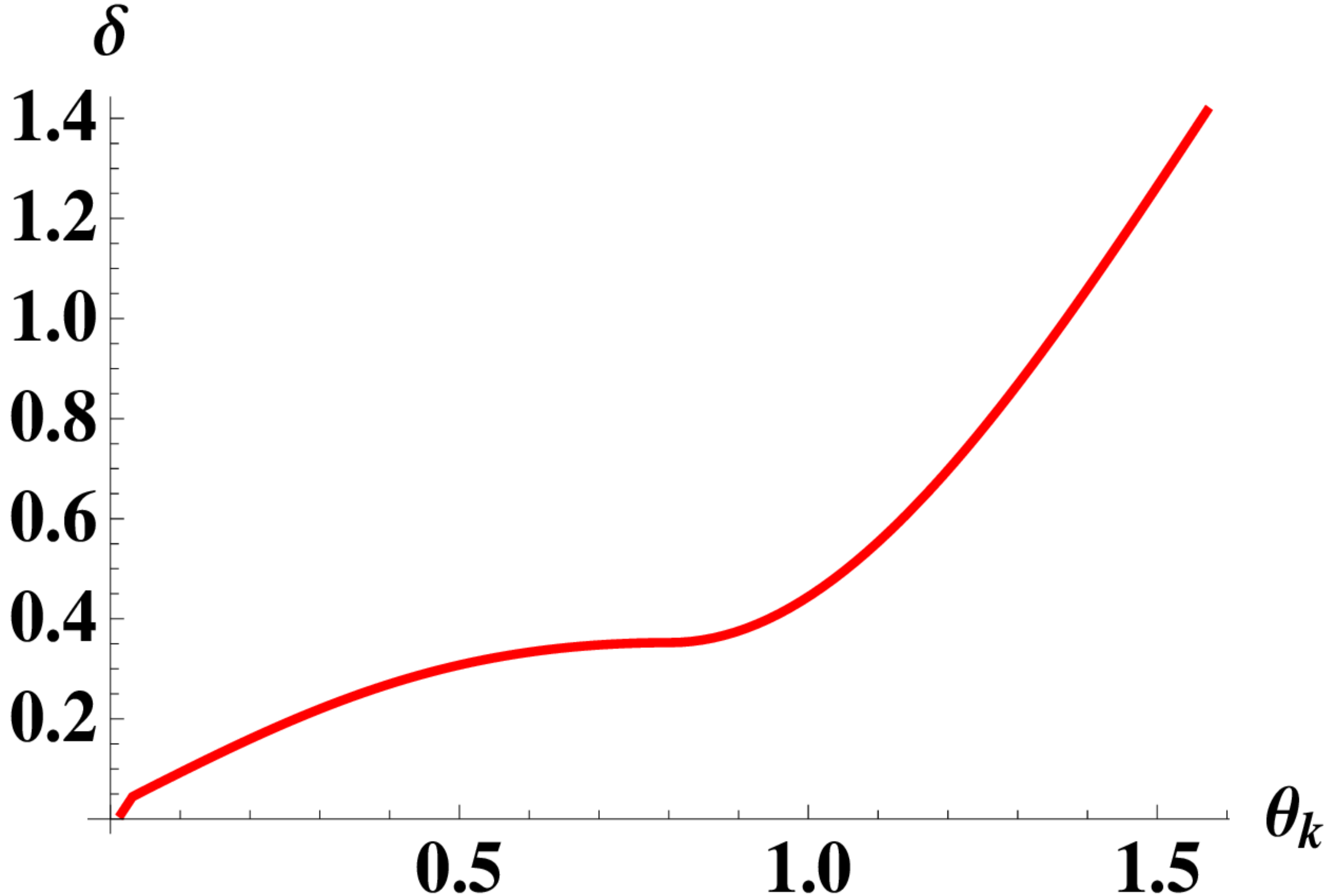


Evolution with Ising Hamiltonian

$$|\Psi(0)\rangle = |\varphi\rangle^{\otimes 3}$$

where

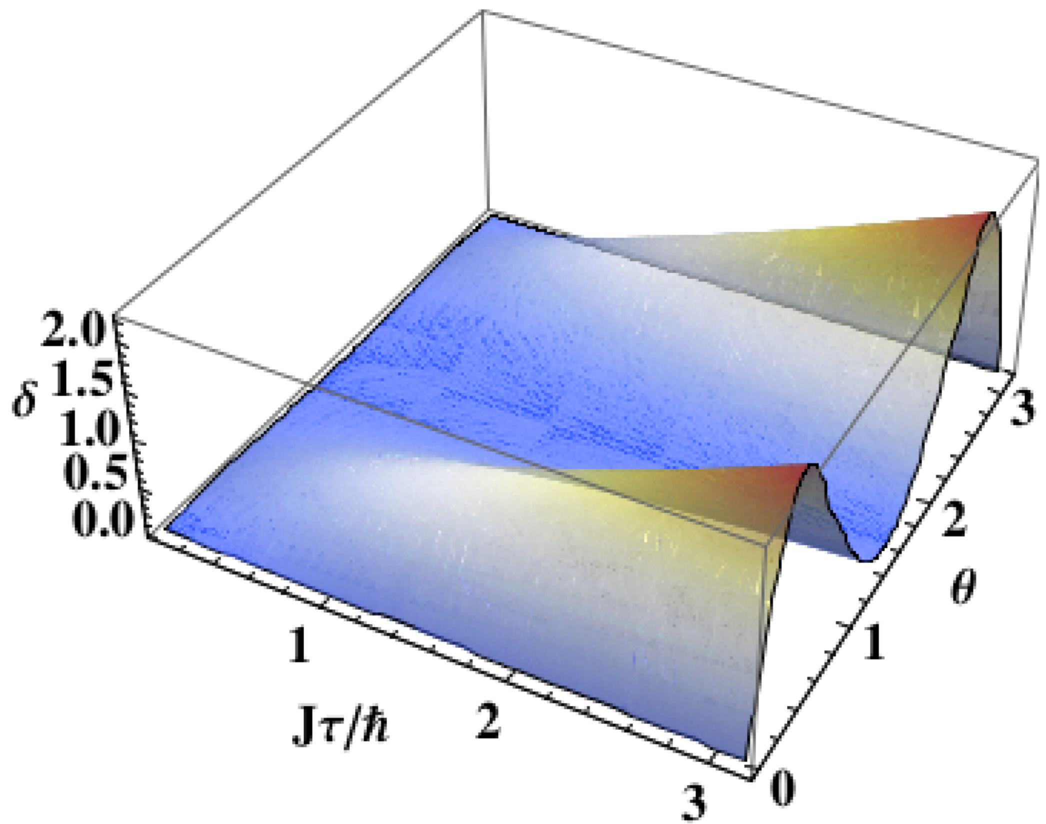
$$|\varphi\rangle = \cos\theta|0\rangle + \exp(-i\phi)\sin\theta|1\rangle,\ \theta \in [0,\pi],\ \phi \in [0,2\pi)$$

Driving with Ising Haniltonian

$$H_I = \frac{J}{4}\left(\sum_{i=1}^{N-1}(I-\sigma_i^z)(I-\sigma_{i+1}^z)\right)$$

Evolution with Ising Hamiltonian

$$\delta = \frac{\tau \Delta H}{\hbar} - \mathcal{G}(\mathcal{E}_G)$$



Mixed states

For mixed states

- *Hilbert-Schmidt distance*

$$dS_{HS}^2 = \mathrm{Tr}\left[\rho(t+dt) - \rho(t)\right]^2$$
$$= \mathrm{Tr}\left(\dot{\rho}\right)^2 dt^2$$

- *Speed of evolution*

$$\frac{dS_{HS}^2}{dt^2} = \frac{4}{\hbar^2}\mathrm{Tr}\left[(\rho^2 H^2) - (\rho H)^2\right]$$

For mixed states

- Hilbert-Schmidt distance

$$dS_{HS}^2 = \mathrm{Tr}\left[\rho(t+dt) - \rho(t)\right]^2$$
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- Speed of evolution

$$\frac{dS_{HS}^2}{dt^2} = \frac{4}{\hbar^2}\mathrm{Tr}\left[(\rho^2 H^2) - (\rho H)^2\right]$$

- Fubini-Study metric

$$dS_{FS}^2 = 4\left(1 - \frac{\mathrm{Tr}\left[\rho(t+dt)\rho(t)\right]}{\mathrm{Tr}\left[\rho(t)^2\right]}\right)$$
$$= 2\, dS_{HS}^2$$

$$\Longrightarrow \quad \frac{dS_{FS}}{dt} = \frac{2}{\hbar}\Delta H_Q(t)$$

Anandan (1991)

Uncertainty relation: mixed state

Bargmann angle: for unitarily connected quantum states

$$\frac{\mathrm{Tr}\left[\rho_1 \rho_2\right]}{\mathrm{Tr}\left[\rho_1^2\right]} = \cos^2\left(\frac{S_0}{2}\right)$$

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GQUR:

$$\frac{1}{\hbar}\int_0^{\tau} \Delta H_Q(t)dt \geqslant S_0 = 2\cos^{-1}\sqrt{\frac{\mathrm{Tr}\left[\rho(0)\rho(\tau)\right]}{\mathrm{Tr}\left[\rho(0)^2\right]}}$$

For a time-independent Hamiltonian

$$\tau\ \Delta H_Q \geqslant \hbar\ S_0$$

Anandan (1991)

GQUR and entanglement

- *Geometric entanglemt measure*

$$\mathcal{E}_{\mathcal{C}}^{FS}(\rho_{A_1 \dots A_N}) = \min_{\rho^S_{A_1 \dots A_N} \in \mathcal{S}_C} \left(1 - \frac{\mathrm{Tr}\left[\rho_{A_1 \dots A_N} \rho^S_{A_1 \dots A_N}\right]}{\mathrm{Tr}\left[\rho^2_{A_1 \dots A_N}\right]} \right)$$

- *Similarly as in the case of pure states*

$$\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right) = \cos^{-1}\sqrt{1 - \mathcal{E}_{\mathcal{C}}^{FS}}$$

GQUR and entanglement

- *Geometric entanglemt measure*

$$\mathcal{E}_{\mathcal{C}}^{FS}(\rho_{A_1\ldots A_N}) = \min_{\rho^S_{A_1\ldots A_N}\in\mathcal{S}_C}\left(1-\frac{\mathrm{Tr}\left[\rho_{A_1\ldots A_N}\rho^S_{A_1\ldots A_N}\right]}{\mathrm{Tr}\left[\rho^2_{A_1\ldots A_N}\right]}\right)$$

- *Similarly as in the case of pure states*

$$\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right) = \cos^{-1}\sqrt{1-\mathcal{E}_{\mathcal{C}}^{FS}}$$

$$\Longrightarrow \quad \int_0^\tau \Delta H_Q(t)dt \geqslant \hbar\,\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right)$$

Conclusion

- *Quantum entanglement plays an important role in setting the limits for the quantum uncertainties.*

- *The geometric time-energy uncertainty relation is shown to be bounded below by the multipartite entanglement.*

- *Examples considered: Grover quantum search, cluster state preparation.*

Thank you