MINIMAL STATE-DEPENDENT PROOF OF MEASUREMENT CONTEXTUALITY FOR A QUBIT

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[Joint work with Ravi Kunjwal [arXiv: 1305.7009]]

- Two typical non-classical features of Quantum Theory:

(1) It does not admit Bell-local hidden variable model

(2) It does not admit Kochen-Specker noncontextual hidden variable model (KS-NCHV)

- (1) is manifest through Bell-nonlocality of QT

  (2) is manifest through KS-contextuality of QT

- Both features arise due to the lack of a global joint probability distribution over measurement outcomes that can reproduce the marginals predicted by QT

- Traditionally, KS-contextuality is shown w.r.to KS-NCHV models involving projective measurements for system Hilbert space of dim. ≥ 3

- Generalized noncontextuality — a generalization of KS-noncontextuality — allows for outcome-inderministic response fns for unsharp measurements, while KS-NCHV models insist on outcome-deterministic response fns

- In the simplest scenario of generalized noncontextuality — considered by Specker — three dichotomic measurements M_1, M_2, M_3 are required to provide three non-trivial contexts $\{M_1, M_2\}$, $\{M_2, M_3\}$, $\{M_1, M_3\}$.

- Specker's consideration of generalized noncontextuality gives rise to — when M_1, M_2, M_3 are unsharp spin-1/2 measurements (with unsharpness parameter η) — the following inequality:

$$R_3 \equiv \frac{1}{3} \sum_{(i\ j) \in \{(1\ 2), (2\ 3), (1\ 3)\}} \text{Prob}(x_i \neq x_j \mid M_{ij}) \leq 1 - \frac{\eta}{3} \qquad (1)$$

with $x_i, x_j \in \{+1, -1\}$ being the outcomes of M_i, M_j respectively, while M_{ij} denotes the joint measurement POVM for M_i, M_j.

- Is there a quantum mechanical violation of the inequality (1)?

- We show:
  (i) No state-independent violation of (1)
  (ii) State-dependent violation of (1) is possible

OUTLINE

- KS – noncontextuality for projective measurements
- Generalized – noncontextuality
- Liang – Spekkens – Wiseman (LSW) generalized noncontextual inequality and their conjecture about non-violation of the inequality by any triple of unsharp spin-1/2 measurements
- No state-independent violation of LSW inequality
- State-dependent violation of LSW inequality with trine spin axes
- Conclusion

KS-noncontextuality for projective measurements

- Outcomes of a projective measurement A is independent of whether A is measured together with B or C where $[A, B] = 0 = [A, C]$ but $[B, C] \neq 0$.

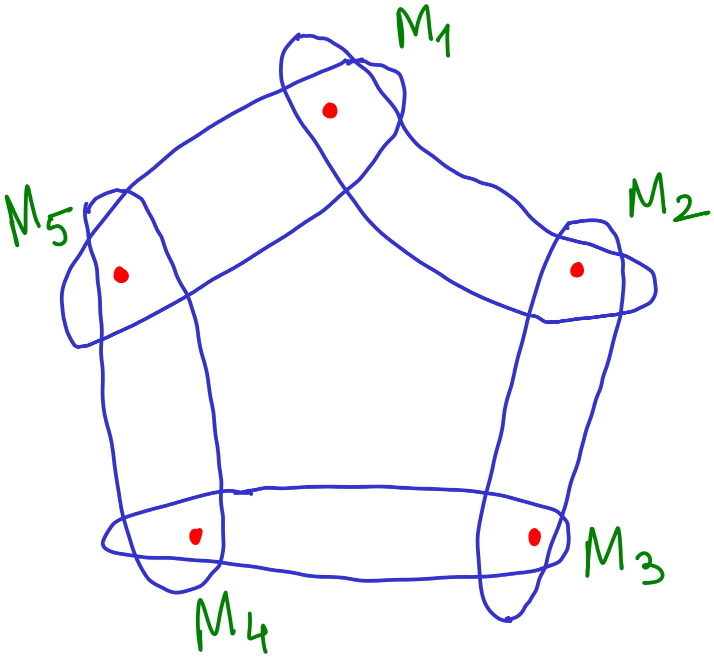
$\Rightarrow$ A KS-noncontextual model for a qubit is always possible

- State-independent proof of KS-contextuality $\Leftrightarrow$ violation of KS-noncontextuality for any state preparation

- State-dependent proof of KS-contextuality $\Leftrightarrow$ violation of KS-noncontextuality for some choice of state preparation

- Minimal state-independent proof of KS-contextuality in QM : qutrit & 13 projectors
[Yu & Oh, PRL (2012); Cabello, arxiv (2011)]

- Minimal state-dependent proof of KS-contextuality in QM : qutrit & 5 projectors
[Klyachko et al. (KCBS), PRL (2008); Kurzynski et al., PRL (2012)]



KCBS Contextuality scenario

- KS-Contextuality for a qubit with YES/NO value assignments of POVMs [Cabello, PRL (2003)]

- Our approach is different

- Based on Liang-Spekkens-Wiseman (LSW) [Phys. Rep. (2011)] by considering a generalized-noncontextuality proposed by spekkens [PRA (2005)]

Generalized – noncontextuality

- KS-noncontextuality entails measurement noncontextuality and outcome-determinism for sharp measurements

- Measurement noncontextuality : The response fn $prob(X_i \mid M_i, \lambda)$ is insensitive to the contexts $\{M_i, M_j, M_k, \ldots\}, \{M_i, M_j', M_k', \ldots\}, \ldots$ where X_i : outcome of M_i, λ : hidden variable associated with the system's preparation

- Outcome-determinism : $prob(X_i \mid M_i, \lambda) \in \{0, 1\}$

- KS-noncontextual inequality : Constraint on measurement statistics under measurement noncontextuality & outcome-determinism assumptions

- Generalized-noncontextual model derives outcome-determinism for sharp measurements as a consequence of preparation noncontextuality

- Generalized-noncontextuality does not imply outcome-determinism for unsharp measurements : modelled by outcome-indeterministic response fn. $prob(x_i \,|\, M_i, \lambda)$

$\Longrightarrow$ A KS-NCHV model is necessarily generalized-noncontextual but the converse is not true

- Generalized-noncontextuality of Specker (1960):

$\rightarrow$ Three $\{+1,-1\}$-valued measurements M_1, M_2, M_3 to allow for three non-trivial contexts $\{M_1, M_2\}$, $\{M_1, M_3\}$, $\{M_2, M_3\}$

$\rightarrow$ Only two anti-correlated contexts possible with single value assignment

$$\Longrightarrow \quad R_3 \equiv \frac{1}{3} \sum_{(i\,j) \in \{(1\,2),\,(1\,3),\,(2\,3)\}} \underbrace{\operatorname{Prob}(X_i \neq X_j \mid M_{ij})}_{\leq \frac{2}{3}}$$

$X_i, X_j \in \{+1, -1\}$: measurement outcomes of M_i, M_j in a joint implementation M_{ij}

$\Longrightarrow$ KS-inequality: $R_3 \leq R_3^{KS} \equiv \frac{2}{3}$

- Specker's scenario precludes projective measurements: three pairwise commuting projective measurements is jointly measurable — can not show contextuality

- Can one have Specker's contextuality scenario in QM?

- LSW showed that Specker's contextuality scenario can be realized using noisy spin-1/2 observables

LSW generalized-noncontextual inequality

- Three unsharp qubit observables:

$$M_k = \left\{ E_+^{(k)} = \frac{1}{2}\left(\mathbb{1} + \eta\, \vec{\sigma}\cdot\hat{n}_k\right),\ E_-^{(k)} = \frac{1}{2}\left(\mathbb{1} - \eta\, \vec{\sigma}\cdot\hat{n}_k\right) \right\}$$

$$(k = 1, 2, 3)$$

- Joint measurement POVM for the context

$$\{M_i, M_j\} : \quad M_{ij} = \{ \underbrace{M^{ij}_{++}, M^{ij}_{+-}, M^{ij}_{-+}, M^{ij}_{--}}_{\text{effects}} \}$$

with
$$\sum_{x_j \in \{+,-\}} M^{ij}_{x_i x_j} = E^{(i)}_{x_i} \quad \text{for } i \in \{1,2,3\}$$

- $$R_3 \equiv \frac{1}{3} \sum_{(ij)\in\{(12),(23),(13)\}} \text{prob}\,(x_i \neq x_j \mid M_{ij}) :$$

average probability of anticorrelation when any one of the three contexts $\{M_1, M_2\}, \{M_1, M_3\}, \{M_2, M_3\}$ is chosen uniformly at random

- LSW inequality: Under generalized-noncontextuality,

$$R_3 \leq R_3^{LSW} \equiv 1 - \frac{\eta}{3}$$

- Question: Does there exist $\{M_k = \{E_+^{(k)} = \frac{1}{2}(\mathbb{1} + \eta\,\vec{\sigma}\cdot\hat{n}_k), E_-^{(k)} = \frac{1}{2}(\mathbb{1} - \eta\,\vec{\sigma}\cdot\hat{n}_k)\} : k = 1, 2, 3\}$ for which $R_3^Q > 1 - \frac{\eta}{3}$ for some state preparation?

- LSW tried with (1) Orthogonal spin axes: $\hat{n}_i \cdot \hat{n}_j = 0 \;\; \forall\, i, j$ (2) trine spin axes: $\hat{n}_i \cdot \hat{n}_j = -\frac{1}{2} \;\; \forall\, i, j$

  $\rightarrow$ Did not seem to show violation of $R_3 \leq 1 - \frac{\eta}{3}$

- Conjecture of LSW: violation of $R_3 \leq 1 - \frac{\eta}{3}$ will not happen in QM

- Our results:

  (1) No triple $\{M_k = \{E_+^{(k)}, E_-^{(k)}\} : k = 1, 2, 3\}$ admits a state-independent violation of $R_3 \leq 1 - \frac{\eta}{3}$

  (2) There is a triple $\{M_k = \{E_+^{(k)}, E_-^{(k)}\} : k = 1, 2, 3\}$ which admits state-dependent violation of $R_3 \leq 1 - \frac{\eta}{3}$

- LSW inequality $R_3 \leq 1-\frac{\eta}{3}$ (Note) is not a KS-inequality but captures more general noncontextuality than KS-noncontextuality

- LSW inequality follows from the assumption of noncontextual probability assignments to measurements:

$$prob(X_k \mid M_k, \lambda) = \eta\,[X_k(\lambda)] + (1-\eta)\left(\tfrac{1}{2}[+1] + \tfrac{1}{2}[-1]\right)$$

with $prob(x) = [X]$ denotes the point distribution $\delta_{x,X}$ and λ: hidden variable associated with system's preparation

- KS-inequality for this scenario follows from the assumption of noncontextuality of value assignment $\equiv$ noncontextuality of deterministic (i.e., $\{0,1\}$-valued) probability assignment:

$$prob(X_k \mid M_k, \lambda) = [X_k(\lambda)]$$

$\Rightarrow$ KS-model is subsumed by LSW model: violation of $R_3 \leq R_3^{LSW} = 1-\frac{\eta}{3}$ implies violation of $R_3 \leq R_3^{KS} = \frac{2}{3}$

Note

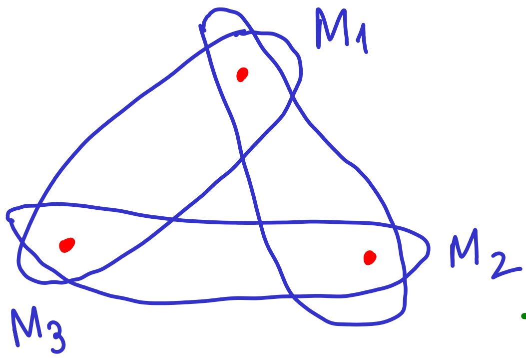
- violation of $R_3 \leq R_3^{KS} = \frac{2}{3}$: not enough to claim non-classicality

[Why: Generalized-noncontextual LSW model, which takes into account unsharp measurement, can simulate a violation of $R_3 \leq R_3^{KS} = \frac{2}{3}$ purely from noise]

- Violation of the LSW inequality $R_3 \leq R_3^{LSW} = 1 - \frac{\eta}{3}$: rules out such explanation involving noise

$\Longrightarrow$ genuine non-classicality

Joint measurability Constraints on η



→ Jointly measurable Contexts: $\{M_1, M_2\}$, $\{M_1, M_3\}$, $\{M_2, M_3\}$

→ But not triplewise jointly measurable

- Sufficient Condition [LSW] : $\eta_\ell < \eta \le \eta_u$

with

$$\eta_\ell = \frac{1}{3} \cdot \max\left\{ \sqrt{3 + 2 \times \sum_{\substack{k,\ell \in \{1,2,3\} \\ k<\ell}} x_k x_\ell \cos\theta_{k\ell}} \; : \; x_1, x_2, x_3 \in \{+1, -1\} \right\}$$

$$\eta_u = \min\left\{ \frac{1}{\sqrt{1 + |\sin\theta_{ij}|}} : (ij) \in \{(12), (23), (13)\} \right\}$$

and $\quad \hat{n}_i \cdot \hat{n}_j = \cos\theta_{ij} \quad \forall\, i, j \in \{1, 2, 3\}$

- The condition $\eta_\ell < \eta \le \eta_u$ is necessary-sufficient for orthogonal as well as trine spin axes

Joint measurement effects

- **Joint measurement POVM [Heinosaari et al. (2008)]:**

→ The joint measurement POVM $M_{ij} = \{ M_{++}^{ij}, M_{+-}^{ij}, M_{-+}^{ij}, M_{--}^{ij} \}$ for jointly measuring $M_i = \{ E_+^{(i)} = \frac{1}{2}(\mathbb{1} + \eta \vec{\sigma}\cdot\hat{n}_i),\ E_-^{(i)} = \frac{1}{2}(\mathbb{1} - \eta\vec{\sigma}\cdot\hat{n}_i) \}$ and M_j :

$$M_{++}^{ij} = \frac{1}{2}\left[\frac{\alpha_{ij}}{2}\mathbb{1} + \vec{\sigma}\cdot\frac{1}{2}\{ \eta(\hat{n}_i + \hat{n}_j) - \vec{a}_{ij} \} \right]$$

$$M_{+-}^{ij} = \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right)\mathbb{1} + \vec{\sigma}\cdot\frac{1}{2}\{ \eta(\hat{n}_i - \hat{n}_j) + \vec{a}_{ij} \} \right]$$

$$M_{-+}^{ij} = \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right)\mathbb{1} + \vec{\sigma}\cdot\frac{1}{2}\{ \eta(-\hat{n}_i + \hat{n}_j) + \vec{a}_{ij} \} \right]$$

$$M_{--}^{ij} = \frac{1}{2}\left[\frac{\alpha_{ij}}{2}\mathbb{1} + \vec{\sigma}\cdot\frac{1}{2}\{ \eta(-\hat{n}_i - \hat{n}_j) - \vec{a}_{ij} \} \right]$$

with $\alpha_{ij} \in \mathbb{R}$, $\vec{a}_{ij} \in \mathbb{R}^3$ and

$$\sqrt{2\eta^2(1+\cos\theta_{ij}) + |\vec{a}_{ij}|^2 + 2\eta\,|(\hat{n}_i + \hat{n}_j)\cdot\vec{a}_{ij}|} \leq \alpha_{ij} \leq$$

$$2 - \sqrt{2\eta^2(1-\cos\theta_{ij}) + |\vec{a}_{ij}|^2 + 2\eta\,|(\hat{n}_i - \hat{n}_j)\cdot\vec{a}_{ij}|} \quad \text{—— (2)}$$

together with $\eta_l < \eta \leq \eta_u$

- Joint measurement effects corresponding to anti-correlation sums to :

$$M_{+-}^{ij} + M_{-+}^{ij} = \left(1 - \frac{\alpha_{ij}}{2}\right)\mathbb{1} + \frac{1}{2}\vec{\sigma}\cdot\vec{a}_{ij}$$

No state-independent violation of LSW inequality

- For any qubit state ρ:

$$R_3^{QM} \equiv \frac{1}{3} \sum_{(ij)\in\{(12),(23),(13)\}} \mathrm{Tr}\left[\rho\left(M_{+-}^{ij} + M_{-+}^{ij}\right)\right]$$

- Condition for violation of LSW inequality:

$$R_3^{QM} > 1 - \frac{\eta}{3} \iff$$

$$\frac{1}{3} \sum_{(ij)} \mathrm{Tr}\left[\rho\left\{\left(1 - \frac{\alpha_{ij}}{2}\right)\mathbb{1} + \frac{1}{2}\vec{\sigma}\cdot\vec{a_{ij}}\right\}\right] > 1 - \frac{\eta}{3}$$

$$\iff \mathrm{Tr}\left[\rho \sum_{(ij)} \left(\alpha_{ij}\mathbb{1} - \vec{\sigma}\cdot\vec{a_{ij}}\right)\right] < 2\eta$$

$$\iff \sum_{(ij)} \alpha_{ij} + \lambda_\rho < 2\eta \qquad \text{—— (3)}$$

with $\lambda_\rho \equiv (1-2q)\,\vec{a}\cdot\hat{n} \in \left[-|\vec{a}|, |\vec{a}|\right]$,

$$\rho = q\cdot\frac{1}{2}\left(\mathbb{1} + \hat{n}\cdot\vec{\sigma}\right) + (1-q)\cdot\frac{1}{2}\left(\mathbb{1} - \hat{n}\cdot\vec{\sigma}\right),$$

and $\vec{a} = \left(a_x = \sum_{(ij)} (\vec{a_{ij}})_x,\ a_y = \sum_{(ij)} (\vec{a_{ij}})_y,\ a_z = \sum_{(ij)} (\vec{a_{ij}})_z\right)$

- For a state-independent violation:

either $\lambda_\rho = 0 \quad \forall \rho$ ——— (4)

or $\sum_{(ij)} \alpha_{ij} + \max_\rho \lambda_\rho < 2\eta$ ——— (5)

- Eqn (3) $\Longleftrightarrow$ $\vec{a} = 0$ ——— (6)

- Eqn (4) $\Longleftrightarrow$ $\sum_{(ij)} \alpha_{ij} + |\vec{a}| < 2\eta$ ——— (7)

- Using eqn (6) and (7) into eqn (2):

$$\sum_{(ij)} \alpha_{ij} > \sqrt{2}\,\eta \sum_{(ij)} \sqrt{1+\cos\theta_{ij}} \qquad \text{——— (8)}$$

- Using eqn (8) into condition (3), we get a <u>necessary</u> condition for state-independent violation of the LSW inequality:

$$\sum_{(ij)} \sqrt{1+\cos\theta_{ij}} < \sqrt{2}$$

$$\Longleftrightarrow \left|\cos\frac{\theta_{12}}{2}\right| + \left|\cos\frac{\theta_{13}}{2}\right| + \left|\cos\frac{\theta_{23}}{2}\right| < 1 \qquad \text{——— (9)}$$

- without loss of generality:

$$\hat{n}_1 = (0, 0, 1), \quad \hat{n}_2 = (\sin\theta_{12}, 0, \cos\theta_{12}),$$

$$\hat{n}_3 = (\sin\theta_{13}\cos\phi_3, \sin\theta_{13}\sin\phi_3, \cos\theta_{13})$$

with $0 < \frac{\theta_{ij}}{2} < \frac{\pi}{2} \quad \forall\ (ij) \in \{(12), (13), (23)\}$

and $0 \leq \phi_3 < 2\pi$ and

$$\cos\theta_{23} = \sin\theta_{12}\sin\theta_{13}\cos\phi_3 + \cos\theta_{12}\cos\theta_{13}$$

which implies:

$$\cos(\theta_{12} + \theta_{13}) \leq \cos\theta_{23} \leq \cos(\theta_{12} - \theta_{13}) \quad \text{—— (10)}$$

$$\Longrightarrow \min_{\theta_{12}, \theta_{13}, \theta_{23}} \left\{ \left|\cos\frac{\theta_{12}}{2}\right| + \left|\cos\frac{\theta_{13}}{2}\right| + \left|\cos\frac{\theta_{23}}{2}\right| \right\} \geq$$

$$\min_{\theta_{12}, \theta_{13}} \left\{ \left|\cos\frac{\theta_{12}}{2}\right| + \left|\cos\frac{\theta_{13}}{2}\right| + \sqrt{\frac{1+\cos(\theta_{12}+\theta_{13})}{2}} \right\} > 1$$

— contradicts eqn. (9) ! $\square$

State-dependent violation of LSW inequality

- From the condition of violation of the LSW inequality, i.e., the condition: $\sum_{(ij)} \alpha_{ij} + \lambda_\rho < 2\eta$,

we have: $\sum_{(ij)} \alpha_{ij} - |\vec{a}| < 2\eta$

for the choice: $\rho = \frac{1}{2}\left(\mathbb{1} + \frac{\vec{a}}{|\vec{a}|} \cdot \vec{\sigma}\right)$

- Does there exist a choice of $\hat{n}_1, \hat{n}_2, \hat{n}_3, \eta, \alpha_{12}, \alpha_{13}, \alpha_{23}, \vec{a}_{12}, \vec{a}_{13}, \vec{a}_{23}$ such that

$$\sum_{(ij)} \alpha_{ij} - |\vec{a}| < 2\eta \ ?$$

- $C \equiv 2\eta - \left\{ \sum_{(ij)} \alpha_{ij} - |\vec{a}| \right\}$

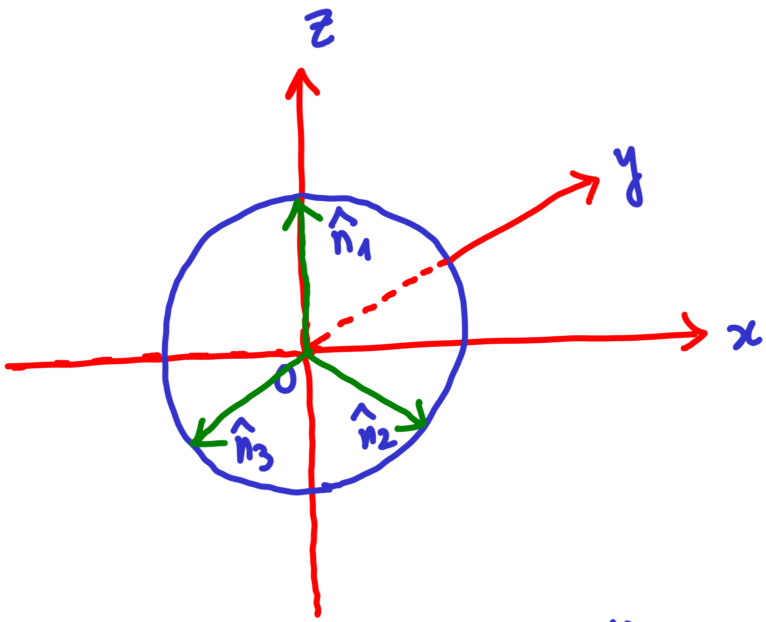
$C > 0$ implies state-dependent violation

- $S \equiv R_3^{QM} - \left(1 - \frac{\eta}{3}\right) = \frac{C}{6}$

So $S > 0$ implies state-dependent violation

- Given a coplanar choice of $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ and η satisfying $\eta_l < \eta \leq \eta_u$:

$$C_{max}(\{\hat{n}_1,\hat{n}_2,\hat{n}_3\},\eta) = 2\eta + \sum_{(ij)} \Big\{ \sqrt{1+\eta^4\cos^2\theta_{ij} - 2\eta^2} + (1+\eta^2\cos\theta_{ij}) \Big\} \quad \text{——} \quad (11)$$

- 

$\hat{n}_i \cdot \hat{n}_j = -\frac{1}{2}$ $\forall\, ij$ (Trine spin axes)

$\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$;

$\eta_l = \frac{2}{3}$, $\eta_u = \sqrt{3} - 1$;

$\alpha_{ij} = 1+\eta^2\cos\theta_{ij} = 1 - \frac{\eta^2}{2}$;

$\vec{a}_{ij} = (0, \sqrt{1+\eta^4\cos^2\theta_{ij} - 2\eta^2}, 0) = (0, \sqrt{1+\eta^4/4 - 2\eta^2}, 0)$;

Optimal violation corresponds to: $\eta = \eta_l$

$$\Rightarrow S^{trine}_{max} = \frac{C^{trine}_{max}}{6} = 0.03364$$

$\square$

Conclusion

- Quantum theory can show contextuality even in two dim. in Specker's scenario.
- Such an issue never arises for sharp measurements
- Thus we have an example of a task which unsharp measurements can only accomplish
- Specker's scenario is minimal : three measurements and they are dichotomic
- Our result rules out generalized-noncontextual models for QM (state-dependent manner)

  — a scenario not encompassed by KS - noncontextuality
- What information - theoretic task might such violation (of the LSW inequality) be useful for ?

THANK YOU !