Sensing and imaging at the quantum limit

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Humphreys/Barbieri/AD/Walmsley, PRL, **111**, 070403, (2013)
Crowley/AD/Barbieri/Walmsley, arxiv:1206.0043
Vidrighin/Donati/Genoni/Jin/Kolthammer/Kim/AD/Barbieri/Walmsley, arxiv: ****.****

QIPA 2013, HRI, Allahabad

December 7, 2013

Enhanced imaging in the real world

Figure: Hooke's microscope

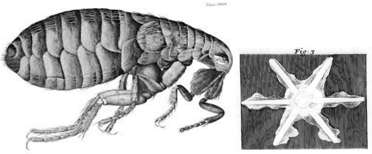
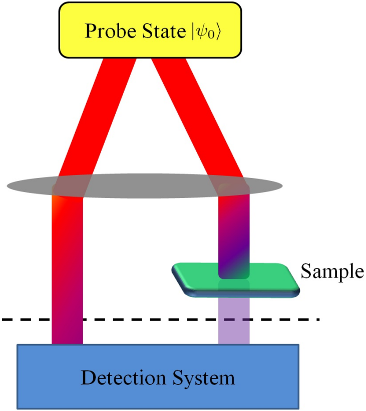


Figure: Hooke's *Micrographia*



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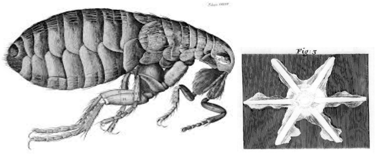

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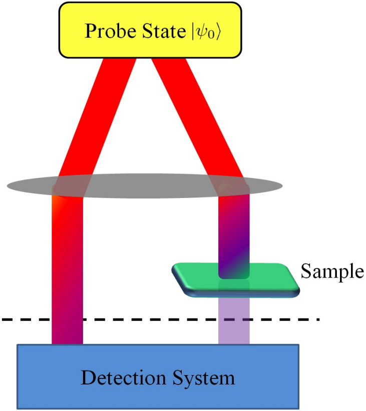

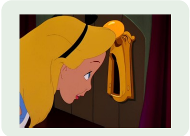

Quantum-enhanced sensing has been used in

- gravitational wave detection LIGO, GEO600
- phase tracking Yonezawa *et al.* Science, **337**, 1514, (2012)
- small displacements Taylor *et al.* Nat. Phot. **7**, 229, (2013)
- concentration measurements Crespi *et al.* APL, **100**, 233704, (2012)

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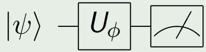
- microscopy and imaging
  - inherently multi-parameter problems
  - study it as such

We will study a discretised model for phase imaging

Enhancement using quantum probes

- A probe made of constituents, qubits - quantum bits - quantum two-level systems
- Using a simple probe $|\Psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and

$$U_\phi = e^{-i\phi Z}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



- $|\Psi\rangle = \frac{|0\rangle+e^{i\phi}|1\rangle}{\sqrt{2}}$
- $M = X$
- $\langle M\rangle \sim \sin^2(\phi/2)$

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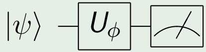

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- $|\Psi\rangle = \frac{|00\rangle + e^{i2\phi}|11\rangle}{\sqrt{2}}$
- $M = X \otimes X$
- $\langle M \rangle \sim \sin^2 \phi$

Enhancement using quantum probes

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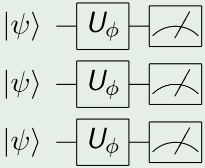
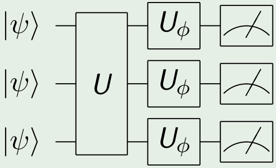
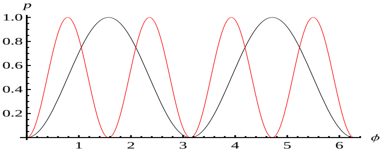
- a Hamiltonian $H_\phi = \phi Z$
- a measurement M



- $|\Psi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes N}$
- $\langle M \rangle \sim \sin^2(\phi/2)$



- $|\Psi\rangle = \frac{|00\cdots000\rangle + e^{iN\phi}|11\cdots111\rangle}{\sqrt{2}}$
- $\langle M \rangle \sim \sin^2(N\phi/2)$



High frequency fringes aren't enough

- The Cramér-Rao bound places a lower bound on the accuracy of estimation

$$\textit{Classical} : \delta^2\phi \geq \frac{1}{F_\phi} \qquad \textit{Quantum} : \delta^2\phi \geq \frac{1}{\mathcal{I}_\phi}$$

- Classical Fisher Information

$$F_\phi = \int \mathrm{d}\phi \; p(\phi) \left(\frac{\partial^2}{\partial \phi^2} \ln p(\phi) \right) = \int \mathrm{d}\phi \frac{1}{p(\phi)} \left(\frac{\partial p(\phi)}{\partial \phi} \right)^2$$

- Quantum Fisher Information

$$\mathcal{I}_\phi = 4(\langle \partial_\phi \psi | \partial_\phi \psi \rangle - |\langle \partial_\phi \psi | \psi \rangle|^2)$$

- FI measures curvature, which determine the precision
- These bounds depend on the probe and dynamics, and are always attainable

Braunstein/Caves, PRL, **72**, 3439, (1994)

States attaining the quantum scaling

Highly correlated states such as

GHZ (Greenberger-Horne-Zeilinger) states

$$\frac{|00\cdots000\rangle + |11\cdots111\rangle}{\sqrt{2}} \qquad \mathcal{I}_\phi = N^2$$

Greenberger *et al.*, arXiv:0712.0921

N00N states

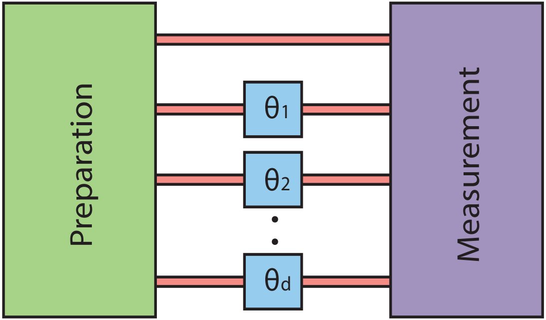
$$\frac{|N,0\rangle + |0,N\rangle}{\sqrt{2}} \qquad \mathcal{I}_\phi = N^2$$

Kok *et al.*, Phys. Rev. A **65**, 052104 (2002)

Classical scaling

$$\mathcal{I}_\phi \sim N$$

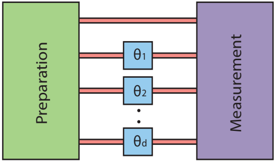
The problem



- N photons across $d+1$ modes
- An image is an estimation of all the d phases
- Minimise the total variance $|\Delta\boldsymbol{\theta}|^2 = \sum_{m=1}^{d} \delta^2\theta_m$

The problem

For a probe $|\psi\rangle = \sum_{k=1}^{D} \alpha_k |N_{k,0}, \cdots, N_{k,d}\rangle \equiv \sum_{k=1}^{D} \alpha_k |\mathbf{N}_k\rangle$, $D = (N+d)!/N!d!$



- $U_{\boldsymbol{\theta}} = \exp(i \sum_{m=1}^{d} \hat{N}_m \theta_m)$
- $|\psi_{\boldsymbol{\theta}}\rangle = U_{\boldsymbol{\theta}} |\psi\rangle$
- Use the Cramer-Rao bound

$$\mathrm{Cov}(\boldsymbol{\theta}) \geq (M\, \mathcal{I}_{\boldsymbol{\theta}})^{-1}$$

The multi-parameter bound can be saturated.

Matsumoto, J. Phys. A 35, 3111 (2002)

Quantum Fisher information

- $\mathcal{I}_{\boldsymbol{\theta}} = 4\sum_i |\alpha_i|^2 \mathbf{N}_i \mathbf{N}_i^T - 4\sum_{ij} |\alpha_i|^2 |\alpha_j|^2 \mathbf{N}_i \mathbf{N}_j^T$
- Exploit the symmetry :

$$|\psi_s\rangle = \alpha(|0, N, 0, 0\rangle + \cdots + |0, 0, N\rangle) + \beta|N, 0, \cdots, 0\rangle,$$

- $[\mathcal{I}_{\boldsymbol{\theta}}]_{l,m} = 4N^2(\delta_{l,m}\alpha^2 - \alpha^4)$
- Minimise : $|\Delta\boldsymbol{\theta}|^2 = \sum_{m=1}^{d} \delta^2\theta_m = \mathrm{Tr}[\mathcal{I}_{\boldsymbol{\theta}}^{-1}]$

  provides $\alpha = 1/\sqrt{d + \sqrt{d}}$,

$$\boxed{|\Delta\boldsymbol{\theta}_s|^2 = \frac{(1 + \sqrt{d})^2 d/4}{N^2}}$$

Comparison

- Simultaneous quantum estimation

$$|\Delta\boldsymbol{\theta}_s|^2 = \frac{(1+\sqrt{d})^2 d/4}{N^2}.$$

- Individual quantum estimation using $\frac{N}{d}00\frac{N}{d}$ states

$$|\Delta\boldsymbol{\theta}_{ind}|^2 = \frac{d^3}{N^2}$$

- Classical scheme using uncorrelated coherent states

$$|\Delta\boldsymbol{\theta}_{clas}|^2 = \frac{d^2}{N}$$

$|\Delta\boldsymbol{\theta}_s|^2 < |\Delta\boldsymbol{\theta}_{ind}|^2 < |\Delta\boldsymbol{\theta}_{clas}|^2$ for $d > 1$ and $d < N$

Comparison

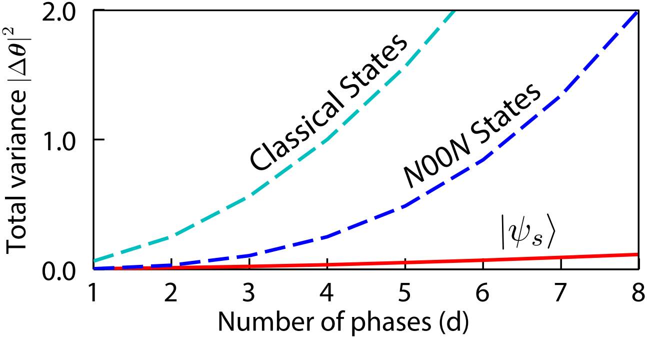


Figure: Strategies for multiple phase estimation using $N = 16$ photons.

With equal resources, multi-parameter estimation is better by $\mathcal{O}(d)$

Practicalities

Probe states

- Interference of n photons on each port of a 'beamsplitter'
- Call it $|\psi(n, d)\rangle$ with $N = n(d + 1)$
- Spagnolo *et al.* explored the QFI for $d = 2, 3$ and $n = 1$

Measurements

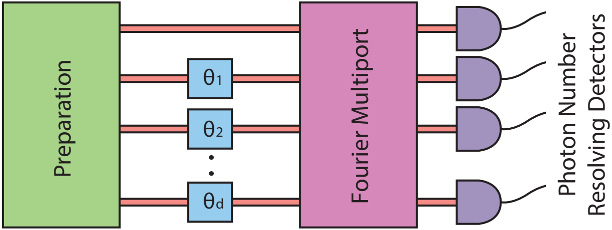


Figure: Schematic of a realistic multi-phase estimation protocol

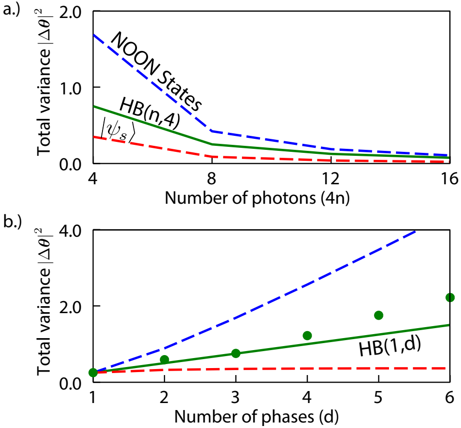


Figure: (a.) QCRB for simultaneous estimation of 4 phases using $|\psi(n,4)\rangle$, N00N and $|\psi_s\rangle$ states. (b.) Green dots : Simultaneous estimation of d phases using a $|\psi(1,d)\rangle$ and a realistic measurement apparatus. QCRB for the same $|\psi(1,d)\rangle$, equivalent N00N and $|\psi_s\rangle$ states.

Losses are inevitable !

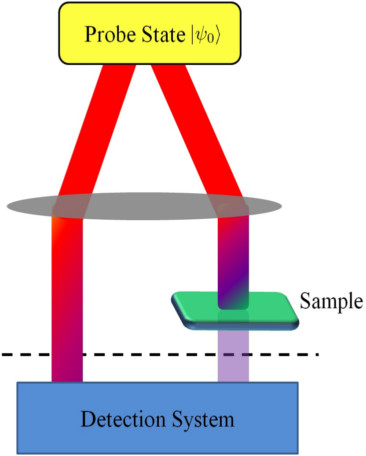


Figure: Sample induces simultaneous dispersion ϕ and absorption η

Crowley/AD/Barbieri/Walmsley, arxiv:1206.0043

Losses are inevitable !

To estimate multiple parameters $\{\boldsymbol{\theta}\}$ simultaneously,

$$\mathrm{Cov}(\boldsymbol{\theta}) \geq (\mathcal{I}_{\boldsymbol{\theta}})^{-1},$$

For estimating phase ϕ and loss η simultaneously,

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{\phi\phi} & 0 \\ 0 & \mathcal{I}_{\eta\eta} \end{pmatrix}.$$

Crowley/AD/Barbieri/Walmsley, arxiv:1206.0043

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But, attainability not guaranteed ☹

- In fact, the only option leads to a trade-off : $I_{\eta\eta} = \mathcal{I}_{\eta\eta} - \frac{1}{4\eta^2}\mathcal{I}_{\phi\phi}$
- Quantum mechanics prevents attainment of the quantum limit

Crowley/AD/Barbieri/Walmsley, arxiv:1206.0043

Dephasing is also a challenge !

- The phase diffusion channel is

$$\rho = \mathcal{N}_\Delta(\rho_{in}) = \frac{1}{\sqrt{2\pi}\Delta} \int d\xi \; e^{-\frac{\xi^2}{2\Delta^2}} U_\xi \rho_{in} U_\xi^\dagger,$$

where $U_\xi = \exp(-i\xi \hat{a}^\dagger \hat{a})$ is the phase shift operator.

- In the Fock basis,

$$\mathcal{N}_\Delta(|n\rangle\langle m|) = e^{-\Delta^2 (n-m)^2} |n\rangle\langle m|.$$

- Start with the probe state

$$\rho_0 = \begin{pmatrix} \cos^2(\frac{\theta}{2}) & \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) & \sin^2(\frac{\theta}{2}) \end{pmatrix}.$$

- At present, the analysis applies to 2d Hilbert spaces

QFI matrix

For simultaneous phase and diffusion estimation

$$\mathbf{H}_\theta(\Delta) = \sin^2\theta \begin{pmatrix} e^{-2\Delta^2} & 0 \\ 0 & \frac{4\Delta^2}{e^{2\Delta^2}-1} \end{pmatrix}.$$

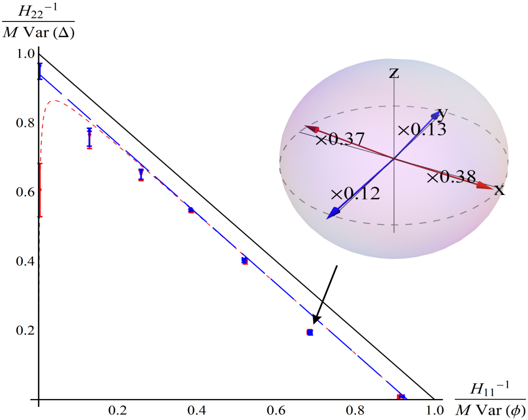
- Optimal probe state is $\theta = \pi/2$
- Joint bound attainable for this state
- Furthermore,

$$\mathbf{H}^{(N00N)}(\Delta) = N^2 \mathbf{H}_{\pi/2}(N\Delta)$$

$$\mathbf{H}^{(coh)}(\Delta) = |\alpha|^2 \mathbf{H}_\theta(\Delta)$$

Attaining the bounds with real measurements

In terms of the statistical variances, $$\frac{\mathbf{H}_{11}^{-1}}{MVar(\phi)} + \frac{\mathbf{H}_{22}^{-1}}{MVar(\Delta)} \leq 1$$



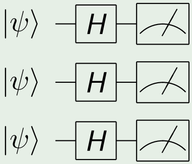
The ultimate, and the attained limit. Effect of small deviations.

Quantum metrology has 3 ingredients

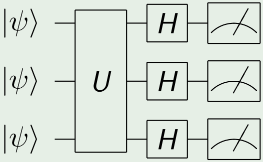
- Design of the probe states
- Dynamics
- Measurements/Detection

Recall : Enhancement using quantum probes

- a Hamiltonian $H_\chi = \chi H$
- a probe made of constituents, say N qubits, in a state $|\Psi\rangle$
- a measurement M



- $|\Psi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes N}$
- $M = X^{\otimes N}$
- $\delta^2\chi \sim \frac{1}{N}$



- $|\Psi\rangle = \frac{|00\cdots000\rangle + |11\cdots111\rangle}{\sqrt{2}}$
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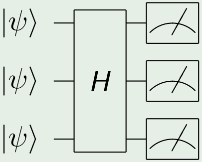
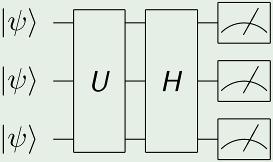
Nonlinear quantum metrology consists of

- a Hamiltonian $H_\chi = \chi H$
- a probe made of constituents, say N qubits, in a state $|\Psi\rangle$
- a measurement M



- $|\Psi\rangle = |\psi\rangle^{\otimes N}$
- $M = X^{\otimes N}$
- $\delta^2\chi \sim ?$



- $|\Psi\rangle$
- $M = X^{\otimes N}$
- $\delta^2\chi \sim ?$

The Cramér-Rao bound

$$\delta^2\chi \geq \frac{1}{\mathcal{I}_\chi}$$

For a k-body Hamiltonian,

$$\delta^2\chi \geq \frac{1}{\binom{N}{k}^2 (\lambda_M^k - \lambda_m^k)^2} \sim \frac{1}{N^{2k}}$$

Boixo *et al.*, PRL, **98**, 090401 (2007)

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Using no correlations : $|\psi\rangle^{\otimes N}$

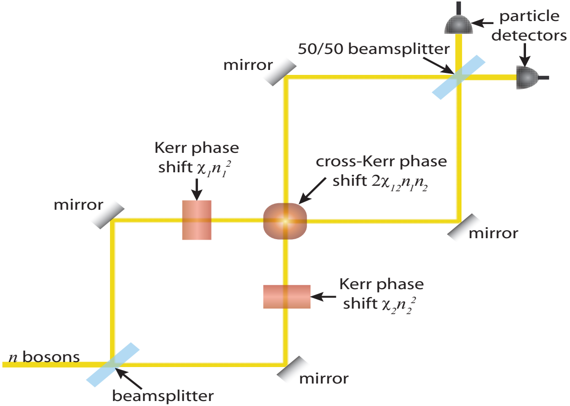
$$|\psi_\beta\rangle = \cos\left(\frac{\beta}{2}\right)|0\rangle + \sin\left(\frac{\beta}{2}\right)|1\rangle \text{ for } \sin\beta = \sqrt{1/k},$$

$$\delta^2\chi \geq \frac{1}{N^{2k-1}}$$

Boixo/AD/Flammia/Shaji/Bagan/Caves , PRA, **77**, 0123017, (2008)

Intuition for quadratic Hamiltonians

$$\mathcal{H}t/\hbar = \chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2$$



With $n = n_1 + n_2$, $J_z = n_1 - n_2$,

$$\mathcal{H}t/\hbar = (\chi_1 + \chi_2 + 2\chi_{12})n^2 + 2(\chi_1 - \chi_2)nJ_z + (\chi_1 + \chi_2 - 2\chi_{12})J_z^2$$

Boixo/AD/Davis/Flammia/Shaji/Caves, PRL, **101**, 040403, (2008)

Boixo/AD/Davis/Shaji/Tacla/Caves, PRA, **80**, 032103, (2009)

- Estimate the self-Kerr effect in polarization maintaining fibre
- Coherent state input in H and V
- H, V split in time domain before the fibre sample ($\chi_{12} \approx 0$)

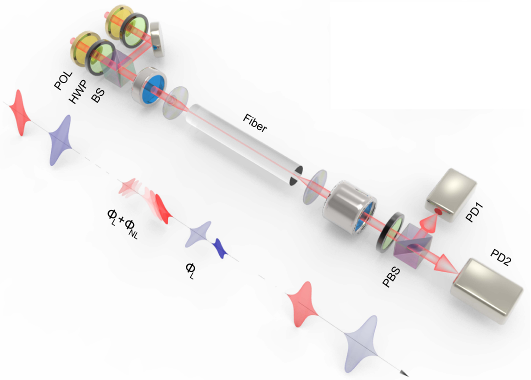


Figure: Setup for surpassing the $1/N$ scaling

Optical nonlinear quantum metrology

$$E_{out} = \chi^{(3)} E_{in}^3 \sim \chi^{(3)} |E_{in}|^2 E_{in}$$

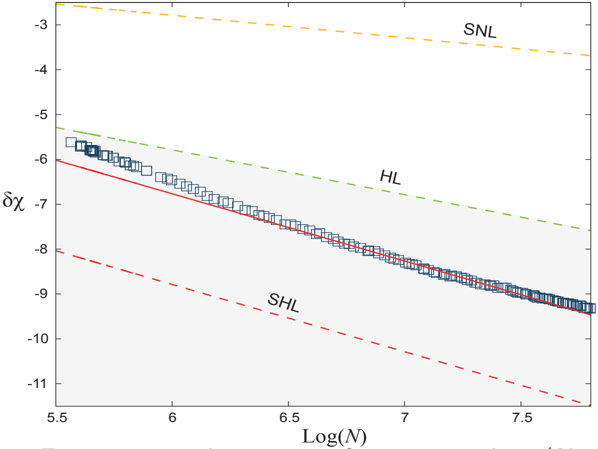


Figure: Experimental observation of surpassing the $1/N$ scaling

Jin/Zhang/AD/Walmsley, In preparation

Onto quantum imaging ... the future

- Biological cells have variations in $\chi^{(3)}$
- Sample is tiny, so the nonlinear phase picked up is $\sim 10^{-6}$ rad

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Recent developments make this feasible

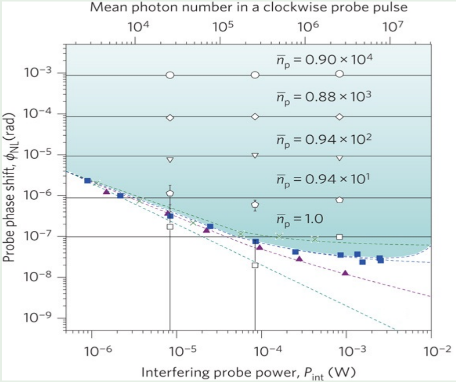

Figure: Matsuda *et al.*, Nature Photonics 3, 95 (2009)

Quantum sensing is at an exciting stage of innovation ...

- ✓ Combining the tools from estimation theory with quantum mechanics and quantum optics
- ✓ Theory work to direct experimental efforts in attaining tangible quantum advantages
- ✓ Issues of practice require a deeper understanding of issues of principle

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Thank you for your time and attention !!