Lower bounding convex roofs

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Motivation

- Convex roof, quantum information framework:
  i) Function $f(|\psi\rangle)$ defined for pure states $|\psi\rangle$
  ii) Extension to mixed state ρ via convex roof

$$f(\rho) = \inf_{\rho=\sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i f(|\psi_i\rangle) \tag{1}$$

- Examples:
  - Bipartite concurrence: $C(|\psi\rangle_{AB}) = \sqrt{2(1 - \operatorname{tr}\rho_A^2)}$
  - Schmidt number: $|\psi\rangle_{AB} = \sum_{i=1}^{r} s_i |ii\rangle$, $s_i > 0$, $R_S(|\psi\rangle) = r$, $\Rightarrow SN(\rho_{AB}) = \inf_{p_i,|\psi_i\rangle} \max_i R_S(|\psi_i\rangle)$
  - 3-tangle: $\tau(|\psi\rangle_{ABC}) = \dots$

$\Rightarrow$ Entanglement quantification, Entanglement classification

- Lower bounds usually desired, but in general difficult to compute.

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► This talk: Solution for polynomial functions f ◄

Main idea - Step I

$\Rightarrow$ explain idea via an example

- Linear entropy/Concurrence^2

$$S_L(|\psi\rangle_{AB}) = [C(|\psi\rangle_{AB})]^2 = 2(1 - \operatorname{tr}\rho_A^2) \tag{2}$$

can be rewritten as a linear operator on 2 copies

$$\begin{aligned} S_l(|\psi\rangle_{AB}) &= 2[1 - \sum_i \operatorname{tr}(\sigma_i^A \otimes \mathbb{1} \,|\psi\rangle\langle\psi|)^2] && (3)\\ &= 2[1 - \sum_i \operatorname{tr}(\sigma_i^A \mathbb{1} \otimes \sigma_i^{A'} \mathbb{1}\, |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|)] && (4)\\ &= \operatorname{tr}[\big(2(\mathbb{1}_{ABA'B'} - \sum_i \sigma_i^A \mathbb{1} \sigma_i^{A'} \mathbb{1})\big)\, |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|] && (5)\\ &= \operatorname{tr}(L_{ABA'B'}\ |\psi\rangle\langle\psi|^{\otimes 2}) && (6) \end{aligned}$$

- More general: Any polynomial of degree n in the density operator can be written as a linear operator on n copies.

Main idea - Step II

- Evaluate $[\rho_{AB} = \sum q_i |\phi_i\rangle\langle\phi_i|]$:

$$S_L(\rho_{AB}) = \sum_i q_i S_L(|\phi_i\rangle) = \sum_i q_i \operatorname{tr}[L\,|\phi_i\rangle\langle\phi_i| \otimes |\phi_i\rangle\langle\phi_i|] \quad (7)$$

$$= \operatorname{tr}[L\Big(\sum_i q_i\,|\phi_i\rangle\langle\phi_i| \otimes |\phi_i\rangle\langle\phi_i|\Big)] = \operatorname{tr}(L\,\omega_{ABA'B'}) \quad (8)$$

- The state $\omega = \omega_{ABA'B'}$ satisfies:
  i) separable state for $AB|A'B'$,
  ii) only on symmetric subspace $Sym(\mathcal{H}_{AB}^{\otimes 2})$,
  iii) reduced state $\rho_{AB} = \operatorname{tr}_{A'B'}(\omega)$.

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  - i) separable state for $AB|A'B'$,
  - ii) only on symmetric subspace $Sym(\mathcal{H}_{AB}^{\otimes 2})$,
  - iii) reduced state $\rho_{AB} = \operatorname{tr}_{A'B'}(\omega)$.

- Thus, if we optimize we obtain

$$S_L(\rho_{AB}) \geq \min\ \operatorname{tr}(L\,\omega) \quad (9)$$
$$\text{s.t. } \omega \text{ satifies } i) - iii)$$

- "$\leq$" Any ω with $i) - iii)$ gives a decomposition as in (8).

Main idea

- Equivalent formulation:

$$
\begin{aligned}
S_L(\rho_{AB}) = \min\ & \operatorname{tr}(L\,\omega) \\
\text{s.t. } & \omega = \omega_{ABA'B'} \text{ separable } AB|A'B', \\
& \omega \in \mathcal{L}(Sym(\mathcal{H}_{AB}^{\otimes 2})),\ \operatorname{tr}_{A'B'}(\omega) = \rho_{AB}
\end{aligned} \tag{10}
$$

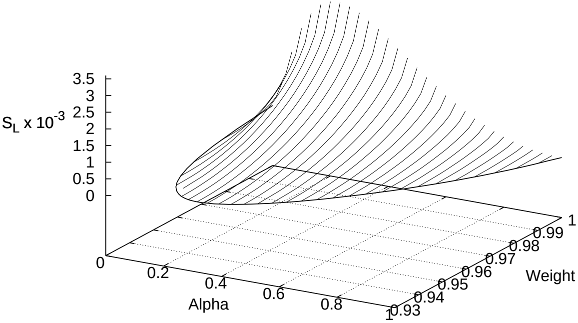
- Relax separability condition via
  - PPT

$$
\begin{aligned}
S_L(\rho_{AB}) \geq \min\ & \operatorname{tr}(L\,\omega) \\
\text{s.t. } & \omega = \omega_{ABA'B'} \text{ PPT } AB|A'B', \\
& \omega \in \mathcal{L}(Sym(\mathcal{H}_{AB}^{\otimes 2})),\ \operatorname{tr}_{A'B'}(\omega) = \rho_{AB}
\end{aligned} \tag{11}
$$

  - Hierarchy of symmetric extensions [Doherty *et al.* PRL (2003)] $\Rightarrow$ Complete solution
- Advantages:
  - Optimization is a semidefinite program (certified solution)
  - Dual provides "quantitative entanglement witnesses"
  - Clear picture about strength

Example I - $3 \otimes 3$ Horodecki state

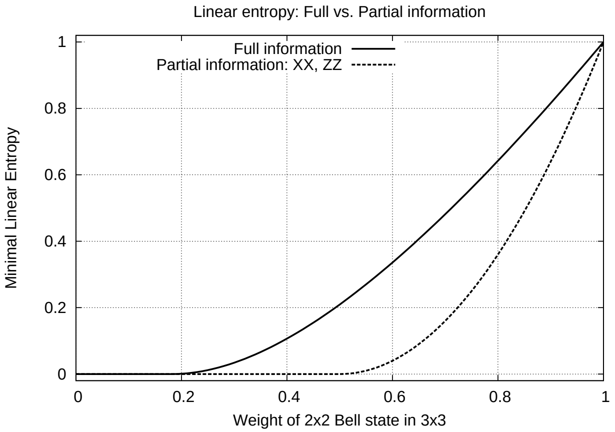
- Family of PPT entangled states: $p\rho_H(\alpha) + (1-p)\mathbb{1}/9$



- Lower bound is positive at least for all
  - NPT entangled state
  - states with no symmetric extensions to 2 copies of A, B
- Zero for states with PPT sym. ext. to 3 copies of A, B

Example II - Partial information

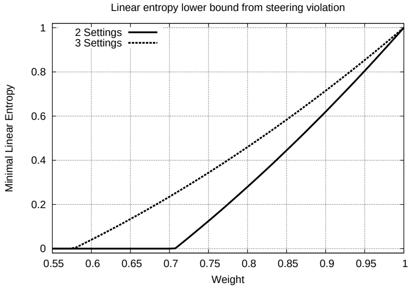
- Estimate entanglement from a few expectation value
- Example: 2-qubit Bell state embedded in $3 \otimes 3$
- Partial info.: only $\langle \sigma_x \otimes \sigma_x \rangle, \langle \sigma_z \otimes \sigma_z \rangle$ on the "qubit"

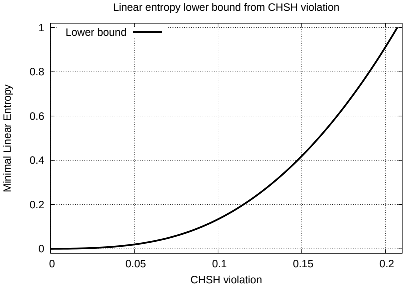


$\Rightarrow$ Just add constraint: $\rho_{AB}(x) = \rho_{AB}^{\text{data}} + \rho_{AB}^{\text{open}}(x) \geq 0$

Example III - Steering & device-independent scenario

- Estimate entanglement from a few measurement results and no knowledge about performed measurement
- Steering: only Alice apparatus uncharacterized
- Full dvi: both uncharacterized $\Rightarrow$ Bell inequality





$\Rightarrow$ Map ρ_{AB} in $?\otimes?$ to fixed finite dim. output state $\chi_{\bar{A}\bar{B}}$ by local maps $\Rightarrow$ Estimate S_L via partial information on $\chi_{\bar{A}\bar{B}}$

Schmidt number

- Measure for the Schmidt number is given by $[\lambda_i = \lambda_i(\rho_A)]$

$$N_2(|\psi\rangle) = \sum_{i<j} \lambda_i \lambda_j = 4 S_L(|\psi\rangle), \;\; N_2(|\psi_{r=1}\rangle) = 0 \tag{12}$$

$$N_3(|\psi\rangle) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k, \;\; N_3(|\psi_{r=2}\rangle) = 0 \tag{13}$$

$$N_4(|\psi\rangle) = \dots \tag{14}$$

- N_3 can be expressed as linear operator on 3 copies

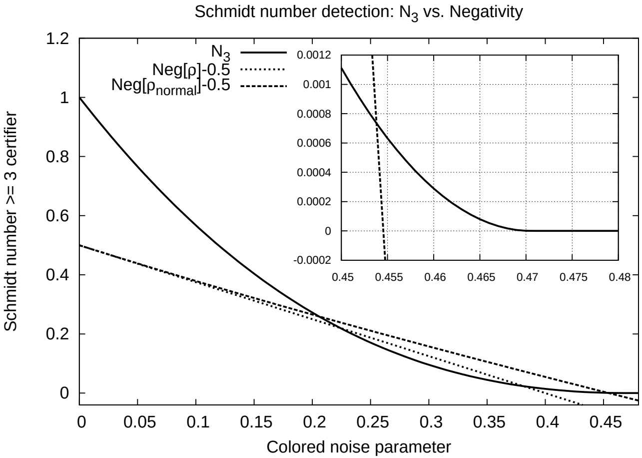
$$N_3(|\psi\rangle) \propto 1 - 3\,\mathrm{tr}(\rho_A^2) + 2\,\mathrm{tr}(\rho_A^3) = \mathrm{tr}(D_3 \, |\psi\rangle \langle\psi|^{\otimes 3}) \tag{15}$$

- Equivalent formulation/Lower bound:

$$\begin{aligned} N_3(\rho_{AB}) = \min \; & \mathrm{tr}(D_3 \, \omega_{123}) \\ \text{s.t.} \; & \omega = \omega_{123} \text{ fully separable} \\ & \omega \in \mathcal{L}(Sym(\mathcal{H}_{AB}^{\otimes 3})), \mathrm{tr}_{23}(\omega) = \rho \end{aligned} \tag{16}$$

Example IV - Schmidt number

- $|\psi\rangle \propto |00\rangle + |11\rangle + |22\rangle$ mixed with colored "qubit" noise



$\Rightarrow$ N_3 beats Schmidt number estimate via negativity

Tangle - GHZ vs. W class

- Distinguishes between the two SLOCC classes of 3 qubit multiparticle entanglement:

$$|W\rangle \propto |001\rangle + |010\rangle + |100\rangle\,, \;\; \tau^2(|W\rangle) = 0 \tag{17}$$

$$|GHZ\rangle \propto |000\rangle + |111\rangle\,, \;\; \tau^2(|GHZ\rangle) = 1 \tag{18}$$

- Tangle^2 can be expressed as linear operator on 4 copies

$$\tau^2(|\psi\rangle) = C^2_{A|BC} - C^2_{AB} - C^2_{AC} = \mathrm{tr}(T \; |\psi\rangle\langle\psi|^{\otimes 4}) \tag{19}$$

- Equivalent formulation/Lower bound:

$$\begin{aligned} \tau^2(\rho_{ABC}) = \min \; & \mathrm{tr}(T\omega_{1234}) \\ \text{s.t. } & \omega = \omega_{1234} \text{ fully separable} \\ & \omega \in \mathcal{L}(Sym(\mathcal{H}^{\otimes 4}_{ABC})), \mathrm{tr}_{234}(\omega) = \rho_{ABC} \end{aligned} \tag{20}$$

Can we still compute it?

- Resources:
  a) $Sym((\mathbb{C}^8)^{\otimes 4}) \cong \mathbb{C}^{330} \Rightarrow 110.000$ parameters
  b) 2 PPT $1|234$, $12|34$: 960×960, 1296×1296 matrix

☹ Just slightly too big!

- Ways to reduce resources:
  - Reduced rank

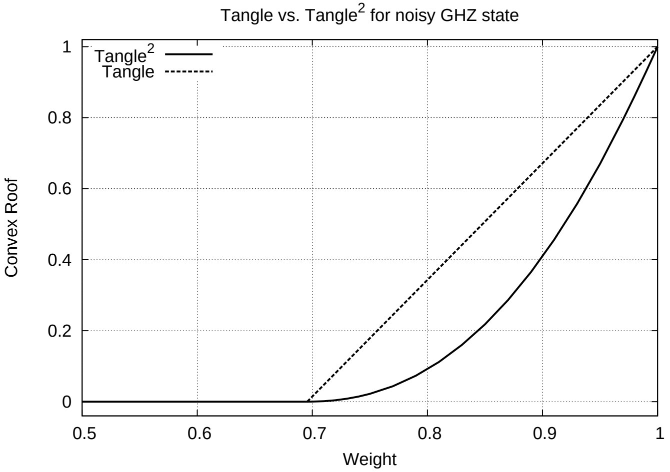
    e.g. $r(\rho_{ABC}) = 2$ then ω_{1234} effectively $Sym((\mathbb{C}^2)^{\otimes 4})] \cong \mathbb{C}^5$
    - Works up to $r(\rho_{ABC}) = 6$: ≈ 15.000 parameters
  - Symmetries

$$g_i \rho g_i = \rho, \forall i, g_i^{\otimes 4} T g_i^{\otimes 4} = T \;\; \Rightarrow \;\; g_i^{\otimes 4} \omega g_i^{\otimes 4} = \omega \qquad (21)$$

    - GHZ diagonal states: ≈ 10.000 parameters
    - Permutationally invariant states: ≈ 20.000 parameters
    - Real input $\rho = \rho^T$: parameters/2
  - Are there other solutions [Carathéodory $r(\omega) \leq 64$, normal form, methods to reduce rank,...]?

Example V - Tangle vs. Tangle^2

- GHZ state mixed with white noise
- Required weight $p \geq 0.6955$ [Eltschka, Siewert PRL (2012)]



Summary

- Convex roof for polynomial functions f:
  i) $f(|\psi\rangle)] = \mathrm{tr}(L\ |\psi\rangle\langle\psi|^{\otimes n})$
  ii) Equivalent formulation/Lower bound:

$$f(\rho) = \min_{p_i, |\psi_i\rangle} \sum_i p_i f(|\psi_i\rangle) \tag{22}$$

$$\begin{aligned} = \ & \min\ \mathrm{tr}(L\,\omega_{1\ldots n}) \\ & \text{s.t. } \omega = \omega_{1\ldots n} \text{ fully separable} \\ & \quad \omega \in \mathcal{L}(Sym(\mathcal{H}^{\otimes n})), \mathrm{tr}_{2\ldots n}(\omega) = \rho \end{aligned} \tag{23}$$

- Examples:
  - Linear entropy [with partial information]
  - Schmidt number
  - GHZ vs. W class for 3 qubits
- Properties:
  - ☺ Semidefinite program (certified optimization)
  - ☹ Numerical expensive, in particular for higher polynomials
  - ☺ Employ lots of known techniques from separability questions