The weak, the strong,
and the pretty strong

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The weak, the strong, and the pretty strong: Converses for quantum channel capacities

Andreas Winter (ICREA & UAB Barcelona)

[...and many others 1999-2013]

Outline

1. Quantum channels and their capacities

2. Entropic capacity formulas; weak converse

3. What is a strong converse?

4. Ideal channel (warm-up); simulation argument

5. Rényi divergence paradigm: classical capacity

6. Min-entropies: "pretty strong" converse

7. End credits

1. Channels & capacity

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Complementary channel:

$$\widehat{N}(\rho) = \mathrm{Tr}_B\, V \rho V^\dagger.$$

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Ex: 1) Noiseless channel = identity id_A.

2) Constant channel $K(\rho) = \omega_0$.

3) Depolarizing channels

4) Amplitude damping channels

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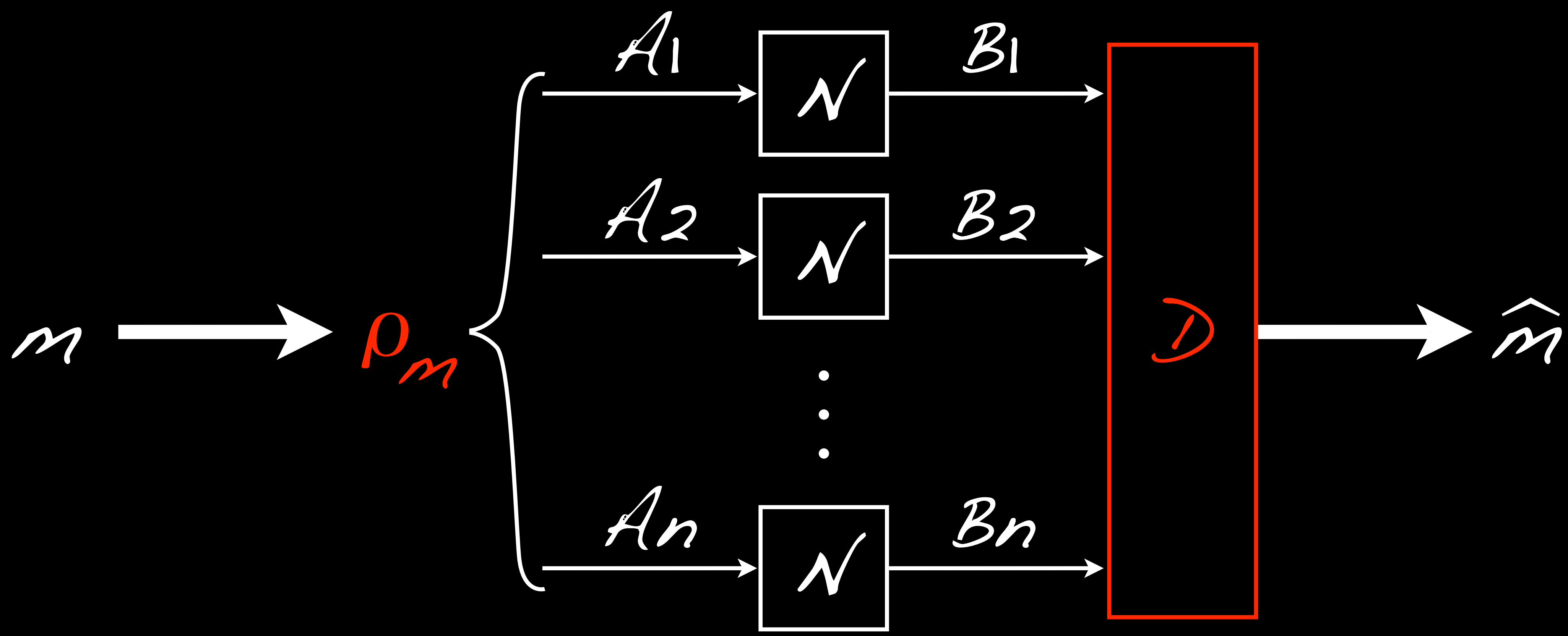
(Later in this talk, we'll look at some special classes: degradable, Hadamard, entanglement-breaking, ...)

Classical capacity $C(\mathcal{N})$:= maximum cbit rate $\frac{k}{n}$ for asymptotically error-free transmission over $\mathcal{N}^{\otimes n}$:



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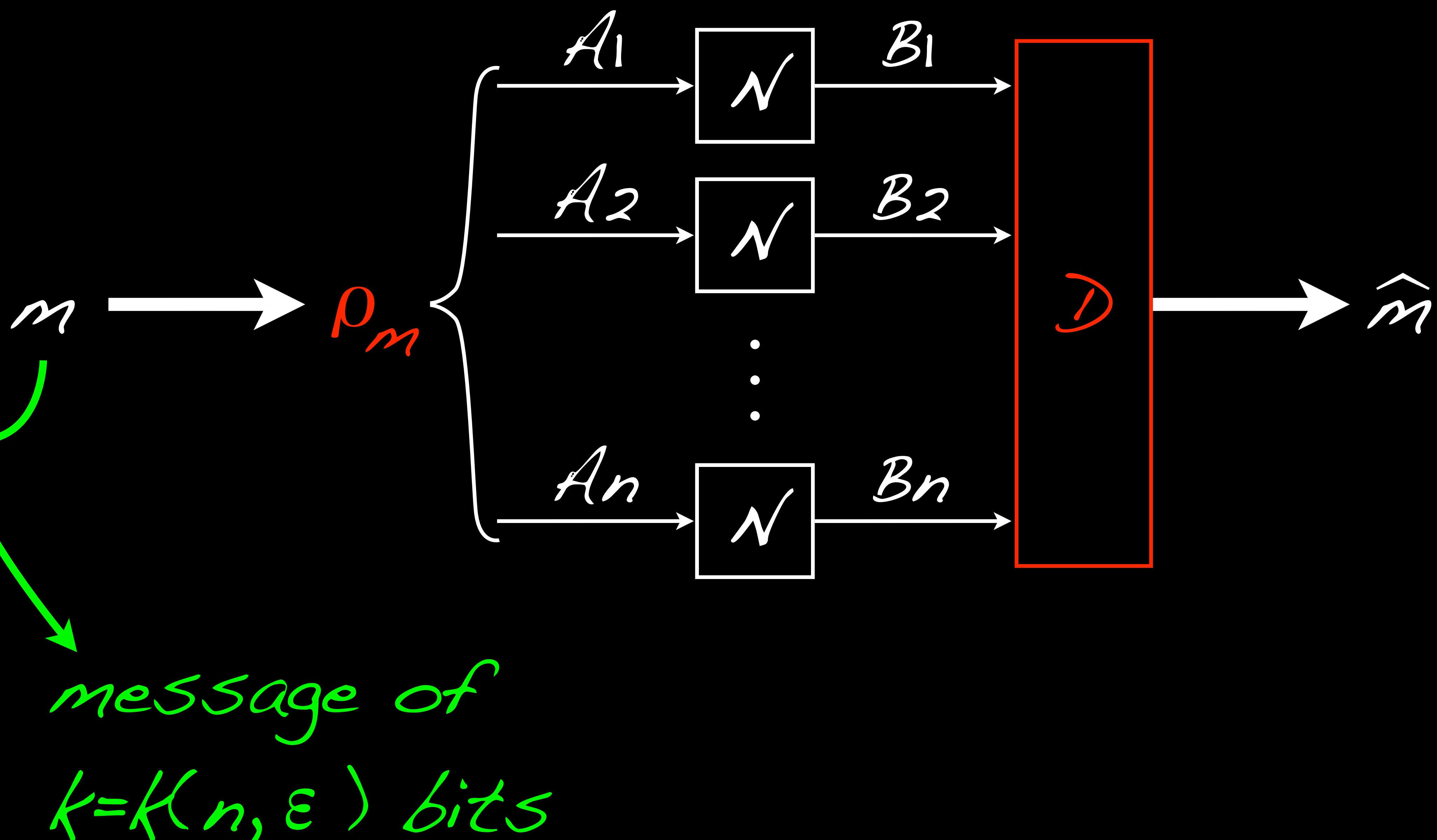

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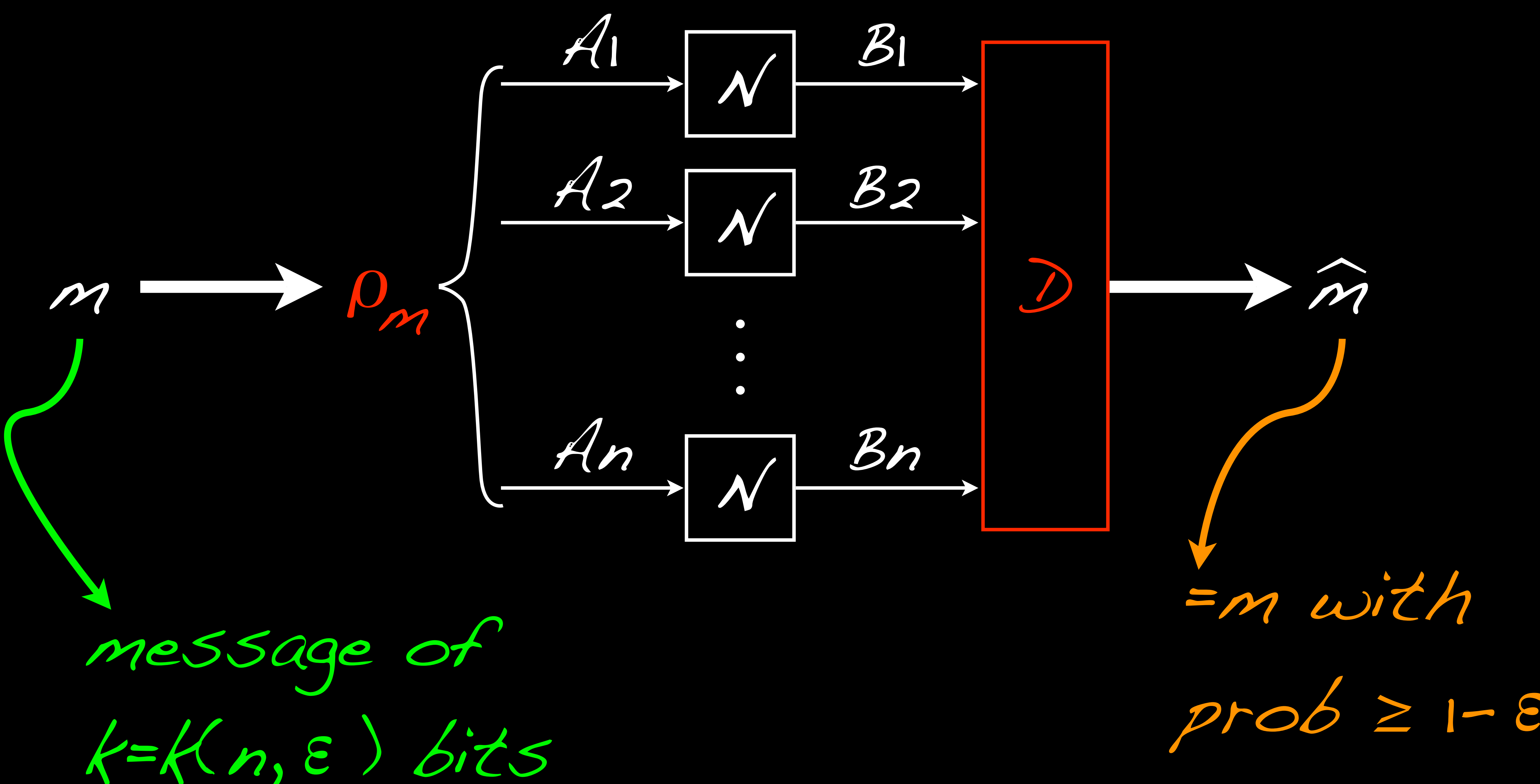

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Quantum capacity $Q(\mathcal{N})$:= maximum qubit rate for asymptotically faithful transmission.

...and a veritable "zoo" when allowing other free resources: E, $\leftarrow$, $\rightarrow$, $\leftrightarrow$, ...

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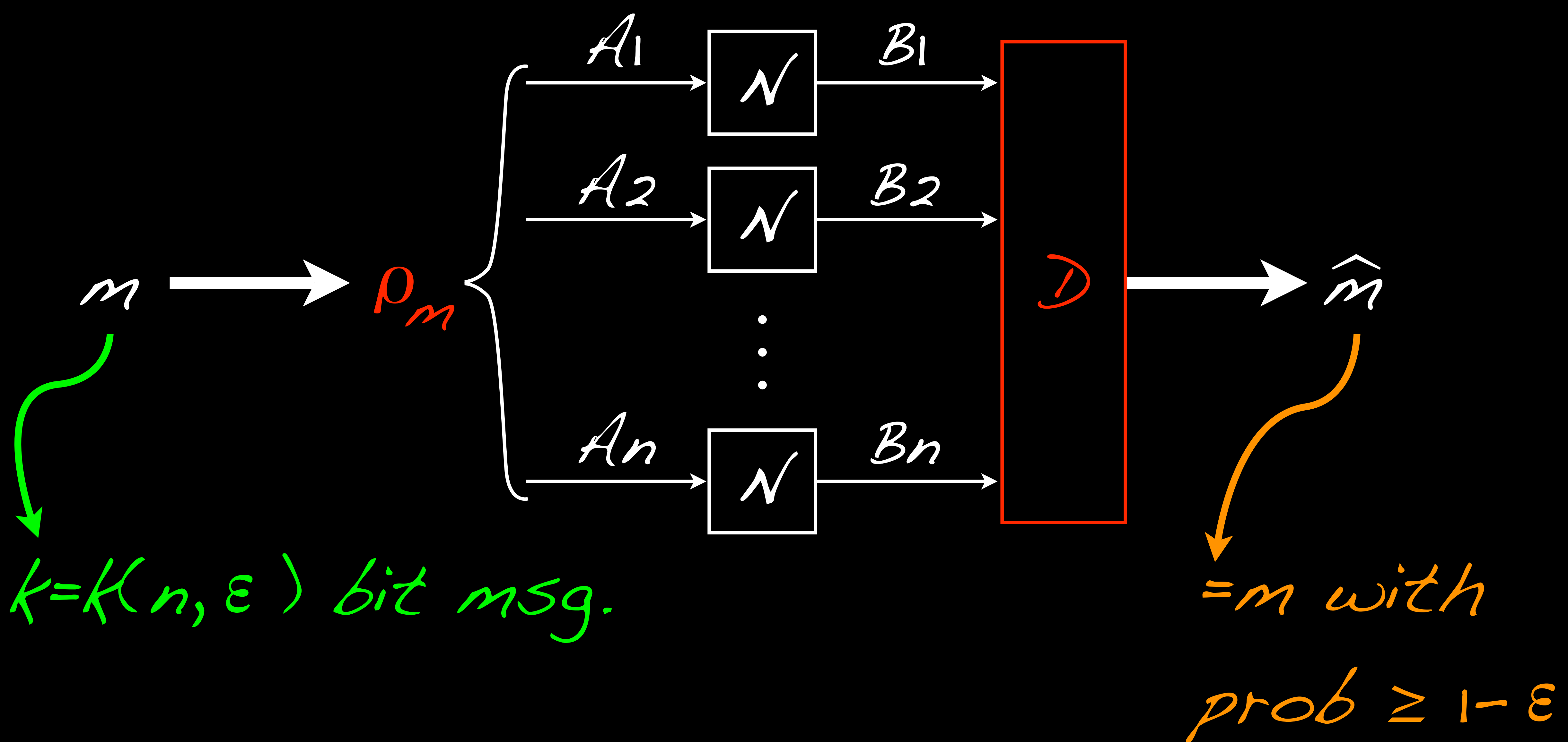

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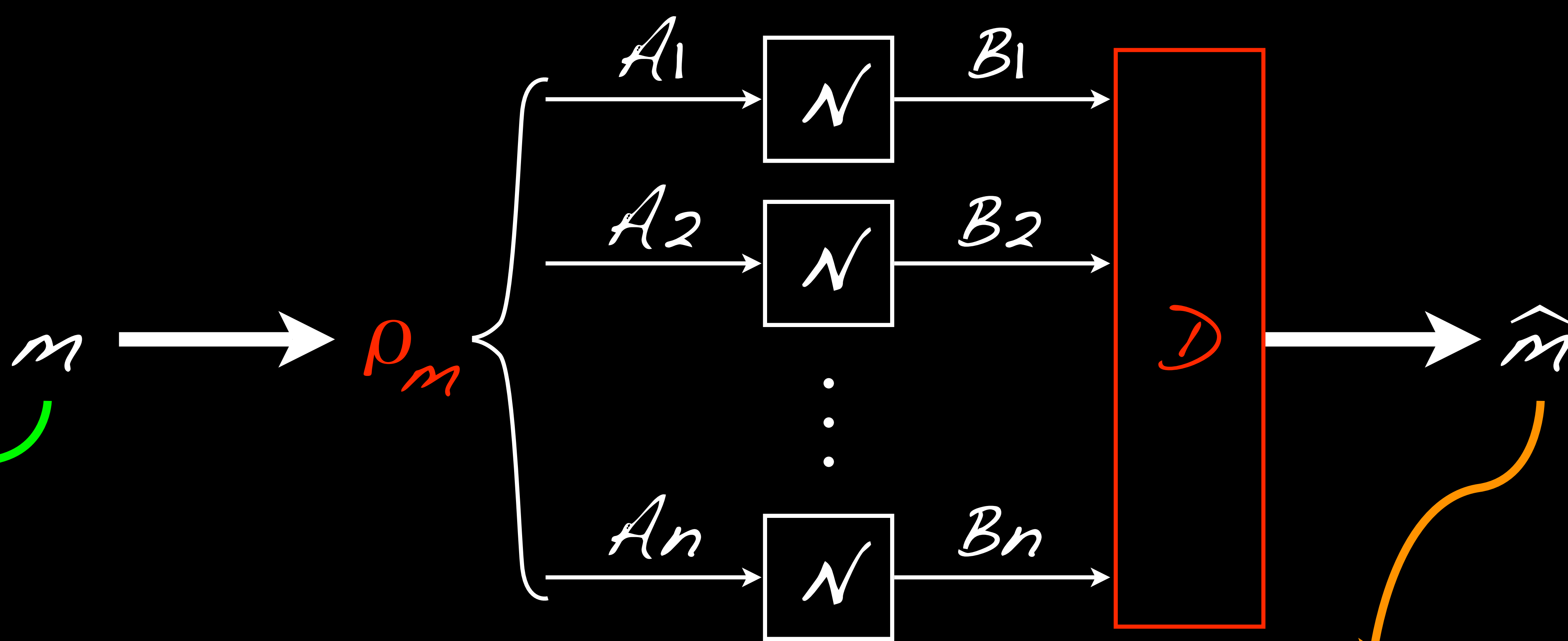
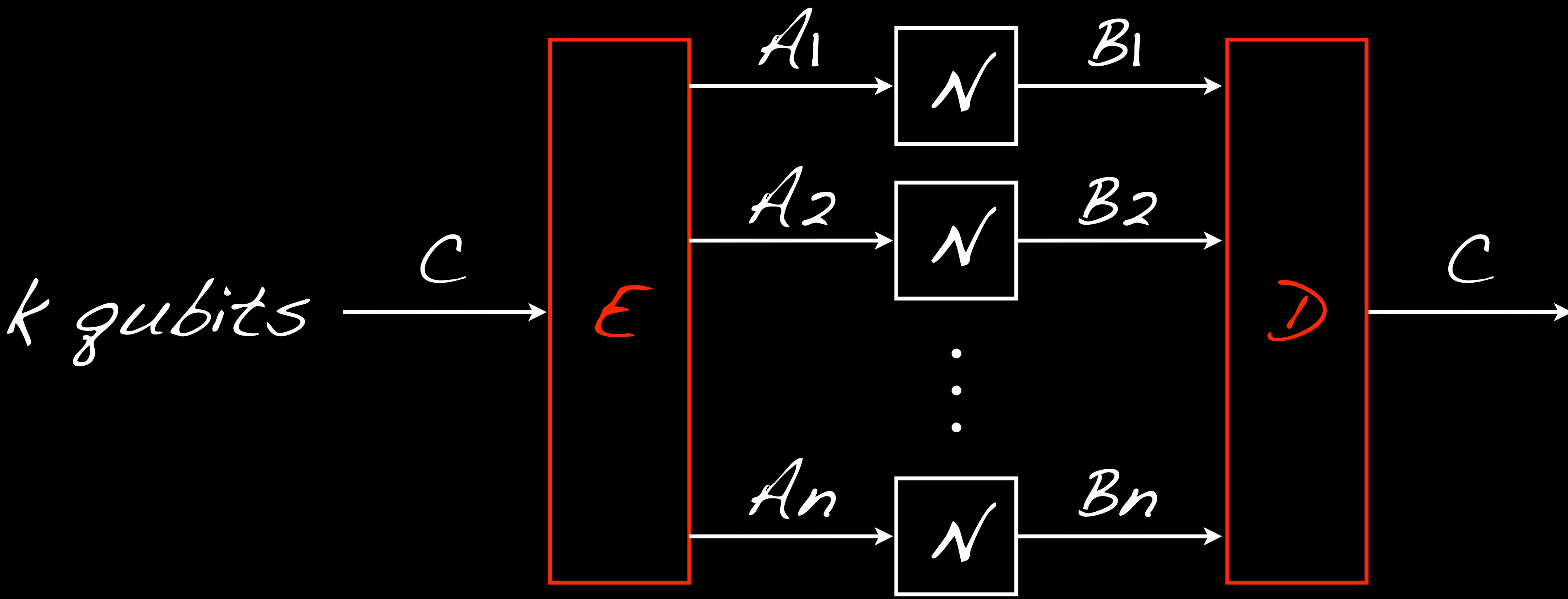


$k = k(n, \varepsilon)$ bit msg.

$= m$ with prob $\geq 1 - \varepsilon$

Secret: $\| \hat{\mathcal{N}}^{\otimes n}(\rho_m) - \omega_0 \|_1 \leq \varepsilon$

Quantum capacity $Q(\mathcal{N})$ requires en- and decoding by cptp maps $\mathcal{E}, \mathcal{D}$:



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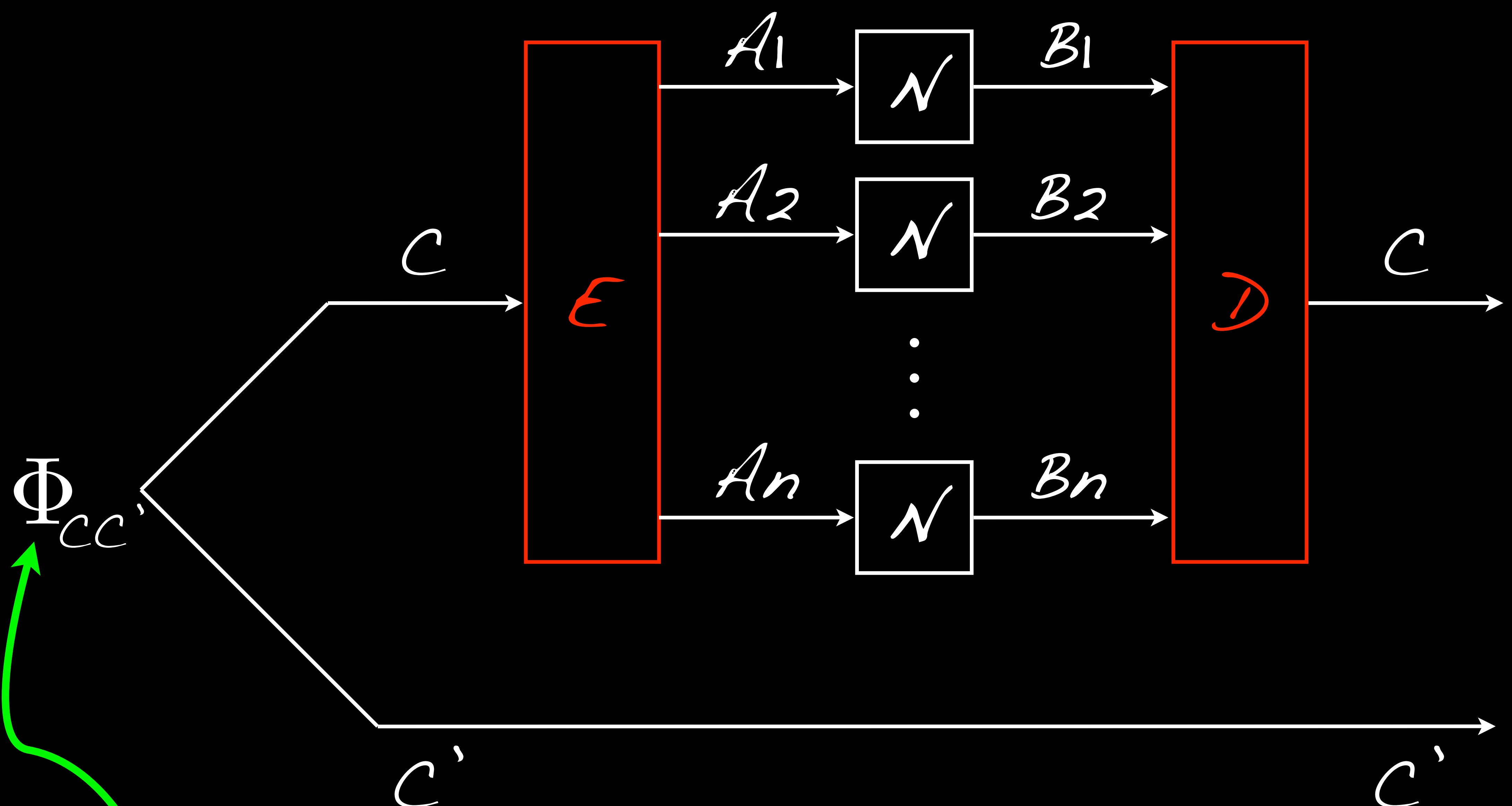

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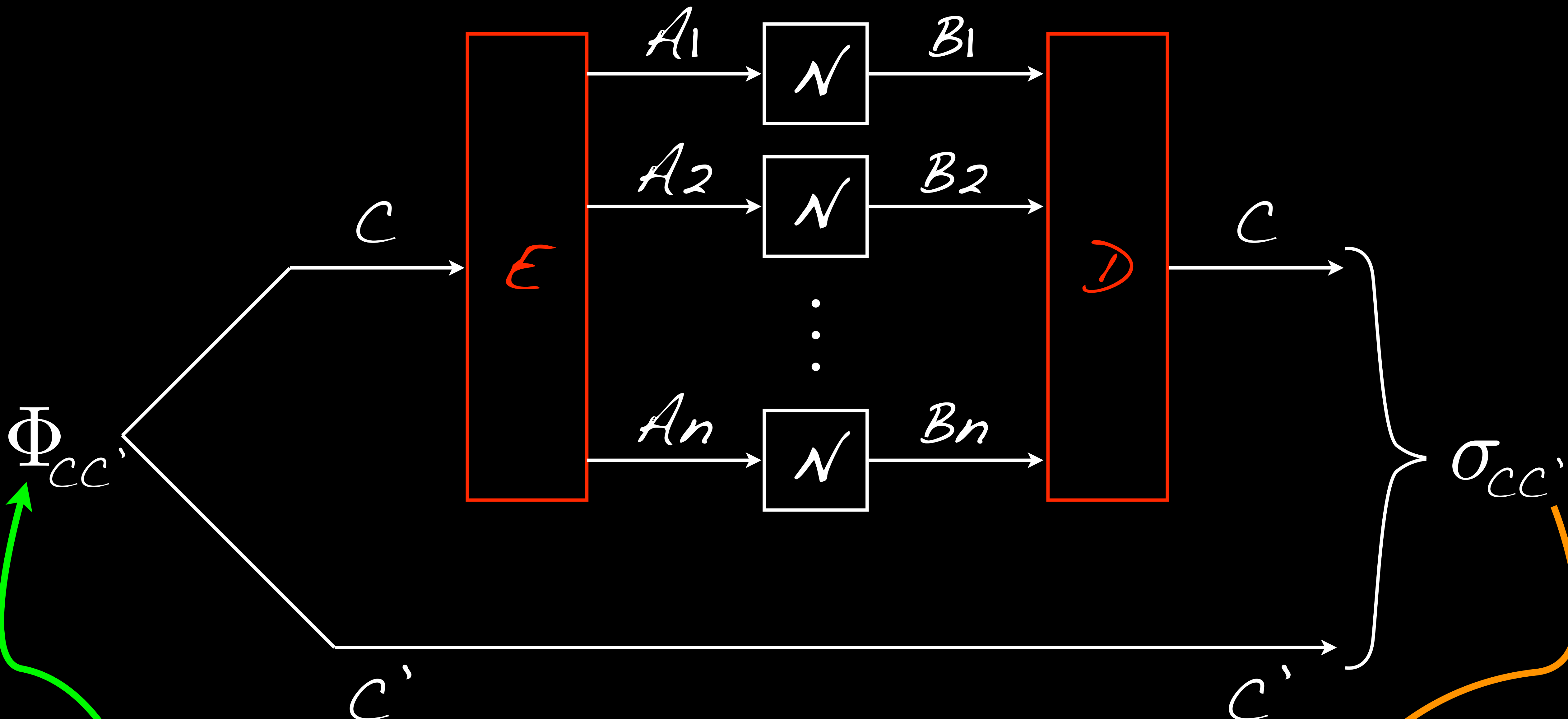

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Approximates input: $\mathcal{P}(\Phi, \sigma) \leq \varepsilon$.

Digression on fidelity:

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$$
$$= \max |\langle\psi|\varphi\rangle| \text{ s.t.}$$
$|\psi\rangle$ purifies ρ, $|\varphi\rangle$ purifies σ.

$P(\rho,\sigma) := \sqrt{1-F(\rho,\sigma)^2}$ is a metric on states;

...and so is $A(\rho,\sigma) := \arcsin P(\rho,\sigma)$.

Note: Both are equivalent to the trace distance $\|\rho-\sigma\|_1$.

[cf. M. Tomamichel, PhD thesis, arXiv:1203.2142]

2. Capacity formulas and weak converse

Thm (Holevo and Schumacher/Westmoreland, 1973 and 1996/7):

$$C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n}),$$ with

$\chi(\mathcal{N}) = \max I(X:B)$ wrt. $\{p_x, \rho_x\}$ and

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \mathcal{N}(\rho_x).$$

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Holevo information $S(\rho_B) - \sum_x p_x S(N(\rho_x))$

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However, for some classes of channels it is, and we know the classical capacity $C(\mathcal{N})$ as $\chi(\mathcal{N})$.

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...is the implied tradeoff real?

Analogous formulas for $P(N)$ and $Q(N)$:

Thm (Devetak and Cai/Yeung/AW, 2003):

$$P(N) = \lim_{n\to\infty} \frac{1}{n} P^{(1)}(N^{\otimes n}), \text{ with}$$

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Thm (Schumacher and Lloyd-Shor-Devetak, 1996-2003):

$$Q(N) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(N^{\otimes n}), \text{ with}$$

$$Q^{(1)}(N) = \max I(A\rangle B)$$

coherent information

$$= \max S(N(\rho)) - S(N(\rho)) \text{ wrt. } \rho$$

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Have analogous weak converses for $\mathcal{P}(\mathcal{N})$ and $Q(\mathcal{N})$, and for much every other capacity we know how to characterize.

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Have analogous weak converses for $P(N)$ and $Q(N)$, and for much every other capacity we know how to characterize.

(Btw: also additivity issue with both!)

3. Strong converse?

The strong converse – in the sense of Wolfowitz [Ill. J. Math. 1:591 (1957)] –, is the statement that there is no rate-error trade-off. Viz., for rates R above the capacity, the error converges to 1.

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By contrapositive: If error < 1, then asymptotically the rate $\frac{k}{n}$ is bounded by the capacity.

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Known in some cases:

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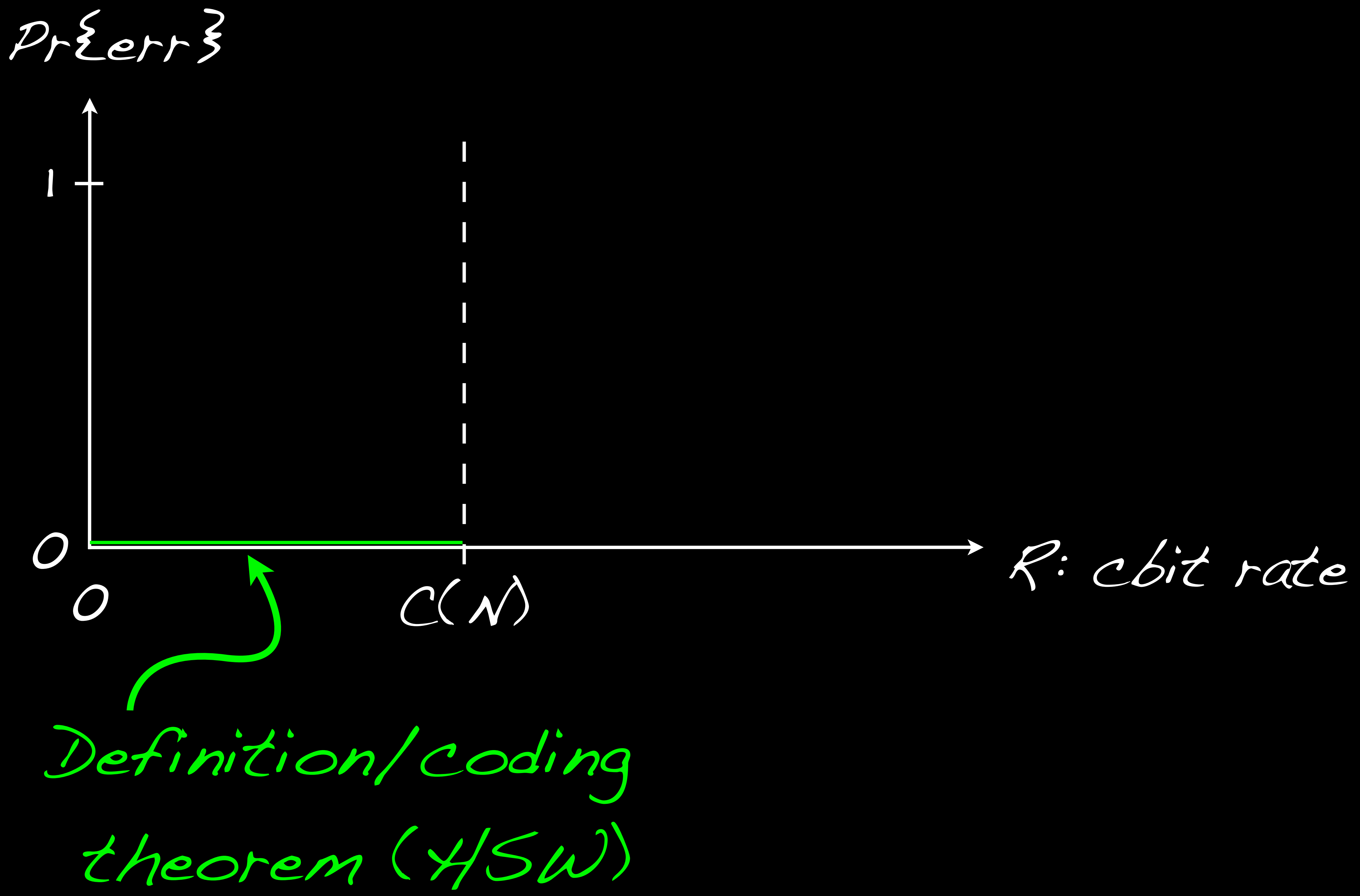
- Classical capacity with product state inputs [Ogawa/Nagaoka; AW, IEEE-IT 45(7), 1999]
- Classical capacity of certain channels [Koenig/Wehner, PRL 103:070504 (2009)]

Rate vs asymptotic error:



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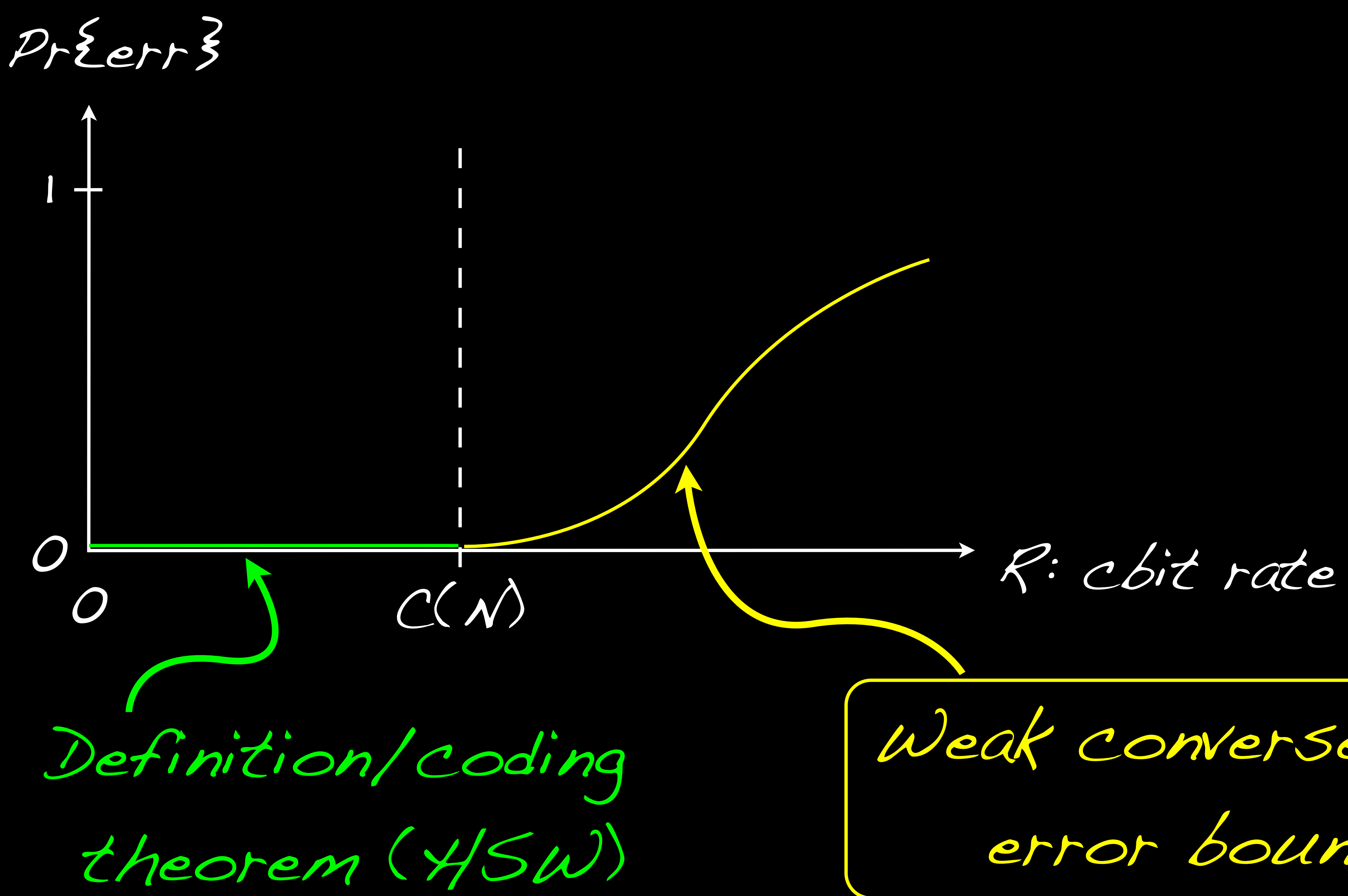

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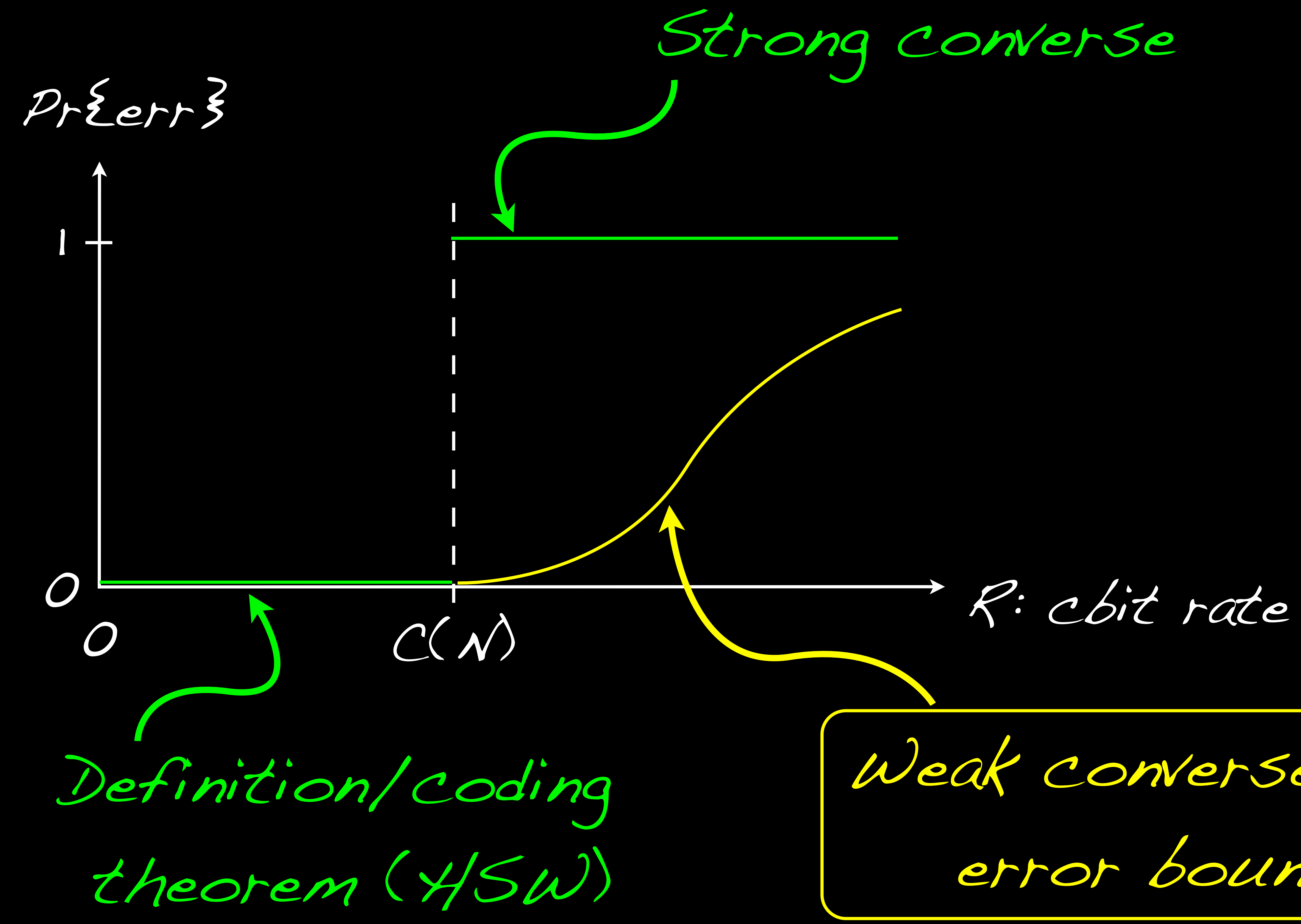

4. Ideal channel

As a warm-up, prove strong converse for the noiseless qubit channel id_2. Note:

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Hence $Q(N) \leq P(N) \leq C(N)$ in general.
Since $Q(\mathrm{id}_2) = P(\mathrm{id}_2) = C(\mathrm{id}_2) = 1$, enough to show it for the classical capacity.

Warm-up: strong converse for the noiseless qubit channel id_2.

Encode M message into id_L via states ρ_m and POVM elements D_m to decode:

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For n uses of the channel and rate $R>1$: $L=2^n$ and $M=2^{nR}$, so $\varepsilon \ge 1 - 2^{-n(R-1)}$. QED

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The simulation argument: If you can simulate a channel N by id_2 at rate K, then $C(N) \leq K$ and for rates $R > K$, the error $\varepsilon \geq 1 - 2^{-n(R-K)}$.

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In particular: If $K = C(N)$, strong converse holds. Almost only trivial cases, except:

Thm (Wilde/AW, 1308.6732): For pure loss optical channel w transmissivity η and maximum mean photon number p, $C = g(\eta p)$, and the strong converse holds.

The simulation argument: If you can simulate a channel $\mathcal{N}$ by id_2 at rate K, then $C(\mathcal{N}) \leq K$ and for rates $R > K$, the error $\varepsilon \geq 1 - 2^{-n(R-K)}$.

More interesting with free resources, eg. $C_E(\mathcal{N})$ = ent.-assisted classical capacity = minimal simulation cost assisted by ent. ("Qu. Reverse Shannon Thm.")

Ie. strong converse holds for C_E.

[Bennett et al., IEEE-IT 48:2637 (2002); Bennett et al. 0912.5537]

[Cf. Berta et al., IEEE-IT 59:6770 (2013) - $P(\mathcal{N})$ bound]

5. Rényi divergences for C

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What can we do for $C(N)$? Nothing general it seems... However, unifying and extending the earlier results of Ogawa/Nagaoka, AW and König/Wehner:

Thm (Wilde/AW/Yang, 1306.1586): If N is entanglement-breaking (EB) or Hadamard ($\mathcal{H}$), then for any code w rate $R > C(N)$, $\Pr\{\text{err}\}$ converges to 1, exponentially fast in the number n of channel uses.

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There exists $t \geq \Omega\left((R - C(N))^2\right)$ s.t.

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In other words, these channels satisfy the strong converse.

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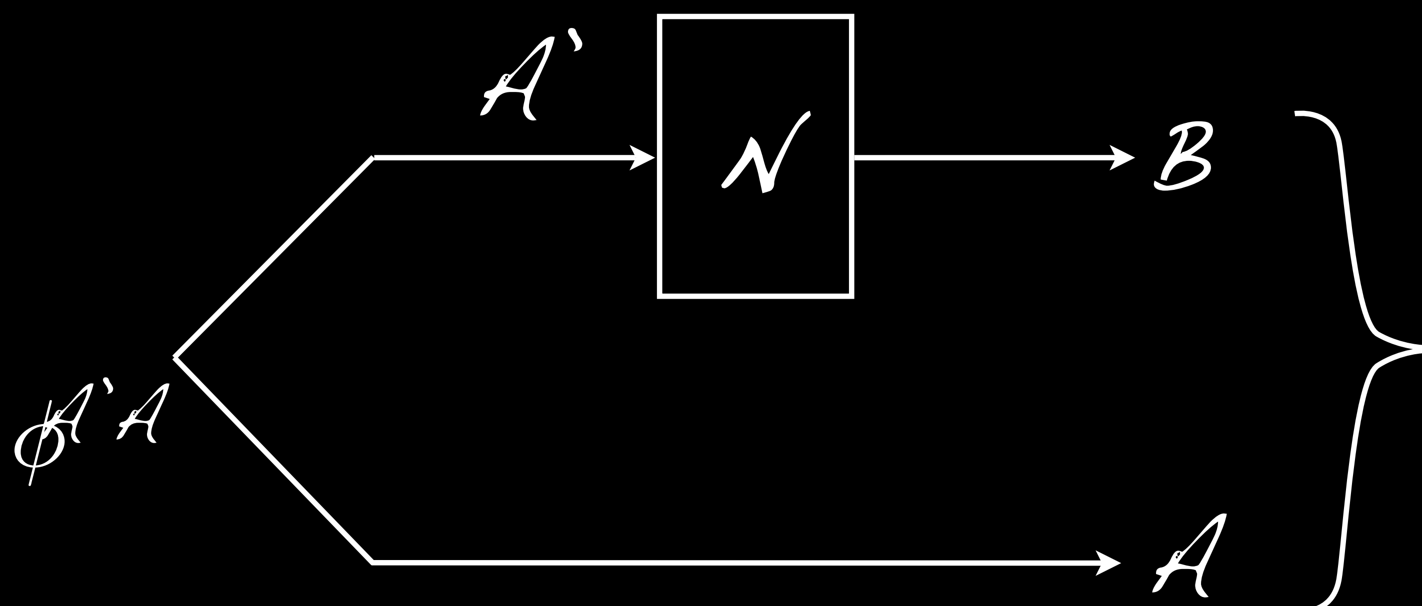
Entanglement-breaking (EB) channels:

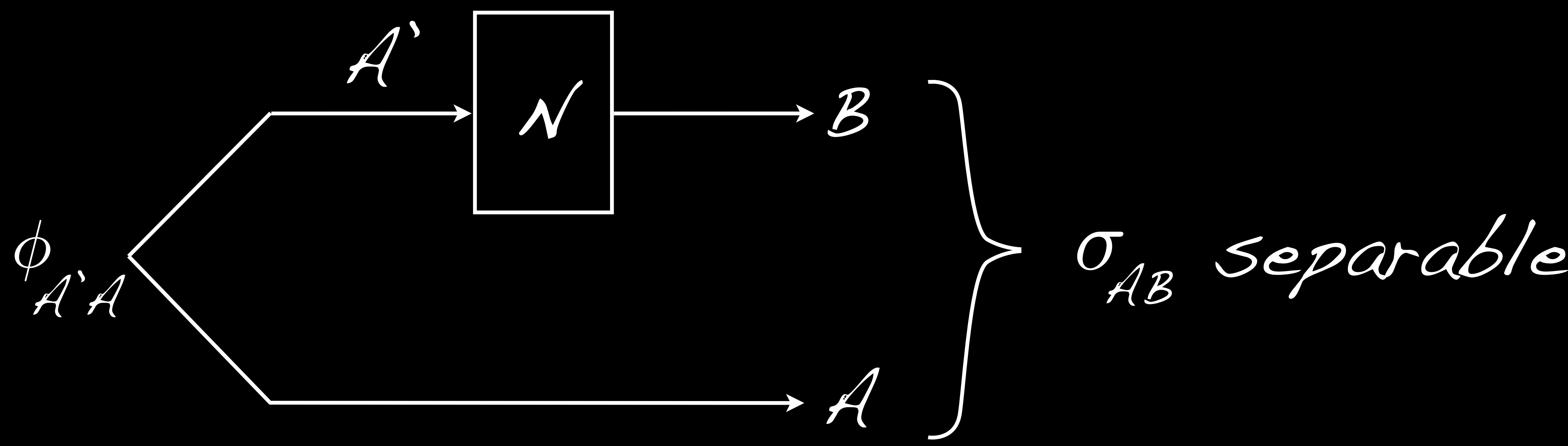


σ_{AB} separable

Complementary to these:

Hadamard channels ($\mathcal{H}$)

Entanglement-breaking (EB) channels:



Fact: $\mathcal{N}$ entanglement-breaking iff

$$\mathcal{N}(\rho) = \sum_i Tr(\rho M_i)\,\sigma_i \quad s.t. \quad \sum_i M_i = \mathbb{1}$$

$$= \sum_j |\beta_j\rangle\langle\alpha_j|\rho|\alpha_j\rangle\langle\beta_j|$$

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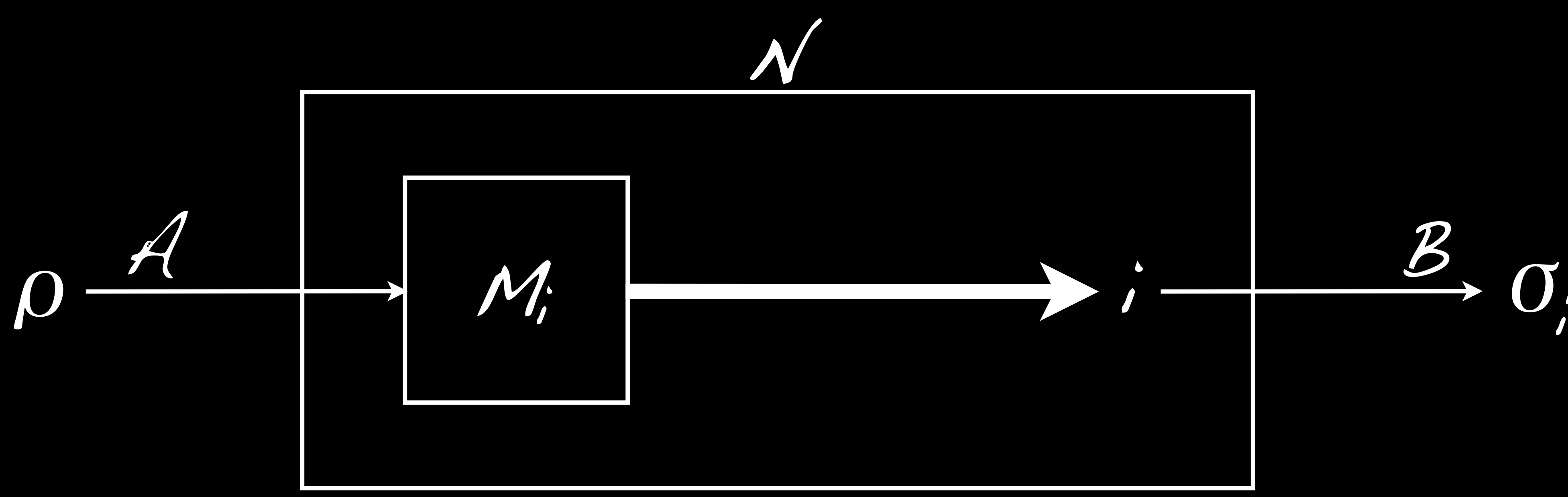

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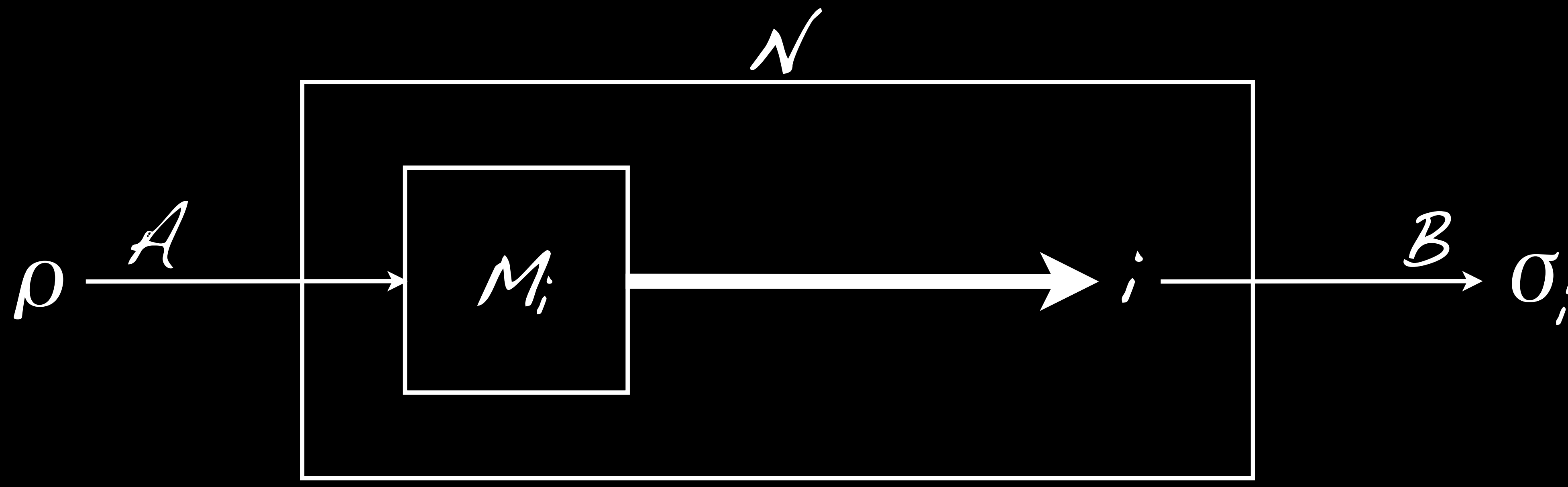

This holds also when $\mathcal{N}$ is only cp (and M_i are only positive)!

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Stinespring: $V : |\phi\rangle_A \to \sum_j \langle\alpha_j|\phi\rangle |\beta_j\rangle_B |j\rangle_E$

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$$\hat{\mathcal{N}}(\rho) = \sum_{jk} |j\rangle\langle k|\,\langle\alpha_j|\rho|\alpha_k\rangle\,\langle\beta_k|\beta_j\rangle$$

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isometry $U = \sum |j\rangle\langle\alpha_j| \; : \; A \to E$

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$$= \qquad U\rho U^\dagger \circ S$$

Schur product

isometry $U = \sum |j\rangle\langle\alpha_j| : A \to E$

$$\mathcal{N}(\rho) = \sum_i \mathrm{Tr}(\rho M_i)\sigma_i \quad \text{s.t.} \quad \sum_i M_i = \mathbb{1}$$

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Channels of this form: Hadamard channels

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4) Cloning channels [cf. Brádler, IEEE-IT 2011]

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Fundamental property is monotonicity: for any cptp map $\mathcal{N}$,

$$\tilde{D}(\rho\,||\,\sigma) \geq \tilde{D}(\mathcal{N}(\rho)\,||\,\mathcal{N}(\sigma)) \geq 0.$$

If it also has a certain "sum" property...

$$\tilde{D}(1-\varepsilon \,\|\, 1/M) \leq \max_{\rho} \tilde{D}(\mathcal{N}(\rho)\,\|\,\sigma) =: \chi_{\tilde{D},\sigma}(\mathcal{N})$$

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$$\tilde{D}_\alpha(\rho \,\|\, \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \, \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha$$

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Crucially additive: $\chi_{\alpha,\sigma}(\mathcal{N}^{\otimes n}) = n \, \chi_{\alpha,\sigma}(\mathcal{N})$.

(Because of identity with some min output entropy.)

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...and converges to $\chi(\mathcal{N})$ as $\alpha \to 1$.

L.h.s.: $\frac{1}{\alpha - 1} \log(1-\varepsilon) - \log M$. ...we win.

6. Min-entropies: "pretty strong" converse for Q

Stinespring: $\mathcal{N}(\rho) = \mathrm{Tr}_E\, V\rho V^{\dagger}$,

with an isometry $V : A \hookrightarrow B \otimes E$.

Complementary channel:

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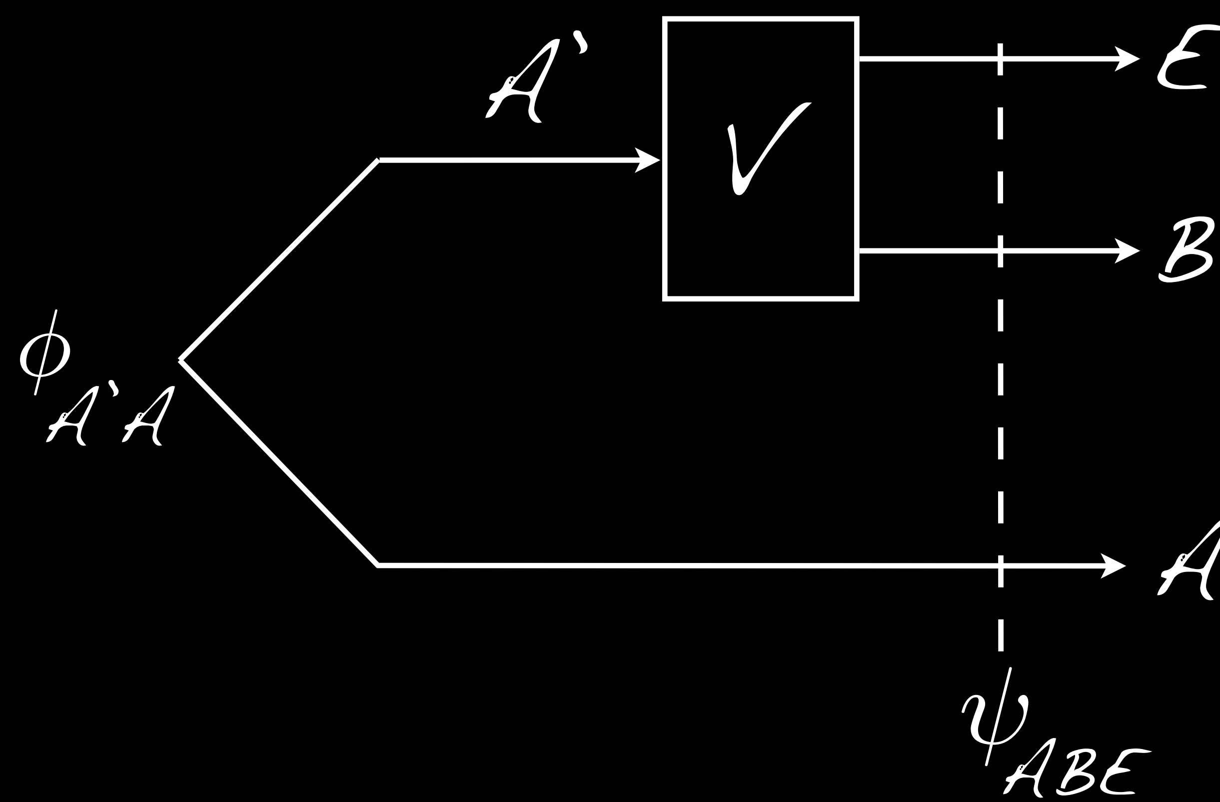
Complementary channel:
$$\widehat{\mathcal{N}}(\rho) = \mathrm{Tr}_B V\rho V^\dagger.$$

$\mathcal{N}$ is degradable if there exists a cptp map $\mathcal{D}$ s.t. $\widehat{\mathcal{N}} = \mathcal{D} \circ \mathcal{N}$. Vice-versa: anti-degradable.

Degradability in the Church of the Larger Hilbert Space:



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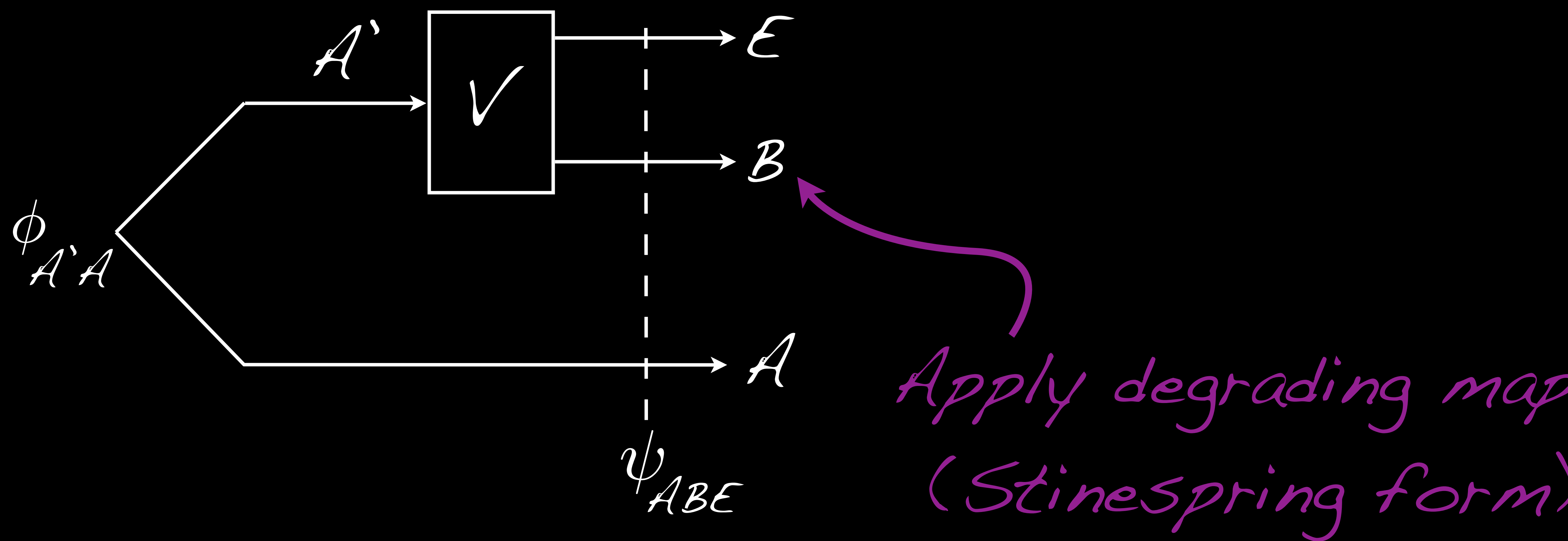

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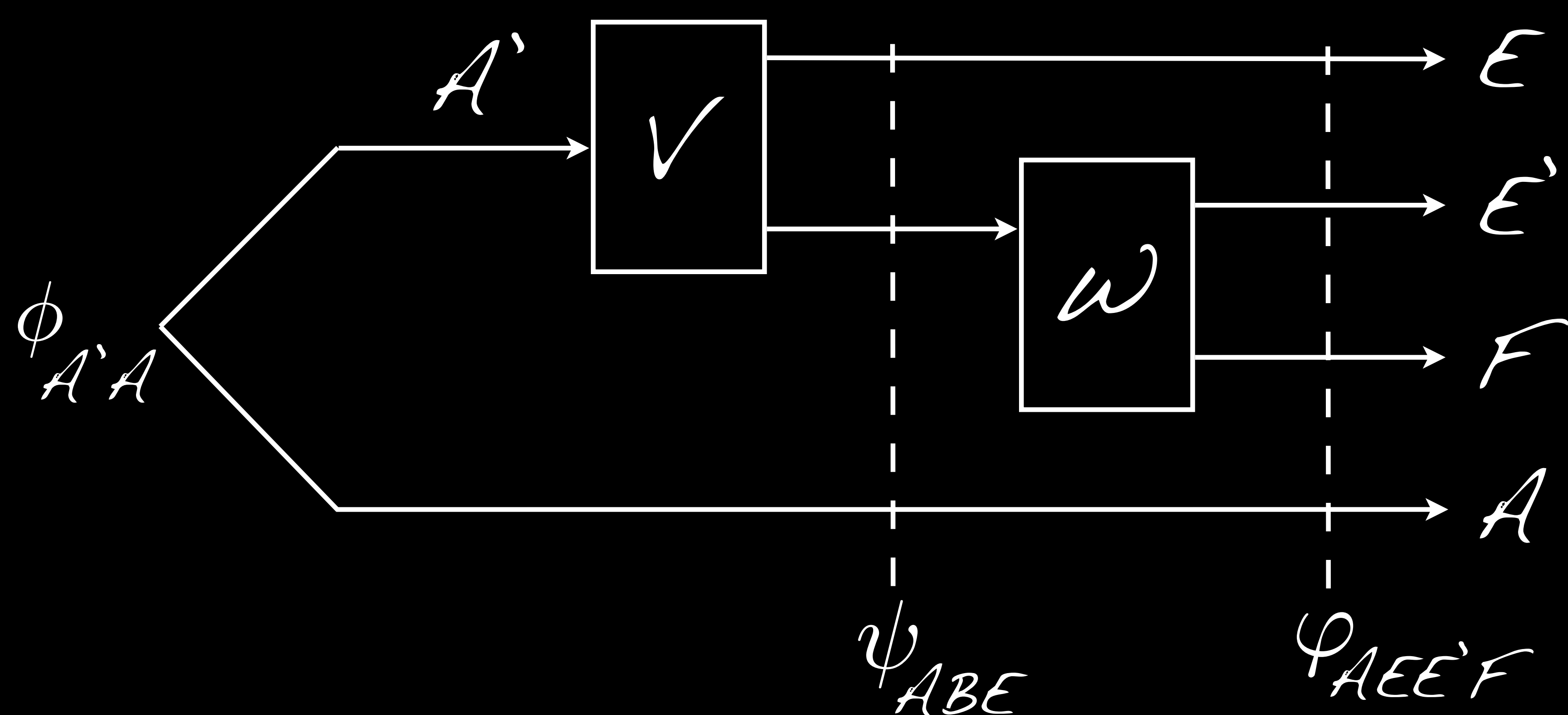

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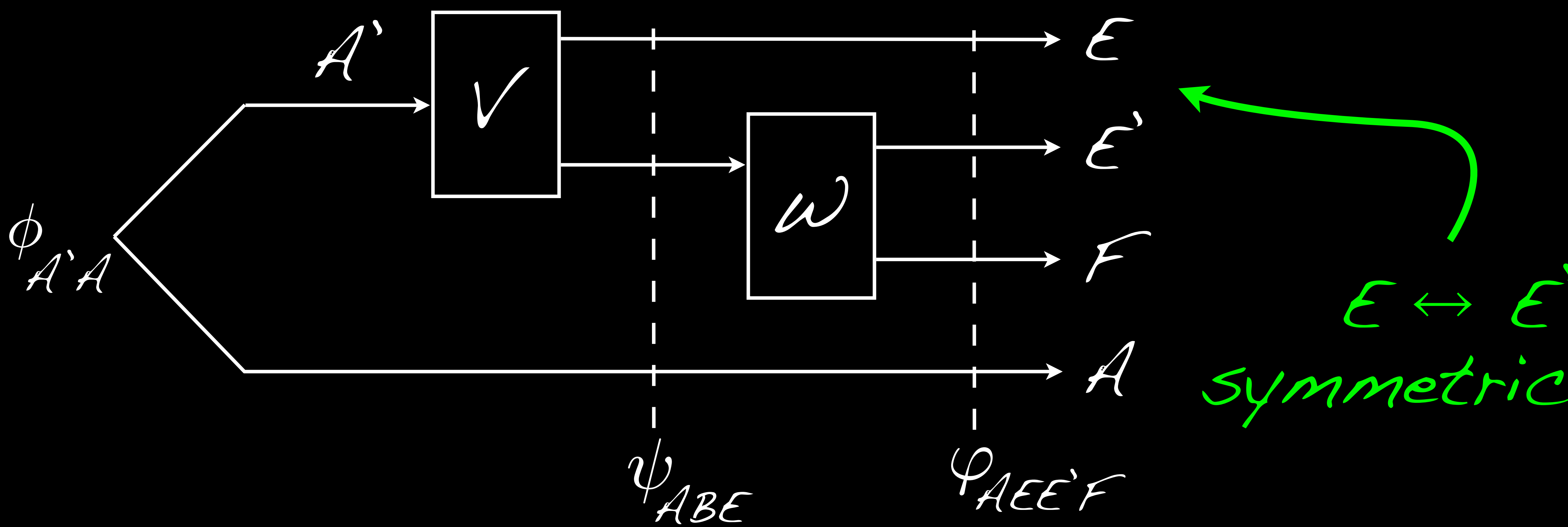
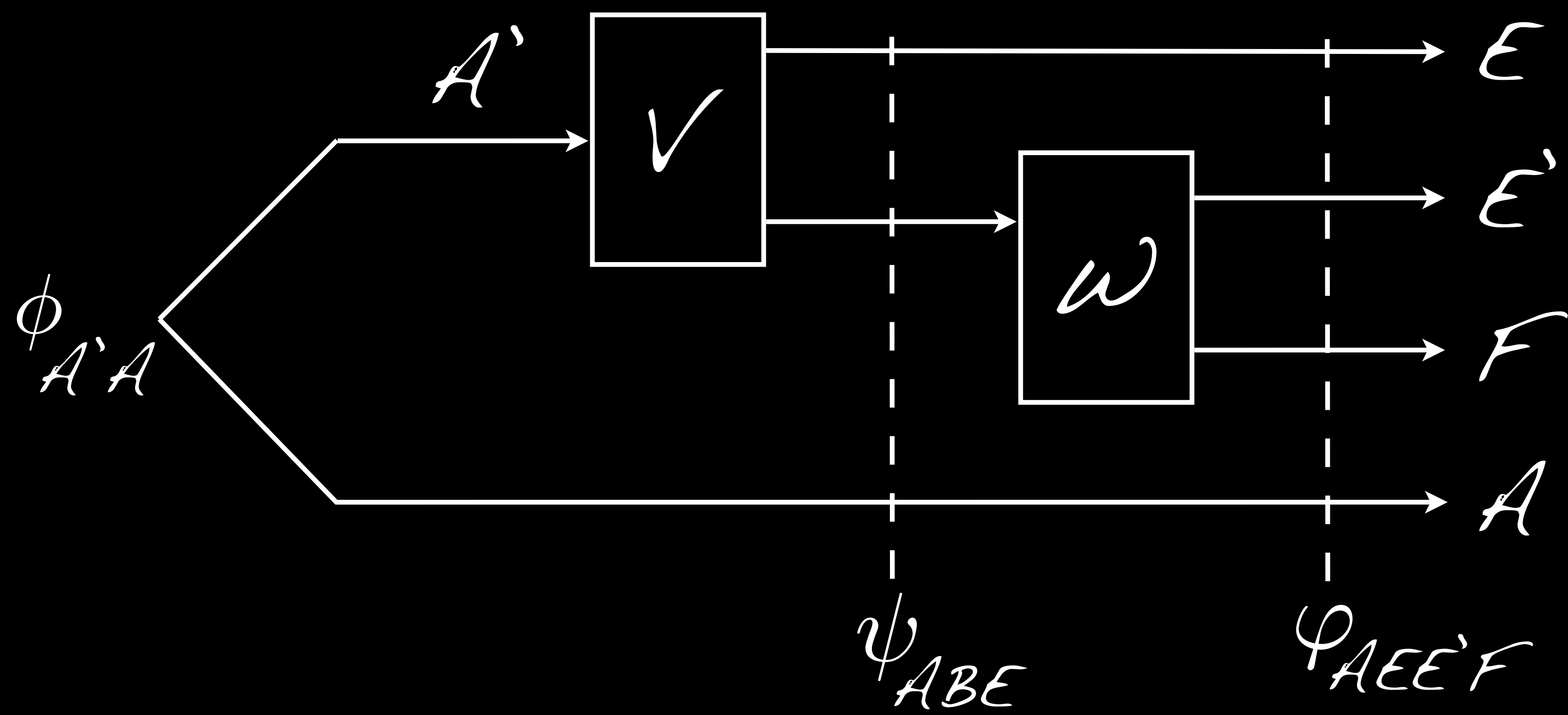
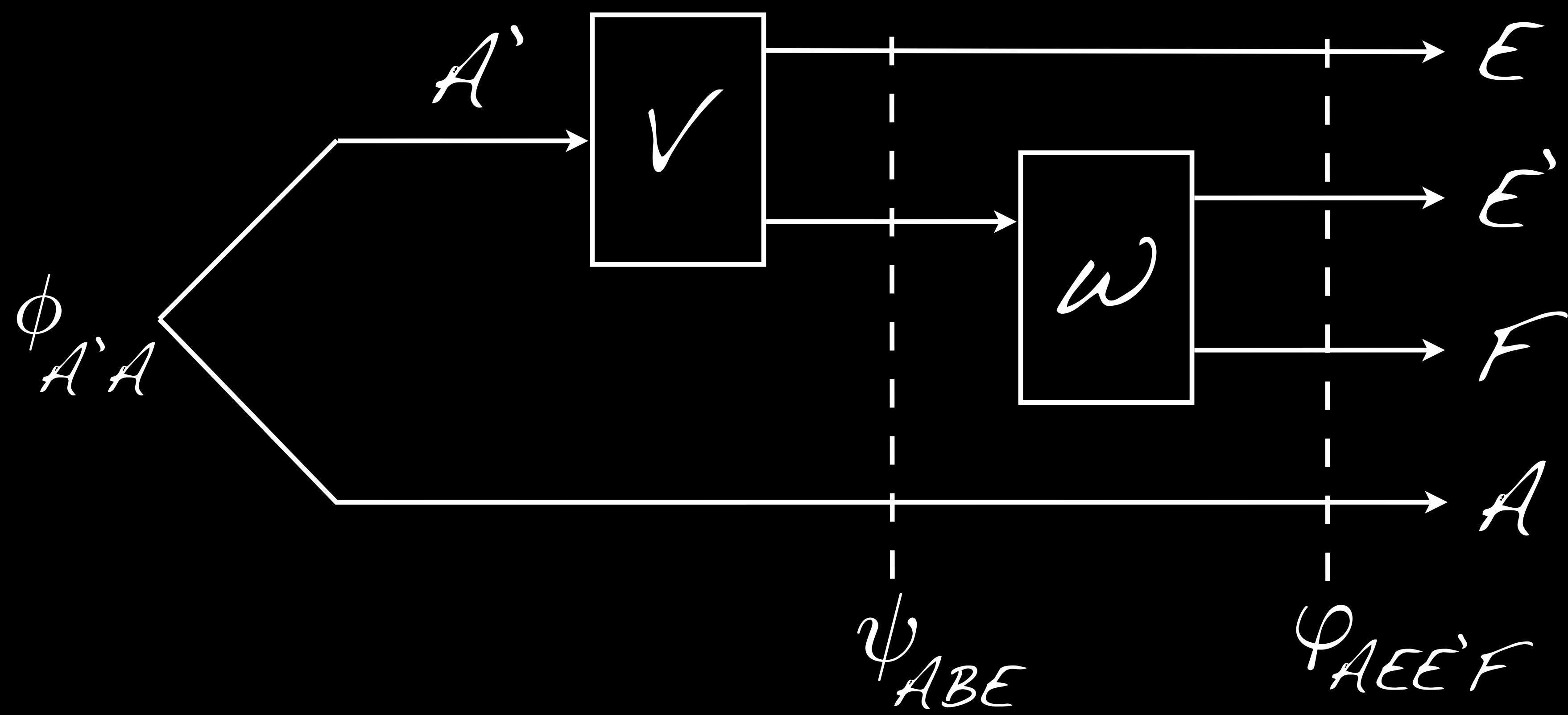




Examples:

1) Phase damping channel, more generally Schur multipliers and Hadamard channels



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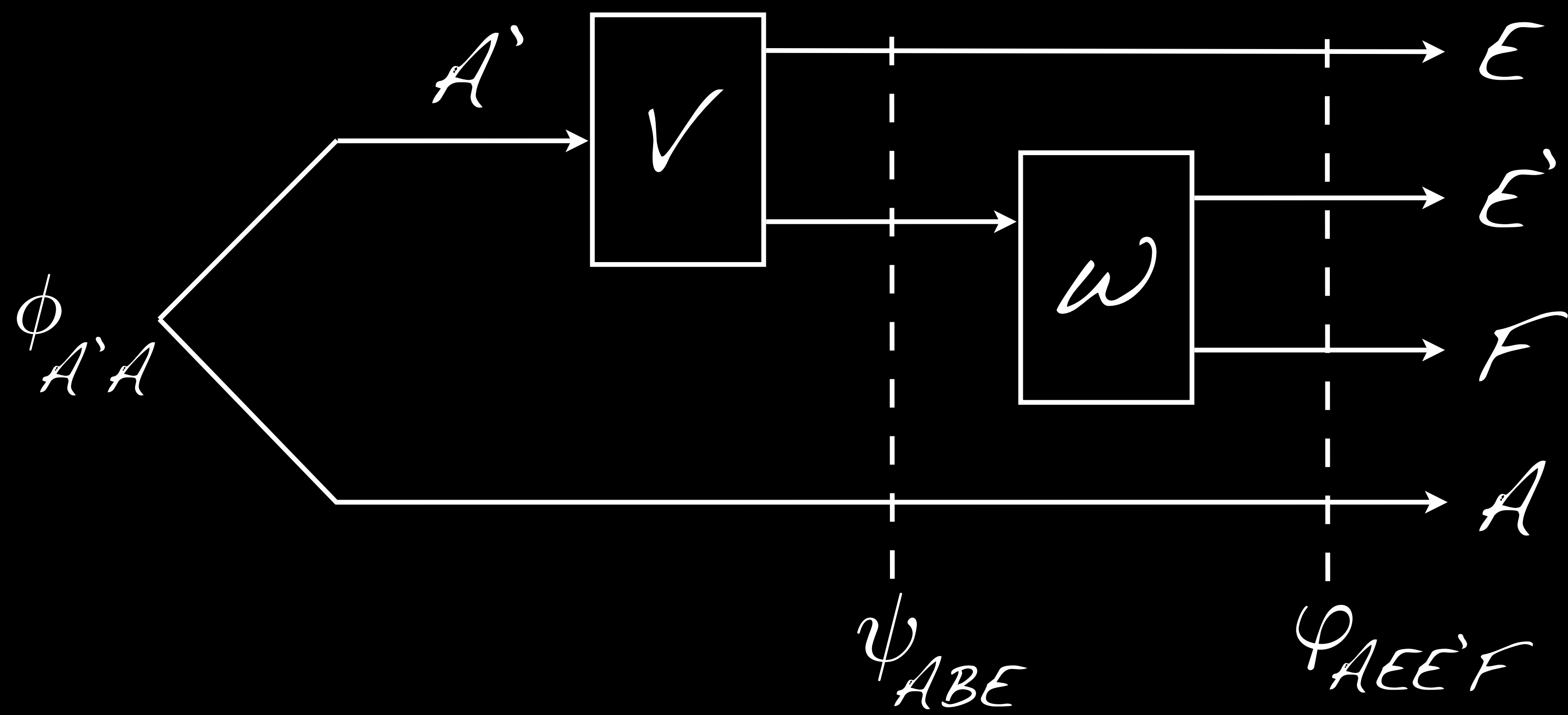

Examples:

1) Phase damping channel, more generally Schur multipliers and Hadamard channels
2) Amplitude damping channel
3) Symmetric channels, i.e. trivial F, for instance 50% erasure channel

A previous result [via E. Rains, IEEE-IT 47(7):2921-2933 (2001)]: If N is PPT entanglement-binding, then of course $Q(N)=0$, and strong converse holds (with error converging exponentially to 1).

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Note: Already for symmetric (degradable & anti-degradable) channels - for which also $Q(N)=0$ - not clear at all.

Thm (Morgan/AW, 1301.4927): For any degradable channel N, and codes with rate $R > Q(N)$ have error at least 0.707, asymptotically.

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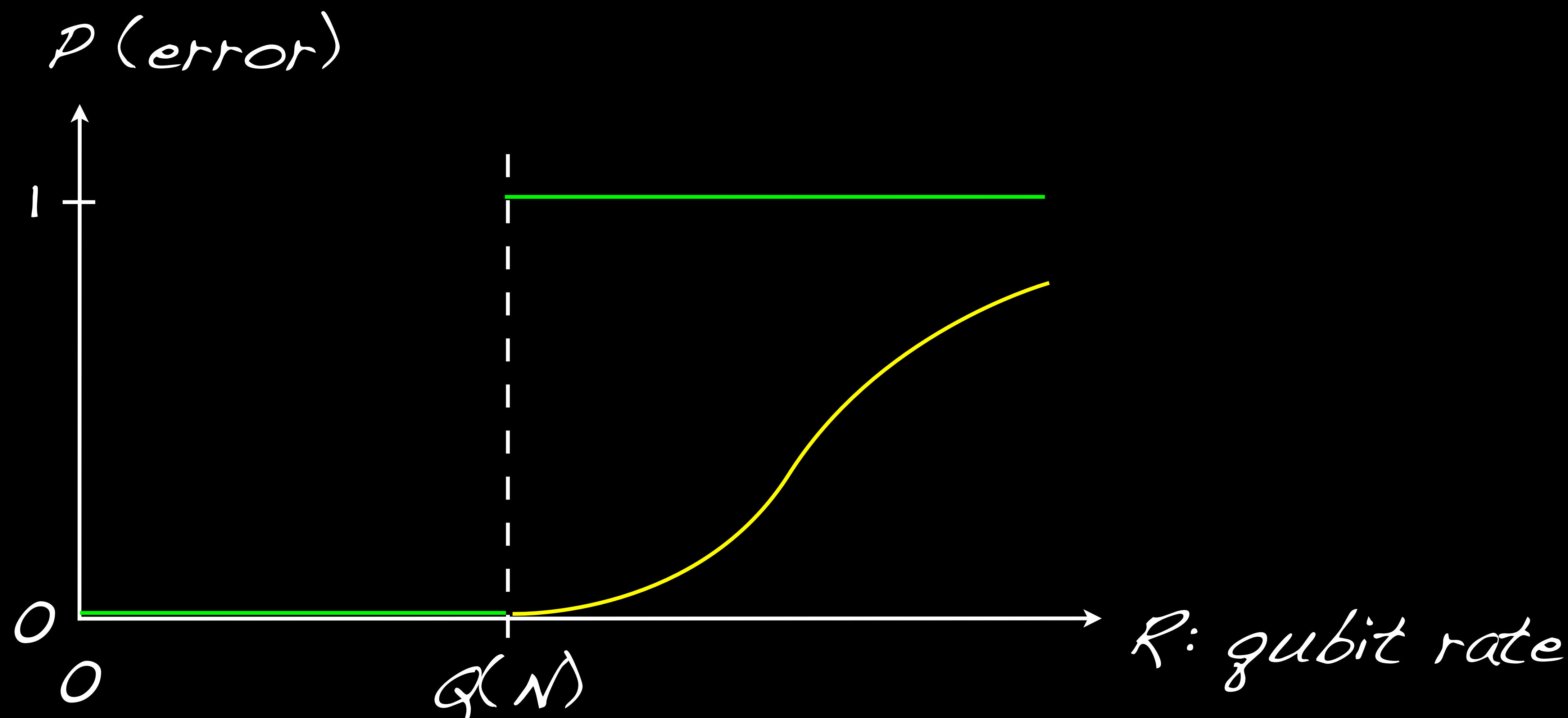

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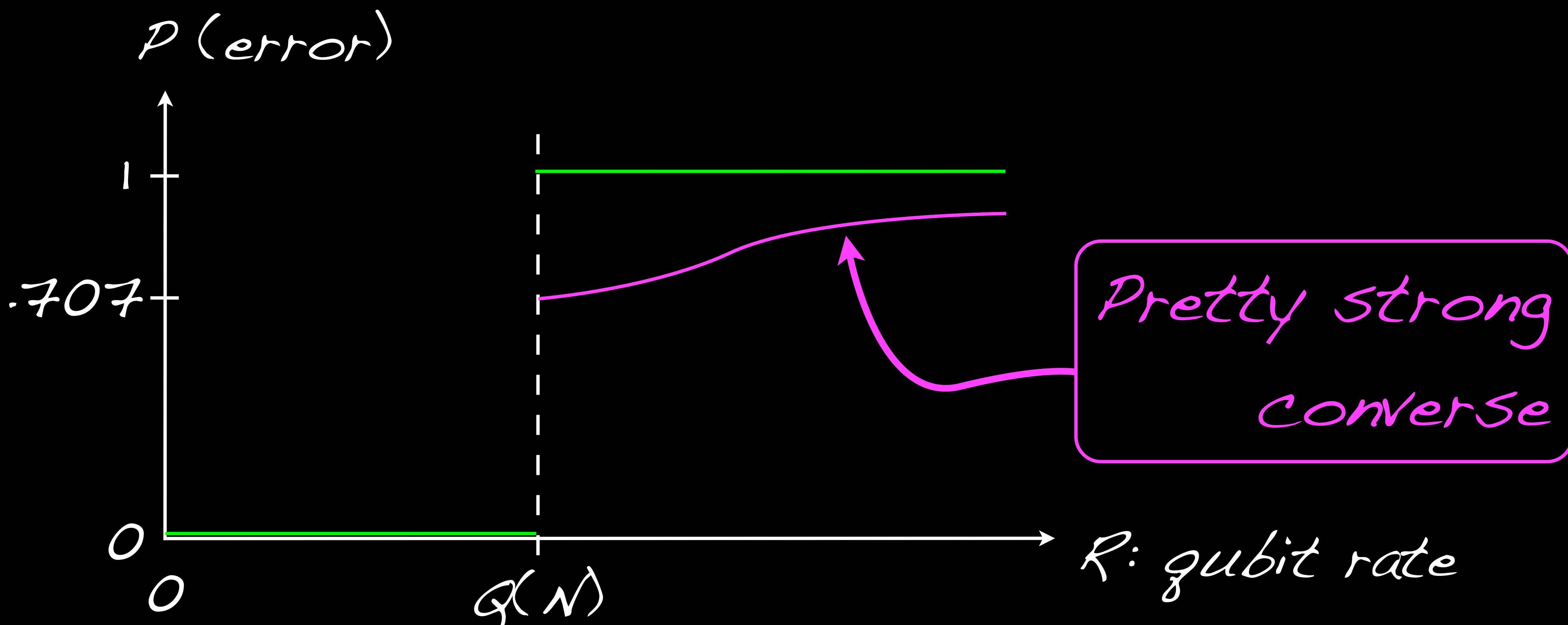

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On the other hand: For larger error, any i.i.d. symmetric channel allows coding of $k = c\sqrt{n}$ qubits, by random codes. More?

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Similar result for private capacity:

Thm (1301.4927): For degradable channel $\mathcal{N}$, if decoding error and distance from perfect privacy are both below some universal threshold, then the rate is asymptotically bounded by $P(\mathcal{N})=Q(\mathcal{N})$.

Thm: For any degradable channel N, codes with rate $R > Q(N)$ have error at least 0.707, asymptotically.

Significance of symmetric channels:

Thm (1301.4927): If symmetric channels (whose quantum capacity is 0) obey a strong converse, then so do all degradable channels N: for error below 1, the rate is asymptotically bounded by $Q(N)$.

Proof uses tight finite block length characterization of P and Q via (smooth) min-entropies & some tricks: symmetrization, de Finetti theorem, asymptotic equipartition property…

[Cf. R. Renner, PhD thesis, quant-ph/0512258 & M. Tomamichel, PhD thesis, arXiv:1203.2142]

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Can be viewed as a complicated version of the proof of additivity: $P(N)=Q(N)=Q^{(1)}(N)$ for degradable N... :-/

[Devetak/Shor, CMP 256:287 (2005)]

7. Conclusion (sort of...)

- The trick with the sandwiched channel reduces the additivity of $\chi(\mathcal{N})$ to that of the minimum output Rényi entropy of an associated family of cp (trace non-preserving) maps. Can it be applied to other channels? Other divergences?
- Can we also get "2nd order" behaviour? [Cf. Tomamichel/Tan, 1308.6503 for cq-channels]

7. Conclusion (sort of...)

- Big open problem: from pretty strong to really strong converse for Q of degradable channels!? Bottleneck are the symmetric channels, e.g. 50% erasure channels...
- How to prove strong converses without additivity? Note that neither P, Q nor $P^{(1)}$, $Q^{(1)}$, χ are generally additive!

(Not known for C.)


FINE

A. Proof ideas for C

A goody first: minimax characterisation of $\chi(\mathcal{N})$: [Schumacher/Westmoreland, PRA 2000]

$$\chi(\mathcal{N}) = \min_{\sigma} \max_{\rho} D(\mathcal{N}(\rho)\|\sigma)$$

Note: For EB and $\mathcal{H}$ channels $\mathcal{N}$ this is additive, and so $C(\mathcal{N}) = \chi(\mathcal{N})$.

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Fundamental property is monotonicity: for any cptp map N,

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Fundamental property is monotonicity: for any cptp map $\mathcal{N}$,

$$\tilde{D}(\rho\|\sigma) \geq \tilde{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq 0. \qquad (*)$$

Notation: for binary distributions $P=(p,1-p)$ and $Q=(q,1-q)$, write $\tilde{D}(P\|Q) = \tilde{D}(p\|q)$.

Assume furthermore that

$$\tilde{D}\Big(\bigoplus_x p_x \rho_x \,\Big\|\, \bigoplus_x p_x \sigma_x\Big) = \sum_x p_x \tilde{D}(\rho_x \| \sigma_x). \quad (+)$$

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$$\begin{aligned}
\tilde{D}(1-\varepsilon \| 1/M) &\overset{(*)}{\leq} \tilde{D}(\rho_{XB} \| \rho_X \otimes \sigma) \\
&\overset{(+)}{\leq} \frac{1}{M} \sum_m \tilde{D}(\mathcal{N}(\rho_m) \| \sigma) \\
&\leq \max_\rho \tilde{D}(\mathcal{N}(\rho) \| \sigma) =: \chi_{\tilde{D},\sigma}(\mathcal{N})
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[Cf. Nagaoka (≈2000); Polyanskiy/Verdú (2010); Sharma/Warsi, 1205.1712.]

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[Cf. Müller-Lennert et al., 1306.3142; Beigi 1306.5920; Frank/Lieb 1306.5358]

It's monotonic, has property (+) and is $\leq D_{\alpha}(\rho \,\|\, \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \rho^{\alpha} \sigma^{1-\alpha}$, with which it coincides when states commute.

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Crucial: $-\chi_{\alpha,\sigma}(\mathcal{N})$ is the minimum α-Rényi output entropy of a perturbed cp map $\mathcal{N}'$,

$$\mathcal{N}'(\rho) = \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}(\rho)\, \sigma^{\frac{1-\alpha}{2\alpha}}.$$

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Key observation: Sandwiched channel is $(\mathcal{N}')^{\otimes n}$, and $\mathcal{N}'$ is EB if $\mathcal{N}$ is.

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$\Rightarrow$ Additivity, $\chi_{\alpha,\sigma}(\mathcal{N}^{\otimes n}) = n\,\chi_{\alpha,\sigma}(\mathcal{N})$.

(Because of identity with min output entropy of $\mathcal{N}'$)

[King, QIC 2003; Holevo, Russ. Math. Surveys 2006]

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$$\log(1-\varepsilon) \leq n\left(1-\frac{1}{\alpha}\right)\left(\chi_{\alpha,\sigma}(\mathcal{N}) - R\right). \qquad (\&)$$

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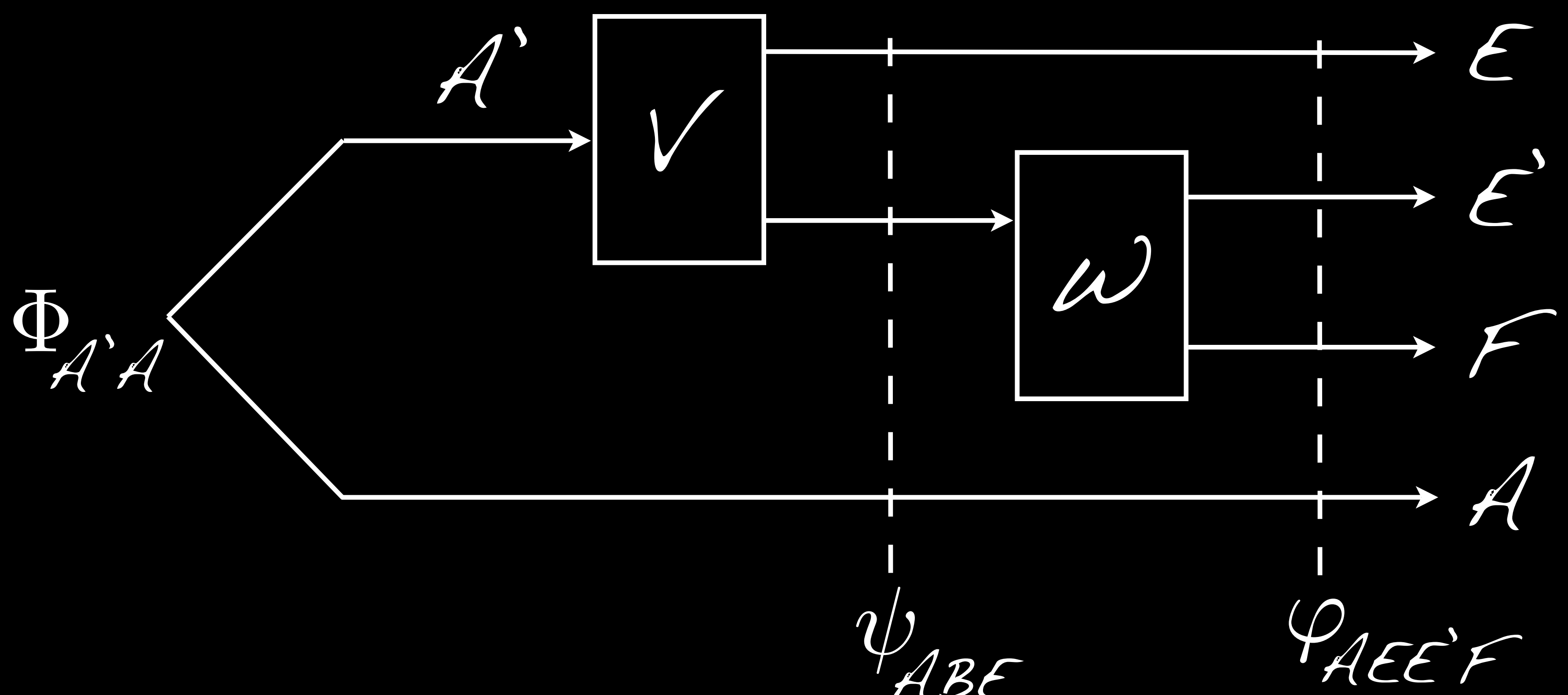
Takes care of EB channels; $\mathcal{H}$ similar but requires another small trick (...) QED

B. Proof ideas for Q & P

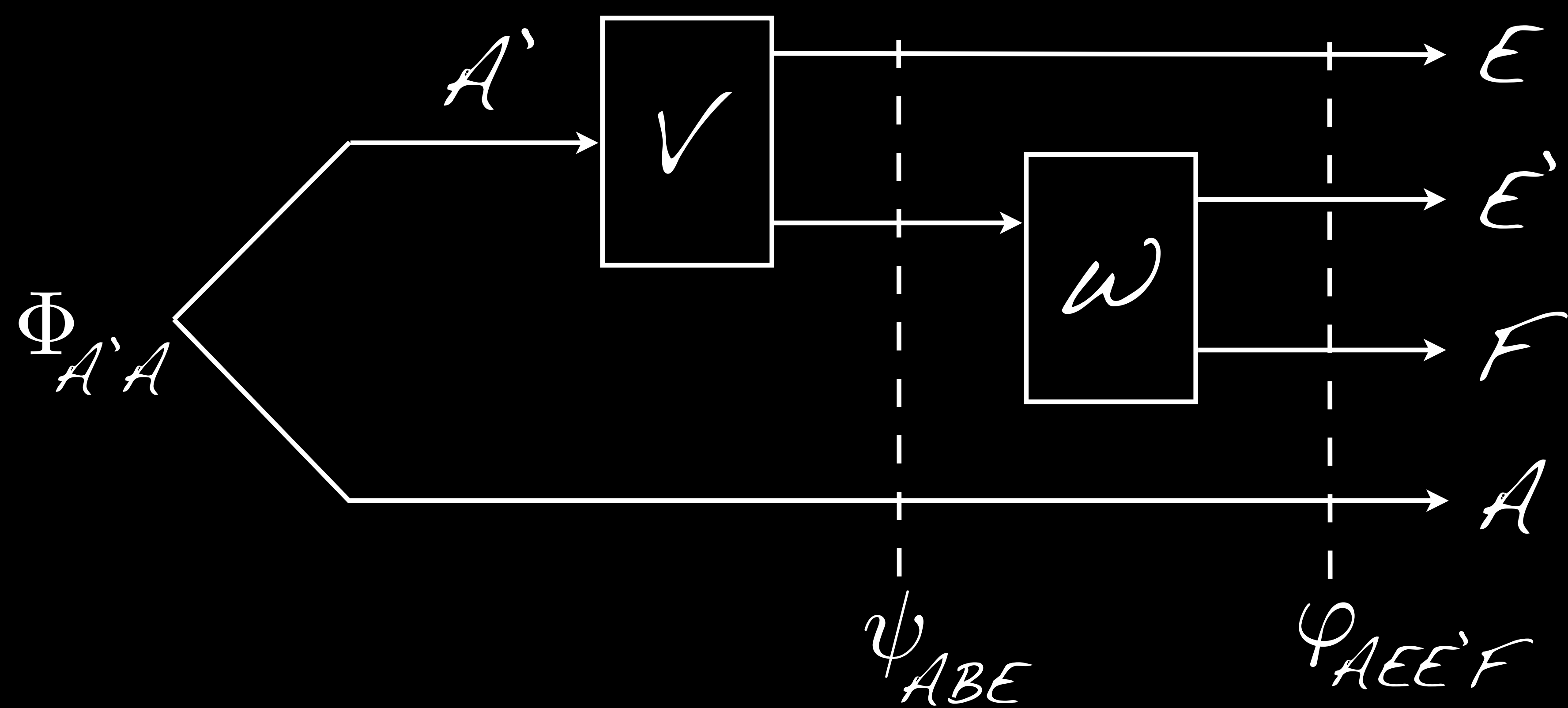
(smooth) min-entropies,
symmetrisation, de Finetti theorem,
AEP

Ideas: (smooth) min-entropies,
symmetrisation, de Finetti theorem,
AEP

1) Use code - for simplicity subspace - with maximally entangled state Φ of k qubits:



Maximally entangled state Φ of k qubits:



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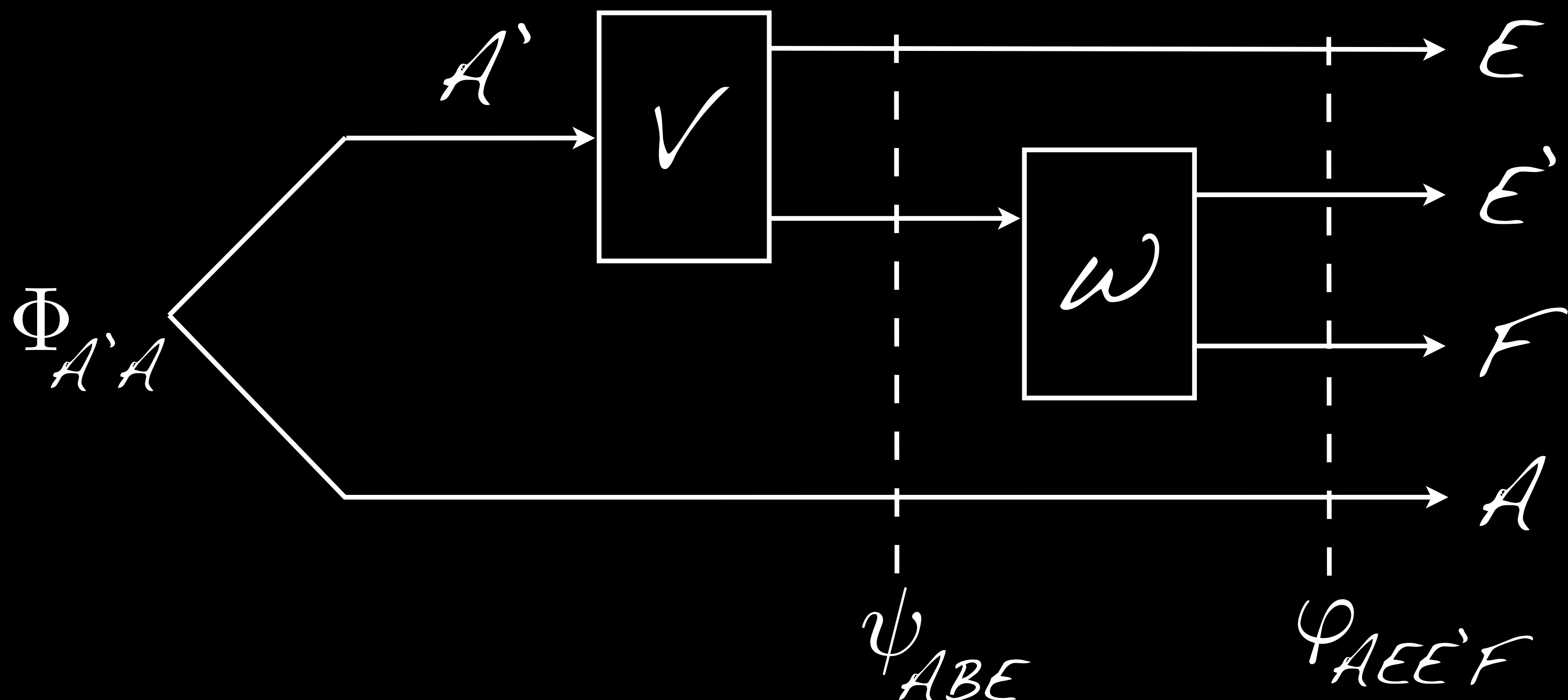

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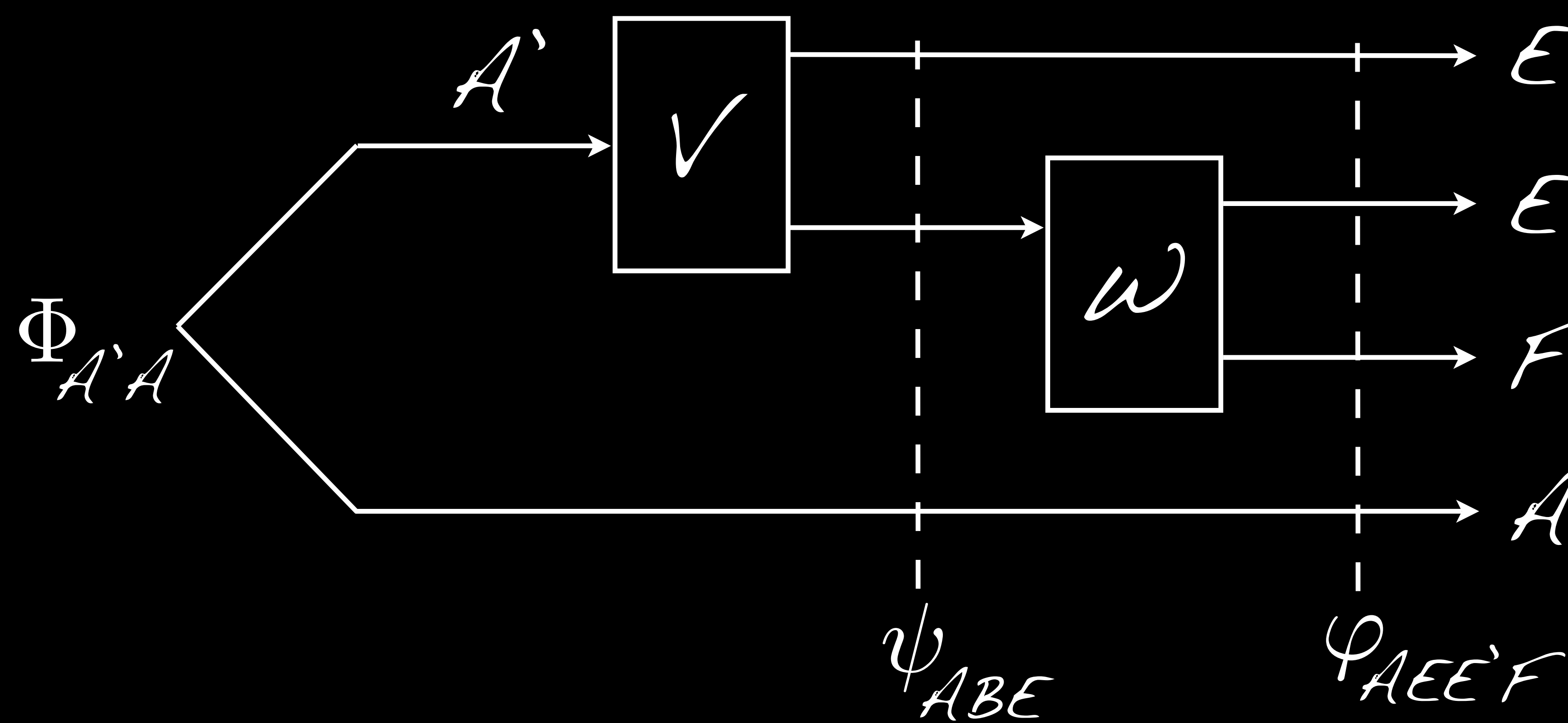

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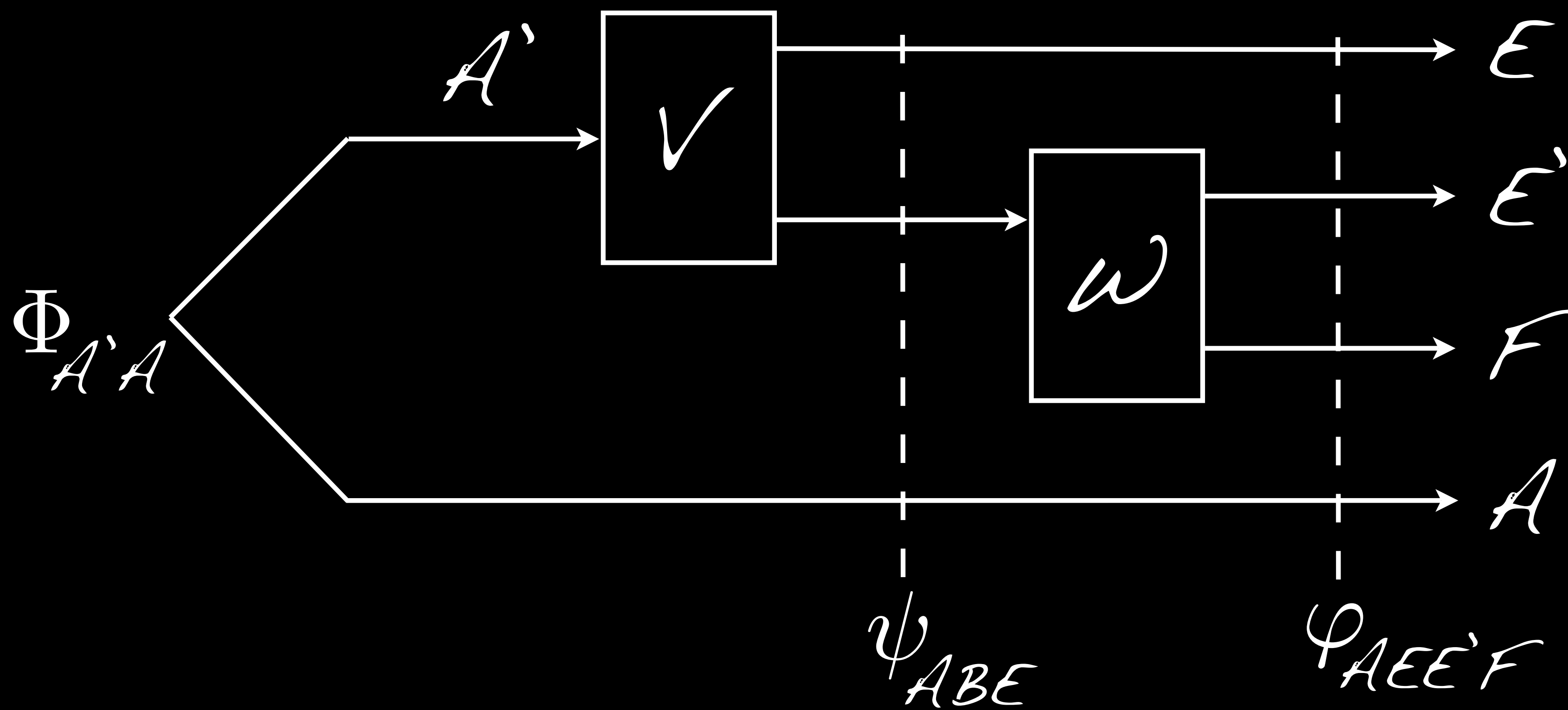

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[For min-entropy calculus, consult
R. Renner, PhD thesis, quant-ph/0512258
& M. Tomamichel, PhD thesis, arXiv:1203.2142]

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[Cf. also Buscemi/Datta, IEEE-IT 56(3), 2010; Datta/Hsieh, 1103.1135]

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Note: If we knew that for n channel uses, the maximum min-entropy is attained on a tensor product input, we'd be done by AEP (= asymptotic equipartition property)...

$$
\begin{aligned}
k &\le H_{min}^{\epsilon}(A|E) \\
&= -H_{max}^{\epsilon}(A|E'F) \\
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...if $\delta < 0.707$, by inequality H_{min} vs. H_{max}, and using symmetry between E and E'...

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w.r.t. a permutation symmetric input state and $\lambda' = \lambda / \sqrt{2}$

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3) By de Finetti theorem

[R. Renner, PhD thesis, quant-ph/0512258]:

$$k \leq \max_{\rho_A} H_{max}^{\lambda''}(F^n | E'^n)_{\rho^{\otimes n}} + o(n)$$

4) By AEP (asymptotic equipartition property) [M. Tomamichel, arXiv:1203.2142]:

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QED