Local Quantum Uncertainty and Bound on Quantumness for orthogonally invariant class of states

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- The basic problem here is: the behavior of quantum systems is not fully understood whenever there are more number of parties involved.
- In other words, there exist a kind of correlation between the parties involved which is not explainable by classical scenario.

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- As entanglement is used as a resource in many information processing tasks, (e.g, teleportation, dense coding, etc.) therefore, the characterization and quantification problems are the some fundamental issues generated in the last two decades. However, there are lot of difficulties.
- As far as bipartite entanglement is concerned we have at least some knowledge how to deal with entangled states. For pure bipartite states there exists a unique measure of entanglement calculated by Von- Neumann entropy of reduced density matrices.

Contd..

- However for mixed entangled states there is no unique measure of entanglement. One has to look on different ways to quantify entanglement. Some of the measures of entanglement are distillable entanglement, entanglement cost, entanglement of formation, relative entropy of entanglement, logarithmic negativity, squashed entanglement, etc.

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- The Difficulty: In most of the cases it is really hard to calculate exactly the measures of entanglement. Only for some few classes of states, actual values are available. A similar problem is that it is hard to find whether a mixed bipartite state is entangled or not.
- The situation in multipartite case is more complex than that of bipartite case. e.g., how could we define a measure of entanglement for multipartite states at least for pure states are concerned. It is also very difficult to define maximally entangled states in multipartite systems.

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- Consider a mixed entangled state in a multipartite system with the property that it has maximal entanglement w.r.t. any bipartite cut (i.e., reduced density matrices corresponding to the cut is proportional to the identity operator), then we observe that for n-qubit ($n \geq 3$) system, there does not exist any maximally entangled states for $n = 4$ and $n \geq 8$.

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- So one has to think how to define maximally entangled states for such situations. Gour and others have defined maximally entangled states in 4-qubit system considering some operational interpretation. A possible way: the average bipartite entanglement w.r.t. all possible bipartite cuts the state is maximal.

Correlation measures beyond entanglement

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- The above state is separable. However, it has non-zero quantum discord which is defined by the difference of measuring mutual information in two different ways, $D(A,B) = I(A:B) - J(A:B)$ where, $I(A:B) = S(A) - S(A|B)$ and $J(A:B) = S(A) - \min_{\Pi_j} \sum_j p_j S(A|j)$

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- The above quantity is a measure of non-classical correlation. It has zero value if and only if there exists a Von Neumann-measurement $\Pi_j = |\psi_j\rangle\langle\psi_j|$ such that the bipartite state $\rho = \Pi_j \otimes I\rho\Pi_j \otimes I$.

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- The set of Classical-Quantum states is non convex. Due to the optimization problem, it is in general very hard to find analytic expression for discord. Exact analytical result is available only for a few classes of states. It was found that Quantum discord is always non-negative and it reduces to Von Neumann Entropy of the reduced density matrix for pure bipartite states.

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- There are other variations of discord and their extensions to multipartite systems have also been proposed. e.g., Geometric discord: $D(\rho) = \min_{\chi} ||\rho - \chi||$ where the minimum is taken over all zero discord state χ.

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- Exact analytical formula for geometric discord is also available for only a few class of states. A tight lower bound is found recently.

Quantum (Un)Certainty

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$$\frac{1}{\sqrt{2}}|00\rangle + |11\rangle$$

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How to measure this uncertainty?
How to extend this idea to mixed state also?

For this we need to build a measure and which does not affected by *classical mixing*.

Introduction

1. Classically, it is possible to measure any two observable with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation gives the statistical nature of errors in these kind of measurement. Measurement of single observable can also help to detect uncertainty of a quantum observable.

Introduction

1. Classically, it is possible to measure any two observable with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation gives the statistical nature of errors in these kind of measurement. Measurement of single observable can also help to detect uncertainty of a quantum observable.

2. For a quantum state ρ, an observable is called *quantum certain* if the error in measurement of the observable is due to only the ignorance about the classical mixing in ρ. A good quantifier of this uncertainty is the skew information.

Skew information

Wigner and Yanase introduced the quantity

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It quantifies non-commutativity between a quantum state and an observable so it serves as a measure of uncertainty of the observable K in the state ρ. This type of measure helps to quantify the quantum part of error in measuring an observable. $I = 0$ indicates quantum certain nature of the observable K. It is also convex and *non-increasing under classical mixing*. It has a nice property that $I(\rho, K) \leq Var(\rho, K)$.

Local Quantum Uncertainty

For a bipartite quantum state ρ_{AB}, Girolami *et.al.* (Phys. Rev. Lett. **110**, 240402 (2013)) introduced the concept of local quantum uncertainty(LQU) and it is defined as

$$\mathcal{U}_A^{\Lambda} := \min_{K^{\Lambda}} I(\rho_{AB}, K^A)$$

The minimization is performed over all local *maximally informative observable* (or non-degenerate spectrum Λ) $K^{\Lambda} = K_A^{\Lambda} \otimes \mathbb{I}$. This quantity quantifies the minimum amount of uncertainty in a quantum state. Non-zero value of this quantity indicates the non existence of any quantum certain observable for the state ρ_{AB}.

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- It reduces to entanglement monotone for pure state. In fact, for pure bipartite states it reduces to linear entropy of reduced subsystems. So, LQU can be taken as a measure of bipartite quantumness.
- For a quantum state ρ of $2 \otimes n$ system, LQU reduces to $1 - \lambda_{max}(\mathcal{W})$ where λ_{max} is the maximum eigenvalue of the matrix $\mathcal{W} = (w_{ij})_{3\times 3}$, $w_{ij} = \mathrm{tr}\{\sqrt{\rho}(\sigma_i \otimes \mathbb{I})\sqrt{\rho}(\sigma_j \otimes \mathbb{I})\}$ and σ_i's are standard Pauli matrices in this case.
- Geometrically, LQU in a state ρ of a $2 \times n$ system is the minimum Hellinger distance between ρ and the state after a least disturbing root-of-unity local unitary operation applied on the qubit.

Orthogonal Invariant State

Any $\mathcal{O} \otimes \mathcal{O}$ invariant state from a $n \times n$ system can be taken as

$$\rho = a\,\mathbb{I}_{n^2} + b\,\mathbb{F} + c\,\hat{\mathbb{F}}$$

with $n(na + b + c) = 1$ (trace condition) and proper positivity constraints. $\mathbb{I}$ is the identity operator, $\mathbb{F}$ is the flip operator and $\hat{\mathbb{F}}$ is the projection on maximally entangled state.
The operators satisfy the algebra $\mathbb{F}^2 = \mathbb{I}, \mathbb{F}\hat{\mathbb{F}} = \hat{\mathbb{F}}\mathbb{F} = \hat{\mathbb{F}}, \hat{\mathbb{F}}^2 = n\,\hat{\mathbb{F}}$, n is the dimension of each subsystem.

Orthogonal Invariant State

The parametrization procedure can be done in another way by considering the expectation values of the operators $\mathbb{I}_n$, $\mathbb{F}$, $\hat{\mathbb{F}}$. Expectation value of $\mathbb{I}_n$ just gives the relation $\operatorname{tr}\rho = 1$ which is obvious. We define two parameters f and $\hat{f}$ as

$$f := \langle \mathbb{F} \rangle_\rho = \operatorname{tr}(\rho \mathbb{F})$$
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We can define three orthogonal projectors U, V and W as,

$$\begin{aligned} U &= \hat{\mathbb{F}}/n \\ V &= (\mathbb{I}_{n^2} - \mathbb{F})/2 \\ W &= (\mathbb{I}_{n^2} + \mathbb{F})/2 - \hat{\mathbb{F}}/n \end{aligned}$$

Orthogonal Invariant State

In terms of this orthogonal basis, ρ can be expressed as,

$$\rho = \frac{\hat{f}}{n}U + \frac{1-f}{n(n-1)}V + \frac{n+nf-2\hat{f}}{n(n-1)(n+2)}W$$

The old parameters a, b, c are connected to the new ones f, $\hat{f}$ by the relation,

$$\begin{pmatrix} 1 \\ f \\ \hat{f} \end{pmatrix} = n \begin{pmatrix} n & 1 & 1 \\ 1 & n & 1 \\ 1 & 1 & n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In terms of the new parameters the positivity conditions on ρ reads,$0 \leq \hat{f}, f \leq 1, \hat{f} \leq n(f+1)/2$

Generators of SU(n)

We can construct traceless, orthogonal generators (*generalized Gell-Mann matrices*) for SU(n), containing $n^2 - 1$ elements as:

$$\lambda_\alpha = \begin{cases} \sqrt{\frac{2}{\alpha(\alpha+1)}}\left(\sum_{k=1}^{\alpha} |k\rangle\langle k| - \alpha|\alpha+1\rangle\langle\alpha+1|\right), \alpha = 1, ..., n-1 \\ |k\rangle\langle m| + |m\rangle\langle k|, 1 \leq k < m \leq n, \alpha = n, ..., \frac{n^2+n}{2} - 1 \\ \mathrm{i}(|k\rangle\langle m| - |m\rangle\langle k|), 1 \leq k < m \leq n, \alpha = \frac{n(n+1)}{2}, ..., n^2 - 1 \end{cases}$$

Generators of SU(n)

Among the (n^2-1) matrices, the first $(n-1)$ are mutually commutative, next $(n^2-1)/2$ are symmetric and rest $(n^2-1)/2$ are antisymmetric. The generators λ_α satisfy the orthogonality relation $\mathrm{tr}(\lambda_\alpha\lambda_\beta) = 2\delta_{\alpha\beta}$. The generators satisfy the following commutation and anti-commutation relations,

$$\begin{aligned}[\lambda_i,\, \lambda_j] =& 2\mathrm{i} \sum_k f_{ijk}\, \lambda_k \\ \{\lambda_i,\, \lambda_j\} =& 2 \sum_k d_{ijk}\, \lambda_k + \frac{4}{n}\delta_{ij}\, \mathbb{I}_n\end{aligned}$$

Hence

$$\lambda_i\, \lambda_j = \mathrm{i} \sum_k f_{ijk}\, \lambda_k + \sum_k d_{ijk}\, \lambda_k + \frac{2}{n}\delta_{ij}\, \mathbb{I}_n$$

LQU for Orthogonal Invariant States

$\sqrt{\rho}$ can expressed as,

$$
\begin{aligned}
\sqrt{\rho} =& \sqrt{\frac{\hat{f}}{n}}U + \sqrt{\frac{1-f}{n(n-1)}}V + \sqrt{\frac{n+nf-2\hat{f}}{n(n-1)(n+2)}}W \\
=& a_1 \, \mathbb{I}_{n^2} + b_1 \, \mathbb{F} + c_1 \, \hat{\mathbb{F}}
\end{aligned}
$$

with a_1, b_1, c_1 are functions of $f, \hat{f}$.

LQU for Fixed spectrum

Particularly we choose an class of non-degenerate A-observable $K_A^\Lambda = \mathbf{s}.\lambda$ with $\mathbf{s} = (s_1, s_2, ..., s_{n^2-1})$, $|\mathbf{s}| = 1$ and $\lambda = (\lambda_1,\ \lambda_2, ..., \lambda_{n^2-1})$. We also want that this observable has following spectrum: $\pm(n-1), (n-2), ..., 0$ when n is even and $\pm(n-2), (n-3), ..., 1$ when n is odd. Hence, s_i's also satisfy other functional relations to satisfy the spectrum condition. In this case

$$\begin{aligned}
\mathcal{U}_A^\Lambda(\rho) &= \min_{K^\Lambda} I(\rho, K^\Lambda) \\
&= \min_{K^\Lambda} \{\mathrm{tr}(\rho(K^\Lambda)^2) - \mathrm{tr}(\sqrt{\rho} K^\Lambda \sqrt{\rho} K^\Lambda)\} \\
&= \min_{\mathbf{s}} \{\mathrm{tr}\{\rho(\mathbf{s}.\lambda \otimes \mathbb{I}_n)^2\} - \mathrm{tr}\{\sqrt{\rho}(\mathbf{s}.\lambda \otimes \mathbb{I}_n)\sqrt{\rho}(\mathbf{s}.\lambda \otimes \mathbb{I}_n)\}\}
\end{aligned}$$

The maximum is over all $\mathbf{s}$ with $|\mathbf{s}| = 1$ and s_i's also satisfies all necessary conditions to build the chosen spectrum.

LQU for Orthogonal Invariant States

For any state with $\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) = 0$,

$$\mathcal{U}_A^{\Lambda}(\rho) = \frac{2}{n} - \max_{\mathbf{s}}(\mathbf{s}.\mathcal{W}.\mathbf{s}^{\dagger})$$

The elements of the matrix $\mathcal{W}$ is defined as

$$w_{ij} = \mathrm{tr}\{\sqrt{\rho}(\lambda_i \otimes \mathbb{I}_n)\sqrt{\rho}(\lambda_j \otimes \mathbb{I}_n)\}$$

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For any $\mathcal{O} \otimes \mathcal{O}$ invariant state $\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) = 0$ and $\mathcal{W}$ is diagonal. It is also possible to choose **s** which satisfies all the constraints. Hence we can simplify the value of $\mathcal{U}_A$ in terms of maximum eigenvalue λ_{max} of $\mathcal{W}$ as

$$\mathcal{U}_A^{\Lambda}(\rho) = \frac{2}{n} - \lambda_{max}(\mathcal{W})$$

The above result can work as a lower bound for the large class of states with $\mathrm{tr}(\rho\lambda_i \otimes \mathbb{I}_n) = 0,\ i = 1, 2, ..., n^2 - 1$

Two Qutrit system

For $\mathcal{O}\otimes\mathcal{O}$ invariant state in two-qutrit system, eigenvalues of $\mathcal{W}$ are $2(3a_1^2 \pm 2b_1c_1 + 2a_1b_1 + 2a_1c_1)$. Hence in this case

$$\mathcal{U}_A^{\Lambda} = \frac{2}{3} - 2(3a_1^2 + 2|b_1c_1| + 2a_1b_1 + 2a_1c_1)$$

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For Werner ($c = 0$) and Isotropic ($b = 0$) class of states in two-qutrit system, the eigenvalues of $\mathcal{W}$ become all equal. Hence the explicit form of LQU are,

$$\mathcal{U}_A^{\Lambda}(\rho^{wer}) = \frac{1}{3}(1 - \sqrt{1-12b}\sqrt{1+6b}) - b$$
$$\mathcal{U}_A^{\Lambda}(\rho^{iso}) = \frac{4}{27}(1 - \sqrt{1-3c}\sqrt{1+24c}) + \frac{14}{9}c$$

The results can be easily transferred in terms of f and $\hat{f}$.

Two Qutrit system

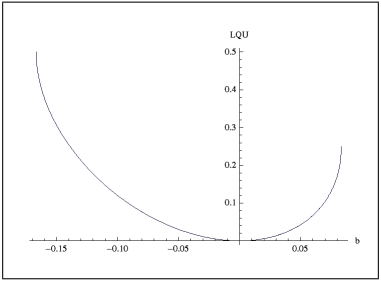

Figure : *LQU for Werner class of states in two-qutrit system for suitable parameter range of b. The class is obtained by putting $c = 0$ in (12). The highest value of $\mathcal{U}_A^{\Lambda}$ reaches 0.5*

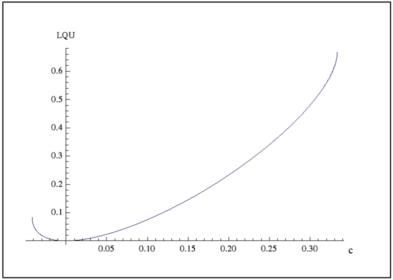

Figure : *LQU for Isotropic class of states in two-qutrit system for suitable range of the parameter c. The class is obtained by putting $b = 0$ in (12). The highest value of $\mathcal{U}_A^{\Lambda}$ reaches 0.66 in this case.*

Two Qutrit system

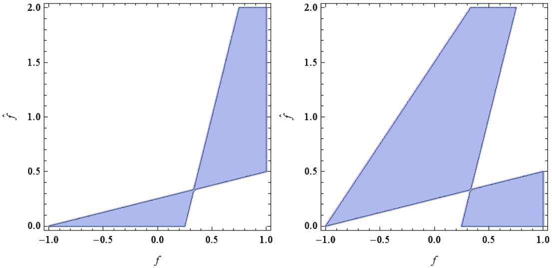

Figure : *Region plot in (f $\hat{f}$)-plane of the eigenvalue of $\mathcal{W}$ for two-qutrit orthogonal invariant class. Both the regions are enclosed by the constraints* $0 \leq \hat{f}, f \leq 1, \hat{f} \leq 3(f+1)/2$. *First figure shows the shaded region where* $b_1 c_1 \geq 0$ *and second one shows the shaded region where* $b_1 c_1 < 0$. *Hence in the first region* $\mathcal{U}_A^\Lambda = \frac{2}{3} - 2(3a_1^2 + 2b_1 c_1 + 2a_1 b_1 + 2a_1 c_1)$ *and in the second* $\mathcal{U}_A^\Lambda = \frac{2}{3} - 2(3a_1^2 - 2b_1 c_1 + 2a_1 b_1 + 2a_1 c_1)$

A General Approach

We can consider d^2 elements of SU(d) as

$$u_{nm} := \sum_{j=0}^{d-1} \exp(\frac{2\pi \mathrm{i} jn}{d})|j\rangle\langle j \oplus m \bmod d|; \;\; n, m = 0, ..., d-1 \qquad (1)$$

Now consider any general observable $K = \mathbf{s}.\boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda} = (\lambda_0, \lambda_1, ..., \lambda_{d-1})$ and λ_i's are d diagonal matrices of order d^2 with only single entry 1 at corresponding ii-th position. λ_i's can be obtained from linear combination of u_{n0}'s. Hence

$$K = \mathbf{s}.\boldsymbol{\Lambda} = \sum_{i=0}^{d-1} s_i\lambda_i = \sum_{i=0}^{d-1} t_i u_{i0}; \; t_i\text{'s are functions of } s_i\text{'s} \qquad (2)$$

A General Approach

For any unitarily connected observable with same spectrum,

$$
\begin{aligned}
VKV^\dagger &= t_0\mathbb{I} + \sum_{i=1}^{d-1} t_i\mathbf{K}_i.\mathbf{\Lambda};\ \text{ where we take } Vu_{i0}V^\dagger = \mathbf{K}_i.\mathbf{\Lambda} \\
&= \mathbf{m}.\mathbf{\Lambda} + t_0\mathbb{I};\ \text{ with } \mathbf{m} = \sum_i t_i\mathbf{K}_i \\
&= \parallel \mathbf{m} \parallel \hat{\mathbf{m}}.\mathbf{\Lambda} + t_0\mathbb{I}
\end{aligned} \tag{3}
$$

Hence we can safely choose any observable(*maximally informative*) as $\hat{\mathbf{m}}.\mathbf{\Lambda}$ and perform the optimization over all unit vector $\hat{\mathbf{m}}$. The amount of LQU are proportional on all such orbits. In fact the optimization problem turns out to be

$$
\mathcal{U}_A^\Lambda(\rho) = \min_{m_i, \sum m_i^2 = 1} g(m_i, f, \hat{f}) \tag{4}
$$

where g is a real valued function of the parameters m_i, f, $\hat{f}$.

Bounds on Geometric Discord and Measurement induced Nonlocality(MIN)

- Due to the same type of optimization problem we are able to find explicit bounds for geometric discord and MIN as follows:
- $0 \leq (n^2 - n)(b^2 + c^2) + 4(n-1)bc \leq D(\rho) \leq N(\rho) \leq (n^2 - n)(b^2 + c^2)$, if $bc \leq 0$
- $0 \leq (n^2 - n)(b^2 + c^2) \leq D(\rho) \leq N(\rho) \leq (n^2 - n)(b^2 + c^2) + 4(n-1)bc$, if $bc \geq 0$

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THANK YOU