ANALOGUES OF BESICOVITCH - WIENER THEOREM FOR THE HEISENBERG GROUP

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Abstract

In this paper we prove Plancherel theorem for measures on the Heisenberg group. We also consider Hermite and special Hermite expansions and prove Plancherel theorem for discrete measure and surface measure for these expansions.

Key words : Elementary spherical functions, Laguerre functions, Hermite functions, Special Hermite functions.

A.M.S Classification : ??A ??, ?? A ??

1 Introduction

Consider the discrete measure μ given by

$$\mu = \sum_{j=0}^{\infty} c_j \delta(x - a_j)$$

with $\sum_{j=0}^{\infty} c_j^2 < \infty$. The Fourier transform of such a measure is given by the almost periodic function

$$F(\xi) = \hat{\mu}(\xi) = \sum_{j=0}^{\infty} c_j e^{ia_j.\xi}.$$

For such functions the Parseval's theorem says

$$lim_{r\to\infty} r^{-n} \int_{B_r(y)} |\hat{\mu}(\xi)|^2 d\xi = c \sum_{j=0}^{\infty} |c_j|^2$$

for any fixed y. The left hand side is the so called Bohr means of the almost periodic function F and it was first proved by H. Bohr for uniformly almost periodic functions and the above general form was proved by A. Besicovitch [1]. Wiener [11] considered finite measures of the form

$$\sum_{j=0}^{\infty} c_j \delta(x - a_j) + \nu$$

where ν is absolutely continuous with respect to the Lebesgue measure. He proved that ν contributes nothing to the Bohr means of $\hat{\mu}$.

For the Fourier transform we also have the Plancherel theorem which can be written as

$$lim_{r\to\infty} \int_{B_r(y)} |\hat{f}(\xi)|^2 d\xi = (2\pi)^n \int |f(x)|^2 dx.$$

If we think of this as the Plancherel theorem for the n-dimensional measure fdx, then the Besicovitch - Wiener theorem can be considered as the 0-dimensional analogue for the discrete measure μ. For the surface measure on

$|x| = t$ there is a result due to Agmon - Hormander which gives the $(n-1)$ - dimensional case of the above theme. Recently Strichartz [5] has made far reaching generalisation of the above theme by considering measures which are fractal in nature. In the general set up equalities have to be replaced by inequalities. Results of the type

$$limsup_{r\to\infty} r^{\alpha-n} \int_{B_r(y)} |(fd\mu)\hat{}\,(\xi)|^2 d\xi \leq c \int |f|^2 d\mu.$$

have been proved by Strichartz for a large class of measures.

Our aim in this paper is to initiate the investigation of similar problems in the case of the Fourier transform on the Heisenberg group. We obtain analogue of Besicovitch - Wiener theorem for the discrete measures and also an analogue of Agmon - Hormander theorem for the surface measure. We also consider the case of Hermite and Laguerre expansions and prove similar results.

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2 Fourier transform on the Heisenberg group

The $(2n+1)$ dimensional Heisenbeg group H^n is the group $\mathbb{C}^n \times \mathbb{R}$ with the operation

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Imz.\bar{w})$$

where $z, w \in \mathbb{C}^n, t, s \in \mathbb{R}$. The Fourier transform on the Heisenberg group is defined using the infinite dimensional Schrodinger representation π_λ indexed by nonzero reals λ. These are all realised on $L^2(I\!R^n)$ and are given by $\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y)$, for φ in $L^2(I\!R^n)$. Consequently the Fourier transform of a function f on H^n is the operator valued function

$$\hat{f}(\lambda) = \int f(z,t)\pi_\lambda(z,t)dzdt.$$

The Plancherel formula then reads

$$\|f\|_2^2 = (2\pi)^{-n-1}\int \|\hat{f}(\lambda)\|_{HS}^2|\lambda|^n d\lambda.$$

where $\|T\|_{HS}$ is the Hilbert - Schmidt norm of the operator T.

Our point of departure is the following variant of the Plancherel formula established by Strichartz [4]. For f in $L^2(H^n)$ one has

$$\|f\|_2^2 = 2\pi\sum_{k=0}^{\infty}\int_{C\!\!\!\!I^n}\int_{-\infty}^{\infty}|f * e_k^\lambda(z,0)|^2 d\lambda dz.$$

Here e_k^λ are the elementary spherical functions defined by

$$e_k^\lambda(z,t) = e^{-i\lambda t}\varphi_k^\lambda(z)$$

where $\varphi_k^\lambda(z) = L_k^{n-1}(\frac{1}{2}|\lambda||z|^2)e^{-\frac{1}{4}|\lambda||z|^2}$, L_k^{n-1} being Laguerre polynomials of type (n-1). For various properties of Fourier transform and spherical functions we refer to [8]. The book of Folland [2] gives a nice introduction to the Heisenberg group.

We now state and prove an analogue of Besicovitch - Wiener theorem for the measure

$$\mu = \sum_{j=0}^{\infty} c_j\delta(z_j,t_j)$$

where $\delta(z_j,t_j)$ is the Dirac measure at the point (z_j,t_j). We need various properties of the Laguerre functions $\varphi_k(z)$. They satisfy the orthogonality relation $\varphi_k\times\varphi_j(z) = (2\pi)^n\varphi_k(z)\delta_{k,j}$ where $\delta_{k,j}$ is the Kronecker delta. Here

$\varphi_k \times \varphi_j(z)$ denote the twisted convolution $\int_{\mathbb{C}^n} \varphi_k(z-w)\varphi_j(w)e^{\frac{i}{2}Imz.\bar{w}}dw$ of φ_k and φ_j. They satisfy the product formula

$$\int_{|w|=r} \varphi_k(z-w)e^{\frac{i}{2}Imz.\bar{w}}d\mu_r = \frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r)\varphi_k(z)$$

where $\varphi_k(r)$ stands for $\varphi_k(w)$ with $|w| = r$, and μ_r is the normalised surface measure on the sphere $|w| = r$. We also need the following generating function identity

$$\sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r)\varphi_k(s)t^{2k} = (n-1)!(1-t^2)^{-1}e^{-\frac{1+t^2}{1-t^2}\frac{r^2+s^2}{2}} \frac{J_{n-1}(\frac{irst}{1-t^2})}{(\frac{irst}{2})^{n-1}}.$$

For these formule we refer to [10]. We frequently use the following Tauberian theorem.

Theorem 1 *Let* α_j, λ_j *be sequences of real numbers such that* $\alpha_j \geq 0$, $\lambda_j = j^{1/n} + o(j^{1/n})$. *Then the following are equivalent :*

(1) $\sum_{j=1}^{\infty} e^{-\epsilon\lambda_j}\alpha_j \approx c_0\epsilon^{-(n-l)}$ *as* $\epsilon \downarrow 0$

(2) $\sum_{j=1}^{N} \alpha_j \approx c_0(\Gamma(n-l+1))^{-1}N^{\ 1-l/n}$.

A proof of this theorem can be seen in [7].

Theorem 2 *Let* $\mu = \sum_{j=0}^{\infty} c_j\delta(z_j, t_j)$, *where the sequence* (c_j) *belongs to* $l^1 \bigcap l^2$ *and* z_j*'s are distinct. Then*

$$\lim_{N\to\infty} N^{-n} \sum_{k=0}^{N} \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 dz = \frac{(2\pi)^n}{|\lambda|^n n!} \sum_{j=0}^{\infty} |c_j|^2.$$

Proof : We have $e_k^{\lambda}(z,t) = e^{-i\lambda t}\ \varphi_k(\sqrt{|\lambda|}z)$. Therefore,

$$\mu * e_k^{\lambda}(z,0) \ = \ \int_{H^n} e_k^{\lambda}(z-w, -s - \frac{1}{2}(Imz.\bar{w}))\ d\mu(w,s)$$

$$= \sum_{j=0}^{\infty} c_j \; e_k^{\lambda}(z - z_j, -t_j - \frac{1}{2}(Imz.\bar{z_j}))$$
$$= \sum_{j=0}^{\infty} c_j \; e^{-i\lambda(t_j + \frac{1}{2}(Imz.\bar{z_j}))} \; \varphi_k(\sqrt{|\lambda|}(z - z_j)).$$

Therefore,

$$\int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 dz$$
$$= \int_{\mathbb{C}^n} \sum_{j,p} c_j \, \bar{c_p} \; e^{-i\lambda(t_j - t_p + \frac{1}{2} Imz.\overline{z_j - z_p} \;)} \; \varphi_k(\sqrt{|\lambda|}(z - z_j)) \; \varphi_k(\sqrt{|\lambda|}(z - z_p))dz$$
$$= \sum_{j,p} c_j \, \bar{c_p} \; e^{-i\lambda(t_j - t_p)} \int_{\mathbb{C}^n} e^{i\frac{\lambda}{2} Im((z_j - z_p).\bar{z})} \; \varphi_k(\sqrt{|\lambda|}(z - z_j)) \; \varphi_k(\sqrt{|\lambda|}(z - z_p))dz.$$

The interchange of the order of integration and the sum is justified by Fubini's theorem since (c_j) is in l^1 and

$$\int_{\mathbb{C}^n} |\varphi_k(\sqrt{|\lambda|}(z - z_j))| \; |\varphi_k(\sqrt{|\lambda|}(z - z_p))| \; dz \leq |\lambda|^{-n} \int_{\mathbb{C}^n} |\varphi_k(z)|^2 \; dz.$$

A simple calculation shows that

$$\int_{\mathbb{C}^n} \; e^{i\frac{\lambda}{2} Im((z_j - z_p).\bar{z})} \; \varphi_k(\sqrt{|\lambda|}(z - z_j)) \; \varphi_k(\sqrt{|\lambda|}(z - z_p))dz.$$
$$= \; |\lambda|^{-n} \; e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \; \varphi_k \times \varphi_k(\sqrt{|\lambda|}(z_j - z_p))$$
$$= \; (2\pi)^n \; |\lambda|^{-n} \; e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \; \varphi_k(\sqrt{|\lambda|}(z_j - z_p)).$$

Thus we have

$$\int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 dz = (2\pi)^n \; |\lambda|^{-n} \; \sum_{j,p} c_j \, \bar{c_p} \; e^{-i\lambda(t_j - t_p)} e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \; \varphi_k(\sqrt{|\lambda|}(z_j - z_p)).$$

Now we consider the sum

$$\sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 dz.$$

As the functions $\varphi_k(z)$ are uniformly bounded (in fact, $|\varphi_k(z)| \leq ck^{n-1}$), we can first sum with respect to k to get

$$\sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 dz$$
$$= (1-t)^{-n}(2\pi)^n |\lambda|^{-n} \sum_{j,p} c_j \bar{c_p} e^{-i\lambda(t_j - t_p)} e^{i\frac{\lambda}{2} Im z_j . \bar{z_p}} e^{-\frac{1}{2}\frac{1+t}{1-t}|\lambda|\,|z_j - z_p|^2} dz$$

In deriving the above we have used the generating function identity,

$$\sum_{k=0}^{\infty} t^k \varphi_k(z) = (1-t)^{-n} e^{-\frac{1}{2}\frac{1+t}{1-t}|z|^2}.$$

Since $|z_j - z_p| \neq 0$ for $j \neq p$ we see that

$$lim_{t\to 1-} e^{-\frac{1}{2}\frac{1+t}{1-t}|\lambda||z_j - z_p|^2} = 0$$

for $j \neq p$. Therefore,

$$lim_{t\to 1-}(1-t)^n \sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \; dz = (2\pi)^n \; |\lambda|^{-n} \; \sum_{j=0}^{\infty} |c_j|^2.$$

By appealing to the Tauberian theorem we get the result.

We next consider the surface measure on the sphere $S_r = \{(z,0) : |z| = r\} \subset H^n$. Let μ_r be the normalised measure on the sphere S_r.

Theorem 3 *Let $g \in L^2(\mathbb{R})$ and let μ be the product of μ_r and and $g\,dt$ on H^n. Then*

$$lim_{N\to\infty} N^{-\frac{1}{2}} \sum_{k=0}^{N} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-\frac{1}{2}} \, dz \, d\lambda = \frac{\pi^{n-1} 2^{2n-\frac{1}{2}} (n-1)!}{r^{2n-1}} \|g\|_2^2.$$

Proof : We have

$$\begin{aligned}
\mu * e_k^\lambda(z,0) &= \int_{H^n} e_k^\lambda(z-w, -s-\frac{1}{2}Imz.\bar{w})\, d\mu(w,s)\\
&= \int_{\mathbb{C}^n\times\mathbb{R}} \varphi_k(\sqrt{|\lambda|}(z-w))\, e^{-i\lambda(-s-\frac{1}{2}Imz.\bar{w})} g(s)\, d\mu_r(w)\, ds\\
&= \int_{|w|=r} \varphi_k(\sqrt{|\lambda|}(z-w))\, e^{-i\lambda\frac{1}{2}Imz.\bar{w}}\, d\mu_r(w) \int_{\mathbb{R}} g(s)\, e^{i\lambda s} ds\\
&= \frac{k!(n-1)!}{(k+n-1)!}\, \varphi_k(\sqrt{|\lambda|}r)\, \varphi_k(\sqrt{|\lambda|}|z|)\, \hat{g}(\lambda).
\end{aligned}$$

Thus

$$\int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2\, dz = |\lambda|^{-n}\, (2\pi)^n \frac{k!(n-1)!}{(k+n-1)!}\, (\varphi_k(\sqrt{|\lambda|}r))^2\, |\hat{g}(\lambda)|^2.$$

As in the previous theorem, we consider

$$\begin{aligned}
&\sum_{k=0}^{\infty} t^{2k} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2\, |\lambda|^{2n-\frac{1}{2}}\, dz\, d\lambda\\
&= \sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!}\, (2\pi)^n\, \Gamma(n) \int_{-\infty}^{\infty} |\varphi_k(\sqrt{|\lambda|}r)|^2\, |\hat{g}(\lambda)|^2\, |\lambda|^{n-\frac{1}{2}}\, d\lambda
\end{aligned}$$

First we claim that the sum can be taken inside the integral. To see this it is enough to check that the integral

$$\int \{\sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!}\, (2\pi)^n\, \Gamma(n)\, (\varphi_k(\sqrt{|\lambda|}r))^2\}\, |\hat{g}(\lambda)|^2\, |\lambda|^{n-\frac{1}{2}}\, d\lambda$$

is finite. The generating function identity for the Laguerre functions give

$$\begin{aligned}
&\sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!}\, (\varphi_k(\sqrt{|\lambda|}r))^2\\
&= (1-t^2)^{-n}\, \{\frac{|\lambda|r^2 t}{2(1-t^2)}\}^{-n+1}\, J_{n-1}(i\frac{|\lambda|r^2 t}{1-t^2})\, e^{-\frac{|\lambda|\,r^2}{2}\frac{1+t^2}{1-t^2}}
\end{aligned}$$

The Bessel function $J_{n-1}(iz)$ has the estimate

$$|J_{n-1}(iz)| \leq c\, z^{-\frac{1}{2}} e^z \qquad\qquad z \geq 1.$$

In view of this the above sum is bounded by

$$c_{r,t}\ |\lambda|^{-n+\frac{1}{2}}\ e^{-\frac{1}{2}\frac{1-t}{1+t}|\lambda|r^2}$$

and hence the integral under consideration is finite. Thus we have shown that

$$\sum_{k=0}^{\infty} t^{2k}\int_{-\infty}^{\infty}\int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2\ |\lambda|^{2n-\frac{1}{2}} dz\, d\lambda = \Gamma(n)\,(2\pi)^n(1-t^2)^{-n}$$
$$\times \int_{-\infty}^{\infty} |\,\hat{g}(\lambda)|^2 \{\frac{|\lambda|r^2 t}{2(1-t^2)}\}^{-n+1} J_{n-1}(i\frac{|\lambda|r^2 t}{1-t^2})\, e^{-\frac{1}{2}\frac{1+t^2}{1-t^2}|\lambda|r^2}\ |\lambda|^{n-\frac{1}{2}} d\lambda$$

consider the function

$$(1-t^2)^{-n+\frac{1}{2}}\ \{\frac{|\lambda|r^2 t}{2(1-t^2)}\}^{-n+1}\ J_{n-1}(i\frac{|\lambda|r^2 t}{1-t^2})\ e^{-\frac{1}{2}\frac{1+t^2}{1-t^2}|\lambda|\,r^2}.$$

Using the integral representation for the Bessel function this is equal to

$$(1-t^2)^{-n+1/2}\ e^{-\frac{|\lambda|r^2}{2}\frac{1+t^2}{1-t^2}}\ \frac{1}{\Gamma(1/2)\Gamma(n-1/2)}\int_{-1}^{1}(1-s^2)^{n-3/2}\ e^{-\frac{|\lambda|r^2 t\ s}{1-t^2}} ds$$

$$= \frac{(1-t^2)^{-n+1/2}}{\Gamma(1/2)\Gamma(n-1/2)}\int_{-1}^{1}(1-s^2)^{n-3/2}\ e^{-\frac{|\lambda|r^2}{2}\frac{1-t}{1+t}}\ e^{-\frac{|\lambda|r^2}{2}\frac{2t}{1-t^2}(1-s)}\ ds.$$

A simple computation shows that this is equal to

$$\frac{e^{-\frac{|\lambda|r^2}{2}\frac{1-t}{1+t}}}{\Gamma(1/2)\Gamma(n-1/2)}\int_0^{2/1-t^2}(y[2-y(1-t^2)])^{n-3/2}\ e^{-|\lambda|r^2 ty}dy.$$

As $t \to 1-$, this converges to

$$\frac{1}{\Gamma(1/2)\Gamma(n-1/2)}\int_0^{\infty}(2y)^{n-3/2}\ e^{-|\lambda|r^2 y}\ dy = 2^{n-3/2}\ \pi^{-1/2}\ (|\lambda|r^2)^{-n+\frac{1}{2}}.$$

Therefore we have proved

$$lim_{t\to 1-}(1-t^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} t^{2k} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \ |\lambda|^{2n-\frac{1}{2}} \ dz \ d\lambda$$
$$= \Gamma(n) \ 2^{2n-\frac{3}{2}} \ \pi^{n-\frac{1}{2}} \ r^{-2n+1} \int_{-\infty}^{\infty} |\hat{g}(\lambda)|^2 \ d\lambda$$

By appealing to Tauberian theorem we complete the proof.

In the above theorem we can replace $g(t)\,dt$ by a discrete measure $\sum c_j\,\delta_{a_j}$, where $a_j's$ are distinct. Then as a corollary we obtain the following result.

Corollary 4 *Let μ be the product of μ_r and the discrete measure $\sum_j c_j\,\delta_{a_j}$, where $a_j's$ are distinct. Then*

$$lim_{M\to\infty} \ lim_{N\to\infty} \frac{N^{-\frac{1}{2}}}{M} \int_{-M}^{M} \sum_{k=0}^{N} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \ |\lambda|^{2n-1/2} \ dz\, d\lambda$$
$$= \frac{\pi^{n-1}(n-1)! \ 2^{2n-1/2}}{r^{2n-1}} \sum_{j=0}^{\infty} |c_j|^2.$$

Proof : In this case

$$\mu * e_k^\lambda(z,0) = \int_{|w|=r} \varphi_k[\sqrt{|\lambda|}(z-w)] \ e^{i\frac{\lambda}{2} Imz.\bar{w}} \ d\mu_r(w) \ F(\lambda),$$

where

$$F(\lambda) = \sum_j c_j \ e^{i\lambda s_j}.$$

Therefore,

$$\mu * e_k^\lambda(z,0) = \frac{k!(n-1)!}{(k+n-1)!} \ \varphi_k(\sqrt{|\lambda|}r) \ \varphi_k(\sqrt{|\lambda|}z) \ F(\lambda).$$

Hence

$$\int_{-M}^{M}\int_{\mathbb{C}^n}|\mu * e_k^\lambda(z,0)|^2\ |\lambda|^{2n-1/2}\ dz\, d\lambda$$
$$= (2\pi)^n \frac{k!(n-1)!}{(k+n-1)!}\int_{-M}^{M}\{\varphi_k(\sqrt{|\lambda|}r)\}^2\ |F(\lambda)|^2\ |\lambda|^{n-\frac{1}{2}} d\lambda.$$

As in the theorem we can show that

$$lim_{N\to\infty} N^{-\frac{1}{2}}\sum_{k=0}^{N}\int_{-M}^{M}\int_{\mathbb{C}^n}|\mu * e_k^\lambda(z,0)|^2\ |\lambda|^{2n-1/2}\ dz\, d\lambda$$
$$= \frac{\pi^{n-1}\ (n-1)!\ 2^{2n-\frac{1}{2}}}{r^{2n-1}}\int_{-M}^{M}|F(\lambda)|^2\, d\lambda.$$

By Wiener's theorem

$$lim_{M\to\infty}\frac{1}{M}\int_{-M}^{M}|F(\lambda)|^2 d\lambda = \sum_{j=0}^{\infty}|c_j|^2.$$

This completes the proof of the corollary.

3 Hermite and special Hermite expansions

Consider the normalised Hermite functions $\Phi_\alpha(x)$ on $\mathbb{R}^n$.These are indexed by multi indices $\alpha \in \mathbb{N}^n$ and are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$. In fact $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$ where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\{\Phi_\alpha\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. The Plancherel theorem for the Hermite expansion reads

$$\int |f|^2 dx = \sum_\alpha |\hat{f}(\alpha)|^2.$$

where $\hat{f}(\alpha) = \int f(x)\Phi_\alpha(x)dx$. The Hermite functions Φ_α satisfies the Mehlers formula

$$\sum_\alpha \Phi_\alpha(x)\Phi_\alpha(y)t^{|\alpha|} = \pi^{-\frac{n}{2}}(1-t^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\frac{1+t^2}{1-t^2}(|x|^2+|y|^2)+\frac{2t}{1-t^2}x.y}.$$

Using the generating function and the Tauberian theorem one can easily prove the following result.

Theorem 5 *Let $\mu = \sum c_j \delta(a_j) + \nu$ where ν is absolutely continuous and $a_j's$ are distinct. Then*

$$lim_{N \to \infty} N^{-\frac{n}{2}} \sum_{|\alpha| \leq N} |\hat{\mu}(\alpha)|^2 = \frac{(2\pi)^{-n/2}}{\Gamma(n/2+1)} \sum |c_j|^2$$

where $\hat{\mu}(\alpha) = \int \Phi_\alpha(x) d\mu$.

For a proof of this see Strichartz [5].

We now consider the case of surface measure ν_r on the sphere $|x| = r$. More generally we consider measures of the form $f d\nu_r$, where f is a square integrable function on the sphere $|x| = r$. To treat such measures we need the following Hecke - Bochner type identity for the Hermite projection operators. Let $P_k(f)$ stand for the projection of f onto the k - th eigenspace spanned by $\Phi_\alpha(x), |\alpha| = k$, that is

$$P_k f(x) = \sum_{|\alpha|=k} \hat{f}(\alpha) \Phi_\alpha(x).$$

Let L_k^δ be the Laguerre polynomial of the type δ and define

$$R_k^\delta(f) = \frac{2\Gamma(k+1)}{\Gamma(k+\delta+1)} \int_0^\infty f(r) L_k^\delta(r^2) e^{-\frac{r^2}{2}} r^{2\delta+1} dr.$$

For radial functions the following proposition has been proved in [9].

Proposition 6 *Assume that $f(x) = f_0(|x|)p(x)$ where $p(x)$ is a solid harmonic of degree m. Then*

$$P_{2k+m} f(x) = F_k(|x|) p(x)$$

where $F_k(r) = R_k^\delta(f) L_k^\delta(r^2) e^{-\frac{1}{2}r^2}$ with $\delta = \frac{n}{2} + m - 1$. For other values of k, $P_k(f) = 0$.

For a measure $d\mu$ on $\mathbb{R}^n$, let $P_k(d\mu)$ be defined by $P_k(d\mu) = \sum_{|\alpha|=k}(\int \Phi_\alpha(y)d\mu)\Phi_\alpha(x)$.

Theorem 7 *Let ν be the normalised surface measure on S^{n-1} and let $f \in L^2(S^{\,n-1}, d\nu)$. Then $lim_{N\to\infty}N^{-\frac{1}{2}}\sum_{k=0}^{N}\|P_k(fd\nu)\|_2^2 = \frac{2}{\pi}\int_{S^{n-1}}|f|^2d\nu$.*

Proof : Expand f in terms of spherical harmonics $f = \sum c_m Y_m$ where Y_m is a spherical harmonic of degree m. In view of the proposition it is easy to see that

$$P_{2k+m}(Y_m d\nu) = \frac{2\,\Gamma(k+1)}{\Gamma(k+\delta+1)} L_k^\delta(1)\, e^{-\frac{1}{2}}\, L_k^\delta(|x|^2)\, e^{-\frac{1}{2}|x|^2}\, Y_m.$$

As various Y_m's are orthogonal to each other it is enough to prove the theorem when $f = Y_m$.

$$\begin{aligned}\sum_{k=0}^{\infty} t^k\|P_{2k+m}(Y_m d\nu)\|_2^2 &= 2\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)}(L_k^\delta(1)e^{-\frac{1}{2}})^2\; t^k \\ &= 2(1-t)^{-1}e^{-\frac{1+t}{1-t}}\frac{J_\delta(\frac{2i\sqrt{t}}{1-t})}{(\sqrt{t})^\delta}.\end{aligned}$$

Now we can proceed as in theorem (3) to conclude the proof.

Finally we briefly consider the case of special Hermite expansions. By the term we mean an expansion of the type

$$f = (2\pi)^{-n}\sum f \times \varphi_k$$

where f is a function on $\mathbb{C}^n$. Analogues of theorems (5) and (7) can be proved in the context of special Hermite expansions. These are in a way particular cases of theorems in section 2 when we consider functions on the Heisenberg group that are independent of t. We leave the formulations and proofs to the interested reader.

References

[1] A. S. BESICOVITCH, Almost periodic functions, *Dover. New York*, (1954).

[2] G. B. FOLLAND, Harmonic analysis in phace space, *Ann. Math. Study., Princeton univ. press*,112 (1989).

[3] N. N. LEBEDEV, Special functions and their applications, *Dover, New York* (1992).

[4] R. STRICHARTZ, Harmonic analysis as spectral theory of Laplacians,*J. Funct. Anal., vol.* 87 (1989), 51-148.

[5] R. STRICHARTZ, Fourier asymptotics of Fractal measures, *J. Funct. Anal., vol.* 89 (1990), 154-187.

[6] G. SZEGO, Orthogonal polynomials, *Amer. Math. Soc., Colloq. Publ.,Providence,* (1967).

[7] M. E. TAYLOR, Pseudodifferential operators. *Princeton Univ. press, Princeton, N.J.* (1981) .

[8] S. THANGAVELU, Spherical means on the Heisenberg group and a restriction theorem for the symplectic Fourier transform, *Revist.Mat.Ibero,* 7, (1991), 135-156.

[9] S. THANGAVELU, Hermite expansion on $\mathbb{R}^n$ for radial functions, *Proc. Amer. Math. Soc.,* 118 (1993),1097 - 1102.

[10] S. THANGAVELU, Lectures on Hermite and Laguerre expansions, *Mathematical notes, 42, Princeton Univ. press, Princeton.*(1993).

[11] N. WIENER, The Fourier integrals and certain of its applications, *Dover, New york*,(1933).