GELFAND PAIRS, K-SPHERICAL MEANS AND INJECTIVITY ON THE HEISENBERG GROUP

By

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Abstract. We study the injectivity properties of the spherical mean value operators associated to the Gelfand pairs (H^n, K), where K is a compact subgroup of $U(n)$. We show that these spherical mean value operators are injective on $L^p(H^n)$ for $1 \leq p < \infty$. For $p = \infty$, these operators are not injective. Nevertheless, if the spherical means $f * \mu_i$ over K-orbits of sufficiently many points $(z_i, t_i) \in H^n$ vanish, we identify a necessary and sufficient condition on the points (z_i, t_i) which guarantees $f = 0$. For $K = U(n)$, this is equivalent to the condition for the two-radius theorem.

1 Introduction

Given a continuous function f on the Heisenberg group H^n, the spherical mean is defined to be

$$M_r f(z) = \int_{|w|=r} f(z-w, t-s-\tfrac{1}{2}\operatorname{Im} z \cdot \overline{w})\, d\mu_r(w),$$

where μ_r is the normalised surface measure on the sphere $\{(z,0) : z \in \mathbb{C}^n, |z| = r\}$ in H^n. The study of injectivity of such mean value operators on $L^p(H^n)$ has been carried out by Thangavelu [12] for $1 \leq p < \infty$ and by Agranovsky et al. [1] for the case $p = \infty$. In [12], the basic tool used to prove the injectivity is the spectral decomposition of a function in terms of the joint eigenfunctions of the operator $T = i\partial/\partial t$ and the sublaplacian $\mathcal{L}$ on the Heisenberg group, due to Strichartz (see [9]).

The method employed in the case $p < \infty$ does not work for $p = \infty$. In [1], Agranovsky et al. studied the injectivity properties for $U(n)$ and $T(n)$ spherical averages for the bounded continuous functions on H^n. Their approach is to exploit the general theory of commutative Banach $*$ algebras for $L^1_K(H^n)$, the space of

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K-invariant functions in $L^1(H^n)$ for $K = U(n)$ or $T(n)$. The basic tool they use is the Wiener–Tauberian theorem for these algebras.

Crucial in the study of injectivity problems for the spherical mean operator is the relation

$$e_k^\lambda * \mu_{r,t} = C_k\, e_k^\lambda(r,t)\, e_k^\lambda,$$

where $\mu_{r,t}$ is the normalised surface measure on the sphere $\{(z,t) : |z| = r\}$ in H^n and

$$C_k = \frac{(k+n-1)!}{k!(n-1)!}.$$

This relation follows from the functional equation

$$\int_K \phi(x\, k \cdot y)\, dk = C_k\, \phi(x)\, \phi(y), \qquad C_k = \frac{(k+n-1)!}{k!(n-1)!},$$

satisfied by the $U(n)$-spherical function $\phi = e_k^\lambda$ on H^n.

Since the functional equation is the characterizing property of spherical functions associated to the Gelfand pairs, one is led to conjecture that these results are valid in the more general set-up of Gelfand pairs.

The aim of this paper is to investigate the above problems in the set-up of Gelfand pairs and prove the injectivity results for more general K-orbital averages on H^n.

Observe that the sphere $\{(w,t) : |w| = r\}$ is the $U(n)$ orbit of a point $(w,t) \in H^n$ with $|w| = r$. In general, let (z_0, t_0) be a point in H^n and let $K(z_0, t_0) = \{(kz_0, t_0) : k \in K\}$ denote the K-orbit of (z_0, t_0). Since K is a compact subgroup of $U(n)$, it is easy to see that (Kz_0, t_0) is a smooth compact manifold in $\mathbb{C}^n \times \{t_0\} \subset H^n$ homeomorphic to $K/I(z_0)$, where $I(z_0)$ is the isotropic subgroup for (z_0, t_0), i.e., $I(z_0) = \{k \in K : kz_0 = z_0\}$. Let μ_{z_0,t_0} denote the normalised surface measure on the K-orbit of the point $(z_0, t_0) \in H^n$.

2 The Heisenberg group and its representations

Recall that the Heisenberg group H^n is defined to be $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = (z + w, t + s + \tfrac{1}{2}\,\mathrm{Im}\,(z.\overline{w})).$$

Under this group law, H^n becomes a nilpotent Lie group, with the Haar measure $dz\, dt$, the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. The corresponding Lie algebra h_n is generated by the $(2n+1)$ left invariant vector fields

$$X_j = \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}\right), \qquad j = 1, 2, \ldots, n,$$
$$Y_j = \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}\right), \qquad j = 1, 2, \ldots, n,$$
$$T = i\frac{\partial}{\partial t}.$$

The operator $\mathcal{L} = -\sum_{j=1}^{n}\left(X_j^2 + Y_j{}^2\right)$ is called the sublaplacian on the Heisenberg group.

Now we give a brief description of the representations of the Heisenberg group. We are primarily interested in the Fock–Bargmann representation. For each $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the Fock space $\mathcal{F}_\lambda$ is defined to be the space of all holomorphic or anti-holomorphic functions on $\mathbb{C}^n$ (depending on whether λ is positive or negative) which are square integrable with respect to the measure $|\lambda|^n e^{-|\lambda||w|^2}\, dw$. Then $\mathcal{F}_\lambda$ is a Hilbert space with the inner product $\langle f, g\rangle_{\mathcal{F}_\lambda} = |\lambda|^n \int_{\mathbb{C}^n} f(w)\,\overline{g(w)}\, e^{-|\lambda||w|^2}\, dw$ for $f, g \in \mathcal{F}_\lambda$.

For each $\lambda \in \mathbb{R}^*$, we define a representation ρ_λ of H^n on $\mathcal{F}_\lambda$ by

$$\rho_\lambda(z,t)f(w) = e^{i\lambda t + \frac{i\lambda}{\sqrt{2}} w\cdot z - \frac{\lambda}{4}|z|^2} f\left(w + \frac{i\bar{z}}{\sqrt{2}}\right) \qquad \text{for } \lambda > 0$$

and

$$\rho_\lambda(z,t)f(w) = e^{i\lambda t - \frac{i\lambda}{\sqrt{2}} w\cdot \bar{z} + \frac{\lambda}{4}|z|^2} f\left(w + \frac{iz}{\sqrt{2}}\right) \qquad \text{for } \lambda < 0$$

for $(z,t) \in H^n$ and $f \in \mathcal{F}_\lambda$. Then ρ_λ is an irreducible unitary representation of H^n. It is well-known that up to unitary equivalence these are all the unitary representations that are non-trivial at the centre (see [10]).

In addition to these ρ_λ, there is another family χ_w of one dimensional representations of H^n, parametrized by $w \in \mathbb{C}^n$, given by

$$\chi_w(z,t) = e^{i\mathrm{Re}\, w\cdot\bar{z}} \quad \text{for } (z,t) \in H^n.$$

This completes the description of the unitary representations of H^n.

Let us now consider the entry functions for the representation ρ_λ. Notice that the functions

$$\zeta_\alpha^\lambda(w) = \left(\frac{|\lambda|^{|\alpha|}}{\alpha!\,\pi^n}\right)^{1/2} w^\alpha, \quad \alpha \in \mathbb{N}^n$$

form an orthonormal basis for $\mathcal{F}_\lambda$. From the definition of the representation ρ_λ, we see that the entry functions are of the form

$$\langle \rho_\lambda(z,t)\zeta_\alpha^\lambda, \zeta_\beta^\lambda\rangle_{\mathcal{F}_\lambda} = e^{i\lambda t}\Phi_{\alpha\beta}\left(\sqrt{|\lambda|}\, z\right).$$

The functions $\Phi_{\alpha\beta}$ are called the special Hermite functions. The functions $\Phi_{\alpha\beta}$ are the eigenfunctions of the special Hermite operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=0}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

with eigenvalue $2|\beta| + n$. The normalised functions $(2\pi)^{-n/2}\Phi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{N}^n$, form an orthonormal basis for $L^2(\mathbb{C}^n)$. The special Hermite functions enjoy the orthogonality property

$$\Phi_{\alpha\beta} \times \Phi_{\mu\nu} = (2\pi)^n\, \delta_{\beta\mu}\, \Phi_{\alpha\nu}, \tag{2.1}$$

where $\times$ denotes the twisted convolution. The twisted convolution of functions f and g on $\mathbb{C}^n$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)\, g(w)\, e^{\frac{i}{2}\operatorname{Im} z\cdot\bar{w}} dw$$

whenever the integral converges. For results concerning special Hermite functions, see [11]. Note that the functions $\Phi_{\alpha,\beta}$ differ from the special Hermite functions considered in [11] by a multiple of $(2\pi)^{-n/2}$. This follows from the fact that the Bargmann transform (see [10]), which intertwines the Schrödinger representation and the Fock–Bargmann representation, takes the Hermite functions h_α^λ into ζ_α^λ.

3 Gelfand pairs and K-spherical functions

Let G be a nilpotent Lie group and K a compact subgroup of $\operatorname{Aut}(G)$. There is a natural action of K on $L^1(G)$ defined by $k \cdot f(g) = f(k \cdot g)$. We say the pair (G, K) is a Gelfand pair if the subalgebra $L^1_K(G)$ of K-invariant functions in $L^1(G)$ under this action is commutative with respect to the usual convolution.

Associated to a Gelfand pair (G, K), we have a class of K-invariant functions called the K-spherical functions. These can be described in many ways. A K-invariant complex valued function ϕ on G is called K-spherical if $\phi(e) = 1$ and ϕ is a joint eigenfunction for all left G-invariant and right K-invariant differential operators on G.

Alternatively, K-spherical functions can be characterised as the non-trivial continuous functions on G satisfying the functional equation

$$\int_K \phi(x\,k\cdot y)\, dk = \phi(x)\,\phi(y), \tag{3.1}$$

where dk denotes the normalized Haar measure on K.

Bounded spherical functions are important from the point of view of $L^1_K(G)$ as a commutative Banach $*$ algebra. In fact, these functions determine all the multiplicative linear functionals on the algebra $L^1_K(G)$. In Section 6, we use this fact to study the injectivity of K-spherical means in the L^∞ case. For more details about spherical functions, see [6].

The unitary group $U(n)$ gives a subgroup of $\mathrm{Aut}(H^n)$ through the action $k \cdot (z,t) = (k \cdot z, t)$. This gives a maximal compact subgroup of $\mathrm{Aut}(H^n)$, which we denote by $U(n)$ again. Conjugating by an element, if necessary, we can assume that every compact subgroup of $\mathrm{Aut}(H^n)$ is contained in $U(n)$. It is well-known that $(H^n, U(n))$ is a Gelfand pair (see [7]) and there are many proper subgroups K of $U(n)$ for which (H^n, K) form a Gelfand pair.

Benson, Jenkins and Ratcliff [2] have classified all the compact subgroups K of $U(n)$ for which (H^n, K) form a Gelfand pair. They also studied the spherical functions associated to these Gelfand pairs and obtained expressions for the spherical functions in terms of certain fundamental invariants. In a series of papers using the method developed in [2], they computed the spherical functions explicitly for some of the Gelfand pairs. But an explicit expression for the spherical function for all the Gelfand pairs (H^n, K) is not yet available.

We prove our injectivity theorems by studying the qualititive properties of the spherical functions. As before, let (G, K) be a Gelfand pair. Let π be a representation of G on a Hilbert space $\mathcal{H}$. Define

$$K_\pi = \{k \in K : \pi \circ k \text{ unitarily equivalent to } \pi\}.$$

Let $\mathcal{H} = \bigoplus_j \mathcal{H}_j$ be the K_π-irreducible decomposition of $\mathcal{H}$. The following theorem was proved in [2].

Theorem 3.1. *If ϕ is a bounded K-spherical function on G, then it is of the form*

$$\phi(g) = \phi_{\pi,v}(g) = \int_K \langle \pi(kg)v, v\rangle dk$$

for some irreducible unitary representation π and a unit vector v in $\mathcal{H}_j$. Moreover, $\phi_{\pi,v} = \phi_{\pi',v'}$ if and only if π' is unitarily equivalent to $\pi \circ k$ for some $k \in K$ and v, v' belong to the same $\mathcal{H}_j$.

When $K_\pi = K$, we have a simpler representation of the spherical functions (see [2]).

Corollary 3.2. *If $K_\pi = K$ and $\{v_1, v_2, \cdots, v_l\}$ is an orthonormal basis for $\mathcal{H}_j$, then*

$$\phi_{\pi,j}(g) = \frac{1}{\dim \mathcal{P}_j} \sum_{j=1}^{l} \langle \pi(g)v_j, v_j\rangle,$$

where $\phi_{\pi,j}(g) = \phi_{\pi,v}(g)$ *with* $v \in \mathcal{H}_j$, $\|v\| = 1$.

Now we use the above theorem and corollary to study the bounded K-spherical functions associated with the Gelfand pair (H^n, K). To begin with, we discuss the bounded spherical functions associated with the infinite dimensional representations ρ_λ. Notice that when K is a compact subgroup of $U(n)$, $K_{\rho_\lambda} = K$.

For each $m > 0$, let $\mathcal{P}_m$ denote the span of all monomials $w^\alpha, |\alpha| = m$. Then each such $\mathcal{P}_m$ is irreducible under the $U(n)$ action, and $\mathcal{F} = \bigoplus_m \mathcal{P}_m$ is the decomposition of the Fock space $\mathcal{F}$ into $U(n)$-irreducible subspaces. Associated to this decomposition, in view of Corollary 3.2, we have, for each $\lambda \in \mathbb{R}^*$ and for $m \in \mathbb{N}$, a bounded $U(n)$-spherical function ψ_m^λ given by

$$\begin{aligned}\psi_m^\lambda(z,t) &= \frac{1}{\dim \mathcal{P}_m} \sum_{|\alpha|=m} \langle \rho_\lambda(z,t)\zeta_\alpha^\lambda, \zeta_\alpha^\lambda \rangle_{\mathcal{F}_\lambda} \\ &= \frac{1}{\dim \mathcal{P}_m} e^{i\lambda t} \sum_{|\alpha|=m} \Phi_{\alpha\alpha}^\lambda(z) \\ &= \frac{1}{\dim \mathcal{P}_m} e^{i\lambda t} \varphi_m(\sqrt{|\lambda|}z),\end{aligned} \tag{3.2}$$

where $\varphi_m(z) = L_m^{n-1}(\frac{1}{2}|z|^2)e^{\frac{-1}{4}|z|^2}$ is the Laguerre function of order $n-1$.

When K is a proper compact subgroup of $U(n)$, $\mathcal{P}_m$ need not be irreducible under the K-action. So it further decomposes into K-irreducibles. Let $\mathcal{P}_m = \bigoplus_{j=1}^{J_m} \mathcal{P}_{m,j}$ be the decomposition of $\mathcal{P}_m$ into K-irreducible subspaces. Thus, for any compact subgroup K of $U(n)$, the Fock space decomposes into K-irreducible subspaces as $\mathcal{F} = \bigoplus_{m=0}^{\infty} \bigoplus_{j=1}^{J_m} \mathcal{P}_{m,j}$; and for each m, j we get the K-spherical function $\psi_{m,j}^\lambda$ given by

$$\psi_{m,j}^\lambda(z,t) = \frac{1}{\dim P_{m,j}} \sum_{i=1}^{n(j)} \langle \rho_\lambda(z,t)v_i^j, v_i^j \rangle,$$

where for each j, $\{v_i^j, i = 1, ..., n(j)\}$ is an orthonormal basis for $\mathcal{P}_{m,j}$. Since each $v_i^j \in \mathcal{P}_m$ is a linear combination of the monomials ζ_α, it follows that $\psi_{m,j}^\lambda(z,t)$ is of the form

$$\psi_{m,j}^\lambda(z,t) = e^{i\lambda t} q_{m,j}\left(\sqrt{|\lambda|}z, \sqrt{|\lambda|}\bar{z}\right) e^{-\frac{|\lambda|}{4}|z|^2},$$

where $q_{m,j}$ is a polynomial in z and $\bar{z}$.

Now we observe the following relation between the $U(n)$-spherical functions and the K-spherical functions. Let us choose a basis $\{v_i^j : i = 1, ..., n(j)\}$ of $\mathcal{P}_{m,j}$ so that the collection $\{v_i^j : i = 1, ..., n(j); j = 1, ..., J_m\}$ forms an orthonormal basis for $\mathcal{P}_m$. Using this basis in (3.2) and grouping the terms for each j, we see that

$$\dim P_m \, \psi_m^\lambda(z,t) = \sum_{j=1}^{J_m} \dim P_{m,j} \, \psi_{m,j}^\lambda(z,t).$$

With an abuse of terminology, calling $e^\lambda_{m,j}(z,t) = \dim \mathcal{P}_{m,j}\ \psi^\lambda_{m,j}(z,t)$ spherical functions, we get the following relation between the $U(n)$-spherical function and the K-spherical functions $e^\lambda_{m,j}$:

$$e^\lambda_m(z,t) = \sum_{j=1}^{J_m} e^\lambda_{m,j}(z,t). \tag{3.3}$$

Again, in view of Theorem 3.1, we see that the spherical functions associated with the representations χ_w are given by

$$\int_K e^{i\mathrm{Re}(kw.\bar{z})} dk.$$

Integrating with respect to the surface measure $d\mu_{Kw}$ on the orbit Kw and using the K-invariance of the surface measure, we see easily that the above integral is the same as

$$\int_{K.w} e^{iRe(w.\bar{z})} d\mu_{Kw},$$

which is nothing but the Fourier transform of the measure $d\mu_{Kw}$, evaluated at z. Thus the spherical functions associated with the representation χ_w are given by

$$\phi_w(z) = \widehat{\mu_{Kw}}(z). \tag{3.4}$$

These spherical functions will be used in Section 6, when we discuss the injectivity result for the L^∞ case.

4 A spectral decomposition in terms of K-spherical functions

Strichartz has given a spectral decomposition of an L^2 function on the Heisenberg group in terms of the joint eigenfunctions of $\mathcal{T} = i\partial/\partial t$ and the sublaplacian $\mathcal{L}$ on the Heisenberg group (see [9]). More precisely, if $f \in L^2(H^n)$, we have

$$f(z,t) = (2\pi)^{-n} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e^\lambda_k(z,t) |\lambda|^n d\lambda, \tag{4.1}$$

where the series converges in the L^2-norm. Here $e^\lambda_k(z,t)$ denotes the $U(n)$-spherical function on H^n given by

$$e^\lambda_k(z,t) = e^{i\lambda t} \varphi^\lambda_k(z)$$

and $\varphi^\lambda_k(z) = L^{n-1}_k(\frac{1}{2}|\lambda|\,|z|^2)\, e^{-\frac{1}{4}|\lambda||z|^2}$ is the Laguerre function of order $n-1$. The functions $e^\lambda_k(z,t)$ are joint eigenfunctions of the operators $\mathcal{L}$ and $\mathcal{T}$ with the eigenvalues $(2k+n)|\lambda|$ and $-\lambda$, respectively.

The above series may not converge for all $f \in L^p$ for $p \neq 2$. So one considers the Abel means

$$A_r f(z,t) = (2\pi)^{-n} \sum_{k=0}^{\infty} r^k \int_{-\infty}^{\infty} f * e_k^\lambda(z,t)|\lambda|^n d\lambda.$$

Strichartz [9] has shown that for each $0 < r < 1$ the Abel means converge in $L^p(H^n)$, $1 < p < \infty$, and also $A_r f \to f$ in the L^p norm as $r \to 1-$, for $1 < p < \infty$.

The Strichartz decomposition (4.1) is a spectral decomposition in terms of $U(n)$ spherical functions $e_k^\lambda(z,t) = e^{i\lambda t}\varphi_k^\lambda(z)$. We are interested in studying a similar spectral decomposition of f in terms of the spherical functions associated with compact subgroups of $U(n)$. Getting a decomposition of this sort is an easy matter in view of the relation (3.3) between the $U(n)$ spherical functions and the K spherical functions. In fact we have the following

Proposition 4.1. *Let $f \in L^2(H^n)$. Then we have the decomposition*

$$f(z,t) = (2\pi)^{-n} \sum_{k=0}^{\infty} \sum_{j=o}^{J_k} \int_{-\infty}^{\infty} f * e_{k,j}^\lambda(z,t)|\lambda|^n d\lambda,$$

where $e_{k,j}^\lambda$ are the K-spherical functions.

Now we prove the orthogonality of K-spherical functions.

Lemma 4.2. *Let $\varphi_{k,j}(z) = e_{k,j}^1(z,0), z \in \mathbb{C}^n$. Then we have $\varphi_{k,j} \times \varphi_{l,i} = (2\pi)^n \delta_{k,l}\, \delta_{i,j}\, \varphi_{k,j}$, where $\times$ denotes the twisted convolution on $\mathbb{C}^n$.*

Proof. Recall that $\varphi_{k,j}$ is given by

$$\varphi_{k,j}(z) = \sum_{p=1}^{n(j)} \langle \rho_1(z,0)v_p^j, v_p^j \rangle,$$

where $\{v_p^j : p = 1, ..., n(j)\}$ is an orthonormal basis for $\mathcal{P}_{k,j}$. Since each element

$$v_p^j \in \mathcal{P}_k = \text{Span}\{\zeta_\alpha\}_{|\alpha|=k},$$

it is of the form $v_p^j = \sum_{|\alpha|=k} a_p^j(\alpha)\zeta_\alpha$. Since $\mathcal{P}_m = \bigoplus_{j=1}^{J_m} \mathcal{P}_{m,j}$ is an orthogonal decomposition and since $\{v_p^j\}_{p=1}^{n(j)}$ is an orthonormal basis for $\mathcal{P}_{k,j}$, we have $v_p^j \perp v_q^i$ for $i \neq j$ or $p \neq q$ and $||v_p^i|| = 1$. This translates in terms of $\{a_p^j\}$ to

$$\sum_{|\alpha|=k} a_p^i(\alpha)\overline{a_q^j(\alpha)} = \delta_{p,q}\, \delta_{i,j}. \tag{4.2}$$

Now, using the above expression for v_p^j, we see that

$$\begin{aligned}\varphi_{k,j}(z) &= \sum_{p=1}^{n(j)} \sum_{|\alpha|=k=|\beta|} a_p^j(\alpha)\overline{a_p^j(\beta)}\,\langle \rho_1(z,0)\zeta_\alpha, \zeta_\beta\rangle \\ &= \sum_{p=1}^{n(j)} \sum_{|\alpha|=k=|\beta|} a_p^j(\alpha)\,\overline{a_p^j(\beta)}\,\Phi_{\alpha\beta}(z).\end{aligned}$$

When $k \neq l$, it follows in view of (2.1) that

$$\varphi_{k,i} \times \varphi_{l,j} = 0.$$

When $k = l$, using (2.1), (3.4) and (4.1), we see that

$$\begin{aligned}\varphi_{k,j} \times \varphi_{k,i} &= \sum_{p,q=1}^{n(j)} \sum_{\substack{|\alpha|=k=|\beta| \\ |\mu|=k=|\nu|}} a_p^i(\alpha)\,\overline{a_p^i(\beta)}\; a_q^j(\mu)\,\overline{a_q^j(\nu)}\,\Phi_{\alpha\beta} \times \Phi_{\mu\nu} \\ &= (2\pi)^n \sum_{p,q=1}^{n(j)} \sum_{|\alpha|=k=|\nu|} a_p^i(\alpha) \sum_{|\beta|=k} \overline{a_p^j(\beta)} a_q^i(\beta)\; \overline{a_q^j(\nu)}\;\Phi_{\alpha\nu} \\ &= (2\pi)^n \delta_{i,j}\,(2\pi)^{n/2} \sum_{p=1}^{n(j)} \sum_{|\alpha|=k=|\nu|} a_p^i(\alpha)\,\overline{a_p^i(\nu)}\,\Phi_{\alpha,\nu} \\ &= (2\pi)^n\,\delta_{i,j}\,\varphi_{k,j}.\end{aligned}$$

□

When f is a Schwartz class function on H^n, an easy computation shows that

$$f * e_{k,j}^{\lambda}(z,t) = e^{i\lambda t} \int_{\mathbb{C}^n} f^{\lambda}(w)\,\varphi_{k,j}^{\lambda}(z-w)\,e^{-i\frac{\lambda}{2}\operatorname{Im} z.\bar{w}}\,dw, \tag{4.3}$$

where f^{λ} denotes the Fourier transform of f in the t-variable and $\varphi_{k,j}^{\lambda}(w) = \varphi_{k,j}(\sqrt{|\lambda|}w)$. Now, as a consequence of the above orthogonality, we see that the operators

$$P_{k,j}f(z,t) = (2\pi)^{-n} \int_{-\infty}^{\infty} f * e_{k,j}^{\lambda}(z,t)\,|\lambda|^n\,d\lambda$$

are projection operators.

Again for Schwartz class functions, it is easy to see that

$$P_{k,j}f(z,t) = (2\pi)^{-n}\, f * \int_{-\infty}^{\infty} e_{k,j}^{\lambda}(z,t)\,|\lambda|^n\,d\lambda.$$

The kernel of the above operator is $K(z,t) = (2\pi)^{-n}\int_{-\infty}^{\infty} e_{k,j}^{\lambda}(z,t)\,|\lambda|^n\,d\lambda$. The content of the next proposition is that these are Calderon–Zygmund (C–Z) kernels. Consequently, these operators extend as principal-value singular integral operators bounded on $L^p(H^n)$.

Proposition 4.3. *The operators $P_{k,j}(z,t)$ are Calderon–Zygmund operators on the Heisenberg group. Consequently, there exists a constant C_k independent of f such that*

$$||P_{k,j}f||_{L^p} \leq C_k(p)||f||_{L^p(H^n)}$$

for all $f \in L^p(H^n)$, $1 < p < \infty$.

Proof. The kernel of the projection operator $P_{k,j}$ is given by

$$K(z,t) = (2\pi)^{-n} \int_{-\infty}^{\infty} e^{\lambda}_{k,j}(z,t)\,|\lambda|^n\,d\lambda.$$

We show that $K(z,t)$ is a C–Z kernel on the Heisenberg group. For this we have to show that $K(z,t)$ is homogeneous of degree $-2n-2$ and that it satisfies the cancellation property $\int_{\mathbb{C}^n} K(z,t)dz = 0$. Since each $\varphi_{k,j}$ is a product of $e^{-\frac{|\lambda|}{4}|z|^2}$ with a polynomial in z and $\bar{z}$ of degree $2k$, we see that

$$e^{\lambda}_{k,j}(z,t) = \left(\sum_{|\alpha|+|\beta|\leq 2k} a^j_{\alpha\beta} \left(\sqrt{|\lambda|}z\right)^{\alpha} \left(\sqrt{|\lambda|}\bar{z}\right)^{\beta} \right) e^{-\frac{\lambda}{4}|z|^2}\, e^{i\lambda t}. \tag{4.4}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\lambda}_{kj}(z,t)|\lambda|^n d\lambda &= \sum_{|\alpha|+|\beta|\leq 2k} a^j_{\alpha\beta}\, z^{\alpha}\, \bar{z}^{\beta} \int_{-\infty}^{\infty} e^{i\lambda t - \frac{|\lambda|}{4}|z|^2} |\lambda|^{n+\frac{|\alpha|+|\beta|}{2}}\, d\lambda \\ &= \sum_{|\alpha|+|\beta|\leq 2k} a^j_{\alpha\beta} \left[g_{\alpha\beta}(z,t) + g_{\alpha\beta}(z,-t) \right],\end{aligned}$$

where

$$\begin{aligned} g_{\alpha,\beta}(z,t) &= z^{\alpha}\,\bar{z}^{\beta} \int_0^{\infty} e^{\frac{-\lambda}{4}(|z|^2-4it)} \lambda^{\frac{|\alpha|+|\beta|}{2}+n}\, d\lambda \\ &= C_{\alpha,\beta} \frac{z^{\alpha}\bar{z}^{\beta}}{\left(\frac{|z|^2-4it}{4}\right)^{\frac{|\alpha|+|\beta|}{2}+n+1}} \end{aligned}$$

and

$$C_{\alpha,\beta} = \Gamma\left(\frac{|\alpha|+|\beta|}{2}+n+1\right).$$

From the above expression for the kernel, it is easy to see that $K(z,t)$ is homogeneous of degree $-2n-2$ with respect to the Heisenberg dilation.

To show that

$$\int_{\mathbb{C}^n} K(z,t)dz = 0$$

in view of the above expression and the homogeneity of K, it is enough to show that

$$\int_{\mathbb{C}^n} [g_{\alpha,\beta}(z,1) + g_{\alpha,\beta}(z,-1)]\, dz = 0.$$

Let us consider $\int_{\mathbb{C}^n} g_{\alpha,\beta}(z,1)dz$. Changing variable $z_i = r_i e^{i\theta_i}$, we have

$$\int_{\mathbb{C}^n} g_{\alpha\beta}(z,1)dz =$$

$$C_{\alpha,\beta}\int_{r_1=0}^{\infty}\cdots\int_{r_n=0}^{\infty}\frac{r_1^{\alpha_1+\beta_1}\cdots r_n^{\alpha_n+\beta_n}\, r_1dr_1\cdots r_ndr_n}{(r_1^2+\cdots+r_n^2-4i)^{\frac{|\alpha|+|\beta|}{2}+n+1}}\,\Pi_{j=1}^n\int_{\theta_j=0}^{2\pi} e^{i(\alpha_j-\beta_j)\theta_j}\,d\theta_j.$$

Consequently,

$$\int_{\mathbb{C}^n} g_{\alpha\beta}(z,1)dz = 0 \quad \text{if } \alpha\neq\beta$$

and

$$\begin{aligned}\int_{\mathbb{C}^n} g_{\alpha\alpha}(z,1)dz &= C_\alpha\int_{r_1=0}^{\infty}\cdots\int_{r_n=0}^{\infty}\frac{r_1^{2\alpha_1}\cdots r_n^{2\alpha_n}\; r_1dr_1\cdots r_ndr_n}{(r_1^2+\cdots+r_n^2-4i)^{|\alpha|+n+1}}\\ &= C_\alpha\int_{w_1=0}^{\infty}\cdots\int_{w_n=0}^{\infty}\frac{w_1^{\alpha_1}\cdots w_n^{\alpha_n}\,dw_1\cdots dw_n}{(w_1+\cdots+w_n-4i)^{|\alpha|+n+1}}.\end{aligned}$$

Applying the calculus of residues to each variable separately shows that the above integral is equal to

$$(-1)^nC_\alpha\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\frac{w_1^{\alpha_1}\cdots w_n^{\alpha_n}\,dw_1\cdots dw_n}{(w_1+\cdots+w_n-4i)^{|\alpha|+n+1}}.$$

Now use the change of variable $w_i = -v_i$ to see that the above integral is equal to

$$-C_\alpha\int_{v_1=0}^{\infty}\cdots\int_{v_n=0}^{\infty}\frac{v_1^{\alpha_1}\cdots v_n^{\alpha_n}dv_1\cdots dv_n}{(v_1+\cdots v_n+4i)^{|\alpha|+n+1}} = -\int_{\mathbb{C}^n} g_{\alpha\alpha}(z,-1).$$

It follows that

$$\int_{\mathbb{C}^n} [g_{\alpha\alpha}(z,1) + g_{\alpha\alpha}(z,-1)]\, dz = 0,$$

which proves the cancellation property. □

As observed in [9], the above series may not converge for all $f\in L^p$, $p\neq 2$. For our purposes, it is enough to consider the Abel means of the above series, which have much better behaviour.

For each $r>0$ we define the Abel means of the above series by

$$A_rf(z,t) = (2\pi)^{-n}\sum_{k=0}^{\infty} r^{2k+n}\sum_{j=0}^{J_k}\int_{-\infty}^{\infty} f * e_{k,j}^{\lambda}(z,t)|\lambda|^n d\lambda.$$

The fact that the above series converges in L^p for each $0 < r < 1$ and that the operator norms of A_r are uniformly bounded follows, in view of (3.3), from the corresponding result for the case of $U(n)$ proved in [9]. We state the following results.

Theorem 4.4. *For $f \in L^p(H^n)$, we have $\|A_r f\|_p \leq C\|f\|_p$, $1 < p < \infty$ for some constant C independent of r for $0 < r < 1$.*

As a corollary to this, we get the following convergence result.

Theorem 4.5. *Let $f \in L^p(H^n), 1 < p < \infty$. Then*

$$\lim_{r \to 1-} (2\pi)^{-n} \sum_{k=0}^{\infty} \sum_{j=1}^{J_k} r^{2k+n} \int_{\mathbb{R}} f * e_{k,j}^{\lambda}(z,t)|\lambda|^n d\lambda = f(z,t)$$

where the limit is in L^p norm.

Remark. To prove Theorem 4.5, we have to prove the convergence on a dense subset of L^p. Because of (3.3), we can use the dense subset constructed in [9] for the case of $U(n)$ in our situation.

5 K-spherical means and injectivity results

In this section, we prove the injectivity theorem for the case $1 \leq p < \infty$. In fact, the results of this section hold for a wider class of K-invariant probability measures on H^n. For simplicity, we prove the results for the K-spherical means and indicate at the end of the section a wider class of K-invariant measures for which the same proof works.

We start with an interesting property satisfied by the K-spherical functions.

Proposition 5.1. *Let μ_{z_0,t_0} be as above. Then*

$$e_{k,j}^{\lambda} * \mu_{z_0,t_0} = C_{k,j}\, e_{k,j}^{-\lambda}(z_0,t_0)\, e_{k,j}^{\lambda},$$

where $C_{k,j} = (e_{k,j}^{\lambda}(0))^{-1} = (\dim \mathcal{P}_{k,j})^{-1}$

Proof. Since $e_{k,j}^{\lambda}$ is a K-spherical function, by (3.1) (with $\varphi = (\dim \mathcal{P}_{k,j})^{-1} e_{k,j}^{\lambda}$)

$$\int_K e_{k,j}^{\lambda}\left((z,t)\cdot k(w,s)\right) dk = (\dim \mathcal{P}_{k,j})^{-1}\, e_{k,j}^{\lambda}(z,t)\, e_{k,j}^{\lambda}(w,s).$$

Let O denote the K-orbit of $(z_0,t_0) \in H^n$. Therefore,

$$\begin{aligned} e_{k,j}^{\lambda} * \mu_{z_0,t_0}(z,t) &= \int_O e_{k,j}^{\lambda}\left((z,t)(w,s)^{-1}\right)\, d\mu_{z_0,t_0}(w,s) \\ &= \int_K \int_O e_{k,j}^{\lambda}((z,t)k(w,s)^{-1}) d\mu_{z_0,t_0}(w,s) dk. \end{aligned}$$

Writing $(w, s) = (kz_0, s)$, $k \in K$ and using $(kz_0, s)^{-1} = k(z_0, s)^{-1}$, we see that the above is equal to

$$\begin{aligned}
&\int_O \int_K e^{\lambda}_{k,j}\left((z,t)\, k\, (z_0, s)^{-1}\right) dk\, d\mu_{z_0,t_0}(\omega, s) \\
&= (\dim \mathcal{P}_{k,j})^{-1} \int_O e^{\lambda}_{k,j}(z,t)\, e^{\lambda}_{k,j}((z_0,s)^{-1})\, d\mu_{z_0,t_0}(\omega, s) \\
&= (\dim \mathcal{P}_{k,j})^{-1} e^{\lambda}_{k,j}(z,t)\, e^{-\lambda}_{k,j}(z_0,t_0).
\end{aligned}$$

The last equality follows from the definition of $d\mu_{z_0,t_0}$ and the fact that $\int_O d\mu_{z_0,t_0}(\omega, s) = 1$. □

Now we are in a position to prove the injectivity theorem for K-spherical means. The following observation is used in the proof of the theorem. Recall that $e^{\lambda}_{k,j}$ is the product of $e^{-\frac{|\lambda|}{4}|z|^2} e^{i\lambda t}$ and a polynomial in λz and $\lambda \bar{z}$. Consequently, the zeros of $e^{-\lambda}_{k,j}(z_0, t_0)$ as a function of λ form a finite set.

Theorem 5.2. *Let μ_{z_0,t_0} be as before. If $f \in L^p(H^n)$, $1 \le p < \infty$, satisfies $f * \mu_{z_0,t_0} = 0$, then $f \equiv 0$.*

Proof. As in [12], we use the Strichartz decomposition to prove the theorem. We present a proof that does not invoke the Hörmander–Mikhlin multiplier theorem. First let us consider the case $p > 1$. Since $f \in L^p(H^n)$, $1 < p < \infty$, we have

$$f(z,t) = \lim_{r \to 1-} (2\pi)^{-n} \sum_{j=0}^{\infty} r^{2m_j+n} \int_{-\infty}^{\infty} f * e^{\lambda}_{k,j}(z,t)\, |\lambda|^n\, d\lambda.$$

Since $f * \mu_{z_0,t_0} = 0$ and convolution with μ_{z_0,t_0} defines a bounded operator on $L^p(H^n)$, we see in view of Proposition 5.1 that

$$\lim_{r \to 1-} (2\pi)^{-n} \sum_{j=0}^{\infty} r^{2m_j+n} \int_{-\infty}^{\infty} e^{-\lambda}_{k,j}(z_0,t_0)\, f * e^{\lambda}_{k,j}(z,t) |\lambda|^n d\lambda = 0.$$

Applying the projection operator $P_{k,j} : g \to (2\pi)^{-n} \int_{-\infty}^{\infty} g * e^{\lambda}_{k,j}(z,t)\, |\lambda|^n\, d\lambda$, which is a bounded operator on L^p, we get

$$\int_{-\infty}^{\infty} e^{-\lambda}_{k,j}(z_0,t_0)\, f * e^{\lambda}_{k,j}(z,t)\, |\lambda|^n\, d\lambda = 0.$$

Since $f \in L^p$, this means by definition of the projection that there exists a sequence of Schwartz class functions f_n converging to f in $L^p(H^n)$ such that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} e^{-\lambda}_{k,j}(z_0,t_0) f_n * e^{\lambda}_{k,j}(z,t) |\lambda|^n d\lambda = 0,$$

where the convergence is in L^p. In view of (4.2), the LHS of the above equation is a Fourier integral operator; and the equation can be written as

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e_{k,j}^{-\lambda}(z_0,t_0)\,\widehat{P_{k,j}f_n}(z,\lambda)e^{i\lambda t}d\lambda = 0,$$

where $\widehat{\ }$ denotes the Fourier transform in the t-variable.

Since the above sequence converges to 0 in L^p, the sequence of Fourier transforms converges to 0 in the sense of distributions. That is,

$$\lim_{n\to\infty} e_{k,j}^{-\lambda}(z_0,t_0)\,\widehat{P_{kj}(f_n)} = 0.$$

Also, since

$$\lim_{n\to\infty}\widehat{P_{k,j}f_n} = \widehat{P_{k,j}f}$$

as distributions, it follows that $e_{k,j}^{-\lambda}(z_0,t_0)\widehat{P_{k,j}f} = 0$. This means that the support of $\widehat{P_{kj}f}$ in the t-variable is contained in the zero set of $e_{k,j}^{-\lambda}(z_0,t_0)$, which is finite. As $P_{k,j}f\in L^p$, this is not possible unless $P_{k,j}f = 0$. Since this holds for all k,j, we see that $f=0$. This settles the case $1<p<\infty$.

To deal with the case $p=1$, choose an approximate identity $g_n\in C_0^\infty(H^n)$. Then $g_n * f\in L^p$ for all $p\geq 1$, and $g_n * f * \mu_{z_0,t_0} = 0$ for every n. As $g_n * f$ belongs to L^p for $p>1$, by the above arguments we see that $g_n * f = 0$ for all n. Also, since $g_n * f\to f$ in $L^1(H^n)$, it follows that $f\equiv 0$. This settles the case $p=1$. □

We observe that the above theorem also holds for a wider class of K-invariant probability measures on H^n. First of all, notice that if μ is a probability measure on H^n, then $f*\mu\in L^p(H^n)$ whenever $f\in L^p(H^n)$.

Let $t_0\in\mathbb{R}$ be fixed. Define an equivalence relation on $\mathbb{C}^n\times\{t_0\}\subset H^n$ by setting $(z_1,t_0)\sim(z_2,t_0)$ if there exists $k\in K$ such that $z_1 = kz_2$. Then the equivalence classes are precisely the K-orbits; and the set of equivalence classes $\mathbb{C}^n\times\{t_0\}/\sim\ = \mathcal{O}$ can be identified with a subset, say Δ, of $\mathbb{C}^n$. Clearly, this is an unbounded set which contains $\mathbb{R}_+ = [0,\infty)$. Let M be a probability measure on Δ such that $\int_\Delta \mu_{(z_0,t_0)}dM(z_0)$ converges weak $*$ in $(C_0(\mathbb{C}^n))^*$. Then $\mu = \int_\Delta \mu_{(z_0,t_0)}dM(z_0)$ defines a K-invariant probability measure on H^n. Note that this construction is in the spirit of Choquet's theorem [3]. In general, the supports of these measures are non-compact. It is easy to see that Proposition 5.1 holds for all such measures. Theorem 5.2 also holds for all such measures μ provided that the zeros of $\mu(e_{k,j}^{-\lambda})$ as a function of λ form a discrete set. In fact, multiplying $\widehat{P_{k,j}f}$ by a compactly supported function of the form $\widehat{\varphi}(rt)$, one can proceed as in the proof of Theorem 5.2 and show that $\varphi_r * P_{k,j}f = 0$. Since φ_r is an approximate identity, by letting $r\to 0$ we get $P_{k,j}f = 0$.

In special cases it is easy to give examples of such measures. For instance, when $K = U(n)$, we have $\mathcal{O} = [0,\infty)$. Let $\mu_{r,t}$ denote the normalised surface measure on the sphere $\{(z,t) : |z| = r\} \subset H^n$. Let M be any probability measure on $[0,\infty)$ such that $\int_0^\infty r^2\, dM(r) < \infty$. Then $\mu_t = \int_\Delta \mu_{r,t}\, dM(r)$ is a $U(n)$-invariant probability measure. In view of the integrability condition on the measure M, it is easy to see that $\mu_t(e_{k,j}^{-\lambda})$ extends to the half planes $\{\mathrm{Re}\ \lambda > 0\}$ and $\{\mathrm{Re}\ \lambda < 0\}$ as a holomorphic function of λ. Consequently, the zeros of $\mu_t(e_{k,j}^{-\lambda})$ form a discrete set.

6 The case of bounded continuous functions

In this section, we investigate the injectivity properties of the K-spherical means for the case $p = \infty$. The crux of the matter is that the Wiener–Tauberian theorem holds for the algebra $L^1_K(H^n)$.

As in the case of $U(n)$ or $T(n)$ (see [1]), the vanishing of the average of a bounded continuous function f over a single orbit does not guarantee that the function is zero. This is a consequence of the functional equation (3.1) satisfied by a bounded K-spherical function. More precisely, let $(z_0,t_0) \in H^n$ be a zero of a bounded K-spherical function ϕ. Then by Proposition 5.1 we have

$$\phi * \mu_{z_0,t_0}(z,t) = \phi(z_0,t_0)\phi(z,t) \equiv 0.$$

As in [1], we prove that if the average of f over sufficiently many K-orbits vanishes, then $f \equiv 0$. To prove this, we need to show that the algebra $L^1_K(G)$ has the Wiener–Tauberian property. In fact, the proof given in [5], [7] for the subalgebra of radial functions in $L^1(H^n)$ holds for the algebra $L^1_K(H^n)$ as well. We state the theorem without proof.

Theorem 6.1. *Let J be a proper closed ideal in $L^1_K(H^n)$. Then there exists ϕ in the maximal ideal space of $L^1_K(H^n)$ such that $\widehat{f}(\varphi) = 0$ for every $f \in J$, where $\widehat{\ }$ stands for the Gelfand transform.*

Remark. Since every element in the maximal ideal space is given by a bounded K-spherical function, this amounts to saying that $\int_{H^n} f(z,t)\phi(z,t)dz\, dt = 0$ for every $f \in J$.

Now we prove the following injectivity result.

Theorem 6.2. *Let $\Re$ be a family of K-invariant compactly supported Radon measures on H^n such that for any bounded K-spherical function ϕ there exists a $\mu \in \Re$ such that $\int \phi d\mu \neq 0$. If f is a bounded continuous function on H^n such that $f * \mu = 0$ for $\mu \in \Re$, then $f \equiv 0$. On the other hand, if the above condition fails to* *hold, there exists a bounded continuous function f not identically zero such that $f * \mu = 0$.*

Proof. In the proof of the theorem we closely follow [1]. Let I be the closed ideal generated by the set

$$\{\mu * \eta : \mu \in \mathcal{R}, \eta \in L^1_K(H^n)\},$$

which is clearly contained in $L^1_K(H^n)$, as both μ and η are K-invariant. As $f * \mu = 0$ for every $\mu \in \mathcal{R}$, it follows that $f * I = 0$.

Now to prove that $f = 0$, it is enough to show that $I = L^1_K(H^n)$. In view of Theorem 6.1, it suffices to show that for any bounded K-spherical function ϕ there exists $\eta \in \mathcal{S}_K(H^n) \subset L^1_K(H^n)$ such that

$$\tilde{\eta}(\phi) = \int_{H^n} \phi.\eta \neq 0. \tag{6.1}$$

Here $\mathcal{S}_K(H^n)$ denotes the space of K-invariant Schwartz class functions on H^n.

We consider the cases in which ϕ is given by an infinite-dimensional representation and by the one-dimensional representation of H^n separately.

In the former case, ϕ is of the form

$$\phi(z,t) = e^\lambda_{k,j}(z,t) = e^{i\lambda t}\varphi^\lambda_{k,j}(z), \quad \text{where } \lambda \in \mathbb{R}^*.$$

In this case, we choose $\eta^\lambda = g(\lambda)\overline{\varphi^\lambda_{k,j}(z)}$ so that

$$\int_{H^n} \phi\eta \, dz\, dt = \int_{\mathbb{C}^n} \eta^\lambda(z)\phi(z)\, dz = \int_{\mathbb{C}^n} g(\lambda)\,|\psi^\lambda_j|^2\, dz \geq 0,$$

where $\eta^\lambda(z)$ denotes the Fourier transform of η in the t variable at λ and g is a smooth non-negative K-invariant function in $L^1(\mathbb{C}^n)$.

On the other hand, if ϕ is given by the one-dimensional representation χ_{z_0}, then by (3.4) ϕ is of the form ϕ_{z_0}, where

$$\phi_{z_0}(\xi) = \widehat{\mu}_{Kz_0}(\xi),$$

the Fourier transform of the K-invariant normalised surface measure on the K orbit Kz_0 of z_0. In this case, choose $\eta \in \mathcal{S}_K(H^n)$ such that $\eta \geq 0$ and $\widehat{h}(z_0) = \int \widehat{\eta}(z_0,t)\, dt \neq 0$. Then

$$\int_{H^n} \widehat{\mu}_{Kz_0}(z)\,\eta(z,t)\, dz\, dt = \int_{\mathbb{C}^n} h(z)\widehat{\mu}_{Kz_0}\, dz = \int_{\mathbb{C}^n} \widehat{h}(z)\, d\mu_{Kz_0}.$$

Since h is K-invariant, $\widehat{h}$ is also K-invariant. Hence $\widehat{h}$ is constant on the orbit Kz_0. So

$$\int_{H^n} \widehat{\mu}_{Kz_0}(z)\,\eta(z,t)\, dz\, dt = \widehat{h}(z_0) \neq 0.$$

On the other hand, if (6.1) fails, then there exists a bounded K-spherical function ϕ on H^n such that $\int \phi d\mu = 0$ for $\mu \in \Re$. Then, in view of the functional equation satisfied by ϕ, it is easy to see that $\phi * d\mu = (\int \phi d\mu)\, \phi = 0$, but ϕ is not identically equal to zero.

Now we prove the injectivity in the L^∞ case, which is an "N radius theorem" for the K-spherical means.

Theorem 6.3. *Let f be a bounded continuous function on H^n satisfying the condition $f * \mu_{z_j,t_j} = 0$ for the N points $(z_1,t_1),\ldots,(z_N,t_N)$ in H^n. Let H and G be the functions given by*

$$H(\lambda,p) = \sum_{j=0}^{N} |\psi_p(\sqrt{|\lambda|}z_j)| \quad \text{and} \quad G(\xi) = \sum_{j=0}^{N} |\widehat{\mu}_{K\,\xi}(z_j)|.$$

Then $f \equiv 0$ if and only if neither H nor G vanish.

Remark. Note that H and G can be interpreted as functions on the maximal ideal space of the Banach algebra $L^1_K(H^n)$. Also, the function G corresponds to the case $\lambda = 0$.

Proof. In view of the previous theorem, it is enough to prove that given any bounded spherical function ϕ there exists a measure μ_{z_i,t_i} for some $i \in \{1,2,\ldots,N\}$ such that $\int \phi\, d\mu_{z_i,t_i} \neq 0$. Now observe that a bounded spherical function is either of the form $e^\lambda_{k,j}(z,t)$ or $\widehat{\mu}_{K\,\xi}(z)$. Since both $e^\lambda_{k,j}(z,t)$ and $\widehat{\mu}_{K\,\xi}$ are K-invariant, the above condition becomes $e^\lambda_{k,j}(z_i,t_i) \neq 0$ and $\widehat{\mu_{K\xi}z_i} \neq 0$, which obviously holds whenever the functions H and G are non-vanishing.

Conversely, if H vanishes at some point (λ_0, p_0), then $e^{\lambda_0}_{p_0,j}(z,t)$ is a non-trivial function such that $\int e^{\lambda_0}_{p_0,j}(z,t)\, d\mu_{z_i,t_i} = e^{\lambda_0}_{p_0,j}(z_i,t_i) = 0$.

Also, if G vanishes at some point $\xi \in \mathbb{C}^n \subset H^n$, then $\widehat{\mu}_{K\xi}$ is a non-trivial function and $\widehat{\mu}_{K\,\xi} * \mu_{Kz_i} \equiv 0$, for $i = 1,2,\ldots,N$.

Remark. It is interesting to observe that this condition is the same as saying that z_i/z_j are not the quotients (with suitable interpretation) of zeros of K-spherical functions. This is analogous to the condition for the two-radius theorem in [1].

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