A LOCALISATION THEOREM FOR LAGUERRE EXPANSIONS

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Abstract

Regularity properties of Laguerre means are studied in terms of certain Sobolev spaces defined using Laguerre functions. As an application we prove a localisation theorem for Laguerre expansions.

Key words: Laguerre means, Laguerre series, Sobolev spaces.

1 Introduction

The Laguerre polynomials $L_n^\alpha(x)$, of type $\alpha > -1$ are defined by the generating function identity

$$\sum_0^\infty L_n^\alpha(x)t^n \quad = \quad (1-t)^{-\alpha-1}e^{-\frac{x\ t}{1-t}}, \qquad |t| < 1. \tag{1.1}$$

The associated Laguerre functions are defined by

$$\tilde{\mathcal{L}}_n^\alpha(x) \quad = \quad L_n^\alpha(x)e^{-\frac{x}{2}}x^{\frac{\alpha}{2}} \tag{1.2}$$

and they are the eigenfunctions of the Laguerre differential operator

$$-\frac{d}{dx}\left\{x\frac{d}{dx}\tilde{\mathcal{L}}_n^\alpha(x)\right\} + \left\{\frac{x}{4}+\frac{\alpha^2}{4x}\right\}\tilde{\mathcal{L}}_n^\alpha(x) \quad = \quad (n+\frac{\alpha+1}{2})\tilde{\mathcal{L}}_n^\alpha(x) \tag{1.3}$$

Moreover the normalised functions $\mathcal{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}}\tilde{\mathcal{L}}_n^\alpha(x)$ form an orthonormal basis for $L^2[(0,\infty),dx]$. Therefore for any $f \in L^2(0,\infty)$ we have the eigenfunction expansion

$$f \quad = \quad \sum_0^\infty a_n\mathcal{L}_n^\alpha(x) \tag{1.4}$$

with $a_n = \int_0^\infty f(x)\mathcal{L}_n^\alpha(x)dx$

Three types of Laguerre expansions have been studied in the literature. The first one is concerned with the Laguerre polynomials $L_n^\alpha(x)$, $\alpha > -1$, which form an orthonormal basis for $L^2[(0,\infty),e^{-x}x^\alpha dx]$. The second type is concerned with the Laguerre functions (1.2) which form an orthogonal family in $L^2[(0,\infty),dx]$. Considering the functions

$$l_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} L_n^\alpha(x)\ e^{-\frac{x}{2}}$$

as an orthonormal family in $L^2[(0,\infty),x^\alpha dx]$, we get a third type of expansion.

Several authors have studied norm convergence and almost everywhere convergence of Riesz means of such expansions. Some references are Askey-Wainger[2], Muckenhoupt[6], Gorlich-Markett[3], Markett[5], Stempak[7], Thangavelu[10]. Various results can also be seen in [12].

Recently by invoking an equiconvergence theorem of Muckenhoupt for Laguerre expansion, Stempak[8] has proved the following almost everywhere convergence result for expansions with respect to $\mathcal{L}_n^\alpha(x)$ as well as $l_n^\alpha(x)$.

(1) $\sum_0^N (g, \mathcal{L}_k^\alpha)_{L^2(dx)} \mathcal{L}_k^\alpha(x) \to g(x)$ for almost every $x \in I\!\!R_+$ as $N \to \infty$ for $\frac{4}{3} < p < 4$ if α ¿ $-\frac{1}{2}$, and for $p \in \left((1+\frac{\alpha}{2})^{-1}, 4\right)$ otherwise.

(2) $\sum_0^N (g, l_k^\alpha)_{L^2(x^\alpha dx)}\, l_k^\alpha(x) \to g(x)$ for almost every $x \in I\!\!R_+$ as $N \to \infty$ for $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{(2\alpha+1)}$ if $\alpha > -\frac{1}{2}$, and for $1 < p < \infty$ otherwise.

In this paper we study the twisted spherical means associated with the Laguerre expansions which we will call Laguerre means. We consider expansions with respect to the system $\varphi_k^\alpha(x) = L_k^\alpha(x^2)\ e^{-\frac{x^2}{2}}$. Then the normalised functions

$$\psi_k^\alpha(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} \varphi_k^\alpha(x) \tag{1.5}$$

form an orthonormal basis for $L^2[(0,\infty), x^{2\alpha+1}dx]$. We have the mapping $T : L^2[x^{2\alpha+1}dx] \to L^2[x^\alpha dx]$ defined by $Tf(x) = \frac{1}{\sqrt{2}} f(\sqrt{x})$, which is a unitary mapping which takes $\psi_k^\alpha(x)$ to $l_k^\alpha(x)$. Therefore the expansion in ψ_k^α is equivalent to the expansion in l_k^α.

We prove a localisation theorem for Laguerre expansion with respect to ψ_k^α without appealing to the equiconvergence theorem. Clearly a localisation theorem follows from the almost everywhere convergence result of Stempak given above, but this result only says that if $f \equiv 0$ in a neighbourhood of a point $z \in (0,\infty)$, then $S_N f(w) \to 0$ for almost every w in this neighbourhood. But using the method of Laguerre means we could identify the set on which $S_N f(w) \to 0$.

The twisted spherical mean of a locally integrable function f on $I\!\!C^n$ is defined to be

$$f \times \mu_r(z) \quad = \quad \int_{|w|=r} f(z-w)\ e^{\frac{i}{2} Im(z.\bar{w})}\ d\mu_r(w), \tag{1.6}$$

where $d\mu_r(w)$ is the normalised surface measure on the sphere $\{|w| = r\}$ in $I\!\!C^n$. Such spherical means have been considered by Thangavelu in [11], where its regularity properties are used to prove a localisation theorem for the special Hermite expansion of L^2 functions on $I\!\!C^n$. The special Hermite expansion of a function f is given by

$$f(z) \quad = \quad (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k(z), \tag{1.7}$$

where $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)\ e^{-\frac{1}{4}|z|^2}$. Here $L_k^{n-1}(r)$ stands for the Laguerre polynomial of type $n-1$. Measuring the regularity of $f \times \mu_r(z)$ using a certain Sobolev space denoted by $W_R^s(\mathbb{R}_+)$, he proved the following localisation theorem:

Theorem 1 *(S.Thangavelu) Let f be a compactly supported function vanishing in a neighbourhood of a point $z \in \mathbb{C}^n$. Further assume that $f \times \mu_r(z) \in W_R^{n/2}(\mathbb{R}_+)$ as a function of r. Then $S_N f(z) \to 0$ as $N \to \infty$.*

By assuming certain regularity of $f \times \mu_r(z)$ as a function of r he could also establish an almost everywhere convergence result for special Hermite expansion. In the study of $f \times \mu_r(z)$ a crucial role is played by the following series expansion:

$$f \times \mu_r(z) \quad = \quad (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) \; f \times \varphi_k(z) \tag{1.8}$$

for the twisted spherical means. Here $f \times \varphi_k$ denotes the twisted convolution of f and φ_k, where twisted convolution of two functions f and g on $\mathbb{C}^n$ is defined by

$$f \; \times \; g(z) \quad = \quad \int_{\mathbb{C}^n} f(z-w) \; g(w) \; e^{\frac{i}{2} Im(z.\bar{w})} \; dw. \tag{1.9}$$

For a radial function f we have

$$f \times \varphi_k(z) \quad = \quad (2\pi)^{-n} R_k(f) \; \varphi_k(z), \tag{1.10}$$

where

$$R_k(f) = \frac{2^{1-n}k!}{(k+n-1)!} \int_0^{\infty} f(s) L_k^{n-1}(\frac{1}{2}s^2) e^{-\frac{1}{4}s^2} s^{2n-1} ds.$$

Therefore from (1.8) it follows that for a radial function f the special Hermite expansion becomes the Laguerre expansion with respect to the family $L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$. The above observation suggests that we can also study the localisation problem for Laguerre expansion with respect to the orthogonal family $L_k^{\alpha}(r^2)e^{-\frac{1}{2}r^2}, \alpha > -1$. What we need is something similar to twisted spherical means. Using the local co-ordinates on the sphere $|z| = r$ in $\mathbb{C}^n$ it is easy to see that

$$f \times \mu_r(z) = c_n \int_0^{\pi} f[(r^2 + |z|^2 + 2r|z|cos\theta)^{1/2}] \frac{J_{n-3/2}(r|z|sin\theta)}{(r|z|sin\theta)^{n-3/2}} sin^{2n-2}\theta d\theta. \tag{1.11}$$

for a suitable constant c_n.
We define the Laguerre means of order α to be

$$T_r{}^{\alpha} f(z) \qquad (1.12)$$
$$= \frac{2^{\alpha}\Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^{\pi} f[(r^2+z^2+2rzcos\theta)^{1/2}] \frac{J_{\alpha-1/2}(rzsin\theta)}{(rzsin\theta)^{\alpha-1/2}} sin^{2\alpha}\theta d\theta$$

Then $T_r{}^{\alpha}$ is a bounded self adjoint operator on $L^2(I\!R_+, x^{2\alpha+1}dx)$.
We have the interesting formula, see [12]

$$T_r{}^{\alpha}\varphi_k^{\alpha}(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\varphi_k^{\alpha}(r)\varphi_k^{\alpha}(z), \qquad (1.13)$$

for $\alpha > -\frac{1}{2}, r \geq 0,\ z \geq 0$. From the series expansion for $T_r{}^{\alpha} f(z)$ in terms of $\varphi_k^{\alpha}(z)$ and using the above formula it is easy to see that $T_r{}^{\alpha} f(z)$ has the series expansion

$$T_r{}^{\alpha} f(z) = \sum_0^{\infty} \left(\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\right)^2 (f\ ,\varphi_k^{\alpha})_{\alpha}\ \varphi_k^{\alpha}(z)\ \varphi_k^{\alpha}(r), \qquad (1.14)$$

$r \geq 0, z \geq 0, \alpha > -1/2$, where $\varphi_k^{\alpha}(r) = L_k^{\alpha}(r^2)e^{-\frac{1}{2}r^2}$. Here $(,)_{\alpha}$ denotes the inner product in the Hilbert space $L^2[R_+, x^{2\alpha+1}]$. Using this notion of Laguerre means we establish a localisation theorem for Laguerre series expansion for $f \in L^2[I\!R_+, x^{2\alpha+1}dx]$ with respect to the orthogonal family $\varphi_k^{\alpha}(r)$. Our main result is the following :

Theorem 2 *Let $f \in L^2[I\!R_+, x^{2\alpha+1}dx], \alpha > -1/2$ be a function vanishing in a neighbourhood B_z of a point $z \in I\!R_+$. If $w \in B_z$ is such that $T_r{}^{\alpha} f(w) \in W_{\alpha}^{\frac{\alpha+1}{2}}(I\!R_+)$, as a function of r, then $S_N f(w) \to 0$ as $N \to \infty$.*

We use the following notation: $L^2_{\alpha}(I\!R_+)$ stands for the space $L^2[I\!R_+, x^{2\alpha+1}dx]$, and the norm and the inner product in this space are denoted by $\|.\|_{\alpha}$ and $(.,.)_{\alpha}$ respectively.

2 The Sobolev space $W_{\alpha}{}^{s}(I\!R_+)$

The usual Sobolev space $H^s(I\!R^n)$, for $s \geq 0$ is defined to be

$$H^{s}(I\!R^n) = \left\{ f \in L^2(I\!R^n) : (-\Delta+1)^s f \in L^2(I\!R^n) \right\}$$

using the operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. Since we are interested in studying the regularity of the function $r \to T_r{}^\alpha f(z)$, motivated by the expansion (1.14) we define the Sobolev space $W_\alpha^s(I\!R_+)$ using the operator $L_\alpha = -[\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx} - x^2]$, which is a positive definite symmetric operator and the φ_k^α's form the family of eigenfunctions with corresponding eigenvalues $4(k + \frac{\alpha+1}{2})$. Also we have the normalised functions $\psi_k^\alpha(z)$ forming an orthonormal basis for $L_\alpha^2(I\!R_+)$. We define for $s \geq 0$

$$W_\alpha{}^s(I\!R_+) \quad = \quad \left\{ f \in L_\alpha^2(I\!R_+) : L_\alpha^s f \in L_\alpha^2(I\!R_+) \right\}, \tag{2.1}$$

where L_α^s is defined using the spectral theorem. In other words

$$f = \sum_{k=0}^{\infty} (f, \psi_k^\alpha)_\alpha \ \psi_k^\alpha$$

belongs to $W_\alpha{}^s$ if and only if,

$$\sum_{k=0}^{\infty} |4^s(k + \frac{\alpha+1}{2})^s (f \ , \psi_k^\alpha)_\alpha \ |^2 < \infty.$$

We now prove the following useful proposition which is needed for the proof of the main theorem.

Proposition 3 *Let $\alpha > -1$ and let φ be a smooth function on $I\!R_+$ which satisfies the following conditions*
(i) $\varphi \equiv 0$ near the origin in $I\!R_+$
(ii) $|(\frac{d}{dr})^j \varphi(r)| = O(\frac{1}{r^{2+j}})$ as $r \to \infty$ for $j = \ 0, 1, 2, 3 \ldots 2m$.
Then the operator $M_\varphi : W_\alpha{}^s \to W_{\alpha+1}^{\ s}$ defined by $M_\varphi \ f = \varphi.f$ is a bounded operator $\forall s$ such that $s \leq m$.

The proof of this proposition needs the following lemmas. Before stating the first lemma we introduce, for each nonnegative integer k, the class C_k, consisting of all smooth functions on $I\!R_+$, vanishing near 0 and which also satisfying the decay condition,$(\frac{d}{dr})^j \varphi = O(\frac{1}{r^{2+k+j}})$ as $r \to \infty$. The class C_k satisfies the following properties: (i) $C_{k+1} \subset C_k$, (ii) If $\varphi \in C_k$,$\frac{1}{r}\varphi \in C_{k+1}$, $r\varphi \in C_{k-1}$, for k¿1, (iii) If $\varphi \in C_k$, $\varphi^{(j)} \in C_{k+j}$.

Lemma 4 *Under the above assumptions on m, φ and α we have $L_{\alpha+1}^m \circ M_\varphi \circ L_\alpha^{-m} = \sum_{t+k \leq m} M_{\varphi_{k,t}} (\frac{d}{dr})^k L_\alpha^{t-m}$ with $\varphi_{k,t} \in C_k$.*

Proof: We claim that $L_{\alpha+1}^m \circ M_\varphi$ can be written as a linear combination of the form

$$L_{\alpha+1}^m \circ M_\varphi = \sum_{t+k \leq m} M_{\varphi_{k,t}} (\frac{d}{dr})^k L_\alpha^t \quad \textit{with} \ \ \varphi_{k,t} \in C_k. \tag{2.2}$$

First we note the following relations

$$L_\alpha M_\varphi = M_\varphi L_\alpha - 2M_{\varphi'}\frac{d}{dr} - M_{\varphi''+\frac{2\alpha+1}{r}\varphi'} \tag{2.3}$$

$$L_{\alpha+1} = L_\alpha - \frac{2}{r}\frac{d}{dr} \tag{2.4}$$

Using this relation in the above we get

$$\begin{aligned} L_{\alpha+1}M_\varphi &= M_\varphi L_\alpha \\ &- 2M_{(\varphi'+\frac{\varphi}{r})}\frac{d}{dr} - M_{(\varphi''+\frac{2\alpha+1}{r}\varphi')}. \end{aligned} \tag{2.5}$$

We also use the relation,

$$\begin{aligned} L_\alpha(\frac{d}{dr})^k &= (\frac{d}{dr})^k L_\alpha + \sum_{j=0}^{k-1} b_j(\frac{1}{r})^j(\frac{d}{dr})^{k-j} \\ &+ c_1 r(\frac{d}{dr})^{k-1} + c_2(\frac{d}{dr})^{k-2} \end{aligned} \tag{2.6}$$

where $b_j, c_1, c_2,$ are constants.This can be easily proved by induction on k. We prove (2.2) by induction on m. (2.2) is clear for $m = 1$ Assume (2.2) for $m = j$. Now,

$$\begin{aligned} L^{j+1}_{\alpha+1}\circ M_\varphi &= (L_\alpha - \frac{2}{r}\frac{d}{dr})(L^j_{\alpha+1}M_\varphi) \\ &= (L_\alpha - \frac{2}{r}\frac{d}{dr})(\sum_{t+k\le j} M_{\varphi_{k,t}}(\frac{d}{dr})^k L^t_\alpha) \\ &= \sum_{t+k\le j} L_\alpha(M_{\varphi_{k,t}}(\frac{d}{dr})^k L^t_\alpha) - 2\sum_{t+k\le j}\frac{1}{r}\frac{d}{dr}M_{\varphi_{k,t}}(\frac{d}{dr})^k L^t_\alpha \\ &= \sum_{t+k\le j}\left[M_{\varphi_{k,t}}L_\alpha - 2M_{\varphi'_{k,t}}\frac{d}{dr} - M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})}\right](\frac{d}{dr})^k L^t_\alpha \\ &- \frac{2}{r}\sum_{t+k\le j}\frac{d}{dr}M_{\varphi_{k,t}}(\frac{d}{dr})^k L^t_\alpha. \\ &= \sum_{t+k\le j} M_{\varphi_{k,t}}L_\alpha(\frac{d}{dr})^k L^t_\alpha - 2\sum_{t+k\le j} M_{\varphi'_{k,t}}(\frac{d}{dr})^{k+1}L^t_\alpha \\ &- \sum_{t+k\le j} M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})}(\frac{d}{dr})^k L^t_\alpha - \frac{2}{r}\sum_{t+k\le j} M_{\varphi'_{k,t}}(\frac{d}{dr})^k L^t_\alpha \\ &- \frac{2}{r}\sum_{t+k\le j} M_{\varphi_{k,t}}(\frac{d}{dr})^{k+1}L^t_\alpha \end{aligned} \tag{2.7}$$

In the above computation we have used (2.3). In view of (2.6), the first term of the above is

$$\begin{aligned}
&= \sum_{t+k\le j} M_{\varphi_{k,t}}(\frac{d}{dr})^k L_\alpha^{t+1} + \sum_{i=0}^{k-1} b_i(\frac{1}{r})^i \sum_{t+k\le j} M_{\varphi_{k,t}}(\frac{d}{dr})^{k-i} L_\alpha^t \\
&+ \sum_{t+k\le j} c_1 r M_{\varphi_{k,t}}(\frac{d}{dr})^{k-1} L_\alpha^t + \sum_{t+k\le j} c_2(\frac{d}{dr})^{k-2} \\
&= \sum_{t+k\le j+1} M_{\varphi_{k,t}}(\frac{d}{dr})^k L_\alpha^t + \sum_{i=0}^{k-1} b_i \sum_{t+k\le j} (\frac{1}{r})^i M_{\varphi_{k,t}}(\frac{d}{dr})^{k-i} L_\alpha^t \\
&+ \sum_{t+k\le j} c_1 r M_{\varphi_{k,t}}(\frac{d}{dr})^{k-1} L_\alpha^t + \sum_{t+k\le j} c_2(\frac{d}{dr})^{k-2} L_\alpha^t
\end{aligned} \tag{2.8}$$

Now by induction hypothesis we have $\varphi_{k,t} \in C_k$. Note that in the second term of the above the coefficient of $(\frac{d}{dr})^{k-i}L_\alpha^t$ is $(1/r)^i\varphi_{k,t}$. We have $(1/r)^i\varphi_{k,t} \in C_{k+i} \subset C_k \subset C_{k-i}$ for $i \ge 0$ and also $r\varphi_{k,t} \in C_{k-1}$. Hence the first term in (2.7) is of the required form. The second term of (2.7) can be written as $-2\sum_{t+k\le j+1} M_{\varphi'_{k-1,t}}(\frac{d}{dr})^k L_\alpha^t$, and $\varphi_{k,t} \in C_k$ by induction hypothesis. Therefore $\varphi'_{k-1,t} \in C_k$ in view of (iii). Hence the second term of (2.7) is also of the required form. In the third term the coefficient of $(\frac{d}{dr})^k L_\alpha^t$ is $M_{\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t}}$ and $\varphi''_{k,t} + \frac{2\alpha+1}{r}\varphi'_{k,t} \in C_{k+2} \subset C_k$ by induction hypothesis and in view of (i),(ii) and (iii). Similarly $\frac{1}{r}\ \varphi'_{k,t}$ occuring in the fourth term belongs to $C_{k+2} \subset C_k$. Also $\frac{1}{r}\varphi_{k,t}$ occuring in the fifth term $\in C_{k+1} \subset C_k$. Therefore (2.2) holds for $m = j+1$ also. Thus we have $T^{\ m} f = L_{\alpha+1}^m \circ M_\varphi \circ L_\alpha^{-m} f = \sum_{t+k\le m} M_{\varphi_{k,t}}(\frac{d}{dr})^k L_\alpha^{t-m} f$. Which proves the first lemma.

Lemma 5 $(\frac{d}{dr})^i L_\alpha^t : L_\alpha^2(I\!R_+) \to L_\alpha^2(I\!R_+)$ *is a bounded operator whenever* i *is a non negative integer and* $i + t \le 0$

Proof: We prove that $\frac{d}{dr}\ L_\alpha^{\ t}$ is a bounded operator on $L_\alpha^2(I\!R_+)$ for $1 + t \le 0$. We first note that

$$\frac{d}{dr}\psi_k^\alpha \;=\; -r\left[k^{\frac{1}{2}}\psi_{k-1}^{\alpha+1} + (k+\alpha+1)^{1/2}\psi_k^{\alpha+1}\right] \tag{2.9}$$

This can be seen as follows. We have

$$\begin{aligned}
\frac{d}{dr}\ L_k^\alpha(r)e^{-\frac{1}{2}r^2} &= \frac{d}{dr}L_k^\alpha(r^2)e^{-\frac{r^2}{2}} - rL_k^\alpha(r^2)e^{-\frac{r^2}{2}} \\
&= -2rL_{k-1}^{\alpha+1}(r^2)e^{-\frac{r^2}{2}} - rL_k^\alpha(r^2)e^{-\frac{r^2}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= (-r)\left[L_{k-1}^{\alpha+1}(r^2) + L_{k-1}^{\alpha+1}(r^2) + L_k^{\alpha}(r^2)\right] e^{-\frac{r^2}{2}} \\
&= (-r)\left[L_{k-1}^{\alpha+1}(r^2) + L_k^{\alpha+1}(r^2)\right] e^{-\frac{r^2}{2}}
\end{aligned}
$$

Here we have used the relations

$$(i) \qquad \frac{d}{dr} L_k^{\alpha}(r) = -L_{k-1}^{\alpha+1}$$

and,

$$(ii) \qquad L_k^{\alpha+1} - L_{k-1}^{\alpha+1} = L_k^{\alpha}$$

Now (2.9) follows from the definition of ψ_k^{α}. Let $f \in L_{\alpha}^2(I\!\!R_+)$. By definition

$$
\begin{aligned}
L_{\alpha}^t f &= 4^t \sum_{k=0}^{\infty} (k + \frac{\alpha+1}{2})^t (f, \psi_k^{\alpha})_{\alpha} \; \psi_k^{\alpha} \\
\frac{d}{dr} L_{\alpha}^t f(r) &= 4^t \sum_{k=0}^{\infty} (k + \frac{\alpha+1}{2})^t (f, \psi_k^{\alpha})_{\alpha} \; \frac{d}{dr} \psi_k^{\alpha}(r),
\end{aligned}
$$

and using(2.9) we get

$$
\begin{aligned}
\frac{d}{dr} L_{\alpha}^t f(r) &= 4^t \sum_{k=1}^{\infty} (k + \frac{\alpha+1}{2})^t k^{\frac{1}{2}} (f, \psi_k^{\alpha})_{\alpha} \; (-r) \; \psi_{k-1}^{\alpha+1}(r) \\
&+ 4^t \sum_{k=0}^{\infty} (k + \frac{\alpha+1}{2})^t (k+\alpha+1)^{1/2} (f, \psi_k^{\alpha})_{\alpha} \; (-r) \; \psi_k^{\alpha+1}(r) \\
&= -rTf(r) - rSf(r) \qquad (2.10)
\end{aligned}
$$

where

$$Tf(r) = 4^t \sum_{k=1}^{\infty} (k + \frac{\alpha+1}{2})^t k^{\frac{1}{2}} (f, \psi_k^{\alpha})_{\alpha} \; \psi_{k-1}^{\alpha+1}(r) \qquad (2.11)$$

and

$$Sf(r) = 4^t \sum_{k=0}^{\infty} (k + \frac{\alpha+1}{2})^t (k+\alpha+1)^{1/2} (f, \psi_k^{\alpha})_{\alpha} \; \psi_k^{\alpha+1}. \qquad (2.12)$$

Therefore,

$$
\begin{aligned}
\|\frac{d}{dr} L_{\alpha}^t f(r)\|_{\alpha}^2 &\leq (\|rTf(r)\|_{\alpha} + \|rSf(r)\|_{\alpha})^2 \\
&\leq 2\left(\|rTf(r)\|_{\alpha}^2 + \|rSf(r)\|_{\alpha}^2\right). \qquad (2.13)
\end{aligned}
$$

Now using the expansion (2.11) we calculate,

$$\begin{aligned} \|rTf(r)\|_\alpha^2 &= \int_0^\infty r^2|Tf(r)|^2 r^{2\alpha+1}dr \\ &= \int_0^\infty |Tf(r)|^2 r^{2\alpha+3}dr \\ &= 4^{2t}\sum_{k=1}^{\infty}(k+\frac{\alpha+1}{2})^{2t}k|(f,\psi_k^\alpha)_\alpha|^2 \\ &\leq \sum_{k=1}^{\infty}4^{2t}(k+\frac{\alpha+1}{2})^{2t+1}|(f,\psi_k^\alpha)_\alpha|^2 \\ &\leq \sum_{k=1}^{\infty}|(f,\psi_k^\alpha)_\alpha|^2 \\ &= \|f\|_\alpha^2 \end{aligned} \tag{2.14}$$

since $1+t \leq 0$. Similarly one can see that

$$\|rSf(r)\|_\alpha^2 \leq \|f\|_\alpha^2 \tag{2.15}$$

Using (2.14) and (2.15) in (2.13) we see that $\|\frac{d}{dr}L_\alpha^t f\|_\alpha \leq 2\|f\|_\alpha$ for $1+t \leq 0$. Similarly one can show that $\|(\frac{d}{dr})^j L_\alpha^t f\|_\alpha \leq c\|f\|_\alpha$ for some constant c, whenever $j+t \leq 0$, which proves the second lemma.

Proof of proposition 3: We have by definition $W_\alpha{}^s = L_\alpha^{-s}(L_\alpha^2(I\!R_+))$. Therefore it is enough to prove that

$$L_{\alpha+1}^s \circ M_\varphi \circ L_\alpha^{-s} : L_\alpha^2(I\!R_+) \to L_{\alpha+1}^2(I\!R_+) \tag{2.16}$$

is a bounded operator. Put

$$T^{\,t}f = L_{\alpha+1}^t \circ M_\varphi \circ L_\alpha^{-t} f \tag{2.17}$$

Where $L_{\alpha+1}^t$ and L_α^{-t} are defined using spectral theorem. Then clearly,

$$\begin{aligned} \|T^0 f\|_{\alpha+1} &= \|\varphi f\|_{\alpha+1} \\ &\leq c_0\|f\|_\alpha, \end{aligned} \tag{2.18}$$

for some constant c_0 independent of f. We will also prove that, for any positive integer m

$$\|T^{\,m}f\|_{\alpha+1} \leq c_1\|f\|_\alpha, \tag{2.19}$$

for some constant c_1 independent of f.

Assuming (2.19) for a moment choose $f_1 \in L^2_\alpha(I\!R_+)$ and $g_1 \in L^2_{\alpha+1}(I\!R_+)$ to be finite linear combinations of ψ_k^α's and $\psi_k^{\alpha+1}$'s, respectively. Consider the function h which is holomorphic in the region $0 < Re(z) < m$ and continuous in $0 \leq Re(z) \leq m$, defined by:

$$h(z) = (T^{\ z} f_1\ ,\ g_1)_{\alpha+1} = (L^z_{\alpha+1} \circ M_\varphi \circ L^{-z}_\alpha f_1\ ,\ g_1)_{\alpha+1} \tag{2.20}$$

Then by (2.18) we have,

$$\begin{aligned} |h(iy)| &= |(L^{iy}_{\alpha+1} \circ M_\varphi \circ L^{-iy}_\alpha f_1\ ,\ g_1)_{\alpha+1}| \\ &= |(\varphi(r)\tilde{f}_1\ ,\ \tilde{g}_1)_{\alpha+1}| \end{aligned}$$

where $\tilde{f}_1 = L^{-iy}_\alpha f_1$, and $\tilde{g}_1 = L^{-iy}_{\alpha+1} g_1$. Therefore,

$$\begin{aligned} |h(iy)| &\leq \|T^0 \tilde{f}_1\|_{\alpha+1} \|\tilde{g}_1\|_{\alpha+1} \\ &\leq c_0 \|\tilde{f}_1\|_\alpha \|\tilde{g}_1\|_{\alpha+1} \end{aligned}$$

and since both L^{-iy}_α and $L^{-iy}_{\alpha+1}$ are unitary operators,we get

$$|h(iy)| \leq c_0 \|f_1\|_\alpha \|g_1\|_{\alpha+1}$$

Similarly by using (2.19) we get

$$\begin{aligned} |h(m+iy)| &= |(L^{m+iy}_{\alpha+1} \circ M_\varphi \circ L^{-m-iy}_\alpha f_1\ ,\ g_1)_{\alpha+1}| \\ &= |(L^m_{\alpha+1} \circ M_\varphi \circ L^{-m}_\alpha \tilde{f}_1\ ,\ \tilde{g}_1)_{\alpha+1}| \\ &\leq \|T^{\ m} \tilde{f}_1\|_{\alpha+1} \|\tilde{g}_1\|_{\alpha+1} \\ &\leq c_1 \|f_1\|_\alpha \|g_1\|_{\alpha+1} \end{aligned}$$

Thus we have

$$|h(iy)| \leq c_0 \|f_1\|_\alpha \|g_1\|_{\alpha+1} \tag{2.21}$$

$$|h(m+iy)| \leq c_1 \|f_1\|_\alpha \|g_1\|_{\alpha+1}. \tag{2.22}$$

Since h is a bounded function we have by three lines theorem

$$|\ h(t+iy)\ | \leq c_0^{1-t/m} c_1^{t/m} \|f_1\|_\alpha \|g_1\|_{\alpha+1}$$

for 0 ¡ t ¡ m. In particular,

$$|h(t)| \ \leq\ c_0^{1-t/m} c_1^{t/m} \|f_1\|_\alpha \|g_1\|_{\alpha+1},$$

that is,

$$|(T^{\ t} f_1\ ,\ g_1)| \leq c_0^{1-t/m} c_1^{t/m} \|f_1\|_\alpha \|g_1\|_{\alpha+1}. \tag{2.23}$$

Now taking supremum over all such $g_1 \in L^2_{\alpha+1}$ with $\|g_1\|_{\alpha+1} \leq 1$ we get $\|T^{\ t} f_1\|_{\alpha+1} \leq c_0^{1-t/m} c_1^{t/m} \|f_1\|_\alpha$. Therefore $T^{\ t}$ is a bounded operator on a dense subset of L^2_α. Therefore it has a norm preserving extension to L^2_α. Thus we have

$$\|T^{\ t} f\|_{\alpha+1} \leq c_t \|f\|_\alpha \quad \forall f \in L^2_\alpha(I\!R_+), \qquad for \ \ 0 < t < m \tag{2.24}$$

which proves (2.16).

To prove (2.19) we proceed as follows. By Lemma (4) we have $T^m f = \sum_{t+k \leq m} M_{\varphi_{k,t}} (\frac{d}{dr})^k L_\alpha^{t-m}$. And by Lemma (5) $(\frac{d}{dr})^k L_\alpha^{t-m}$ is a bounded operator on $L^2_\alpha(I\!R_+)$, whenever $k + (t - m) \leq 0$. Also note that since $\varphi_{k,t}$ satisfies the conditions(1) and (2) of the proposition 3 for $j = 0$, $M_{\varphi_{k,t}}$ maps $L^2_\alpha(I\!R_+) \to L^2_{\alpha+1}(I\!R_+)$ boundedly. Thus we get $\|T^m f\|_{\alpha+1} \leq c_1 \|f\|_\alpha$. This completes the proof of the proposition.

3 Regularity of $T_r^{\ \alpha} f(z)$

In this section we prove that the Laguerre means $T_r^{\ \alpha} f(z)$ are slightly more regular than f, for $z \neq 0$. To prove this fact we use the series expansion (1.14) for $T_r^{\ \alpha} f(z)$. Let $f \in W_\alpha^{\ s}$ .Then

$$4^s \sum_0^\infty (k + \frac{\alpha+1}{2})^s \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} (f, \varphi_k^\alpha)_\alpha \ \varphi_k^\alpha(r) \tag{3.1}$$

converges in $L^2_\alpha(I\!R_+)$. We also use the following asymptotic estimates, (see[4])

$$\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \approx k^{-\alpha} \tag{3.2}$$

$$\psi_k^\alpha(z) \approx k^{-1/4} |z|^{-\alpha-\frac{1}{2}} cos(2\sqrt{k} z - \frac{\alpha\pi}{2} - \frac{\pi}{4}), \quad z \neq 0 \tag{3.3}$$

$$\psi_k^\alpha(0) \approx k^{\alpha/2} \ \text{ as } \ k \ \to \infty \tag{3.4}$$

From (1.14) we have

$$\begin{aligned} \int_0^\infty |T_r^\alpha f(z)|^2 r^{2\alpha+1} dr &= \Gamma(\alpha+1)^4 \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} |(f, \psi_k^\alpha)_\alpha|^2 |\psi_k^\alpha(z)|^2 \\ &\leq c(z) \sum_{k=0}^\infty (1+k)^{-\alpha} (1+k)^{-\frac{1}{2}} |(f, \psi_k^\alpha)_\alpha|^2 \end{aligned} \tag{3.5}$$

for $z \neq 0$, in view of (3.2) and (3.3). Also

$$\int_0^\infty |T_r^\alpha f(z)|^2 r^{2\alpha+1} dr \approx \sum_{k=0}^\infty |(f, \psi_k^\alpha)_\alpha|^2 \quad \text{for } \ z = 0 \tag{3.6}$$

in view of (3.2) and (3.4). Comparing (3.1) and (3.5) we see that $f \in {W_\alpha}^s \Rightarrow r \rightarrow {T_r}^\alpha f(z) \in {W_\alpha}^{s+\frac{\alpha}{2}+\frac{1}{4}}$. Comparing (3.1) and (3.6) we see that $f \in {W_\alpha}^s$ if and only if ${T_r}^\alpha f(z) \in {W_\alpha}^s$. Thus we have proved the following :

Lemma 6 *(i)* $f \in {W_\alpha}^s \Rightarrow r \rightarrow {T_r}^\alpha f(z) \in {W_\alpha}^{s+\frac{\alpha}{2}+\frac{1}{4}}, z \neq 0.$
(ii) $f \in {W_\alpha}^s$ *if and only if* $r \rightarrow {T_r}^\alpha f(0) \in W_\alpha^s$.

Now we prove some properties of Laguerre means $T_r^\alpha f$.

Lemma 7 *(i) If* f *is supported in* $z \leq b$ *,then* ${T_r}^\alpha f(z)$ *as a function of* r *is supported in* $r \leq b + z$.
(ii) If f *vanishes in a neighbourhood of* z *then* ${T_r}^\alpha f(z)$ *as a function of* r *vanishes in a neighbourhood of origin in* $\mathbb{R}_+$.

Proof: (i) If f is supported in $z \leq b$ then the integral (1.12) vanishes unless $(r^2 + z^2 + 2rzcos\theta)^{1/2} \leq b$ .This implies $(r - z)^2 \leq b^2$. Therefore the integral (1.12) vanishes unless $|r - z| \leq b$ or $r \leq b + z$
(ii) Again if f vanishes in a neighbourhood $\{|y - z| < a\}, a > 0$ of z, the above integral (1.12) is zero if $|(r^2 + z^2 + 2rzcos\theta)^{1/2} - z| \leq a$. Since z is fixed this says that the above inequality holds for r in a neighbourhood of 0. Now consider the continuous function

$$g(r) = |(r^2 + z^2 + 2rzcos\theta)^{1/2} - z| - a,$$

defined on $\mathbb{R}_+$. We have $g(0) = -a < 0$ Therefore $g < 0$ in a neighbourhood of 0 as well. This means that for r in some neighbourhood of 0 we have $|(r^2 + z^2 + 2rzcos\theta)^{1/2} - z| < a$. Thus ${T_r}^\alpha f(z) \equiv 0$ in that neighbourhood.

4 A localisation Theorem for Laguerre expansions

Now we are in a position to prove Theorem (2) stated in the introduction, From (1.14) using the orthogonality of ψ_k^α we get

$$\int_0^\infty {T_r}^\alpha f(z)\ \varphi_k^\alpha(r)\ r^{2\alpha+1} dr = \Gamma(\alpha+1)^2 (f, \psi_k^\alpha)_\alpha\ \psi_k^\alpha(z). \tag{4.1}$$

Again from (1.14) we get,

$$S_N^\alpha f(z) \;=\; \sum_{k=0}^{N} (f, \psi_k^\alpha)_\alpha \psi_k^\alpha(z)$$

$$= \quad (\Gamma(\alpha+1))^{-2} \int_0^\infty T_r{}^\alpha f(z) \sum_{k=0}^{N} \varphi_k^\alpha(r) r^{2\alpha+1} dr$$

$$= \quad (\Gamma(\alpha+1))^{-2} \int_0^\infty T_r{}^\alpha f(z) \varphi_N^{\alpha+1}(r) r^{2\alpha+1} dr. \qquad (4.2)$$

Here we have used the relation $\sum_0^N L_k^\alpha(x) = L_N^{\alpha+1}(x)$. We use the above representation for $S_N^\alpha f(z)$ to prove Theorem (2). The proof uses the following fact: If $g \in L^2_\alpha(I\!R_+)$, then the Fourier-Laguerre coefficients $(g, \psi_k^\alpha)_\alpha \to 0$ as $k \to \infty$. Recalling the definition of ψ_k^α this means that

$$\int_0^\infty g(r) \varphi_k^\alpha(r) r^{2\alpha+1} dr \quad = \quad \circ(k^{\frac{\alpha}{2}}) \qquad \text{as} \;\; k \to \infty. \qquad (4.3)$$

Also if $g \in W_\alpha{}^s(I\!R_+)$ then,

$$\int_0^\infty g(r) \varphi_k^\alpha(r) r^{2\alpha+1} dr \quad = \quad \circ(k^{-s+\frac{\alpha}{2}}) \qquad \text{as} \;\; k \to \infty. \qquad (4.4)$$

From (4.2) we get

$$S_N^\alpha f(z) \quad = \quad (\Gamma(\alpha+1))^{-2} \int_0^\infty \frac{T_r{}^\alpha f(z)}{r^2} \varphi_N^{\alpha+1}(r) r^{2\alpha+3} dr. \qquad (4.5)$$

Let $\tilde{h}$ be a smooth function on $(I\!R_+)$ such that $\tilde{h}(r) \equiv 1$ on the support of $T_r{}^\alpha f(z)$ and $\tilde{h}(r) \equiv 0$ in a neighbourhood of the origin in $I\!R_+$. Put $h(r) = \frac{\tilde{h}(r)}{r^2}$. Thus we get

$$S_N^\alpha f(z) \quad = \quad (\Gamma(\alpha+1))^{-2} \int_0^\infty h(r) \; T_r{}^\alpha f(z) \; \varphi_N^{\alpha+1}(r) r^{2\alpha+3} dr \qquad (4.6)$$

Now if $T_r{}^\alpha f(z) \in W_\alpha^{\frac{\alpha+1}{2}}$, we have by Proposition 3 $h(r) T_r{}^\alpha f(z) \in W_{\alpha+1}^{\frac{\alpha+1}{2}}$ Therefore by (4.3),

$$S_N^\alpha f(z) = \circ(N^{\,(-\frac{\alpha+1}{2}+\frac{\alpha+1}{2})}) = \circ(1),$$

as $N \to \infty$. Therefore $S_N^\alpha f(z) \to 0$ as $N \to \infty$, which proves the theorem.

In view of Lemma 6, if $f \in W_\alpha^{1/2}$, then $T_r{}^\alpha f(z) \in W_\alpha^{\frac{\alpha+1}{2}}$, for $z \neq 0$. Thus we have the following corollary to the above theorem.

Corollary 8 *If $f \in W_\alpha^{1/2}$ then the conclusion of Theorem 2 holds at points $z \neq 0$.*

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