A restriction theorem for the Heisenberg motion group

P.K. Ratnakumar, Rama Rawat and S. Thangavelu
Statistics and Mathematics unit
Indian Statistical Institute
8th mile, Mysore Road
Bangalore -560 059
INDIA

Abstract

In this paper we prove a restriction theorem for the class -1 representations of the Heisenberg motion group. This is done using an improvement of the restriction theorem for the special Hermite projection operators proved in [13]. We also prove a restriction theorem for the Heisenberg group.

Key words : Hermite function, special Hermite function, Laguerre function, class-1 representation, Heisenberg motion group.

AMS Classification
Primary : 42 C 15, 43 A 80
Secondary : 43 A 99

1 Introduction

The inversion formula for the Fourier transform on $I\!R^n$ can be written in the form

$$f(x) = C_n \int_0^\infty f * \varphi_\lambda(x) \lambda^{n-1} d\lambda$$

where φ_λ is the Bessel function given by

$$\varphi_\lambda(x) = (\lambda|x|)^{-\frac{n}{2}+1} J_{\frac{n}{2}-1}(\lambda|x|).$$

Then for $f \in L^p(I\!R^n), 1 \leq p \leq \frac{2(n+1)}{n+3}$ there follows the inequality

$$\|f * \varphi_\lambda\|_p \leq C_\lambda \|f\|_p.$$

From this one gets the Stein-Tomas restriction theorem for the Fourier transform [11]:

$$\int\limits_{|\xi|=1} |\hat{f}(\xi)|^2 d\sigma \leq C\|f\|_p^2,$$

for $f \in L^p(I\!R^n), 1 \leq p \leq \frac{2(n+1)}{n+3}$. The restriction theorem finds applications in the study of Bochner-Riesz means for the Laplacian.

Analogues of the above restriction theorem have been studied in various set ups. As $f * \varphi_\lambda$ are eigenfunctions of the Laplacian Δ on $I\!R^n$, it is natural to study the $L^p - L^2$ mapping properties of projection operators associated to eigenfunction expansions. In the case of spherical harmonics and eigenfunction expansions on compact Riemannian manifolds such theorems have been proved by Sogge in [9] and [10]. In the non-compact set up, restriction theorems for Hermite and special Hermite projection operators have been studied by Thangavelu in [13].

Restriction theorems have been studied in the case of Heisenberg group also. If one considers the Heisenberg group H^n and let

$$f = \int_0^\infty P_\lambda f \; d\lambda$$

stand for the Strichartz's decomposition [12] of f in terms of eigenfunctions of the sublaplacian $\mathcal{L}$ on H^n, then in [5] Muller has studied mapping properties of P_λ. Some extensions have been treated in the papers [14] and [15] and the restriction theorem has been found useful in the study of Bochner-Riesz means for the sublaplacian [6].

Our aim in this note is to prove a restriction theorem for class-1 representations of the Heisenberg motion group. The main theorem should be compared with the corresponding theorem for the spherical harmonic projections stated and proved in Sogge [9] in the language of representation theory. To prove the main theorem we need a restriction theorem for special Hermite projection operators proved in [13]. We take this opportunity to present a simpler proof of a crucial estimate used in [13] and also to show that the restriction theorem is valid in a slightly bigger range of p than established in [13]. In the last section we also prove a restriction theorem for the Heisenberg group by considering individual projections.

For many facts we use regarding the Heisenberg group and special Hermite expansions we refer to the monographs [1] and [16] and also the paper of Strichartz [12].

2 A restriction theorem for the Heisenberg motion group

Consider the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Im(z.\overline{w})).$$

The group $U(n)$ of $n \times n$ complex unitary matrices acts on H^n by the automorphisms

$$\sigma(z,t) = (\sigma z, t), \quad \sigma \in U(n).$$

The Heisenberg motion group is then the semi-direct product $G = H^n \ltimes U(n)$ which acts on H^n in the following way :

$$(\sigma, z, t)(w,s) = (z+\sigma w, t+s+\frac{1}{2}Im\sigma w.\overline{z}).$$

Functions on H^n can be viewed as right $U(n)$-invariant functions on the Heisenberg motion group G. To formulate our restriction theorem for G we need to recall a family of class -1 represention of G which have been studied in [8] and [12].

For each $\lambda \in I\!R, \lambda \neq 0$ we have an irreducible unitary representation π_λ of H^n which is realised on $L^2(I\!R^n)$ and acts by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y),$$

for $\varphi \in L^2(I\!R^n)$. Upto unitary equivalence these π_λ give all the infinite dimensional irreducible representations of H^n. Let $\Phi_\alpha, \alpha \in I\!N^n$ be the normalised Hermite functions on $I\!R^n$. (For the explicit definition of Φ_α refer [16].) For $\lambda \neq 0$ define $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\ \Phi_\alpha(|\lambda|^{\frac{1}{2}}x)$ and let

$$E_{\alpha,\beta}^\lambda(z,t) = (\pi_\lambda(z,t)\Phi_\alpha^\lambda, \Phi_\beta^\lambda)$$

be the entry functions of the representation π_λ. The functions $\Phi_{\alpha,\beta}(z) = (2\pi)^{-\frac{n}{2}} E_{\alpha,\beta}^1(z,0)$ are called the special Hermite functions and it is well known that $\{\Phi_{\alpha\beta} : \alpha, \beta \in I\!N^n\}$ forms an orthonormal basis for $L^2(C\!\!\!\!I^n)$.

We recall some general facts about the class -1 representations. Let N be a locally compact topological group and K_0 be a compact subgroup of N. Let π be an irreducible unitary representation of N on a Hilbert space H. We say that π is a class-1 representation for the pair (N, K_0) if the space H_0, of K_0- fixed vectors in H i.e. $H_0 = \{v \in H : \pi(k)v = v\ \forall k \in K_0\} \neq \{0\}$.

In case (N, K_0) is a Gelfand pair, i.e. if the algebra $\{f \in L^1(N) : f(k_1 x k_2) = f(x)\ \forall k_1, k_2 \in K_0, x \in N\}$ is commutative with respect to the usual convolution on N, it is known that, (see[2]), for π, H, H_0 as above, $dim H_0 = 1$.

We now list a family of class -1 representations for the pair $(G, U(n))$. For each $\lambda \neq 0$ and $k \in I\!N$, let $\mathcal{H}_k^\lambda$ be the Hilbert space for which an orthonormal basis is given by

$$\{E_{\alpha,\beta}^\lambda(z,t) : \alpha, \beta \in I\!N^n, |\beta| = k\}$$

and the inner product being

$$(f,g) = (2\pi)^{-n}|\lambda|^n \int_{C\!\!\!\!I^n} f(z,0)\overline{g}(z,0)dz.$$

The space $\mathcal{H}_k^\lambda$ can be characterised as certain eigenspace of the sublaplacian, see Strichartz [12]. On $\mathcal{H}_k^\lambda$ define a representation ρ_k^λ of G by

$$\rho_k^\lambda(\sigma,z,t)\varphi(w,s) = \varphi((\sigma,z,t)^{-1}(w,s))$$

for $\varphi \in \mathcal{H}_k^\lambda$ and (w,s) *in* H^n. Then ρ_k^λ is an irreducible unitary representation of G. As noticed in [8], the vector

$$e_k^\lambda(z,t) = (2\pi)^{-\frac{n}{2}} \sum_{|\mu|=k} (\pi_\lambda(z,t)\Phi_\mu^\lambda, \Phi_\mu^\lambda)$$

is a $U(n)$ fixed vector. As $(G, U(n)$ is a Gelfand pair (see [3]), we conclude that e_k^λ is the unique (upto a scalar multiple) $U(n)$ fixed vector in H_k^λ.

Let $f \in L^1(H^n)$; identifying f as a right $U(n)$-invariant function on G we can define the operator

$$\rho_k^\lambda(f) = \int_G f(z,t)\rho_k^\lambda(\sigma,z,t)d\sigma dz dt$$

which acts on the Hilbert space $\mathcal{H}_k^\lambda$. It is easy to calculate the action of $\rho_k^\lambda(f)$ on a function $\varphi \in \mathcal{H}_k^\lambda$. In fact, letting

$$\varphi^\#(z) = \int_{U(n)} \varphi(\sigma z, 0)d\sigma$$

be the radialisation of φ,

$$f^\lambda(z) = \int e^{i\lambda t} f(z,t)dt$$

be the inverse Fourier transform of f in the t-variable and

$$g *_\lambda h(z) = \int_{\mathbb{C}^n} g(z-w)h(w)e^{i\frac{\lambda}{2}Im z.\overline{w}}dw$$

be the λ-twisted convolution of g and h we can show that

$$\rho_k^\lambda(f)\varphi(z,t) = e^{i\lambda t} f^{-\lambda} *_\lambda \varphi^\#(z).$$

It is easy to see that $\rho_k^\lambda(f)$ is a bounded operator on $\mathcal{H}_k^\lambda$. In fact since ρ_k^λ is a unitary operator we have the following norm estimate

$$\|\rho_k^\lambda(f)\|_\infty \leq \int_{H^n} |f(z,t)|dzdt \qquad (*)$$

where we have used $\|\cdot\|_\infty$ to denote the operator norm.

When $f \in L^1 \cap L^2(H^n)$ we can say more about the operator $\rho_k^\lambda(f)$. Let L_k^{n-1} be the k^{th} Laguerre polynomial of type $(n-1)$ and let

$$\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$$

be the Laguerre function. Let $\varphi_k^\lambda(z) = \varphi_k(|\lambda|^{\frac{1}{2}}z)$.

Proposition 2.1 : For $f \in L^1 \cap L^2(H^n)$, $\rho_k^\lambda(f)$ is a Hilbert-Schmidt operator on $\mathcal{H}_k^\lambda$ and

$$\| \rho_k^\lambda(f) \|_2^2 = (2\pi)^{-n}|\lambda|^n \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} |f^{-\lambda} *_\lambda \varphi_k^\lambda(z)|^2 dz.$$

Here $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm.

Proof : We calculate the norm of $\rho_k^\lambda(f)\varphi$ when $\varphi = E_{\alpha,\beta}^\lambda$, $|\beta| = k$. Since

$$e_k^\lambda(z,t) = (2\pi)^{-\frac{n}{2}} \sum_{|\mu|=k} (\pi_\lambda(z,t)\Phi_\mu^\lambda, \Phi_\mu^\lambda)$$

is the essentially unique $U(n)$-invariant function in $\mathcal{H}_k^\lambda$, the radialisation

$$e^{i\lambda t}\varphi^{\#}(z) = \int_{U(n)} E_{\alpha,\beta}^\lambda(\sigma z, t) d\sigma$$

should be a constant multiple of $e_k^\lambda(z,t)$. From the definition of $E_{\alpha,\beta}^\lambda$ we infer that $E_{\alpha,\beta}^\lambda(0,t) = 0$ for $\alpha \neq \beta$ and consequently

$$\rho_k^\lambda(f) E_{\alpha,\beta}^\lambda = 0, \quad \alpha \neq \beta.$$

When $\alpha = \beta$ we have

$$\int_{U(n)} E_{\alpha,\alpha}^\lambda(\sigma z, 0) d\sigma = A \; e_k^\lambda(z,0)$$

where A is a constant. We evaluate the constant by taking $z = 0$.

$$A(2\pi)^{-\frac{n}{2}}(\sum_{|\mu|=k} 1) = \int_{U(n)} d\sigma = 1.$$

This gives

$$\int_{U(n)} E^\lambda_{\alpha,\alpha}(\sigma z, 0)d\sigma = (2\pi)^{\frac{n}{2}} \frac{k!(n-1)!}{(k+n-1)!} e^\lambda_k(z,0).$$

It is well known that (see [8])

$$e^\lambda_k(z,t) = (2\pi)^{-\frac{n}{2}} e^{i\lambda t} \varphi^\lambda_k(z)$$

and consequently

$$\rho^\lambda_k(f) E^\lambda_{\alpha,\alpha}(z,t) = \frac{k!(n-1)!}{(k+n-1)!} e^{i\lambda t} f^{-\lambda} *_\lambda \varphi^\lambda_k(z),$$

for $|\alpha| = k$. Finally

$$\| \rho^\lambda_k(f) \|_2^2 = \sum_{|\alpha|=k} \|\rho^\lambda_k(f) E^\lambda_{\alpha,\alpha}\|^2_{\mathcal{H}^\lambda_k}$$

and after simplification we obtain

$$\| \rho^\lambda_k(f) \|_2^2 = \frac{k!(n-1)!}{(k+n-1)!} |\lambda|^n (2\pi)^{-n} \int_{\mathbb{C}^n} |f^{-\lambda} *_\lambda \varphi^\lambda_k(z)|^2 dz.$$

This proves the proposition.

For $0 < q \le \infty$, let S_q stand for the Schatten-von Neumann class of operators on $\mathcal{H}^\lambda_k$ whose singular numbers belong to ℓ^q. In particular, S_2 will denote the class of Hilbert-Schmidt operators. Let $\| \cdot \|_q$ stand for the norm in S_q. We are now ready to state the following restriction theorem for ρ^λ_k. Let $L^{(p,1)}(H^n)$ stand for the space of all functions on H^n for which

$$\|f\|_{(p,1)} = \left(\int_{\mathbb{C}^n} \left(\int_{-\infty}^{\infty} |f(z,t)| dt \right)^p dz \right)^{1/p} < \infty.$$

Theorem 2.1 : Let $f \in L^{(p,1)}(H^n), 1 \le p < \frac{2(3n+1)}{3n+4}$, and let $q < \frac{3n-2}{n+1} p'$. Then $\rho^\lambda_k(f) \in S_q$ and

$$\| \rho^\lambda_k(f) \|_q \le C\, |\lambda|^{\frac{3n+4}{3n+1}\frac{n}{q}}\, k^{-\frac{3n-2}{3n+1}\frac{n}{q}} \|f\|_{(p,1)}.$$

To prove the theorem we need the following restriction theorem for the special Hermite projections. For functions f on $\mathbb{C}^n$ let

$$f \times \varphi_k(z) = \int_{\mathbb{C}^n} f(z-w) e^{\frac{i}{2} Im z\cdot \overline{w}} \varphi_k(w) dw$$

which is called the twisted convolution of f with φ_k. We need

Proposition 2.2. Let $f \in L^p(\mathbb{C}^n)$, $\ 1 \leq p < \frac{2(3n+1)}{3n+4}$. Then we have

$$\|f \times \varphi_k\|_2 \leq C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\ \|f\|_p.$$

We postpone the proof of this proposition to the next section. Assuming it for a moment we will prove the theorem. From equation $(*)$ we have

$$\|\rho_k^\lambda(f)\|_\infty \leq \|f\|_{(1,1)}.$$

Assuming $\lambda > 0$ for definiteness we see that

$$f^{-\lambda} *_\lambda \varphi_k^\lambda(z) = \lambda^{-n} f_\lambda^{-\lambda} \times \varphi_k(\lambda^{\frac{1}{2}} z)$$

where

$$f_\lambda^{-\lambda}(z) = f^{-\lambda}(\lambda^{-\frac{1}{2}} z).$$

Applying the proposition we get

$$\begin{aligned} \|f^{-\lambda} *_\lambda \varphi_k^\lambda\|_2 &\leq C\ |\lambda|^{n(\frac{1}{p}-\frac{1}{2})}\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f^\lambda\|_p \\ &\leq C\ |\lambda|^{n(\frac{1}{p}-\frac{1}{2})}\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_{(p,1)}. \end{aligned}$$

Using this estimate in proposition 2.1 we get

$$\|\rho_k^\lambda(f)\|_2 \leq C\ |\lambda|^{\frac{n}{p}} k^{\frac{n}{p}-n} \|f\|_{(p,1)}.$$

We pretend as if proposition 2.2 is true at the end point $p_0 = \frac{2(3n+1)}{(3n+4)}$. (A slight modification required is left to the reader.)

$$\|\rho_k^\lambda(f)\|_2 \leq C\ |\lambda|^{\frac{n(3n+4)}{2(3n+1)}}\ k^{-\frac{n(3n-2)}{2(3n+1)}} \|f\|_{(p_0,1)}.$$

Appealing to the noncommutative interpolation theorem of Peetre-Sparr [7] we obtain for $1 \leq p \leq \frac{2(3n+1)}{3n+4}$

$$\|\rho_k^\lambda(f)\|_q \leq C\ |\lambda|^{\frac{n(3n+4)}{2(3n+1)}\theta}\ k^{-\frac{n(3n-2)}{2(3n+1)}\theta} \|f\|_{(p,1)}$$

where p, q and θ are related by $\frac{1}{q} = \frac{\theta}{2}, \theta = \frac{2(3n+1)}{3n-2}\frac{1}{p'}$. Simplifying we see that $q = \frac{3n-2}{3n+1} p'$ thus completing the proof of the theorem.

3 Special Hermite projection operators

In this section we prove proposition 2.2. By the term special Hermite expansion we mean the series

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k(z)$$

which converges in L^2 norm for $f \in L^2(\mathbb{C}^n)$. The above is the compact form of the expansion in terms of the special Hermite functions, namely

$$f(z) = \sum_{\alpha} \sum_{\beta} (f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}.$$

Summability and multipliers for the above expansions have been studied in [13]. A crucial ingredient for proving summability results is the $L^p - L^2$ restriction theorem stated in proposition 2.2.

The proposition was proved in [13] for the slightly smaller range $1 \leq p \leq \frac{2n}{n+1}$. The main idea of the proof is to embed the operator $f \to f \times \varphi_k$ into an analytic family of operators, get estimates at the end points and then appeal to Stein's analytic interpolation theorem. The analytic family used for this purpose is the one given by twisted convolution with the Laguerre function

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}.$$

Here these functions can be defined even for complex α with $Re\ \alpha > -1$. In [13] it was shown that $\psi_k^\alpha(z)$ are bounded uniformly in k and z provided $Re\ \alpha \geq 0$. In the following proposition we show that the same is true as long as $Re\ \alpha > -\frac{1}{3}$.

Proposition 3.1 : Let $\alpha = \sigma + i\tau$ with $-1 < \sigma \leq n$. Then with a constant C independent of k we have

$$\sup_z |\psi_k^\alpha(z)| \leq C(1+|\tau|)^{\frac{2}{3}}$$

provided $\sigma > -\frac{1}{3}$.

Proof : This proposition was proved in [13] for $\sigma \geq 0$ by expressing L_k^α in terms of Hermite functions and then using estimates for the Hermite functions. Here we use the following formula which connects Laguerre polynomials of different types :

$$L_k^{\mu+\nu}(t) = \frac{\Gamma(k+\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \int_0^1 s^{\mu}(1-s)^{\nu-1} L_k^{\mu}(ts)ds.$$

From the above formula we have

$$\psi_k^{\alpha}(z) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{-1}{3})\Gamma(\alpha+\frac{1}{3})} \int_0^1 s^{-\frac{1}{3}}(1-s)^{\alpha-\frac{2}{3}} \psi_k^{-\frac{1}{3}}(\sqrt{s}z) e^{-\frac{1}{4}(1-s)|z|^2} ds$$

, for $\alpha > -\frac{1}{3}$ and it is clear that the above can be defined even for complex α provided $Re\ \alpha > -\frac{1}{3}$.

Using Stirling's formula for the Gamma function we can show that

$$\left| \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{3})} \right| \leq C\ (1+|\tau|)^{\frac{2}{3}}.$$

For the Laguerre functions $\psi_k^{\alpha}(z)$ various L^p estimates are known (see Markett [4]). From Lemma of [4] we can infer that

$$\sup_z |\psi_k^{-\frac{1}{3}}(z)| \leq C$$

where C is independent of k. This completes the proof of the proposition.

Once we have the proposition we look at the analytic family of operators

$$G_k^{\alpha} f = f \times \psi_k^{-\frac{1}{3}+(n+\frac{1}{3})\alpha}.$$

Then by interpolating between the cases $Re\ \alpha > 0$ and $Re\ \alpha = 1$ we get the desired result. For details we refer to [13].

As we have already mentioned the restriction theorems are useful in the study of Bochner-Riesz means. Recall that

$$S_R^{\delta} f = (2\pi)^{-n} \sum (1 - \frac{2k+n}{R})_+^{\delta} f \times \varphi_k$$

are called Bochner-Riesz means of order $\delta \geq 0$ associated to the special Hermite expansions. Using proposition 2.2 we can prove

Theorem 3.1 : Let $1 \leq p < \frac{2(3n+1)}{(3n+4)}$ and $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then $S_R^{\delta} f$ are uniformly bounded on $L^p(\mathbb{C}^n)$.

The theorem was proved in [13] for the smaller range $1 \leq p \leq \frac{2n}{n+1}$. The same proof yields the above theorem in view of proposition 2.2.

In the case of radial functions the estimate of the proposition remains true in the bigger range $1 \leq p < \frac{4n}{2n+1}$. This has been observed in [13]. Based on that it was conjectured that the same is true for all functions. In what follows we show that the proposition is not true above a certain value of p. More precisely we have the following theorem:

Theorem 3.2 : The estimates of the proposition 2.2 are not valid for $p > \frac{2(n+1)}{n+2}$.

Proof : The proof of this theorem is by contradiction. Assume that the estimate

$$\|f \times \varphi_k\|_2 \leq C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2})}\|f\|_p \qquad (**)$$

is valid in the range $\frac{4}{3} < p < \frac{4n}{2n+1}$. Recall that when f is polyradial i.e., $f(z) = f(|z_1|, |z_2|, \cdots, |z_n|)$

$$f \times \varphi_k(z) = \sum_{|\alpha|=k} (\int f(w)\Phi_{\alpha\alpha}(w)dw)\Phi_{\alpha\alpha}(z).$$

and $\Phi_{\alpha\alpha}(z)$ are expressible in terms of Laguerre functions of type 0:

$$\Phi_{\alpha\alpha}(z) = \prod_{j=1}^{n} \psi^0_{\alpha_j}(z_j)$$

By taking

$$f(z) = g(|z_1|)e^{-\frac{1}{4}|z'|^2}, \quad z = (z_1, z')$$

and using the orthogonality properties of the Laguerre functions we get

$$f \times \varphi_k(z) = C\left(\int_0^\infty g(r)L^0_k(\frac{1}{2}r^2)e^{-\frac{1}{4}r^2}rdr\right)\psi^0_k(z_1)^{-\frac{1}{4}|z'|^2}.$$

The estimate $(**)$ now gives us

$$\left|\int_0^\infty g(r)L^0_k(\frac{1}{2}r^2)e^{-\frac{1}{4}r^2}rdr\right| \leq C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\left(\int_0^\infty |g(r)|^p rdr\right)^{\frac{1}{p}}.$$

Taking supremum over all $g \in L^p(\mathbb{C})$ with unit norm we get

$$\left(\int_0^\infty |L^0_k(r)e^{-\frac{r}{2}}|^{p'}dr)\right)^{\frac{1}{p'}} \leq C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}.$$

From Lemma of [4] already mentioned we infer that the left - hand side of the above equation behaves like $k^{\frac{1}{2}-\frac{1}{p}}$ and hence the estimate cannot be valid unless $p \leq \frac{2(n+1)}{n+2}$.

4 Heisenberg group revisited

In this last section we briefly recall Muller's restriction theorem for the Heisenberg group and then prove a slightly different restriction theorem for individual projections. The spectral decomposition of a function on H^n is given by

$$f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e_k^{\lambda}(z,t) |\lambda|^n d\lambda.$$

Defining

$$\tilde{e}_k^{\lambda}(z,t) = e_k^{\lambda/2k+n}(z,t)$$

we can write the above as

$$f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} (2k+n)^{-n-1} \int_{-\infty}^{\infty} f * \tilde{e}_k^{\lambda}(z,t) |\lambda|^n d\lambda$$

or in a more compact form

$$f(z,t) = \int_{-\infty}^{\infty} P_{\lambda} f(z,t) |\lambda|^n d\lambda$$

where

$$P_{\lambda} f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} (2k+n)^{-n-1} f * \tilde{e}_k^{\lambda}(z,t).$$

The restriction theorem of Muller states that

$$\|P_{\lambda} f\|_{(p',\infty)} \leq c_{\lambda} \|f\|_{(p,1)}, \quad 1 \leq p < 2.$$

Instead of considering all $f * e_k^{\lambda}$ together we would like to consider them separately. So we define

$$P_{k,a} f(z,t) = \int_{-a}^{a} f * e_k^{\lambda}(z,t) |\lambda|^n d\lambda$$

and would see what $L^p - L^2$ mapping properties these projections possess. Following is our result:

Theorem 4.1: Let $f \in L^p(H^n), 1 \le p < \frac{2(3n+1)}{3n+4}$. Then we have

$$\|P_{k,a}f\|_2 \le C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\ a^{(n+1)(\frac{1}{p}-\frac{1}{2})}\ \|f\|_p.$$

For the proof we need the following simple lemma :

Lemma 4.1 : For $f \in L^p(\mathbb{R}), 1 \le p < 2$ one has

$$\left(\int_{-a}^{a} |\hat{f}(\lambda)|^2 d\lambda\right)^{\frac{1}{2}} \le C\ a^{\frac{1}{p}-\frac{1}{2}} \|f\|_p.$$

Proof : Let χ_a be the characteristic function of the interval $-a \le t \le a$ so that $\widehat{\chi_a}(\lambda) = \lambda^{-1} \sin a\lambda$. By Plancherel and Young

$$\begin{aligned}\left(\int_{-a}^{a} |\hat{f}(\lambda)|^2 d\lambda\right)^{\frac{1}{2}} &= C\left(\int |f * \widehat{\chi_a}(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le C\|f\|_p \|\widehat{\chi_a}\|_q\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{2}$. But

$$\left(\int |\widehat{\chi_a}(t)|^q dt\right)^{\frac{1}{q}} = a\left(\int \left|\frac{\sin at}{at}\right|^q dt\right)^{\frac{1}{q}}$$

which equals a constant times $a^{1-\frac{1}{q}}$. Since $\frac{1}{q} - 1 = \frac{1}{2} - \frac{1}{p}$ the lemma follows.

Coming to the proof of the theorem we have, by a simple calculation,

$$\int_{-a}^{a} f * e_k^\lambda(z,t)|\lambda|^n d\lambda = \int_{-a}^{a} e^{i\lambda t} f^\lambda *_\lambda \varphi_k^\lambda(z)\ \ |\lambda|^n d\lambda.$$

Therefore,

$$\|\int_{-a}^{a} f * e_k^\lambda\ \ |\lambda|^n d\lambda\|_2^2 = C \int_{\mathbb{C}^n} \int_{-a}^{a} |f^\lambda *_\lambda \varphi_k^\lambda(z)|^2 |\lambda|^{2n} d\lambda dz.$$

In view of proposition 2.2 a simple calculation shows that

$$\int_{\mathbb{C}^n} |f^\lambda *_\lambda \varphi_k^\lambda(z)|^2 dz \le C\ k^{2n(\frac{1}{p}-\frac{1}{2})-1}\ \lambda^{-3n+\frac{2n}{p}} \left(\int |f^\lambda(z)|^p dz \right)^{\frac{2}{p}}.$$

Thus

$$\| \int_{-a}^{a} f * e_k^\lambda\, |\lambda|^n d\lambda \|_2^2 \le C\ k^{2n(\frac{1}{p}-\frac{1}{2})-1} \int_{-a}^{a} |\lambda|^{-n+\frac{2n}{p}} \left(\int |f^\lambda(z)|^p dz \right)^{\frac{2}{p}} d\lambda.$$

Now applying Minkowski's integral inequality we get

$$\begin{aligned}
&\left(\int_{-a}^{a} |\lambda|^{-n+\frac{2n}{p}} \left(\int |f^\lambda(z)|^p dz \right)^{\frac{2}{p}} d\lambda \right)^{\frac{p}{2}} \\
&\qquad \le \int dz \left(\int_{-a}^{a} |\lambda|^{-n+\frac{2n}{p}} |f^\lambda(z)|^2 d\lambda \right)^{\frac{p}{2}} \\
&\qquad \le a^{n-np/2} \int dz \left(\int_{-a}^{a} |f^\lambda(z)|^2 d\lambda \right)^{\frac{p}{2}} \\
&\qquad \le C\ a^{n-\frac{np}{2}}\ a^{1-\frac{p}{2}} \int \int |f(z,t)|^p dz dt
\end{aligned}$$

where we have used the lemma. Finally,

$$\| \int_{-a}^{a} f * e_k^\lambda |\lambda|^n d\lambda \|_2 \le C\ k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\ a^{(n+1)(\frac{1}{p}-\frac{1}{2})} \|f\|_p$$

follows. **Corollary :** Let $Q = 2n+2$ be the homogeneous dimension of H^n. Then for $f \in L^p(H^n), 1 \le p < \frac{2(3n+1)}{(3n+4)}$ we have

$$\| \int_{-k^{2/Q}}^{k^{2/Q}} f * e_k^\lambda\, |\lambda|^n d\lambda \|_2 \le k^{\frac{Q}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_p.$$

References

1. G.B. Folland, Harmonic analysis in phase space, Ann. Math. Stud. Vol. 122, Princeton Univ. Press, Princeton, N.J. (1989).

2. R. Gangoli, Spherical functions on semi simple Lie groups,: in Symmmetric spaces (eds), W. Boothby and G. weiss (1972), New York, Marcel Dekker Inc.

3. Hulanicki and Ricci, A Tauberian theorem and tangential convergence for bounded Harmonic functions on balls in $\mathbb{C}^n$, Inv. Math. 62 (1980) 325-331.

4. C. Markett, Mean Cesaro summability of Laguerre expansions and norm estimates with shifted parameter, Analysis Math. 8 (1982), 19-37.

5. D. Muller, A restriction theorem for the Heisenberg group, Ann. Math. 131 (1990), 567-587.

6. D. Muller, On Riesz means of eigenfunction expansions for the Kohn-Laplacian, J. reine angew. Math, 401 (1989), 113-121.

7. J. Peetre and G. Sparr, Interpolation and non commutative integration, Annali di Mat. pura ed appli. Vol CIV (1975), 187-207.

8. R. Rawat, A Theorem of the Wiener-Tauberian type for $L^1(H^n)$, Proc. Indian Acad. Sci.(Math Sci.) 106 (1996) 369-377.

9. C. Sogge, Oscillatory integrals and spherical harmonics, Duke Math. J. 53 (1986), 43-65.

10. C. Sogge, Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), 123-134.

11. E. Stein, Harmonic Analysis, Princeton Univ. Press, Princeton, N.J. (1993).

12. R. Strichartz, L^p harmonic analysis and Radon transforms on the Heisenberg group, J. Funct. Anal. 96 (1991), 350-406.

13. S. Thangavelu, Weyl multipliers, Bochner-Riesz means and special Hermite expansions, Ark. Mat. 29 (1991), 307-321.

14. S. Thangavelu, Restriction theorems for the Heisenberg group, J. reine angew, Math, 414 (1991), 51-65.

15. S. Thangavelu, Some restriction theorems for the Heisenberg group, Stud. Math. 99, (1991), 11-21.

16. S. Thangavelu, Lectures on Hermite and Laguerre expansions, Math. Notes. No. 42, Princeton Univ. Press, Princeton, N.J. (1993).

e-mail : ratna@isibang.ernet.in
rawat@isibang.ernet.in
veluma@isibang.ernet.in