Indag. Mathem., N.S., 14 (1), 63–79 March 31, 2003

Spherical maximal operator on symmetric spaces, an end point estimate

by P.K. Ratnakumar

Dept. of Mathematics and Statistics, University of Hyderabad, P.O. Central University, Hyderabad 500 046, India.

Communicated by Prof. M.S. Keane at the meeting of February 24, 2003

ABSTRACT

We establish an endpoint weak-type maximal inequality for the spherical maximal operator applied to radial functions on real rank-1 symmetric spaces of dimension $n \geq 2$. More explicitly, we prove the Lorentz space estimate

$$\|\mathcal{M}f\|_{n',\infty} \leq C_n \|f\|_{n',1}\ , \quad n' = \frac{n}{n-1}$$

for every radial function in the Lorentz space $L^{n',1}(X)$ associated with the natural isometry-invariant measure on X. The proof uses only geometric arguments and volume estimates and applies uniformly in every dimension.

1. INTRODUCTION

Let X be a noncompact symmetric space of real rank 1. Then X can be realised as the homogeneous space G/K, where G is the group of isometries of X and K the subgroup that fixes a specified point 0 called the origin in X. Then K is a maximal compact subgroup of G. Let $S_t(x)$ denotes the geodesic sphere in X centered at x and of radius t. Let μ_t denotes the normalised surface measure on the sphere $S_t(x)$ induced by the G-invariant measure on X. Then the spherical mean value operator on X is defined to be

$$\mathcal{A}_t f(x) = \int_{y \in S_t(x)} f(y) d\mu_t(y), t > 0.$$

Consider the maximal operator defined by

$$\mathcal{M}f(x) = \sup_{t>0} |\mathcal{A}_t f(x)|.$$

The L^p mapping properties of $\mathcal{M}$ has been first studied by E.L. Kohen, in 1980, where he considered the case of hyperbolic spaces (see [Ko]). He showed that $\mathcal{M}$ is bounded on $L^p(X)$ for $p > n/(n-1), n \geq 3$, where n is the dimension of the hyperbolic space X. This is the analogue of Stein's spherical maximal theorem in $\mathbb{R}^n, n \geq 3$, proved way back in 1976, see [S1]. That the result also holds for the Euclidean plane is due to Bourgain [B1] which was established a decade later.

The above maximal operator is significant in various contexts. For instance, in connection with the study of pointwise ergodic properties of radial measures on symmetric spaces, Amos Nevo and E.M. Stein [NS], studied the L^p mapping properties of $\mathcal{M}$. They showed that the L^p boundedness of $\mathcal{M}$ holds in the same range i.e. for $p > n/(n-1), n \geq 3$ for more general symmetric spaces X, of real rank one, where n is the dimension of the symmetric space X. Also the case of the circular maximal function on the two dimensional symmetric spaces has been settled by A.D. Ionescu [I]. In all these cases, the range of p, i.e. $\frac{n}{n-1} < p \leq \infty$ is optimal in the sense that for every $1 \leq p \leq n/(n-1)$, there exist an $f \in L^p(X)$ such that $Mf(x) = \infty$ for a.e. $x \in X$, as shown by the counter example in [N2].

The above results however, do not say anything about the behavior of the maximal function $\mathcal{M}$ at the end point $n/(n-1)$. As the counter example in [N2] shows, the operator fails to map even $L^{n/(n-1)}$ to weak $L^{n/(n-1)}$.

In this paper we take up the question of end point estimate for the maximal operator $\mathcal{M}$ at the end point $n/(n-1)$. We show that $\mathcal{M}$ is restricted weak type $(n/(n-1), n/(n-1))$ on radial functions on the symmetric space X. The motivation for this is a previous work, see [KR], where the authors proved an end point estimate for the fractional maximal operator associated with the spherical mean value operator on $\mathbb{R}^n$. The present work also extends the end point estimate proved in [NR] for the constant curvature spaces, to the set up of general real rank one symmetric spaces of non compact type.

To state the results more precisely, let us consider the Lorentz space $L^{p,q}(X, d\mu)$ on the symmetric space X, with respect to the G-invariant measure μ. Let $\|\cdot\|_{p,q}$ denote the norm in $L^{p,q}(X, d\mu)$. See section 4 for the definition of $L^{p,q}$ spaces. Our main result is the following

Theorem 1.1. *Let X be a noncompact symmetric space of real rank-1 and of real dimension n. Then for every $n \geq 2$ there exist a constant C_n independent of f such that the inequality*

$$\|\mathcal{M}f\|_{n',\infty} \leq C_n \|f\|_{n',1}$$

holds for all radial functions in $L^{n',1}(X, d\mu)$. Here $n' = n/(n-1)$ is the index conjugate to n.

By interpolation with the obvious estimate $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$ we get, as a corollary, the L^p boundednes of $\mathcal{M}$ on radial functions in $L^p(X), p > n/(n-1)$, for all $n \geq 2$.

Our approach is to exploit the geometry of the symmetric spaces to estimate the maximal function directly. Here we depart from the traditional way of harmonic analysis using the spectral theory or spherical functions. The novelty of our method is the use of the convolution algebra structure of radial measures on the rank one symmetric spaces, which has been fruitfully employed in the case of constant curvature symmetric spaces, see [NR].

The convolution structure of radial functions on rank-1 symmetric spaces has been studied extensively for a long time. For a detailed account of this see the survey article by Koornwinder [K] and the references therein.

The convolution algebra structure of radial functions on the noncompact symmetric spaces of real rank-1 has been explicitly computed in [K], see also [FJ-K]. In fact the convolution structure of radial measures is the same as the convolution structure of radial functions on the rank-1 symmetric spaces. However the form of the structure constants derived in the above papers is not suitable for the purpose of our analysis. So we derive suitable explicit form of the structure constants, and that is precisely the kernel in the formula (3.2), for the spherical mean of a radial function on X. Moreover, the kernel we present is geometric in the sense that it is given in terms of three geodesic distances r, t and u, (see the formulas (2.3) and (3.2)).

ACKNOWLEDGEMENT

This work has been carried out while the author was a research fellow associated to the Poorna Prajna Institute of Scientific Research, Bangalore. He wishes to thank Poorna Prajna Institute, for the financial support. The author also wishes to thank Amos Nevo for his useful suggestions and comments.

2. GEOMETRY OF THE NON COMPACT RANK-1 SYMMETRIC SPACES

Recall that if X is a symmetric space of real rank 1, then X can be realised as the homogeneous space G/K, where G is the group of isometries of X and K, the maximal compact subgroup of G. In our situation G is a simple non compact group of finite centre.

The structure theory of the noncompact (as well as compact) symmetric spaces is well known, see for instance [H1]. The basic structure theory that we require involves the Cartan decomposition and the knowledge of the structural elements, namely the maximal compact subgroup K, the maximal abelian subgroup A and the centraliser of A in K, denoted by M. These have been explicitly computed in the survey article by Koornwinder, [K]. We adopt these materials and the basic structure theory required for our analysis, from the above survey article.

The non compact symmetric spaces of real rank-1 have the following classification. Let (G, K) be a symmetric pair associated with the symmetric space X. Then up to an equivariant (global) isometry, the noncompact symmetric spaces of real rank-1 correspond to one of the following pairs (G, K)

1. $G = SO_0(1, n), \quad K = SO(n)\ n \in \mathbb{Z}_+, n \geq 2$

2. $G = SU(1,n)$, $K = S(U(1) \times U(n))$ $n \in \mathbb{Z}_+, n \geq 2$
3. $G = SP(1,n)$, $K = SP(1) \times SP(n)$ $n \in \mathbb{Z}_+, n \geq 2$.
4. $G = F_4(-20)$, $K = spin\ 9$

consisting of three series of classical one's and an exceptional one. Here $SO_0(1,n)$ denotes the connected component of $SO(1,n)$ containing the identity. The first three series of symmetric spaces can be studied uniformly by looking at them as hyperbolic spaces over suitable fields, see [K]. In this paper we only consider these symmetric spaces. However the arguments of this paper are also valid for the exceptional case once the kernel estimates (4.2) are established for this case, which corresponds to $d = 8$ and $n = 2$ in our analysis.

Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the skew field of quarternians with real dimension 1,2, or 4. Let $U(m,\mathbb{F})$ denote the unitary group on $\mathbb{F}^m$ and $U(1,n;\mathbb{F})$ the Lie group of linear operators on $\mathbb{F}^{1+n}$ which leave invariant the hermitian form $x_0\bar{y}_0 - x_1\bar{y}_1 \cdots - x_n\bar{y}_n, x, y \in \mathbb{F}^{1+n}$. Then $U(1,n;\mathbb{F})$ is a non compact simple group of finite centre, see [K].

Consider the pair (G,K) where $G = U(1,n;\mathbb{F})$ and $K = U(1,\mathbb{F}) \times U(n,\mathbb{F})$ which is the subgroup of $GL(n,\mathbb{F})$ given by

$$(2.1) \qquad K = \left\{ \begin{pmatrix} v & 0 \\ 0 & W \end{pmatrix} : v \in U(1,\mathbb{F}), W \in U(n,\mathbb{F}) \right\}$$

For $\mathbb{F} = \mathbb{H}$, the pair (G,K) matches the third series. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the groups $U(1,n;\mathbb{F})$ are reductive groups bigger than $SO_0(1,n)$ and $SU(1,n)$ respectively. However they yield the same symmetric spaces as the pairs in (1) and (2), since these groups have the same simple part as $U(1,n;\mathbb{R})$ and $U(1,n;\mathbb{C})$ respectively. We observe that the symmetric space $U(1,n;\mathbb{F})/(U(1,\mathbb{F}) \times U(n,\mathbb{F}))$ is of dimension dn where d is the dimension of $\mathbb{F}$ over $\mathbb{R}$.

Now we note down some of the structural elements of $U(1,n;\mathbb{F})$. The general form of an element in the maximal compact subgroup of $U(1,n;\mathbb{F})$ is given by (2.1). The group $U(1,n;\mathbb{F})$ is of (real)rank 1, i.e. the maximal abelian subgroup A of $U(1,n;\mathbb{F})$ is one dimensional. In fact any element a_t in A is given by

$$(2.2) \qquad a(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Let A_+ denote the semigroup $\{a_t : t \geq 0\}$. Denote by M, the centraliser of A in K, i.e. $M = \{k \in K : ka = ak, \forall a \in A\}$. Then M is given by

$$M = \left\{ \begin{pmatrix} v & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & v \end{pmatrix} : v \in U(1,\mathbb{F}), V \in U(n-1,\mathbb{F}) \right\}$$

With K and A_+ as above, the group $G = U(1,n;\mathbb{F})$ has the Cartan decomposition $G = KA_+K$.

We use the above structure theory to derive a distance formula on the rank-1 symmetric space $X = G/K$. First we observe that, since G has the KAK decomposition, the symmetric space G/K has the geodesic polar decomposition KA_+. The orbits of A in X are two sided unit speed geodesics and for any fixed t, the K orbit $\{ka_t.K : k \in K\}$ are spheres in X centered at the origin. It follows that the K- orbit of the point $x = gK$ is the geodesic sphere centered at the origin and radius t, if $g = k_1 a_t k_2$.

The geodesic distance of a point $x = gK$ from the origin $0 = K$, in $X = G/K$ is given by $d(K, gK) = t$, if g has the Cartan decomposition $k_1 a_t k_2$. More generally define $d(g_1K, g_2K) = d(K, g_1^{-1}g_2K)$. i.e. the geodesic distance between g_1K and g_2K is given by the $A-$ part of $g_1^{-1}g_2$ in its Cartan decomposition. Clearly d defines a G-invariant metric on X. Since K fixes the origin we also have $d(0, x) = d(0, kx), \forall k \in K, x \in X$. Moreover since M commutes with A, the geodesic distance d depends only on the coset of k in K/M.

Now we prove a distance formula for the geometry of the rank one symmetric spaces. Recall that the distance between two points x_1 and x_2 in the Euclidean space is given by the cosine formula

$$|x - y|^2 = |x|^2 + |y|^2 - 2|x||y|cos\theta$$

where θ is the angle between the straight lines from the origin passing through x and y respectively.

Proposition 2.1. *Let X be a rank-1 symmetric space of non compact type. Let $x_1 = k_1 a_r K, x_2 = k_2 a_t K$ are two points in X identified with G/K. Then the geodesic distance u between x_1 and x_2 is given by*

$$\cosh u = |\cosh r \cosh t - \bar{v} w_{n,n} \sinh r \sinh t| \tag{2.3}$$

where v is the (0,0)th entry and $w_{n,n}$ the (n,n)th entry in the matrix of $k_1^{-1}k_2$ as given in (2.1). Here $|\cdot|$ denotes the modulus in the field $\mathbb{F}$ given by $|w| = (w\bar{w})^{\frac{1}{2}}$.

Remark 1. Notice that $r = d(0, x_1)$ and $t = d(0, x_2)$. Therefore this is the analogue of the cosine formula in $\mathbb{R}^n$. The proof of the above geometric identity becomes very elementary using group theory. The case of the real hyperbolic spaces, i.e. the spaces of constant curvature, has already been dealt with in [N1] and the proof follows along the same line of arguments.

Proof. Recall that the geodesic distance between g_1K and g_2K is given by the a-part of $g_1^{-1}g_2$. Thus the geodesic distance between x_1 and x_2 is given by the a-part of $g = a_{-r}k_1^{-1}k_2 a_t$. First we observe that a-part of g is given by the (0,0)th entry of $g \in G = U(1, n; \mathbb{F})$. In fact the modulus of the (0,0)th entry of $g \in U(1, n; \mathbb{F})$ is K bi-invariant and is $\cosh t$ if $g = ka_tk', k, k' \in K$. A direct matrix computation using the form of the structural elements $k_1^{-1}k_2, a_r$ and a_t given by (2.1) and (2.2) shows that the $(0,0)$th entry of $a_{-r}k_1^{-1}k_2a_t$ is $v \cosh r \cosh t - w_{n,n} \sinh r \sinh t$ if $k_1^{-1}k_2$ is as in (2.1). Since $|v| = 1$, we get (2.3) upon taking modulus in $\mathbb{F}$. $\square$

Remark 2. Setting $s = |\bar{v}w_{n,n}|$, we can write $\bar{v}w_{n,n}$ as $s\omega$ with $0 \le s \le 1, |\omega| = 1$. When $\mathbb{F} = \mathbb{C}, \omega = (cos\theta + \mathbf{i}\sin\theta)$ and for $\mathbb{F} = \mathbb{H}$,

$$(2.4) \qquad \omega = (\cos\theta + \sin\theta\cos\varphi\,\mathbf{i} + \sin\theta\sin\varphi\cos\psi\,\mathbf{j} + \sin\theta\sin\phi\sin\psi\,\mathbf{k}),$$

where $0 \le \theta, \varphi < \pi, 0 \le \psi \le 2\pi$, using the standard notation for basis in $\mathbb{C}$ and $\mathbb{H}$. With these notations, the formula (2.3) can be re written as

$$(2.5) \qquad \begin{aligned} \cosh^2 u = (\cosh r\, \cosh t)^2 + s^2(\sinh r\, \sinh t)^2 \\ - 2s\, \sinh r\, \sinh t\, \cosh r\, \cosh t \cos\theta \end{aligned}$$

where $\cos\theta = Re(\bar{v}w_{n,n})$. Thus for fixed r and t, the distance u depends only on $s = |\bar{v}w_{n,n}|$ and the angle θ. When $\mathbb{F} = \mathbb{R}, v = 1$ and $w_{n,n} = cos\theta, 0 \le \theta < \pi$ and $\bar{v}w_{nn}$ in (2.3) becomes just $\cos\theta$, see [N1], [NR]. Thus in the case of the constant curvature spaces, the distance between the points $k_1a_t.0$ and $k_2a_r.0$ depends only on the angle between the geodesics $\gamma_1(t) = \{a_t.0\}_{t\ge 0}$ and $\gamma_2(t) = \{a_r.0\}_{r\ge 0}$ at the origin see [NR]. However in the case of non constant curvature the distance between $x_1 = k_1a_rK$ and $x_2 = k_2a_tK$ also depends on the sectional curvature at the origin determined by the infinitesimal generators of a_r and a_t. This explains the presence of two variables s and θ in the distance formula.

3. SPHERICAL MEANS OF RADIAL FUNCTIONS

The spherical mean value operator on the symmetric space is obtained by averaging over geodesic spheres centered at x. Since geodesic spheres centered at origin are K-orbits, the spherical mean value operator is also given by the orbital average

$$\mathcal{A}_t f(x) = \int_{k\in K} f(xka_tK)dk, \quad x \in X = G/K,$$

where dk is the haar measure on K. It follows immediately from the above formula that $\mathcal{A}_t f$ is K-invariant whenever f is. Consequently for a radial function f, we have the formula

$$(3.1) \qquad \mathcal{A}_t f(x) = \mathcal{A}_t f(r) = \int_{k\in K} f(a_rka_tK)dk \quad \text{if } x = k_1a_rk_2$$

$r = |x| := d(0, x)$.

Notice that if f is a K-invariant function on $X = G/K$, then f is a function that depends only on $d(a_tK, K)$. Thus there exists a function F on $\mathbb{R}_+$ such that $F(cosht) = f(a_tK)$. With an abuse of notation, here and elsewhere, we write $f(t)$ for $F(\cosh t)$. First we prove the following formula for the spherical mean of a radial function on X.

Proposition 3.1. *Let f be a radial function on X. Then $\mathcal{A}_t f$ is given by*

$$(3.2) \qquad \mathcal{A}_t f(r) = \int_{|r-t|}^{r+t} \mathcal{K}_d(r, t, u) f(u)\, du$$

where $\mathcal{K}_d(r, t, u)$ is given by

$$(3.3)\qquad \mathcal{K}_d(r,t,u) = C(d,n)\sinh u\cosh u \int_{|\frac{a-c}{b}|}^{1} \frac{[\Delta_s(r,t,u)]^{d-3}(1-s^2)^{\frac{d(n-1)}{2}-1}}{ab^{d-2}} sds$$

where $C(d,n) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{dn}{2})}{\Gamma(\frac{d(n-1)}{2})\Gamma(\frac{d-1}{2})}$ *and*

$$(3.4)\qquad \Delta_s(r,t,u) = \frac{\sqrt{[(bs)^2-(a-c)^2][(a+c)^2-(bs)^2]}}{2a}.$$

with $a = \cosh r\cosh t, b = \sinh r\sinh t$ *and* $c = \cosh u$, *for* $d = 2$ *or* 4*. Here* d *denotes the dimension of the field* $\mathbb{F}$*. When* $d = 1$

$$\mathcal{K}_1(r,t,u) = \frac{2^{n-3}\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})} \frac{\Delta_1(r,t,u)}{(\sinh r\ \sinh t)^{\frac{n-1}{2}}}.$$

Proof. Since f is radial, as observed before, $f(a_r k a_t K)$ depends only on the distance of $a_r k a_t K$ from the origin. Rewriting (3.1) as

$$\mathcal{A}_t f(r) = \int_{k\in K} f(a_u K)dk$$

where u is given by the cosine formula (2.3) for the geometry of X.

Since $a_u K = a_r k a_t K$ depends only on K/M, the above integral reduces to an integral over the quotient space K/M which can be identified with $S(\mathbb{F}^n)$, the unit sphere in $\mathbb{F}^n$. Therefore writing $f(a_u K)$ as $F(\cosh u)$, we have in view of (2.3)

$$(3.5)\qquad \mathcal{A}_t f(r) = \int_{S(\mathbb{F}^n)} F(|\cosh r\cosh t - y_n \sinh r\sinh t|)d\mu^{dn-1}(y)$$

where $y = (y_1, y_2, \cdots, y_n) \in S(\mathbb{F}^n), d = 1, 2$ or 4 and $y_n = \bar{v}w_{n,n}$ as in (2.3). Notice that $S(\mathbb{F}^n)$ is the $U(n,\mathbb{F})$ orbit of a unit vector in $\mathbb{F}^n$, which is a sphere in $\mathbb{F}^n = \mathbb{R}^{dn}$.

Since F depends only on the last variable, it is constant on the $dn-d$ dimensional spheres $S_{(1-|y_n|^2)^{1/2}}(\mathbb{F}^n) = \{y \in S(\mathbb{F}^n) : \Sigma_{i=1}^{n-1}|y_i|^2 = 1-|y_n|^2\}$ of radius $(1-|y_n|^2)^{1/2}$ for each fixed $|y_n|, 0 \le |y_n| \le 1$. Hence the integral reduces to an integral over the unit disc $|y_n| \le 1$ in $\mathbb{F}$ with a Jacobian factor $(1-|y_n|^2)^{\frac{dn-d}{2}-1}$:

$$(3.6)\qquad \frac{\Gamma(\frac{dn}{2})}{\Gamma(\frac{d(n-1)}{2})\pi^{\frac{d}{2}}} \int_{|y_n|\le 1} F(|\cosh r\ \cosh t - y_n \sinh r\ \sinh t|)\,(1-|y_n|^2)^{\frac{d(n-1)-2}{2}} dy_n$$

When $d = 1$ as we know, $y_n = \cos\theta$ and the above becomes a one dimensional integral. When $d = 2$ or 4, set $y_n = s\omega$ where ω is given by (2.4). Since

$$|\cosh r\cosh t - s\omega\sinh r\sinh t|$$

depends only on s and $\cos\theta = Re\,\omega$ as observed before, using polar co ordinates in $\mathbb{F} = \mathbb{R}^d$ the above integral can be reduced to an integral of the form

$$\mathcal{A}_t f(r) = \frac{\Gamma(\frac{dn}{2})}{\Gamma(\frac{d(n-1)}{2})\pi^{\frac{d}{2}}} \int_0^1 \int_0^\pi F(|\cosh r \cosh t - s\omega \sinh r \sinh t|)\, J_n(s,\theta) ds\, d\theta$$

for $d = 2$ or 4, where

$$J_n(s,\theta) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})} (1-s^2)^{\frac{d(n-1)-2}{2}} s^{d-1} (\sin\theta)^{d-2}.$$

Now we take s and u given by (2.5) as the new variables. The Jacobian of this change of variable is given by

$$d\theta ds = \frac{\sinh u \cosh u}{abs \sin\theta} du ds$$

for $d = 2$ or 4. When $d = 1$ we take u given by (2.5) (with $s = 1$) as the new variable and the Jacobian of the change of variable in this case is given by $d\theta = \frac{\sinh u\, du}{b \sin\theta}$.

Set $\Delta = \Delta_s = bs \sin\theta$ for $d = 2, 4$ and $\Delta = b\sin\theta$ for $d = 1$. Eliminate θ using (2.5) we get (3.4). It follows from (2.5) that $u \in (|r-t|, r+t)$ as θ varies between 0 and π. Also since $\Delta_s^2 = (bs\sin\theta)^2 \geq 0$, from (3.4) we see that $s \geq |\frac{a-c}{b}|$. Thus we get (3.2). □

Remark 3. When $d = 1$, by eliminating θ using the cosine formula (2.3), we get the following 'symmetric' formula for Δ (see [NR]):

$$\Delta(r,t,u) = \Big[\sinh\Big(\frac{r+t+u}{2}\Big) \sinh\Big(\frac{r+t-u}{2}\Big) \sinh\Big(\frac{r-t+u}{2}\Big) \sinh\Big(\frac{-r+t+u}{2}\Big)\Big]^{1/2}.$$

4. THE LORENTZ SPACE $L^{p,q}(X)$ AND THE KERNEL ESTIMATES

In this section we recall the definition of Lorentz space and discuss some auxiliary results that are needed in the proof of the main theorem. Let $(X, \mathcal{B}, \mu)$ be a sigma finite measure space. Given $1 \leq p < \infty, 1 \leq q < \infty$, we say that a function f belongs to $L^{p,q}(X, d\mu)$ if

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty .$$

And given $1 \leq p \leq \infty$, we say that $f \in L^{p,\infty}$ if

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty.$$

Here f^* denotes the non increasing rearrangement of f, namely

$$f^*(t) = inf\{s : \lambda_f(s) \leq t\}$$

where

$$\lambda_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\})$$

denotes the distribution function of f.

We recall that $\|\cdot\|_{p,q}$ is not a norm, as it fails to satisfy the triangle inequality, but $\|.\|_{p,q}$ is equivalent to a norm satisfying the triangle inequality, (see SW Theorem 3.21, page 204). We need to recall two more facts from Lorentz space theory. The first one concerns the norm of the characteristic function; namely $\|\chi_E\|_{p,q} = \mu(E)^{\frac{1}{p}}$, whenever E is a measurable subset of X with finite measure. The second one is the Hölder's inequality for Lorentz spaces:

$$\left| \int_X fg\, d\mu \right| \leq \|f\|_{p,q,\mu} \|g\|_{p',q',\mu}.$$

whenever $f \in L^{p,q}(X, d\mu)$ and $g \in L^{p',q'}(X, d\mu), \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$, (see [BS]). Another fundamental result that we use is the following theorem from Stein and Weiss' book [SW, Theorem 3.13, p. 195].

Theorem 4.1. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose T is a linear operator which maps the finite linear combinations of characteristic functions χ_E of sets $E \subset X$ of finite measure into a vector space V, that is endowed with an order preserving norm $\|\cdot\|$. If*

$$\|T\chi_E\| \leq C\|\chi_E\|_{p,1}$$

where C is independent of E, then there exists a constant A such that

$$\|Tf\| \leq A\|f\|_{p,1}$$

for all f in the domain, $D(T)$, of T.

The measure space (X, μ) that we are concerned with, consists of a rank one symmetric space X and the G- invariant measure on X. Since the group G has a decomposition of the form $G = KA_+K$, the symmetric space has the associated geodesic polar coordinates KA_+. Consequently the G-invariant measure on X decomposes as $d\mu = \sinh^{dn-1} t \cosh^{d-1} t\, dt\, dk$, (see [K]). Moreover since we are dealing with K invariant functions on X, the Lorentz space that we are dealing with is equivalent to the weighted Lorentz space on $\mathbb{R}_+$, namely $L^{p,q}(\mathbb{R}_+, w(t)dt)$, with weight $w(t) = \sinh^{dn-1} t \cosh^{d-1} t$. In the rest of this paper $d\mu(t)$ refers to the measure on $\mathbb{R}_+$ with the weight $\sinh^{dn-1} t \cosh^{d-1} t$.

It is easy to see that the above theorem also holds for sub linear operators. We need to use only a special case of this theorem with $V = L^{p,\infty}(\mathbb{R}_+, d\mu(t))$.

Now we prove the following kernel estimate. Set $K_d(r,t,u) = \frac{\mathcal{K}_d(r,t,u)}{\sinh^{dn-1}(u)\cosh^{d-1}(u)}$ where $\mathcal{K}_d(r,t,u)$ be as in Proposition 3.1.

Proposition 4.2. *Let $K_d(r,t,u)$ be as above with $d = 1, 2$ or 4. Then we have the following estimate.*

$$K_d(r,t,u) \leq \frac{4^{\frac{d-3}{2}}}{\sinh r \sinh t \sinh u (\cosh r \cosh t \cosh u)^{\frac{d-1}{2}}} \left\{ \frac{1}{(\sinh u)^{dn-3}} \bigwedge \frac{1}{(\sinh r)^{dn-3}} \right\} \tag{4.1}$$

for $d = 1, 2$ or 4. Here $\bigwedge$ denotes the minimum.

Proof. The case $d = 1$ has already been dealt with in [NR]. However for the sake of completeness and to highlight the uniformity of the kernel estimates in all the cases $d = 1, 2$ and 4 we brief this case as well. When $d = 1$,

$$K_1(r, t, y) = \frac{[\Delta(r, t, u)]^{n-3}}{[\sinh r \, \sinh t \, \sinh u]^{n-2}}.$$

Since $\Delta = \sinh r \, \sinh t \sin\theta$ as seen in the proof of Proposition 3.1, we have the inequality

$$\Delta \leq \sinh r \, \sinh t.$$

More over, by the symmetry of Δ in r, t and u, as seen in Remark 3, we also have

$$\Delta \leq \sinh r \, \sinh u \quad \text{and} \quad \Delta \leq \sinh t \, \sinh u$$

which yields the inequality

$$\frac{\Delta}{\sinh u} \leq \sinh r \wedge \sinh t.$$

The required estimate, for the case $d = 1$ follows immediately from these two inequalities.

Now consider the cases $d = 2$ and 4. For simplicity we use the notations $a = \cosh r \cosh t, b = \sinh r \sinh t$ and $c = \cosh u$. Equation (3.4) can be written as

$$2a\Delta_s(r, t, u) = b^2\left(\left[s^2 - \left(\frac{a-c}{b}\right)^2\right]\left[\left(\frac{a+c}{b}\right)^2 - s^2\right]\right)^{1/2}.$$

Since $(\frac{a-c}{b})^2 \leq s^2 \leq 1, (\frac{a+c}{b})^2 - s^2 \approx \frac{ac}{b^2}$, it follows from (3.3) and (3.4), that

$K_d(r, t, u)$

$$\approx \frac{1}{(\sinh u)^{dn-2}(\cosh u)^{d-2}} \frac{(b\sqrt{ac})^{d-3}}{(ab)^{d-2}} \int_{|\frac{a-c}{b}|}^{1} (1 - s^2)^{\frac{d(n-1)}{2}-1}\left[s^2 - \left(\frac{a-c}{b}\right)^2\right]^{\frac{d-3}{2}} sds$$

By a change of variable the above integral can be reduced to a gamma integral, i.e.

$$\int_{|\frac{a-c}{b}|}^{1} (1 - s^2)^{\frac{d(n-1)}{2}-1}\left[s^2 - \left(\frac{a-c}{b}\right)^2\right]^{\frac{d-3}{2}} sds$$
$$= \frac{1}{2}\left[1 - \left(\frac{a-c}{b}\right)^2\right]^{\frac{dn-3}{2}} \int_0^1 (1 - w)^{\frac{d(n-1)}{2}-1} w^{\frac{d-3}{2}} dw.$$

Consequently,

$$K_d(r, t, u) \approx \frac{\Gamma(\frac{d(n-1)}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{dn-1}{2})} \frac{\left[1 - (\frac{a-c}{b})^2\right]^{\frac{dn-3}{2}}}{b(ac)^{\frac{d-1}{2}}} \frac{1}{(\sinh u)^{dn-2}}. \tag{4.2}$$

Since $\left[1 - \left(\frac{a-c}{b}\right)^2\right]^{\frac{dn-3}{2}} \leq 1$, it follows from (4.2), that

$$K_d(r,t,u) \leq \frac{\Gamma(\frac{d(n-1)}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{dn-1}{2})} \frac{1}{b(ac)^{\frac{d-1}{2}}} \frac{1}{(\sinh u)^{dn-2}}.$$

We also claim that

$$(4.3) \qquad 1 - \left(\frac{a-c}{b}\right)^2 \leq \frac{c^2-1}{\sinh^2 r} = \frac{\sinh^2 u}{\sinh^2 r}.$$

In view of this inequality, from (4.2), we also get the estimate

$$\begin{aligned} K_d(r,t,u) &\leq \frac{\Gamma(\frac{d(n-1)}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{dn-1}{2})} \frac{1}{b(ac)^{\frac{d-1}{2}}} \frac{1}{\sinh^{dn-2} u} \left(\frac{\sinh^2 u}{\sinh^2 r}\right)^{\frac{dn-3}{2}} \\ &\leq \frac{\Gamma(\frac{d(n-1)}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{dn-1}{2})} \frac{1}{b} \frac{1}{(ac)^{\frac{d-1}{2}}} \frac{1}{\sinh u \sinh^{dn-3}(r)}. \end{aligned}$$

This proves the estimate (4.1).

To complete the proof of the Proposition we need to prove the claim (4.3). Since $b = \sinh r \sinh t$, (4.3) is equivalent to showing

$$(4.4) \qquad (1+\sinh^2 t)c^2 - 2ac + a^2 - b^2 - \sinh^2 t \geq 0.$$

A direct verification shows that $a^2 - b^2 - \sinh^2 t = \cosh^2 r$. Since $1 + \sinh^2 t = \cosh^2 t$ the LHS of (4.4) is $(c \cosh t + \cosh r)^2$ which is clearly non negative. This proves the claim. □

Now we prove the following lemma which contains the essential volume estimates involved in the proof of the Proposition 5.1.

Lemma 4.3.

$$\begin{aligned} \left\|\frac{\chi(|r-t|,r+t)}{\sinh u}\right\|_{dn,\infty,\mu} &\leq [\cosh(r+t)]^{\frac{d-2}{dn}} \text{ for } d = 2,4 \\ &\leq [\cosh(|r-t|)]^{\frac{-1}{n}} \text{ for } d = 1. \end{aligned}$$

Proof. Recall that

$$\left\|\frac{\chi(r-t,r+t)}{\sinh u}\right\|_{dn,\infty,\mu} = sup_{\alpha>0}\, \alpha\, [\lambda(\alpha)]^{\frac{1}{dn}}$$

where λ is the distribution function of $\frac{\chi(|r-t|,r+t)}{\sinh u}$ given by

$$\lambda(\alpha) = \begin{cases} \mu(|r-t|, r+t) & \text{if } \; 0 \leq \alpha < \dfrac{1}{\sinh(r+t)} \\ \mu\left(|r-t|, \sinh^{-1}\left(\dfrac{1}{\alpha}\right)\right) & \text{if } \; \dfrac{1}{\sinh(r+t)} \leq \alpha < \dfrac{1}{\sinh(|r-t|)} \\ 0 & \text{if } \; \alpha \geq \dfrac{1}{\sinh(|r-t|)} \end{cases}$$

It is easy to see, after a change of variable that

$$\alpha^{dn}\mu\left(|r-t|,\sinh^{-1}\left(\frac{1}{\alpha}\right)\right)=\alpha^{dn}\int_{\sinh(|r-t|)}^{\frac{1}{\alpha}}s^{dn-1}(1+s^2)^{\frac{d-2}{2}}ds \tag{4.5}$$

which is a decreasing function of α for $d=2$ and 4. Therefore for $\frac{1}{\sinh(r+t)}\leq\alpha<\frac{1}{\sinh(|r-t|)}$

$$\alpha\left[\mu\left(|r-t|,\sinh^{-1}\left(\frac{1}{\alpha}\right)\right)\right]^{\frac{1}{dn}}\leq\frac{\mu(|r-t|,r+t)^{\frac{1}{dn}}}{\sinh(r+t)}.$$

It follows that

$$\left\|\frac{\chi(r-t,r+t)}{\sinh u}\right\|_{dn,\infty,\mu}=\frac{\mu(|r-t|,r+t)^{\frac{1}{dn}}}{\sinh(r+t)}.$$

Now

$$\begin{aligned}\frac{\mu(|r-t|,r+t)}{[\sinh(r+t)]^{dn}}&=\int_{|r-t|}^{r+t}\frac{\sinh^{dn-1}u\cosh^{d-1}u\,du}{\sinh^{dn}(r+t)}\\&\leq\frac{1}{\sinh^2(r+t)}\int_{|r-t|}^{r+t}\sinh u\cosh^{d-1}u\,du\\&\leq\frac{\cosh^{d-2}(r+t)}{\sinh^2(r+t)}\int_{|r-t|}^{r+t}\sinh u\cosh u\,du\\&\leq\frac{\cosh^{d-2}(r+t)}{2\sinh^2(r+t)}\,\frac{\cosh[2(r+t)]-\cosh[2(r-t)]}{2}\\&\leq\cosh^{d-2}(r+t).\end{aligned}$$

This proves the Lemma 4.2 for the case $d=2$ and 4. The case $d=1$ is slightly more complicated. In fact when $d=1$, (4.5) does not define a decreasing function of α. However using the fact that $\frac{1}{(1+s^2)^{1/2}}\approx 1$ for $0\leq s\leq 1$ and $\frac{1}{(1+s^2)^{1/2}}\approx\frac{1}{s}$ for $s>1$, we can directly estimate (4.5) which gives the required estimates. (see [NR]). □

5. THE PROOF OF THE MAIN THEOREM

In this section we prove some auxiliary results and then prove Theorem 1.1. Let M be an auxiliary maximal function on R_+ defined by

$$Mf(r)=\sup_{\tanh t<\frac{1}{3}\tanh r}\frac{\|f\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu}}{\|\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu}}.$$

Recall that μ is the measure on $\mathbb{R}_+$ with density $\sinh^{dn-1}u\cosh^{d-1}u$. The main step in the proof of Theorem 4.1 is the following control of the spherical maximal operator.

Proposition 5.1. *Let $\mathcal{M}$ be as in Theorem 1.1 and let f be a radial function. Then we have the following inequality*

$$\mathcal{M}f(r) \leq Mf(r) + h(r)\, \|f\|_{\frac{dn}{dn-1},1,\mu}$$

where M is the maximal function defined above and

$$h(r) = \frac{\chi_{(0,1)}}{(\sinh r)^{dn-1}} + \frac{\chi_{(1,\infty)}}{(\sinh r)^{\frac{dn-1}{dn}(dn+d-2)}} \tag{5.1}$$

for $d = 1, 2$ or 4.

Proof. Clearly

$$\sup_{t>0} |\mathcal{A}_t f(r)| \leq \sup_{\tanh t < \frac{1}{3}\tanh r} |\mathcal{A}_t f(r)| + \sup_{\tanh t > \frac{1}{3}\tanh r} |\mathcal{A}_t f(r)|.$$

Let us consider each of these terms separately. Rewrite the formula (3.2) as

$$\mathcal{A}_t f(r) = \int_{|r-t|}^{r+t} K_d(r,t,u) f(u) d\mu(u) \tag{5.2}$$

where $K_d(r,t,u)$ is as given in Proposition 4.2. Using Hölders inequality, we get

$$\begin{aligned} |\mathcal{A}_t f(r)| &\leq \int_{|r-t|}^{r+t} \left|K_d(r,t,u) f(u) \chi_{(|r-t|,r+t)}\right| d\mu(u) \\ &\leq \|f\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu}\, \|\chi_{(|r-t|,r+t)} K_d(r,t,u)\|_{dn,\infty,\mu} \\ &\leq C(r,t) Mf(r) \end{aligned}$$

where

$$C(r,t) = \|\chi_{(|r-t|,r+t)} K_d(r,t,u)\|_{dn,\infty,\mu} \|\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu}.$$

When $\tanh t < \frac{1}{3}\tanh r$, using the estimate

$$K_d(r,t,u) \leq \frac{4^{\frac{d-3}{2}}}{b(ac)^{\frac{d-1}{2}} \sinh^{dn-2}(u)}$$

of Proposition 4.2, and that $c = \cosh u \geq \cosh(|r-t|)$, we see that $C(r,t)$ is at most

$$\begin{aligned} &\frac{4^{\frac{d-3}{2}}}{b[a\cosh(|r-t|)]^{\frac{d-1}{2}}} \left\| \frac{\chi_{(|r-t|,r+t)}}{\sinh^{dn-2}(u)} \right\|_{dn,\infty,\mu} \|\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu} \\ \leq\ &\frac{4^{\frac{d-3}{2}}}{b[a\cosh(|r-t|)]^{\frac{d-1}{2}}} \frac{1}{\sinh^{dn-2}(|r-t|)} \|\chi_{(|r-t|,r+t)}\|_{dn,\infty,\mu} \|\chi_{(|r-t|,r+t)}\|_{\frac{dn}{dn-1},1,\mu}. \end{aligned}$$

Now observe that $\tanh t < \frac{1}{3}\tanh r$ if and only if $\sinh(r+t) < 2\sinh(r-t)$ if and only if $\cosh(r+t) < 2\cosh(r-t)$. Using this and the fact that $\|\chi_E\|_{p,q,\mu} = \mu(E)^{\frac{1}{p}}$, we see that the RHS of the above is at most

$$\frac{2^{dn-2}\mu((|r-t|,r+t))}{\sinh^{dn-2}(r+t)}\frac{2^{\frac{d-1}{2}+d-3}}{\cosh^{\frac{d-1}{2}}(r+t)ba^{\frac{d-1}{2}}}.$$

Since both sinh and cosh are increasing functions, we see that

$$\frac{\mu((|r-t|,r+t))}{\sinh^{dn-2}(r+t)} \leq \cosh^{d-1}(r+t)\int_{|r-t|}^{r+t}\sinh u\,du$$
$$= 2\cosh^{d-1}(r+t)\sinh r\sinh t.$$

It follows that $C(r,t) \leq 2^{dn+d-4+\frac{d+1}{2}}$, whenever $\tanh t < \frac{1}{3}\tanh r$. Consequently

$$\sup_{\tanh t<\frac{1}{3}\tanh r} |\mathcal{A}_t f(r)| \leq 2^{dn+d-4+\frac{d+1}{2}} Mf(r).$$

Now let us consider the case $\tanh t > \frac{1}{3}\tanh r$. Again in this case, using Hölders inequality in (5.2), we see that

$$|\mathcal{A}_t f(r)| \leq \|f\|_{\frac{dn}{dn-1},1,\mu}\|K_d(r,t,u)\chi_{(|r-t|,r+t)}\|_{dn,\infty,\mu}.$$

As before using the estimate

$$K_d(r,t,u) \leq \frac{4^{\frac{d-3}{2}}}{b(ac)^{\frac{d-1}{2}}}\frac{1}{\sinh^{dn-3}(r)\sinh(u)}$$

of Proposition 4.2, and the fact that $c = \cosh u \geq \cosh(|r-t|)$, we see that

$$\|K_d(r,t,u)\chi_{(|r-t|,r+t)}\|_{dn,\infty,\mu}$$
$$\leq \frac{4^{\frac{d-3}{2}}}{b(a\cosh(|r-t|))^{\frac{d-1}{2}}}\frac{1}{\sinh^{dn-3}r}\left\|\frac{\chi(|r-t|,r+t)}{\sinh u}\right\|_{dn,\infty,\mu}$$
$$\leq \frac{3}{\sinh^{dn-1}r}\frac{2^{\frac{d-1}{2}}4^{\frac{d-3}{2}}}{e^{\frac{d-1}{2}(r+t)}[\cosh(|r-t|)]^{\frac{d-3}{2}}}\left\|\frac{\chi(|r-t|,r+t)}{\sinh u}\right\|_{dn,\infty,\mu}.$$

Here we have used the fact that $a \approx e^{r+t}$ and $b\cosh(|r-t|) = b\cosh(r-t) \geq \frac{1}{3}\sinh^2 r$ whenever $\tanh t > \frac{1}{3}\tanh r$. When $d = 2$ or 4, using the estimates of Lemma 4.3 we see that the above is at most

$$\frac{4^{\frac{d-3}{2}}}{\sinh^{dn-1}r}\frac{6\,e^{\frac{d-2}{dn}(r+t)}}{e^{\frac{d-1}{2}(r+t)}e^{\frac{d-3}{2}|r-t|}}$$
$$\leq \frac{4^{\frac{d-3}{2}}}{\sinh^{dn-1}r}\frac{2e^{\frac{d-2}{dn}(r+t)}}{e^{(d-2)r\vee t}e^{t\wedge r}}$$
$$\leq \frac{4^{\frac{d-3}{2}}}{\sinh^{dn-1}r}\frac{2}{e^{[d-2-\frac{d-2}{dn}](r\vee t)}}\frac{1}{e^{(t\wedge r)(1-\frac{d-2}{dn})}}.$$

We also observe that

$$\frac{1}{e^{(d-2)\frac{dn-1}{dn}(r)}} \approx \begin{cases} 1 & \text{for} \quad r<1 \\ \dfrac{1}{(\sinh r)^{\frac{(d-2)(dn-1)}{dn}}} & \text{for} \quad r \geq 1 \end{cases}$$

It follows from these observations that

$$\|K_d(r,t,u)\chi_{(|r-t|,r+t)}\|_{dn,\infty,\mu} \leq \frac{\chi_{(0,1)}}{(\sinh r)^{dn-1}} + \frac{\chi_{(1,\infty)}}{(\sinh r)^{\frac{(dn-1)(dn+d-2)}{dn}}}.$$

This completes the proof of the proposition, for $d = 2$ and 4. Similarly using the estimate of Lemma 4.3 for $d = 1$, one can get (5.1) for $d = 1$ as well. (See [NR]). □

Now we prove the main theorem.

Proof. (of Theorem 1.1) To prove that $\mathcal{M}$ defines a bounded operator from $L^{\frac{dn}{dn-1},1}(\mathbb{R}_+, d\mu)$ to $L^{\frac{dn}{dn-1},\infty}(\mathbb{R}_+, d\mu)$, in view of Proposition 5.1, we need only to show that

(5.3) $$\|Mf\|_{\frac{dn}{dn-1},\infty;\mu} \leq C\|f\|_{\frac{dn}{dn-1},1;\mu}$$

(5.4) $$\|h(r)\|_{\frac{dn}{dn-1},\infty;\mu} \leq \infty.$$

First we show (5.4), i.e. the function $h(r) \in L^{\frac{dn}{dn-1},\infty}(\mathbb{R}_+, d\mu)$. In fact we show that the functions

$$h_1(r) = \frac{\chi_{(0,1)}}{\sinh^p(r)} \in L^{\frac{dn}{dn-1},\infty}(\mathbb{R}_+, d\mu)$$

if and only if $p \leq dn - 1$ and

$$h_2(r) = \frac{\chi_{(1,\infty)}}{\sinh^q(r)} \in L^{\frac{dn}{dn-1},\infty}(\mathbb{R}_+, d\mu)$$

if and only if $q \geq \frac{(dn-1)(dn+d-2)}{dn}$.

Let λ_i denotes the distribution function of h_i, i.e.

$$\lambda_i(\alpha) = \mu(\{u > 0 : h_i(u) > \alpha\}), \quad i = 1, 2.$$

We have

$$\lambda_1 = \begin{cases} \mu(0,1) & \text{if} \quad 0 \leq \alpha < \dfrac{1}{\sinh^p(1)} \\ \mu\left(0, \sinh^{-1}\left(\alpha^{-\frac{1}{p}}\right)\right) & \text{if} \quad \dfrac{1}{\sinh^p(1)} \leq \alpha < \infty. \end{cases}$$

By a simple change of variable we see that

$$\begin{aligned} \mu\left(0, \sinh^{-1}\left(\alpha^{-\frac{1}{p}}\right)\right) &= \int_0^{\sinh^{-1}(\alpha^{-\frac{1}{p}})} \sinh^{dn-1}(u)\cosh^{d-1}(u)du \\ &= \alpha^{-\frac{dn}{p}} \int_0^1 y^{dn-1}\left(1 + \alpha^{-\frac{2}{p}}y^2\right)^{\frac{d-2}{2}} dy. \end{aligned}$$

Therefore for $\alpha \geq \frac{1}{\sinh^p(1)}$ and $0 \leq y \leq 1$,

$$\mu(0, \sinh^{-1}(\alpha^{-\frac{1}{p}})) \approx \alpha^{-\frac{dn}{p}}.$$

Consequently

$$\|h_1\|_{p,\infty;\mu} = \sup_{\alpha > \frac{1}{\sinh^p(1)}} \alpha[\lambda_1(\alpha)]^{\frac{dn-1}{dn}} < \infty$$

if and only if $p \leq dn - 1$.

Similarly one can show that $\lambda_2(\alpha) \approx \alpha^{-\frac{dn+d-2}{q}}$, for α near 0. Hence

$$\|h_2\|_{q,\infty;\mu} = \sup_{\alpha > \frac{1}{\sinh^q(1)}} \alpha[\lambda_2(\alpha)]^{\frac{dn-1}{dn}} < \infty$$

if and only if $q \geq \frac{(dn-1)(dn+d-2)}{dn}$.

To prove (5.3), in view of Theorem 1.2, it suffices to consider only $F = \chi_E, E \subseteq \mathbb{R}_+$, with $\mu(E) < \infty$. For notational convenience, we assume $r > t$, so that $|r - t| = r - t$. Now,

$$(M\chi_E)(r) = \sup_{0<t<r}\left[\frac{\mu(E \cap (r-t, r+t))}{\mu((r-t, r+t))}\right]^{\frac{dn-1}{dn}}$$

it follows that

$$\{r > 0 : M\chi_E\,(r) > \alpha\} = \left\{r > 0 : M_{d,n}\chi_E\,(r) > \alpha^{\frac{dn}{dn-1}}\right\}$$

where

$$M_{d,n}F\,(r) = \sup_{0<t<r}(\mu(r-t, r+t))^{-1}\int_{r-t}^{r+t}|F\,(u)|d\mu(u)$$

which is the "truncated" Hardy Littlewood maximal operator on $\mathbb{R}_+$ with measure $d\mu(u) = \sinh^{dn-1}(u)\cosh^{d-1}(u)\,du$. Observe that the measure μ does not satisfy the doubling condition as it has exponential growth. However $M_{d,n}$ is still weak type $(1,1)$, as proved by Strömberg, see [S]. It follows that

$$\mu(\{r > 0 : M_{d,n}\chi_E\,(r) > \alpha\}) \leq \frac{C_1}{\alpha^{\frac{dn}{dn-1}}}\mu(E).$$

Therefore

$$\begin{aligned}\|M\chi_E\|_{\frac{dn}{dn-1},\infty;\mu} &= \sup_{\alpha>0} \alpha\,[\mu(\{r > 0 : M\chi_E\,(r) > \alpha\})]^{\frac{dn-1}{dn}} \\ &\leq [C_1\,\mu(E)]^{\frac{dn-1}{dn}} = C\|\chi_E\|_{\frac{dn}{dn-1},1;\mu}\,.\end{aligned}$$

This proves (5.3) and completes the proof of Theorem 1.1. □

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