Parton Distributions : A simple-minded Introduction

Lecture 1

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Prehistory : Naive Parton Model

- $e\mu \rightarrow e\mu$ vs $eP \rightarrow eP$:
  Proton is not point-particle $\Longrightarrow$ Introduction of Electric and Magnetic Form factors.
- Inelastic eP scattering : Structure factors $W_i(\nu, Q^2)$
  Q : momentum transfer ; ν : energy transfer to electron
- Deep Inelastic scattering :
  $W_i(\nu, Q^2) \rightarrow F_{1,2}(x)$ $x \equiv 2M_P\nu/Q^2$
  - Proton as a collection of quasi-free partons, each with a momentum fraction x.
  - Partons massless (at the scale of the scattering problem)
  - Callan-Gross relation $\Longrightarrow$ Most partons are spin-half objects
  - Sum rules $\Longrightarrow$ Non-charged partons carry nearly half the momentum.
- Theoretical development : Asymptotic Freedom in non-abelian theories.
  - Charged partons identified as quarks.
  - Neutral partons identified as gluons.
- Experimental development : $e^+e^- \rightarrow 3$ jets
  - Existence of Gluons.
  - Gluons are spin-1.
  - $SU(3)$

Then, why do we talk of $u(x, Q^2)$?

Prehistory : Naive Parton Model

- Rutherford scattering :
  spin-0 non-relativistic particle on heavy target (no recoil)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth.}} = \frac{\alpha^2\, Z^2}{4\, E^4\, \sin^4(\theta/2)}$$

- Mott scattering :
  spin-0 relativistic particle on heavy target (with recoil)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott.}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth.}} \frac{E'}{E}\, \cos^2(\theta/2)$$

- $e\mu \longrightarrow e\mu$

$$\left(\frac{d\sigma}{d\Omega}\right)_{e\mu} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth.}} \frac{E'}{E} \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2\, M^2}\, \sin^2\frac{\theta}{2}\right] = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left[1 - \frac{q^2}{2\, M^2}\, \tan^2\frac{\theta}{2}\right]$$

- Rosenbluth scatt ($eP \rightarrow eP$) :
  Proton is not point-particle $\Longrightarrow$ Introduction of Electric and Magnetic Form factors.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rosen}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left[\left\{F_1^2 - \frac{q^2\, F_2^2}{4\, M^2}\right\} - \frac{q^2}{2M^2}(F_1 + F_2)^2 \tan^2\frac{\theta}{2}\right]$$

$F_i = F_i(q^2)$

- Inelastic eP scattering : Structure factors $W_i(\nu, Q^2)$
  Q : momentum transfer ; ν : energy transfer to electron

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Then, why do we talk of $u(x, Q^2)$?

Running coupling

- Consider a dimensionless physical observable $\mathcal{O}$. (No directional dependence) Therefore,

$$\mathcal{O} = \mathcal{O}(E_i/E_j,\, m_i/m_j,\, E_i/m_j)$$

- If only a single energy scale E relevant, then $\mathcal{O} = \mathcal{O}(m_i/E,\, m_i/m_j)$.

- Let $E \gg m_i \;\; \forall m_i$. Consider limit $m_i \to 0$.
  Then $\mathcal{O}$ independent of E !

- Not true in QFT !
- Perturbative calculation of $\mathcal{O}$ (series in coupling const, α_s) requires renormalization to remove ultraviolet divergences.
  Introduces a second mass scale μ (point where subtractions are made) !

- $\mathcal{O} = \mathcal{O}(E/\mu)$ and not constant in E!
  Renormalized α_s also μ-dependent.

- μ is arbitrary!
  If bare coupling held fixed, $\mathcal{O}$ cannot depend on μ.

- Dimensionless $\mathcal{O}$ can only depend on E/μ and the renormalized α_s. Hence

$$0 = \mu^2 \frac{d}{d\mu^2} \mathcal{O}\left(\frac{E^2}{\mu^2}, \alpha_s\right) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] \mathcal{O}$$

- Define

$$t \equiv \ln \frac{E^2}{\mu^2}\,, \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

  Then,

$$0 = \left[\frac{-\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right] \mathcal{O}$$

- RG equation solved by defining running coupling $\alpha_s(E)$:

$$t = \int_{\alpha_s(\mu)}^{\alpha_s(E)} \frac{dx}{\beta(x)}$$

- Then,

$$\frac{\partial \alpha_s(E)}{\partial t} = \beta\left(\alpha_s(E)\right)\,, \qquad \frac{\partial \alpha_s(E)}{\partial \alpha_s(\mu)} = \frac{\beta\left(\alpha_s(E)\right)}{\beta\left(\alpha_s(\mu)\right)}$$

- $\mathcal{O}\left(E^2/\mu^2, \alpha_s(\mu)\right) = \mathcal{O}\left(1, \alpha_s(E)\right)$ and all scale dependence comes only from running of α_s
- QCD is asymptotically free: $\alpha_s(E) \to 0$ as $E \to \infty$.
  For large E, can safely use perturbation theory.
  Knowledge of $\mathcal{O}(1, \alpha_s)$ to fixed order allows prediction of $\mathcal{O}(E)$

$$\beta(\alpha_s) = -b_0\,\alpha_s^2 - b_1\,\alpha_s^3 + \mathcal{O}(\alpha_s^4)$$

$$b_0 = \frac{11\,C_A - 2\,N_f}{12\,\pi}$$

$$b_1 = \frac{17\,C_A^2 - (5\,C_A + 3\,C_F)\,N_f}{24\,\pi^2}$$

with $C_A = 3 \qquad (f_{abc}\,f_{dbc} = C_A\,\delta_{ad})$ and $C_F = 4/3 \qquad (T_a\,T_a = C_F \text{ in fundamental})$.

Upto 2-loop order [neglecting $\mathcal{O}(\alpha_s^4)$],

$$\ln\frac{Q^2}{\mu^2} = b_0^{-1}\left[\alpha_s^{-1}(Q) - \alpha_s^{-1}(\mu)\right] + b_1\left[\ln\frac{b_0\,\alpha_s(Q)}{b_0 + b_1\,\alpha_s(Q)} - \ln\frac{b_0\,\alpha_s(\mu)}{b_0 + b_1\,\alpha_s(\mu)}\right]$$

Given $\alpha_s(E)$, can calculate $\alpha_s(Q)$

To $\mathcal{O}(\alpha_s^3)$

$$\frac{d\alpha_s}{dt} = -b_0\,\alpha_s^2 - b_1\,\alpha_s^3$$

Scheme dependence:

$$\alpha_s \rightarrow \widetilde{\alpha}_s = \alpha_s\,(1 + c\,\alpha_s)$$

Then,

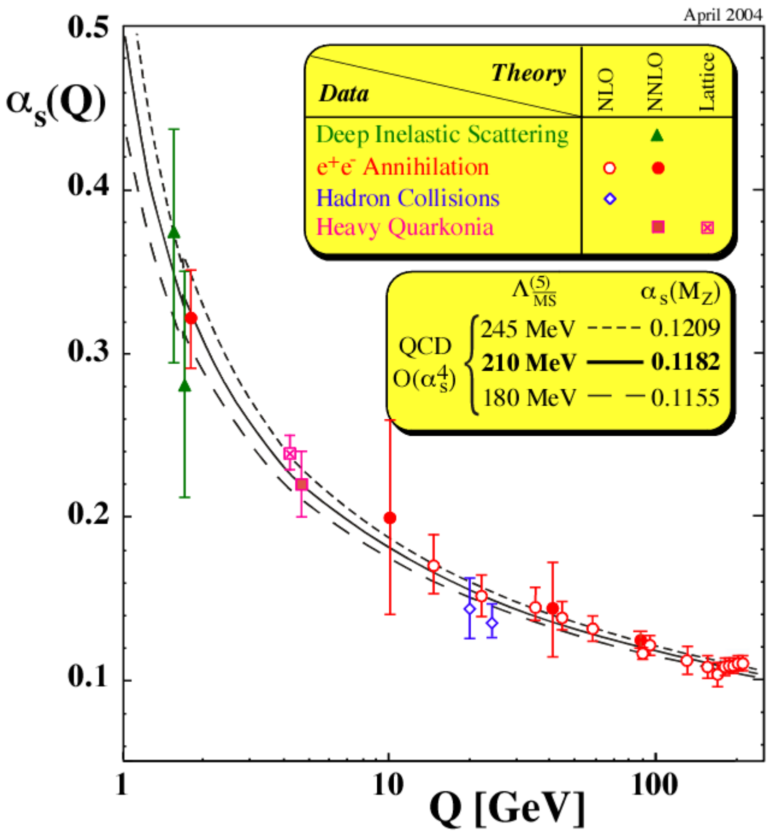
$$\frac{d\widetilde{\alpha}_s}{dt} = -b_0\,\widetilde{\alpha}_s^2 - b_1\,\widetilde{\alpha}_s^3$$

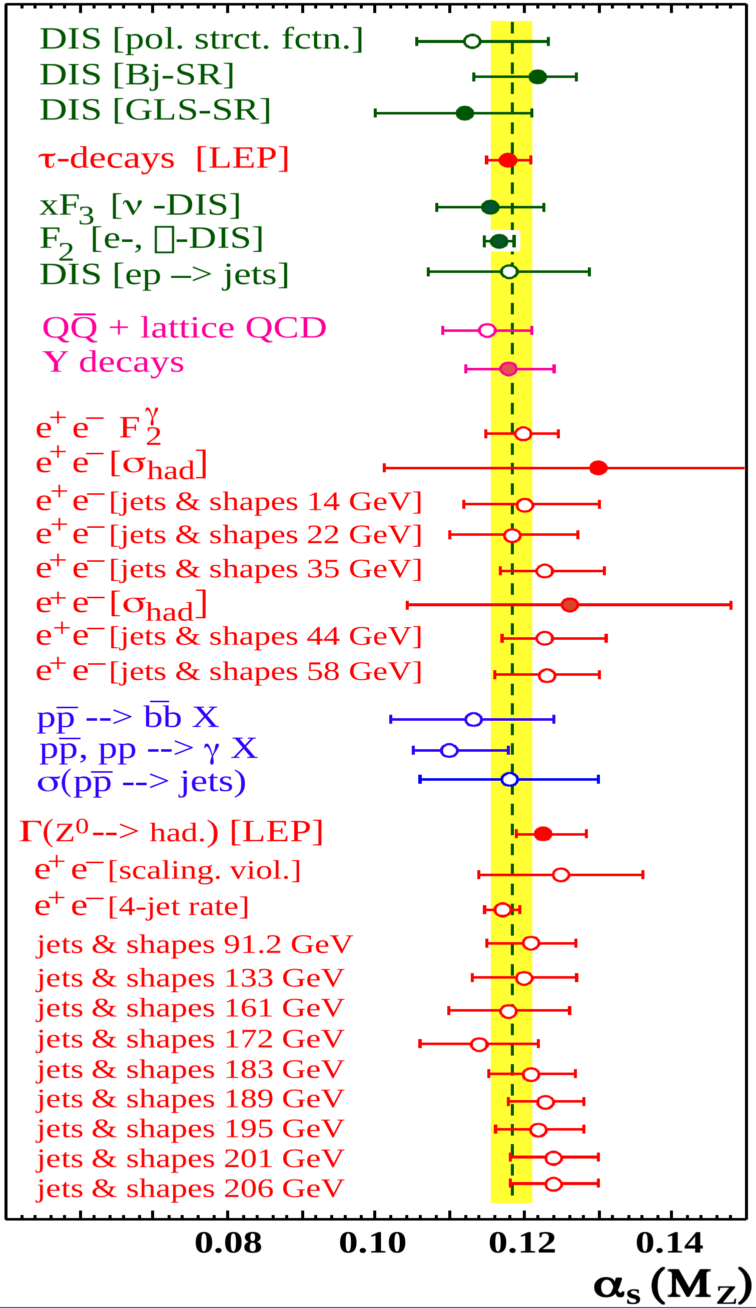
Thus, b_0 and b_1 are scheme independent.

Current experimental results:

$\alpha_s(M_Z) = 0.1182 \pm 0.0027$ in $\overline{\mathrm{MS}}$ at NNLO

[Bethke,hep-ph/0407021]



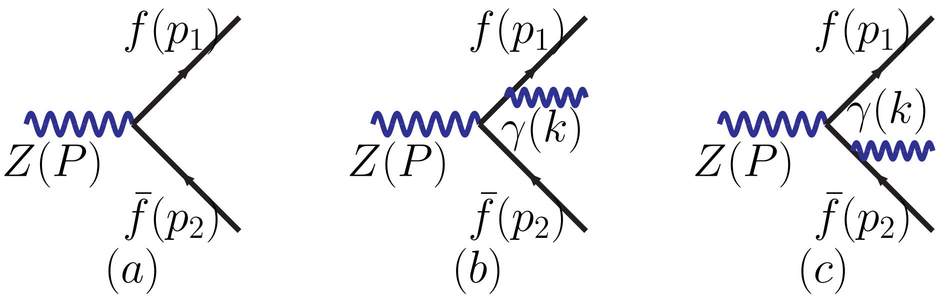


Infrared divergences

Other infinities exist !
Even in high-energy regime, long-distance aspects cannot be ignored.

Soft or collinear gluon emission $\Longrightarrow$ infrared divergences in perturbation theory.
Light quarks ($m_q \ll \Lambda_{QCD}$) $\Longrightarrow$ divergences in the $m_q \to 0$ limit (mass singularities).

Infrared and Collinear Divergences:



2-body :

$$\mathcal{M}_{Z\to f\bar{f}} = \frac{g}{2\,\cos\theta_W}\,\epsilon^\mu(P)\,\bar{u}(p_1)\,\gamma_\mu\,(v_f + a_f\,\gamma_5)\,v(p_2)\;.$$

3-body :

$$\mathcal{M}_{Z\to f\bar{f}\gamma} \;=\; \frac{-g\,e\,Q_f}{4\,\cos\theta_W}\,\epsilon^\mu(P)\,\epsilon^\nu(k)\bar{u}(p_1)\,\Big[\gamma_\nu\,\frac{1}{\not{p}_1 + \not{k} - m}\,\gamma_\mu\,(v_f + a_f\,\gamma_5) + \gamma_\mu\,(v_f + a_f\,\gamma_5)\,\frac{-1}{\not{p}_2 + \not{k} + m}\gamma_\nu\Big]\,v(p_2)$$

Use

$$\bar{u}(p_1)\,\gamma_\nu\,\frac{1}{\not{p}_1 + \not{k} - m} \;=\; \bar{u}(p_1)\,\gamma_\nu\,\frac{\not{p}_1 + \not{k} + m}{2\,p_1\cdot k} = \bar{u}(p_1)\,\frac{2\,p_{1\nu} + \gamma_\nu\,\not{k}}{2\,p_1\cdot k}$$

For soft photons, neglect the terms proportional to k^μ in the numerator:

$$\mathcal{M}_{Z\to f\bar{f}\gamma} \approx \frac{-g\,e\,Q_f}{4\,\cos\theta_W}\,\epsilon^\mu(P)\,\epsilon^\nu(k)\left[\frac{p_{1\nu}}{p_1\cdot k}-\frac{p_{2\nu}}{p_2\cdot k}\right]\bar{u}(p_1)\,\gamma_\mu\,(v_f+a_f\,\gamma_5)\,v(p_2)$$

$$= -e\,Q_f\,\epsilon^\nu(k)\left[\frac{p_{1\nu}}{p_1\cdot k}-\frac{p_{2\nu}}{p_2\cdot k}\right]\mathcal{M}_{Z\to f\bar{f}}\,.$$

Diverges for soft photons ($k^\mu \to 0$).

Velocity of f : β_f angle with photon : $\theta_{f\gamma}$

$$p_1\cdot k = E_f\,k_0\,(1-\beta_f\,\cos\theta_{f\gamma})$$

$\mathcal{M}_{Z\to f\bar{f}\gamma}$ diverges when

- photon is soft ($k_0 = 0$)
- For massless fermions, photon collinear with f or $\bar{f}$.

$\Gamma(Z\to f\bar{f}) \ll \Gamma(Z\to f\bar{f}\gamma)$ even accounting for the extra factor of α_{em}.

Once all such higher order decays are taken into consideration,

$$Br(Z \to f\bar{f}) \approx 0$$

for the exclusive decay mode!

Costs virtually nothing to radiate soft or collinear photons.

In an unbroken gauge theory, only gauge invariant states can exist as asymptotic states.

For an appropriately defined (inclusive) gauge-invariant state, the branching fraction would be finite and non-zero.

Origin of this infinity related to that in the lowest order QED expression for the annihilation of a massless electron-positron pair, viz.

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to \gamma\gamma) \sim \frac{1+\cos^2\theta}{\sin^2\theta}\ . \qquad (1)$$

Infinities vanish once higher order effects are taken into account.

Amplitude for two-body decay $Z \to f\,\bar{f}$, but at one-loop.

$$i\,\Gamma_\mu^{(1)} \;=\; \frac{g\,e^2\,Q_f^2}{4\,\cos\theta_W}\int\frac{d^4k}{(2\,\pi)^4}\,(i\,\gamma_\rho)\,\frac{i}{\not{p}_1+\not{k}-m}\,\gamma_\mu\,(v_f+a_f\,\gamma_5)\,\frac{i}{\not{k}-\not{p}_2-m}\,(i\,\gamma_\eta)\,\frac{-i\,g^{\rho\eta}}{k^2}$$

- UV divergence (as $k_\mu \to \infty$) : absorbed through counterterms;
- Infrared and/or collinear divergence as $k_\mu \to 0$ or when k_μ parallel to either of external momenta.

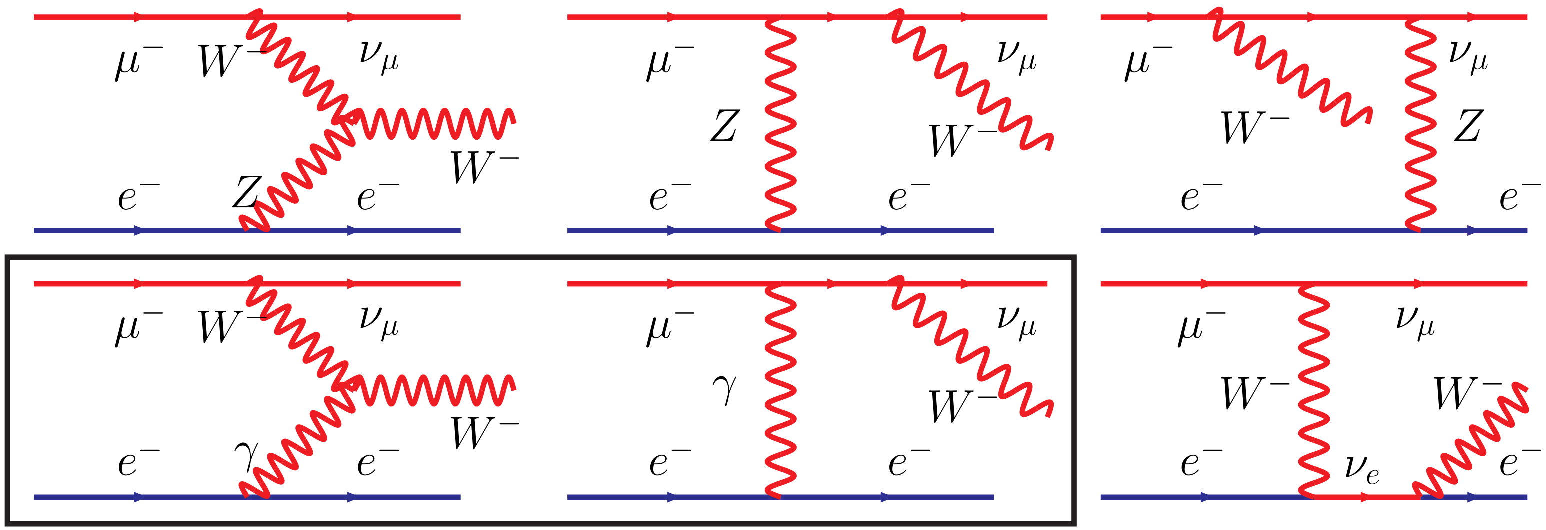
$$i\,\Gamma_\mu^{(1)} \;=\; \frac{-i\,g\,e^2\,Q_f^2}{4\,\cos\theta_W}\int\frac{d^4k}{(2\,\pi)^4}\,\gamma_\rho\,\frac{\not{p}_1+\not{k}}{2\,p_1\cdot k}\,\gamma_\mu\,(v_f+a_f\,\gamma_5)\,\frac{\not{k}-\not{p}_2}{-2\,p_2\cdot k}\,\gamma^\rho\,\frac{1}{k^2}$$

- Interference between $\Gamma_\mu^{(1)}$ and $\Gamma_\mu^{(0)}$: $\mathcal{O}(\alpha)$ correction to the tree level exclusive decay $Z \to f\,\bar{f}$, with a soft/collinear divergence.
- Divergence similar to that in the exclusive $Z \to f\,\bar{f}\gamma$ process.

$$i\,\Gamma_\mu^{(1)} \;\sim\; \frac{-2\,e^2\,Q_f^2}{4\,\cos\theta_W}\int\frac{d^4k}{(2\,\pi)^4}\,\frac{\not{k}-\not{p}_2}{-2\,p_2\cdot k}\,(-i\,g)\,\gamma_\mu\,(v_f+a_f\,\gamma_5)\,\frac{\not{p}_1+\not{k}}{2\,p_1\cdot k}\,\frac{1}{k^2}$$

- Cannot be differentiated in a physical context.
  Soft/collinear photon not registered as separate entities by any realistic detector.
- Only inclusive processes are physically meaningful.
  Bloch-Nordsieck theorem: all IR and collinear divergences cancel (once this inclusive set is considered).
- **QCD : Kinoshita-Lee-Nauenberg theorem.**

A more complicated example : $e^- + \mu^- \to e^- + W^- + \nu_\mu$



Dominating diagrams : γ in the "t-channel". Collinear photons

Need to treat this carefully keeping track of m_e.

Treat the electron as a "source" for the γ !! (Weizsäcker-Williams spectrum)

Starting with an electron of energy E, (Tutorial)

$$\text{Prob(collinear photon of energy } x\,E) = P_{\gamma/e}(x) = \frac{\alpha}{2\pi}\,\frac{1+(1-x)^2}{x}\,\ln\frac{E^2}{m_e^2},$$

x^{-1} : infrared behaviour of γ m_e : regularizes collinear singularity

Dominant contribution :

$$\sigma(e^-\mu \to e^- X) \approx \int dx\, P_{\gamma/e}(x)\, \sigma(\gamma\mu \to X).$$

Effective photon approximation.

Basic Philosophy of the Parton Model:

If q_μ be momentum transfer in electon-hadron scattering,

$$\sigma_{eh}(q, P) = \sum_{\mathcal{P} \in \text{partons}} \int_0^1 dx\, \sigma_{e\mathcal{P}}(q, xP)\, f_{\mathcal{P}/h}(x)$$

- $\sigma_{eh}(q, P)$: inclusive cross section $e(k) + h(P) \to e(k - q) + X(p + q)$
- $\sigma_{e\mathcal{P}}(q, xP)$:
  elastic cross section for $e(k) + \mathcal{P}(xP) \to e(k - q) + \mathcal{A}(xP + q)$
  where $\mathcal{A}$ is massless : $(xP + q)^2 = 0 \quad \Longrightarrow \quad x = \dfrac{-q^2}{2\,P \cdot q} = \dfrac{Q^2}{2\,P \cdot q}$
- $f_{\mathcal{P}/h}(x)$ is the distribution of parton $\mathcal{A}$ in hadron h
  probability for a parton of type $\mathcal{A}$ to have momentum xP
  Independent of details of the hard scattering
- the hallmark of factorization.
- Hadronic *inelastic* c.s. $\equiv$ convolution of partonic *elastic* c.s. with parton distributions
- Quantum mechanical incoherence of large-q scattering and the partonic distributions !
  Multiply probabilities without adding amplitudes.
- Justification:
  nucleon binding : long-time processes
  do not interfere with short-distance scattering.

- Approximately factorize a $2 \to n$ process with massless propagator (t/u-channel) in terms of a $2 \to (n-1)$ process.
- $P_{\rm split}$
  - $\propto g_s^2\, C$ where C is a colour factor.
  - function of the fraction x of the momentum of the parent particle that the daughter carries
  - possibly additional dependence on the spin polarizations of the parent and the two daughters
- In the *collinear* limit, may approximate the *leading* behaviour :

$$|\mathcal{M}_n|^2 \approx \left[\frac{4\, g_s^2}{t}\, C\, f(x;\{s_i\}) + \text{non-singular terms}\right]\, |\mathcal{M}_{n-1}|^2$$

  'Mandelstam variable' t : nothing but the propagator

- Singularity structure cannot depend on the azimuthal angle ϕ.
- Averaging over ϕ and any other unmeasured attribute $\Longrightarrow$ 'Splitting function'

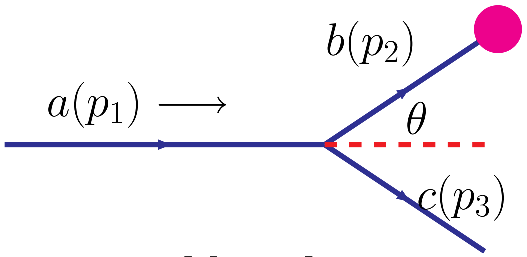
$$\hat{P}(z) \equiv \sum_{s_i} \int \frac{d\phi}{2\pi} f(z;\{s_i\})\ .$$

- $\hat{P}(z)$: dynamics-driven ; determined through explicit computations.

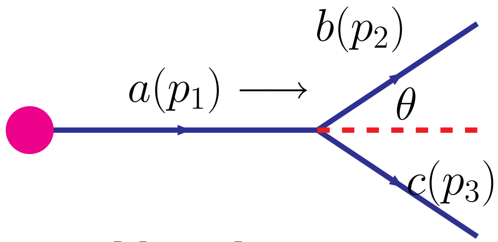
- In terms of phase space elements,

$$d\sigma_n = d\sigma_{n-1}\,\frac{dt}{t}\,dx\,\frac{\alpha_s}{2\pi}\,\hat{P}(x)\ .$$

- Approximation has served to eliminate one phase space integration.



Spacelike splitting



Timelike splitting

Spacelike splitting

gluon splitting on incoming line (a)

$$p_2^2 = -2\,E_1\,E_3\,(1 - \cos\theta).$$

Propagator ($1/p_2^2$) diverges as $E_3 \to 0$ (soft singularity), or $\theta \to 0$ (collinear or mass singularity).

If "a" were a massive quark,

$$p_2^2 - m_q^2 = -2\,E_1\,E_3\,(1 - v_1\,\cos\theta)$$

$E_3 \to 0$ (soft) divergence remains.

Collinear enhancement; becomes divergence as $v_1 \to 1$ ($m_q \to 0$)

If emitted parton c is a quark, vertex factor cancels $E_3 \to 0$ divergence.

Timelike splitting

splitting on outgoing line (b)

$$p_1^2 = 2\,E_2\,E_3\,(1 - \cos\theta).$$

Propagator diverges when either emitted parton is soft ($E_{2,3} = 0$) or when opening angle $\theta = 0$

If b and/or c are quarks, collinear/mass singularity as $m_q \to 0$.

Soft quark divergences cancelled by vertex factor.

- Loop diagrams :
  possibility of soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
  $\Longrightarrow$ infrared divergences in loop diagrams.

- IR divergences indicate dependence on long-distance aspects of QCD (not described by PT).
- Divergent propagators imply propagation of partons over long distances.
- For distances comparable with hadron size, quasi-free partons (PT) are confined/hadronized (non-perturbative), and apparent divergences disappear.

Can still use PT, provided we limit ourselves to

- Infrared safe quantities
  IR divergences either cancel between real and virtual contributions or are removed by kinematic factors.

  Determined primarily by hard, short-distance physics;

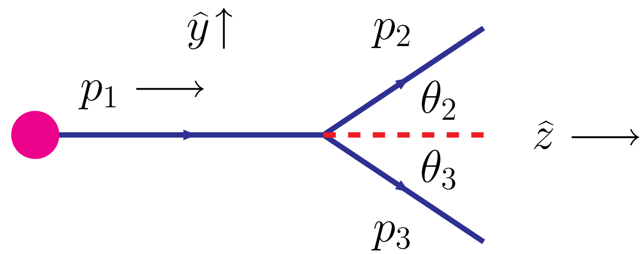
  long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
- Factorizable quantities,
  where IR sensitivity can be absorbed into overall non-perturbative factor (determined experimentally).

However, IR divergences must be **regularized** in PT
(irrespective of whether they cancel or factorize finally)

Ways to regularize:

- Gluon mass regularization: breaks gauge invariance.
- Dimensional regularization: must **increase** dimension of space-time, $\epsilon = 4 - D < 0$.
  Divergences are replaced by powers of $1/\epsilon$.

Branching probabilities



Matrix element squared for $(n+1)$ partons
in the small-angle region
in terms of MEsq for n partons

$$|\mathcal{M}_{n+1}|^2 = \left[\frac{g_s^2}{t}\, C\, F\big(z;\{s_1,s_2,s_3\}\big) + \text{nonsingular terms}\right] |\mathcal{M}_n|^2$$

C : colour factor
F : momentum dependence of the branching probabilities.
$t \equiv p_1^2$

After azimuthal averaging, **splitting functions**

$$\widehat{P}_{ba}(z) = \sum_{s_1,s_2,s_3} \int \frac{d\phi}{2\,\pi}\, F\big(z;\{s_1,s_2,s_3\}\big)$$

and

$$d\sigma_{n+1} = d\sigma_n\, \frac{dt}{t}\, dz\, \frac{\alpha_s}{2\,\pi}\, \widehat{P}_{ba}(z)$$

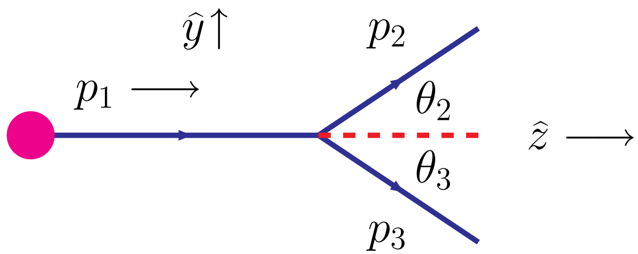
Kinematics:
All momenta defined as outgoing ($p_1 + p_2 + p_3 = 0$)

$$p_1^\mu = \left(E_1 + \frac{p_1^2}{4\,E_1}, 0, 0, E_1 - \frac{p_1^2}{4\,E_1}\right) ,$$

$$p_2^\mu = (E_2, 0, E_2\,\sin\theta_2\,, E_2\,\cos\theta_2)\ ,$$
$$p_3^\mu = (E_3, 0, -E_3\,\sin\theta_3\,, E_3\,\cos\theta_3)$$

Assuming that $p_2^2\,, p_3^2 \ll p_1^2 \equiv t$

parton (1) is outgoing
$\Longrightarrow$ timelike branching/splitting.
Opening angle : $\theta = \theta_2 + \theta_3$



Energy fraction

$$z \equiv \frac{E_2}{E_1} \equiv 1 - \frac{E_3}{E_1}$$

For small angles,

$$t = 2\, E_2\, E_3\, (1 - \cos\theta) = z\,(1-z)\, E_1^2\, \theta^2 \qquad \Longrightarrow \qquad \theta = E_1^{-1} \sqrt{\frac{t}{z\,(1-z)}}$$

Transverse momentum conservation:

$$\theta = E_1^{-1} \sqrt{\frac{t}{z\,(1-z)}} = \frac{\theta_2}{1-z} = \frac{\theta_3}{z}$$

Consider case of all three partons being gluons. Triple-gluon vertex :

$$g_s\, f^{abc}\, [g_{\mu\nu}\,(p_1 - p_2)_\lambda + g_{\nu\lambda}\,(p_2 - p_3)_\mu + g_{\lambda\mu}\,(p_3 - p_1)_\mu]$$

To be multiplied by t^{-1} (propagator), and polarization vectors $\epsilon_1^\mu\, \epsilon_2^\nu, \epsilon_3^\lambda$.

Using $\epsilon_i \cdot p_i = 0$,

$$V_{3g} = -2\, g_s\, f^{abc}\, [\epsilon_1 \cdot \epsilon_2\; \epsilon_3 \cdot p_2 - \epsilon_2 \cdot \epsilon_3\; \epsilon_1 \cdot p_2 - \epsilon_1 \cdot \epsilon_3\; \epsilon_2 \cdot p_3]$$

Gluons almost on mass-shell $\Longrightarrow$ polarization vectors nearly pure transverse.

Resolve into plane polarization states:

ϵ_i^I : in the plane of branching (y-z)

ϵ_i^O : out of the plane of branching

$$\epsilon_i^I \cdot \epsilon_j^I = \epsilon_i^O \cdot \epsilon_j^O = -1 \qquad \epsilon_i^I \cdot \epsilon_j^O = \epsilon_i^I \cdot p_j = 0$$

To $\mathcal{O}(\theta)$,

$$\epsilon_1^I \cdot p_2 = -E_2\,\theta_2 = -z\,(1-z)\,E_1\,\theta \qquad \epsilon_2^I \cdot p_3 = E_3\,\theta = (1-z)\,E_1\,\theta \qquad \epsilon_3^I \cdot p_2 = -E_2\,\theta = -z\,E_1\,\theta$$

Each combination $\propto \theta$. But propagator has $t^{-1} \propto \theta^{-2}$.

$\Longrightarrow$ θ^{-1} singularity in the amplitude

In small angle regime,

$$|\mathcal{M}_{n+1}|^2 = \left[\frac{4\,g_s^2}{t}\,C_A\,F(z;\{\epsilon_1,\epsilon_2,\epsilon_3\}) + \text{nonsingular terms}\right]\,|\mathcal{M}_n|^2$$

where $C_A = 3$ ($f^{abc}\,f^{dbc} = C_A\,\delta^{ad}$) and

F : put above dot products into V_{3g}

F:

ϵ_1	ϵ_2	ϵ_3	$F(z;\{\epsilon_1,\epsilon_2,\epsilon_3\})$
in	in	in	$(1-z)/z + z/(1-z) + z\,(1-z)$
in	out	out	$z(1-z)$
out	in	out	$(1-z)/z$
out	out	in	$z/(1-z)$

Others zero.

Average with respect to ϵ_1 and sum over $\epsilon_{2,3}$:

$$\widehat{P}_{gg}(z) = C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$

unregularized gluon splitting function.

Are neglecting angular correlations
(configurations in which the polarization of the branching gluon lies in plane of branching).

Quite weak: coefficient $(1-z)$ vanishes in the enhanced regions $z \to 0, 1$

Maximum at $z = 1$ (still only $1/9$ of the unpolarized contribution).

Splitting function for fermions

Using Weyl representation of Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Solution of Dirac eqn:

$$u_+(p) \equiv u_R(p) = \left[\sqrt{p_+}\,,\sqrt{p_-}\,e^{i\theta_p}\,,0\,,0\,\right]^T \qquad u_-(p) \equiv u_L(p) = \left[0\,,0\,,\sqrt{p_-}\,e^{-i\theta_p}\,,-\sqrt{p_+}\,\right]^T$$

where

$$p_\pm = p^0 \pm p^3 \qquad e^{\pm i\theta_p} = \frac{p^1 \pm i\,p^2}{\sqrt{p_1^2 + p_2^2}} = \frac{p^1 \pm i\,p^2}{\sqrt{p_+ p_-}}$$

and normalization is

$$u_\pm^\dagger(p)\,u_\pm(p) = 2\,p_0$$

Gives

$$\overline{u_+}(p) = \left[0\,,0\,,\sqrt{p_+}\,,\sqrt{p_-}\,e^{-i\theta_p}\right]^T \qquad \overline{u_-}(p) = \left[\sqrt{p_-}\,e^{i\theta_p}\,,-\sqrt{p_+}\,,0\,,0\,\right]^T$$

Note

$$\overline{u_+}(p_i)\,u_+(p_j) = 0 = \overline{u_-}(p_i)\,u_-(p_j)$$

Not so for $\overline{u_+}(p_i)\,u_-(p_j)$ or $\overline{u_-}(p_i)\,u_+(p_j)$

Consider $g(p_1) \longrightarrow q(p_2) + \bar{q}(p_3)$ where

$$p_1 = \left(E_1 + \frac{p_1^2}{4E_1}, 0, 0, E_1 - \frac{p_1^2}{4E_1}\right) \;, p_2 = (E_2, E_2\,\theta_2, 0, E_2) \;\;, p_3 = (E_3, -E_3\,\theta_3, 0, E_3)$$

Then,

$$u_+(p_3) = v_-(p_3) = \sqrt{2\,E_3}\left[1\,, \frac{-\theta_3}{2}\,, 0\,, 0\right]^T \,, \qquad u_+^\dagger(p_2) = \sqrt{2\,E_2}\left[1\,, \frac{\theta_2}{2}\,, 0\,, 0\right]$$

and the polarization vector choices are

$$\epsilon_1 = (0,1,0,0) \qquad \text{(in plane)} \qquad\qquad \epsilon_2 = (0,0,1,0) \qquad \text{(out of plane)}$$

The interaction vertex is $g_s\,\bar{u}_2\,\gamma_\mu\,T_a\,v_3\,\epsilon^\mu$

Then,

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g_s^2}{t}\,T_R\,F(z;\epsilon_1\,\lambda_2,\lambda_3)\,|\mathcal{M}_n|^2$$

where $T_R = tr(T_a\,T_a)/8 = 1/2$ and

ϵ_1	λ_2	λ_3	$F(z;\{\epsilon_1,\lambda_2,\lambda_3\})$
1	$\pm$	$\mp$	$(1-2z)^2$
2	$\pm$	$\mp$	1

Summing (averaging) over polarizations,

$$\hat{P}_{qg} = T_R\,\frac{(1-2\,z)^2+1}{2} = \frac{z^2+(1-z)^2}{2}$$

In this case, strong anticorrelation between the polarization and the plane.

Relations between Splitting Functions

Consider, for example, the splitting

$$\Pi_1(P) \longrightarrow \Pi_2(x\,P) + \Pi_3\,((1-x)\,P) \ \ .$$

Hence,

$$\hat{P}_{\Pi_1\to\Pi_2}(x) = \hat{P}_{\Pi_1\to\Pi_3}(1-x)\ .$$

Isolate the colour factors and write $\hat{P}_i(x)$ in terms of "reduced" splitting functions $\hat{P}_i^R(x)$.

$$\begin{aligned} \hat{P}_{q\to q}(x) &= C_F\,\hat{P}^R_{q\to q}(x) & \hat{P}_{q\to g}(x) &= C_F\,\hat{P}^R_{q\to g}(x) \\ \hat{P}_{g\to q}(x) &= T_R\,\hat{P}^R_{g\to q}(x) & \hat{P}_{g\to g}(x) &= N_c\,\hat{P}^R_{g\to g}(x) \end{aligned}$$

Crossing symmetry : $q_i \longrightarrow g + X$ splitting $\Longleftrightarrow$ $g \longrightarrow \bar{q}_i + X$ splitting

$$\hat{P}^R_{\Pi_2\to\Pi_1}(x) = (-1)^{2\,s_1+2\,s_2+1}\ \ x\ \ \hat{P}^R_{\Pi_1\to\Pi_2}(x^{-1})\ ,$$

s_i : spins of the partons.

Paradoxical? ($x < 1 \Longrightarrow x^{-1} > 1$):

The two processes have opposite temporal ordering !

Consider pure susy-QCD : only gluons and gluinos

all massless and part of the same vector superfield.

Four possible splittings;

Expressible in terms of supergraphs;

SUSY relates all the four reduced splitting functions

$$\hat{P}^R_{g\to\tilde{g}}(x) + \hat{P}^R_{g\to g}(x) = \hat{P}^R_{\tilde{g}\to g}(x) + \hat{P}^R_{\tilde{g}\to\tilde{g}}(x)$$

Relation independent of the fermion representation!

Generalizes to real QCD (in the massless limit)

$$\boxed{\hat{P}^R_{g\to q}(x) + \hat{P}^R_{g\to g}(x) = \hat{P}^R_{q\to g}(x) + \hat{P}^R_{q\to q}(x)\ .}$$

Thus,

- Weizsäcker-Williams calculation $\Longrightarrow \hat{P}^R_{q\to g}(x)$
- $\Longrightarrow \hat{P}^R_{q\to q}(x)$.
- Crossing symmetry : $\hat{P}^R_{g\to q}(x)$
- SUSY : $\hat{P}^R_{g\to g}(x)$.
- Put in colour factors.

Do not need to calculate anything beyond QED!

Finally, the unregulated branching probabilities:

$$\widehat{P}_{gg}(z) = C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$

$$\widehat{P}_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right]$$

$$\widehat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right]$$

$$\widehat{P}_{qq}(z) = C_F\,\frac{1+z^2}{1-z}$$

$C_F = 4/3\,, C_A = 3\,, T_R = 1/2$

Unregulated because they contain singularities. (bad things)