Parton Distributions : A simple-minded Introduction

Lecture 2

Sangam–2015@HRI

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February 18, 2016

Finally, the unregulated branching probabilities:

$$
\begin{aligned}
\widehat{P}_{gg}(z) &= C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] \\
\widehat{P}_{gq}(z) &= C_F \left[\frac{1+(1-z)^2}{z}\right] \\
\widehat{P}_{qg}(z) &= T_R \left[z^2 + (1-z)^2\right] \\
\widehat{P}_{qq}(z) &= C_F \, \frac{1+z^2}{1-z}
\end{aligned}
$$

$C_F = 4/3\,, C_A = 3\,, T_R = 1/2$

Unregulated because they contain singularities. (bad things)

- Higher-order contributions ? Suppression by more powers of α_s
  But multiple small-angle parton emissions $\longrightarrow$ enhancement!!
- Consider DIS. Parton from target hadron :
  Starts with, say, a fraction x_0 of the hadron energy and
  is associated with a momentum transfer $t_0\,(\equiv -Q^2)$
- First radiation : left with (x_1, t_1) Then $(x_2, t_2)\cdots$
- If hard scattering at stage (x_n, t_n),
  cross section will depend on momentum fraction distribution of partons
  seen by virtual photon at this scale
  $f(x, Q^2)$ where $x = x_n, \quad Q^2 = -t_n$
- How to calculate this ?
- Consider change in the parton distribution $f(x,t)$ when (x,t) changed to $(x+\delta x, t+\delta t)$.

$$\frac{N[\text{partons arriving in element } (x,t)] - N[\text{partons leaving in element } (x,t)]}{\delta x}$$

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$$\frac{N[\text{partons arriving in element } (x, t)] - N[\text{partons leaving in element } (x, t)]}{\delta x}$$

- N[arriving] : [branching probability] $\times$ [parton density] integrated over all momenta $y > x$

$$\begin{aligned} N[\text{arriving}] &= \delta t \int_x^1 dy \int_0^1 dz \left[\frac{\alpha_s}{2\,\pi\,t}\,\widehat{P}(z)\right] f(y, t)\,\delta(x - z\,y) \\ &= \frac{\delta t}{t}\,\frac{\alpha_s}{2\,\pi}\int_0^1 \frac{dz}{z}\,\widehat{P}(z)\,f\left(\frac{x}{z}, t\right) \end{aligned}$$

- N[leaving] :[current density] × [branching prob.] integrated over all momenta $y < x$

$$N[\mathrm{leaving}] \;=\; \delta t \;\int_0^x dy \int_0^1 dz \;\left[\frac{\alpha_s}{2\,\pi\,t}\;\widehat{P}(z)\right] f(x,t)\,\delta(y - z\,x)$$

$$=\; \frac{\delta t}{t}\,\frac{\alpha_s}{2\,\pi}\,f(x,t)\;\int_0^1 \frac{dz}{z}\;\widehat{P}(z)$$

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$$N[\text{leaving}] \;=\; \delta t \int_0^x dy \int_0^1 dz \left[\frac{\alpha_s}{2\,\pi\,t}\,\widehat{P}(z)\right] f(x,t)\,\delta(y - z\,x)$$
$$= \frac{\delta t}{t}\,\frac{\alpha_s}{2\,\pi}\,f(x,t)\,\int_0^1 \frac{dz}{z}\,\widehat{P}(z)$$

- Change in distribution in the element :

$$\delta f(x,t) = N[\text{arriving}] - N[\text{leaving}] \;=\; \frac{\delta t}{t}\,\frac{\alpha_s}{2\,\pi}\int_0^1 dz\,\widehat{P}(z)\left[\frac{1}{z}\,f\left(\frac{x}{z},t\right) - f(x,t)\right]$$
$$= \frac{\delta t}{t}\,\frac{\alpha_s}{2\,\pi}\int_0^1 dz\,\widehat{P}(z)\left[\frac{1}{z}\,f\left(\frac{x}{z},t\right) - \left\{\frac{1}{z}\,f\left(\frac{x}{z},t\right)\right\}_{z=1}\right]$$

- Plus-prescription:

$$\int dx\;h(x)\;f(x)_+ \equiv \int dx\;[h(x) - h(1)]\;f(x)$$

Defined only under integral sign.

- Includes some of the effects of virtual diagrams

- Regularized splitting function : $P(z) = \widehat{P}(z)_+$

- Then

$$t\,\frac{\partial f(x,t)}{\partial t} = \frac{\alpha_s}{2\,\pi}\int_0^1 \frac{dz}{z}\,P(z)\,f\left(\frac{x}{z},t\right)$$
$$= \frac{\alpha_s}{2\,\pi}\int_x^1 \frac{dz}{z}\,P(z)\,f\left(\frac{x}{z},t\right)$$

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation.

- For timelike branching,
$f(x,t)$ represents **hadron momentum fraction distribution** produced by an outgoing parton.

Evolution eqn. remains the same, but boundary conditions and direction of evolution different

- Several types of partons $\Longrightarrow$ must take into account different processes
coupled DGLAP equations

$$t\,\frac{\partial f_i(x,t)}{\partial t} = \frac{\alpha_s}{2\,\pi}\sum_j \int_x^1 \frac{dz}{z}\,P_{ij}(z)\,f_j\left(\frac{x}{z},t\right)$$

- Use data to fix $f_i(x,t)$.
Usually by taking moments (Mellin transforms).

Scale dependence of PDFs : Relook

Consider DIS : $\gamma^* q \longrightarrow q$

Tree level :

$$x^{-1}\, F_2^{\mathrm{LO}}(x,Q^2) = \sum_q e_q^2 \int dy\, f_q(y)\, \delta\left(y - \frac{Q^2}{2\,p\cdot q}\right) = \sum_q e_q^2\, f_q(x)$$

- One-loop QCD (virtual, after UV renorm):

$$x^{-1}\, F_2^{\mathrm{virt.}}(x,Q^2) \;=\; \frac{-\alpha_s\, C_F}{2\,\pi}\left[\frac{4\,\pi\,\mu^2}{Q^2}\right]^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\,\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8+\frac{\pi^2}{3}\right)$$
$$\sum_q e_q^2 \int dydz\, f_q(y)\, \delta\,(y-z)\;\delta(x-z\,y)$$

Includes both soft and collinear divergences.

- $\gamma^* q \longrightarrow qg$ (real radiation)

$$x^{-1}\, F_2^{\mathrm{real}}(x,Q^2) \;=\; \frac{\alpha_s\, C_F}{2\,\pi} \left[\frac{4\,\pi\,\mu^2}{Q^2}\right]^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\,\epsilon)}$$

$$\sum_q e_q^2 \int dydz\; f_q(y)\,\delta(x - z\,y)$$

$$\left\{\left(\frac{2}{\epsilon^2} + \frac{3}{2\,\epsilon} + \frac{7}{2}\right)\,\delta(1-z) + 3 + 2\,z - \frac{1+z^2}{1-z}\,\ln z \right.$$

$$\left. -\left(\frac{1+z^2}{\epsilon} + \frac{3}{2}\right)\left[\frac{1}{1-z}\right]_+ + (1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ \right\}$$

Again, includes both soft and collinear divergences.

- Add all three

$$x^{-1}\, F_2^{\mathrm{NLO}}(x,Q^2) \;=\; \sum_q e_q^2 \int_x^1 \frac{dy}{y}\, f_q(y) \left[\delta(1-x/y) - \frac{\alpha_s}{2\,\pi}\, P_{qq}(x/y)\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{Q^2} + \mathrm{finite}\right)\right]$$

$$P_{qq}(z) \;\equiv\; C_F \left\{(1+z^2)\left[\frac{1}{1-z}\right]_+ + \frac{3}{2}\,\delta(1-z)\right\}$$

ϵ^{-1} : Unbalanced IR collinear divergence!

- But, consider the difference

$$x^{-1}\left[F_2^{\rm NLO}(x,Q^2)-F_2^{\rm NLO}(x,Q_0^2)\right]=\frac{\alpha_s}{2\,\pi}\sum_q e_q^2\int_x^1\frac{dy}{y}\,f_q(y)\,P_{qq}(x/y)\,\ln\frac{Q^2}{Q_0^2}$$

and is finite !

- Naive (tree-level) definition was

$$x^{-1}\,F_2(x,Q^2)=\sum_q e_q^2\,f_q(x,Q^2)=\sum_q e_q^2\,f_q(x)$$

Same continues !

$$\frac{d\,f_q(x,Q^2)}{d\,\ln Q^2}=\frac{\alpha_s(Q^2)}{2\,\pi}\int_x^1\frac{dy}{y}\,f_q(y,Q^2)\,P_{qq}(x/y)$$

- Have missed something !
  We have considered $\gamma^*+q\longrightarrow q$ and $\gamma^*+q\longrightarrow q+g$.
  Inclusive set also includes $\gamma^*+g\longrightarrow q+\bar{q}$.

- Must add

$$x^{-1}\,F_2^{\rm glue}(x,Q^2) \;=\; \frac{\alpha_s}{2\,\pi}\,\sum_q e_q^2\int dydz\,f_g(y)\,\delta(x-z\,y)$$

$$\left[\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\epsilon}\;\frac{-\Gamma(1-\epsilon)}{\epsilon\,\Gamma(1-2\,\epsilon)}\;\{P_{qg}(z)+P_{\bar{q}g}(z)\}\right.$$
$$\left.+\left\{z^2+(1-z)^2\right\}\left[\ln\frac{1-z}{z}\right]_+ +6\,z\,(1-z)\right]$$

$$P_{qg}(z) \;=\; z^2+(1-z)^2$$

Again, consider

$$x^{-1}\left[F_2^{\rm glue}(x,Q^2)-F_2^{\rm glue}(x,Q_0^2)\right]$$

Finally,

$$\frac{df_q(x,Q^2)}{d\,\ln Q^2} \;=\; \frac{\alpha_s(Q^2)}{2\,\pi}\int_x^1\frac{dy}{y}\left[f_q(y,Q^2)\,P_{qq}(x/y)+f_g(y,Q^2)\,P_{qg}(x/y)\right]$$

$$\frac{df_g(x,Q^2)}{d\,\ln Q^2} \;=\; \frac{\alpha_s(Q^2)}{2\,\pi}\int_x^1\frac{dy}{y}\left[f_g(y,Q^2)\,P_{gg}(x/y)+f_q(y,Q^2)\,P_{gq}(x/y)\right]$$

Solving DGLAP

- $(2\,n_f + 1)$ coupled differential eqns.
- Often, useful to choose different basis
  - $g(x, Q^2)$
  - Singlet : $q_S(x, Q^2) \equiv \sum_{i=1}^{n_f} \left[q_i(x, Q^2) + \bar{q}_i(x, Q^2)\right]$
  - Valence (non-singlet) : $q_V(x, Q^2) \equiv \sum_{i=1}^{n_f} \left[q_i(x, Q^2) - \bar{q}_i(x, Q^2)\right]$
  - Flavour-dep (non-singlet) : $q_{ij}^{\pm}(x, Q^2) \equiv \left[q_i(x, Q^2) \pm \bar{q}_i(x, Q^2)\right] - \left[q_j(x, Q^2) \pm \bar{q}_j(x, Q^2)\right]$
- Then

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q_S \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_S \\ g \end{pmatrix}$$

$$\frac{d\,q_V}{d\ln Q^2} = P_V \otimes q_V$$

$$\frac{d}{d\ln Q^2}\,q_{ij}^{\pm} = P_{\pm} \otimes q_{ij}^{\pm}$$

- P's have perturbative expansions (in α_s)
- At LO, $P_{\pm} = P_V = P_{qq}$

- DGLAP could be solved numerically, either in x-space or in Mellin space.

- Need $(2\,n_f + 1)$ boundary conditions, each as a function of x !

- Unknown *a priori* ! Intrinsically non-perturbative.

- Use data and pQCD.

- Need experimental input to get rid of IR collinear divergences (just like renormalization)

Procedure: (flow chart)

1. Decide on experimental data to be used.
2. Choose order of QCD calculation (same for all processes to be used).
3. Perform hard scattering calculations to this order.
4. Choose functional form of boundary conditions for PDFs.
5. Choose parameters.
6. Gives PDFs at some $Q = Q_0$.
7. Use DGLAP (at that QCD order) to evolve to arbitrary Q.
8. Convolute with hard scattering calculations to get inelastic scattering c.s.
9. Compare with data (say, χ^2)
10. Go to step 5 until χ^2 is minimized.
11. Is χ^2_{min} acceptable? If yes, stop. If not, goto step 4.

- Data used: DIS, ν–DIS, Drell-Yan, Dijet(!) …
- Typically, many different choices give very similar χ^2 and similar PDFs.
- Problem appears in extrapolating !

Points to Remember

Different experiments sensitive to :

- sensitive to different kinematic regions (x, Q^2)
- sensitive to different components: g, q_S, q_V, $q_{ij}^{\pm}$

Neutral Current DIS

HERA & (fixed target) SLAC, NMC, BCDMS

- low Q^2 : dominated by photon exchange
- high Q^2 : Z becomes progressively more imp

$$F_{1,2} = F_{1,2}^{\gamma\gamma} + \frac{Q^2}{Q^2+M_Z^2} F_{1,2}^{\gamma Z} + \frac{Q^4}{(Q^2+M_Z^2)^2} F_{1,2}^{ZZ}$$

$$F_3 = \frac{Q^2}{Q^2+M_Z^2} F_3^{\gamma Z} + \frac{Q^4}{(Q^2+M_Z^2)^2} F_3^{ZZ}$$

where

$$F_2^{\gamma\gamma} = x \sum_q e_q^2 \left[q(x,Q^2) + \bar{q}(x,Q^2)\right]$$

$$F_2^{\gamma Z} = x \sum_q b_i \left[q(x,Q^2) + \bar{q}(x,Q^2)\right]$$

$$F_3^{\gamma Z} = \sum_q c_i \left[q(x,Q^2) - \bar{q}(x,Q^2)\right]$$

- quarks & anti-quarks enter with different weights
- Could do both electron and positron scattering
- Could polarize beams !

Neutrino DIS

$\nu_\mu + N \to \mu^- + X$ (CDHSW, CHORUS, NuTeV)

- Contributes to both $F_2(x, Q^2)$ and $F_3(x, Q^2)$
- different combinations of densities.

$$F_2^\nu = x \sum_q \left[q(x, Q^2) + \bar{q}(x, Q^2) \right]$$

$$F_3^\nu = \sum_q \left[q(x, Q^2) - \bar{q}(x, Q^2) \right]$$

- ν DIS on nucleons difficult (WA21/WA22)
- Could do on nuclei; Nuclear effects ?

Charged Current DIS

Opposite of ν-DIS

$$F_2(W^\pm) = x \left[\bar{u}(x, Q^2) + \bar{c}(x, Q^2) \right] \pm x \left[d(x, Q^2) + s(x, Q^2) \right]$$

Very challenging

What about the gluon in DIS?

- NLO effect
- Scaling violation at small x!
- Violation of Callan-Gross !

  Longitudinal structure function:

$$F_L(x, Q^2) \equiv F_2(x, Q^2) - 2\,x\,F_1(x, Q^2)$$

- Need to use differential c.s.

Drell-Yan

All of $p + p(\bar{p}) \longrightarrow \mu^+ + \mu^-, \mu^+ + \nu_\mu, \mu^- + \bar{\nu}_\mu$

- NC DY : At low parton C.M. energy (away from Z resonance), dominated by photon

$$\frac{d^2\sigma}{dQ^2 dy} = \frac{4\,\pi\,\alpha^2}{9\,s_{hh}\,Q^2} \sum_i e_i^2 \left[q_{i/h_1}(x_1, Q^2)\,\bar{q}_{i/h_2}(x_2, Q^2) + (1 \leftrightarrow 2)\right]$$

$$Q^2 \equiv \left[p(\ell) + p(\bar{\ell})\right]^2$$

$$x_{1,2} \equiv \sqrt{Q^2/s_{hh}}\,\exp(\pm y)$$

- Generic formula more complicated.
- At resonance ($Q^2 = m_Z^2$ or m_W^2)

$$\frac{d}{dy}\sigma_Z = \frac{\pi\,\sqrt{2}\,G_F\,m_Z^2}{3\,s_{hh}} \sum_i e_i^2\,(v_i^2 + a_i^2)\,\left[q_{i/h_1}(x_1, Q^2)\,\bar{q}_{i/h_2}(x_2, Q^2) + (1 \leftrightarrow 2)\right]$$

$$\frac{d}{dy}\sigma_W = \frac{\pi\,\sqrt{2}\,G_F\,m_W^2}{3\,s_{hh}} \sum_{i,j} |\mathrm{CKM}_{ij}|^2\,\left[q_{i/h_1}(x_1, Q^2)\,\bar{q}_{j/h_2}(x_2, Q^2) + (1 \leftrightarrow 2)\right]$$

- Different combinations of structure functions !

Drell-Yan (contd.)

- Particularly good for identifying combinations.
- Use ratios and asymmetries
- For proton-antiproton initiated process,

$$\frac{\sigma(W^+)}{\sigma(W^-)} \longrightarrow \frac{u(x_1)\,\bar{d}(x_2)}{d(x_1)\,\bar{u}(x_2)} \approx \frac{u(x_1)}{d(x_1)}$$

- Similarly,

$$\frac{\sigma(W^+) - \sigma(W^-)}{\sigma(W^+) + \sigma(W-)} \longrightarrow \frac{u_V(x_1) - d_V(x_1)}{u(x_1) + d(x_1)}$$

- And

$$\frac{\sigma(W^+) + \sigma(W^-)}{\sigma(Z)} \longrightarrow \frac{u(x_1) + d(x_1)}{0.29 \times u(x_1) + 0.37 \times d(x_1)}$$

Jet data

- Leading order :

$$
\begin{array}{lll}
qq \to qq \;, & q\bar{q} \to q\bar{q} \;, & q\bar{q} \to gg \;, \\
qg \to qg \;, & \bar{q}g \to \bar{q}g \;, & \\
gg \to gg \;, & gg \to q\bar{q} &
\end{array}
$$

- At high E_T, quark–(anti-)quark subprocesses dominate
- Still, gluon processes imp. enough to constrain large-x gluon PDF
- combine with low-x constraints on gluon PDF from DIS;
- and sum rules !
- Need additional direct probes of gluon PDF for NewPhys@LHC.

PDF parametrizations

Typically,

- $Q_0 \sim m_c$

- $x\, f_k(x, Q_0) \sim A_0\, x^{A_1}\, (1-x)^{A_2}\, P_k(x)$

  $P_k(x)$: polynomial in x or even exponentials

  A_i : species dependent

- MSTW (2009):

$$Q_0 = 1.0 \text{ GeV}$$

$u_V, d_V, s+\bar{s}$,

$$2(\bar{u}+\bar{d})+s+\bar{s} \quad : \quad x\, f_k(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2}\, \left[1 + a_3 \sqrt{x} + a_4\, x\right]$$

$$g \quad : \quad x\, f_k(x, Q_0) = a_5\, x^{a_6}\, (1-x)^{a_7}\, \left[1 + a_8 \sqrt{x} + a_9\, x\right] + a_{10}\, x^{a_{11}}\, (1-x)^{a_{12}}$$

- CTEQ6 (2010):

$$Q_0 = 1.3 \text{ GeV}$$

$$u_V, d_V, g, \bar{u} + \bar{d}: \quad x\, f_k(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2}\, e^{a_3\, x}\, (1 + e^{a_4 x})^{a_5}$$

$$\bar{d}(x, Q_0)/\bar{u}(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2} + (1 + a_3\, x)\, (1-x)^{a}_{4}$$

$$s(x, Q_0) = \bar{s}(x, Q_0) = 0.2\, \left[\bar{u}(x, Q_0) + \bar{d}(x, Q_0)\right]$$

- CTEQ6.5 (2015):

$$Q_0 = 1.3 \text{ GeV}$$

$$u_V, d_V, g, : \quad x\, f_k(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2}\, \exp\left[-a_3\, (1-x)^2 + a_4\, x^2\right]$$

$$\bar{u} + \bar{d}: \quad x\, f_k(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2}\, e^{a_3\, x}\, (1 + e^{a_4 x})^{a_5}$$

$$\bar{d}(x, Q_0)/\bar{u}(x, Q_0) = a_0\, x^{a_1}\, (1-x)^{a_2} + (1 + a_3\, x)\, (1-x)^{a}_{4}$$

$$s(x, Q_0) = \bar{s}(x, Q_0) = \kappa\, \left[\bar{u}(x, Q_0) + \bar{d}(x, Q_0)\right]$$