Fundamentals of signals and noise

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Plan of this lecture

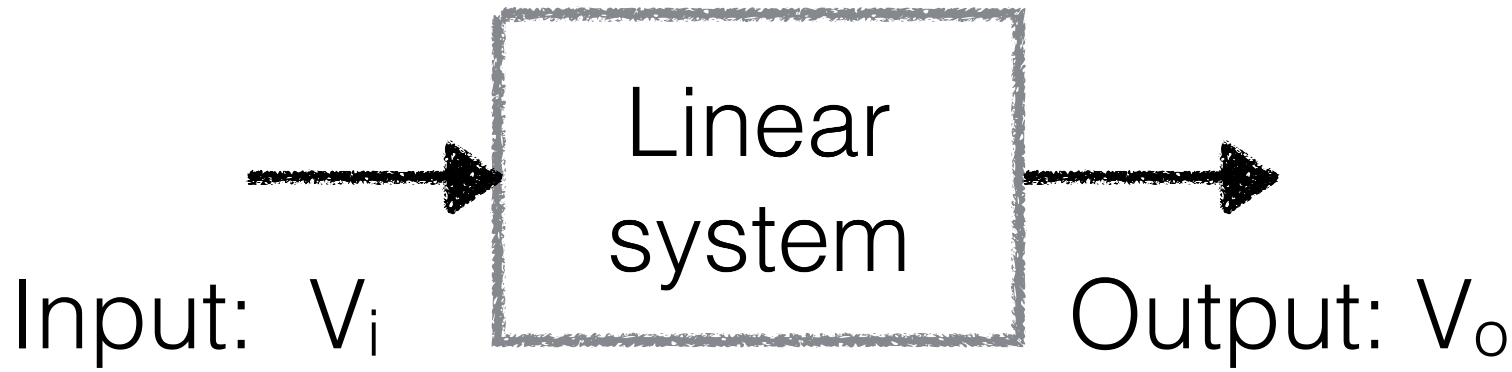
- ***A. Introduction or review of some needed mathematics. e.g. linear systems and Fourier transforms.***
- ***B. Examples of some common electronics and mechanical systems and how to solve them.***
- ***C. Introduction to noise and related phenomena.***

I have left a lot for you to investigate on your own.

This work combines three separate disciplines It is a bit heavy and requires several sittings.

The goal is to present it in an informal way so that the connections are obvious and can be later explored in a deeper way.

1) ***Understanding and modeling physical systems such as electronics or mechanical systems.***
2) ***Elementary Mathematical analysis and Fourier transforms.***
3) ***Some elements of probability and statistics.***



- ***There are many practical mechanical or electrical systems in which the output is linearly related to the input excitation.***

- ***If the excitation is multiplied by a constant then the response is also. (homogeneity). If V_i ->a V_i then V_o-> a V_o***

- ***If there are multiple sources of excitation then the response due to each one adds linearly in the total response. (superposition). If V_i -> V_1+V_2 then V_o->V_1+V_2***

- ***The frequency of the response will be the same as the frequency of the excitation. This property allows analysis of such systems using Laplace or Fourier transforms.***

Impulse response and transfer functions.

If input pulse is a delta function then the output is called the impulse response

$V_i(t) = \delta(t)$ then $V_o(t) = h(t)$

The transfer fuction of a physical device (am amplifier or shaper) is the Fourier transform of this impulse response.

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} h(t)\,dt \qquad \text{and} \qquad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} h(t)\,dt$$

This is the engineering asymmetric convention for Fourier transforms. Often physicists and mathematicians will use $1/\sqrt{2\pi}$ as the normalization for both sides (symmetric).

Dirac's delta function and its Fourier transform

δ(t)=0 if t $\neq$ 0 and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \text{ and } \mathrm{D}(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = 1$$

For the inverse transform $\delta(\mathrm{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$

Properties of the Fourier transform of a real function

Let $g(t)$ be a real function.

$$G(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt \qquad G^*(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i(-\omega)t}\,dt$$

Therefore $G^*(\omega)=G(-\omega)$... is Hermitian

This property allows us to only display the transform for positive frequencies.

Examine the symmetry of $g(t)$ based on $G(\omega)=|G|e^{iArg(G)}$

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{+i\omega t}\,d\omega$$

$$g(t)=\frac{1}{\sqrt{2\pi}}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{-\infty}^{0} G(\omega)e^{+i\omega t}\,d\omega\right)$$

$$g(t)=\frac{1}{2\pi}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{\infty}^{0} G(-\omega)e^{-i\omega t}(-d\omega)\right)$$

$$g(t)=\frac{1}{2\pi}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{0}^{\infty} G^*(\omega)e^{-i\omega t}\,d\omega\right)$$

$$g(t)=\frac{1}{2\pi}\int_{0}^{\infty} (G(\omega)e^{+i\omega t}+G^*(\omega)e^{-i\omega t})\,d\omega$$

$$g(t)=\frac{1}{2\pi}\int_{0}^{\infty} |G(\omega)|2\cos(\omega t+Arg(G(\omega)))\,d\omega$$

$|G(\omega)|^2$ is called the power spectrum.

if g(t) is in volts and we imagine it is applied across 1 Ω resistance, then $|G(\omega)|^2$ is the amount of power per unit frequency

Notice that if G(ω) is a real function then g(t) is a symmetric function.

g(t) = g(-t) iff G(ω) is real.

Some more properties

The following properties allow us to transform differential equations that govern linear systems into algebraic problems in frequency space.

Parseval's Theorem: $\int_{-\infty}^{\infty} g(t)^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|G(\omega)\right|^2 d\omega$

Time shift: $g(t - t_0) \xrightarrow{F} e^{-i\omega t_0} G(\omega)$

Differentiation: $g'(t) \xrightarrow{F} i\omega G(\omega)$

Integration: $\int_{-\infty}^{t} g(\tau)\, d\tau \xrightarrow{F} \frac{1}{i\omega} G(\omega) + \pi G(0)\delta(\omega)$

Linearity: $a g(t) + b h(t) \xrightarrow{F} a G(\omega) + b H(\omega)$

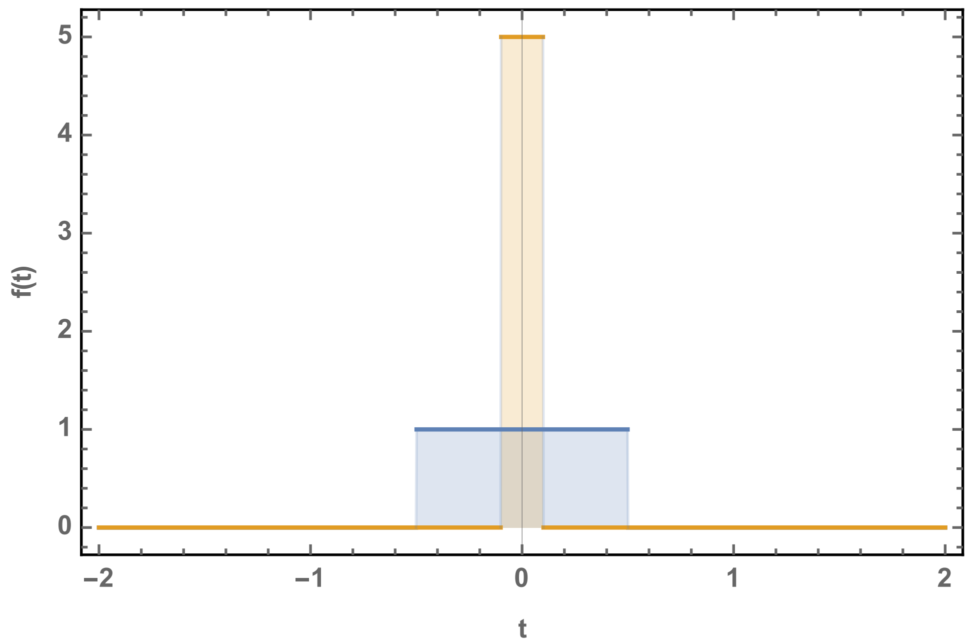
Convolution: $x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(t')h(t-t')\, dt' \xrightarrow{F} X(\omega) H(\omega)$

$\delta(t) \rightarrow \boxed{h(t),\ \mathrm{H}(\omega)} \rightarrow h(t)$ a delta function causes the impulse reseponse of filter

$e^{i\omega_0 t} \rightarrow \boxed{h(t),\ \mathrm{H}(\omega)} \rightarrow H(\omega_0) e^{i\omega_0 t}$ a sinusoidal single frequency amplitude is modified

Fourier and Laplace transforms are examples of more general integral transforms. They allow us to organize a set of numbers (such as probabilities or electrical currents) so that they can be easily manipulated, combined, etc. These transforms are the essential ingredients of quantum mechanics.

little more about a delta pulse



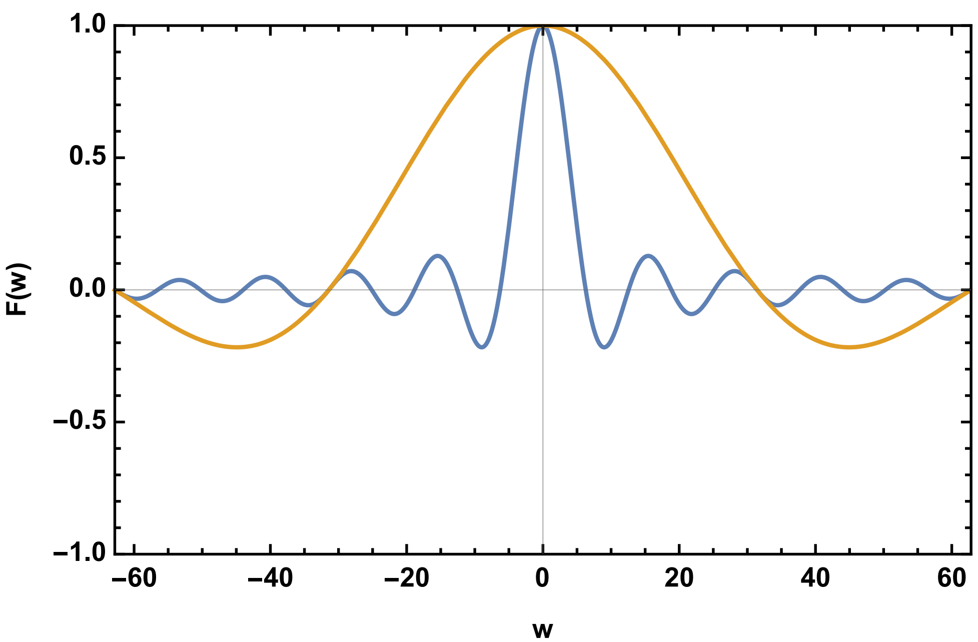
A unit step pulse of duration a centered at 0 is given by

$$f(t) = \frac{1}{a}\left[u(t-\frac{a}{2}) - u(t+\frac{a}{2})\right]$$

Fourier transform of this is

$$F(\omega) = 2\frac{\sin(\frac{a\omega}{2})}{a\omega}$$

As $a \to 0,\quad F(\omega) \to 1$, as the contributions from all frequencies spread.



The the inverse Fourier transform is

$$\int_{-\infty}^{\infty} d\omega \frac{1}{\pi} \frac{\sin(\frac{a\omega}{2})}{a\omega} \times (\cos(\omega t) + i\sin(\omega t))$$

Notice that the "sin" terms cancel each other out as we add complex conjugate pairs giving us a symmetric function $f(t)$.

Cosine terms keeping adding at $t = 0$ giving an infinity at 0 as a $\to 0$

Discrete Fourier Transform

$g(t)$ is a real function.

A symmetric form of Fourier and Inverse Fourier transforms.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt; \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

Fourier transform

If units of $g(t)$ are volts then $G(\omega)$ has units of volts/Hz

set time domain 0 to T with M samples.

$$\Delta = \frac{T}{M};\ \text{define index } k = 0, \ldots, M-1;\ \ t_k = \Delta \bullet k;\ g_k = g(t_k)$$

Use asymmetric form of Discrete Fourier Transform

$$G_l = \sum_{k=0}^{M-1} g_k e^{-i2\pi\frac{l \cdot k}{M}}; \quad g_k = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{i2\pi\frac{l \cdot k}{M}}$$

Discrete Fourier transform

g_k and G_l have the same units. What is the relation between $G(\omega)$ and G_l?

$$G(2\pi f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t}\, dt \quad \ldots f = \frac{l}{N \cdot \Delta}$$

$$G(2\pi f_l) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi \frac{l}{N} \frac{t}{\Delta}}\, dt \Rightarrow \tilde{G}_l = \sum_{-M/2}^{M/2} g_k e^{-i2\pi\frac{l \cdot k}{N}} \Delta$$

$$G_l = \tilde{G}_l \cdot \frac{1}{\Delta}$$

It is useful to pay attention to the units when plotting G_l

If the normalization is chosen to be symmetric, then ratio is $\left(\sqrt{\frac{2\pi}{M}}.1/\Delta\right)$

some more simple observations about the DFT

$x_0,\ldots,x_{M-1}$ are real numbers. Imagine it is a waveform.

$$X_l = \sum_{k=0}^{M-1} x_k e^{-i2\pi k\, l/M}$$

$$x_k = \frac{1}{M}\sum_{l=0}^{M-1} X_l e^{i2\pi k\, l/M}$$

$U_{lk} = \left(e^{-i2\pi/M}\right)^{l\bullet k}, \quad l,k = 0,\ldots M-1$, is a unitary matrix, and the DFT is a linear matrix transform

Both of these are M-periodic: $X_{l+M} = X_l$, $x_{k+M} = x_k$

using M-periodicity and that x_k are real: $X_{-l} = X_l^* = X_{M-l}$

Take the case of M to be even:

$X_0 \in$ Real and $X_{M/2} \in$ Real;

X_1 to $X_{M/2-1}$ are complex and X_{M-1} to $X_{M/2+1}$ are conjugate.

This means there are only $(M/2-1)2+2 = M$ independent numbers.

Take the case of M to be odd:

$X_0 \in$ Real

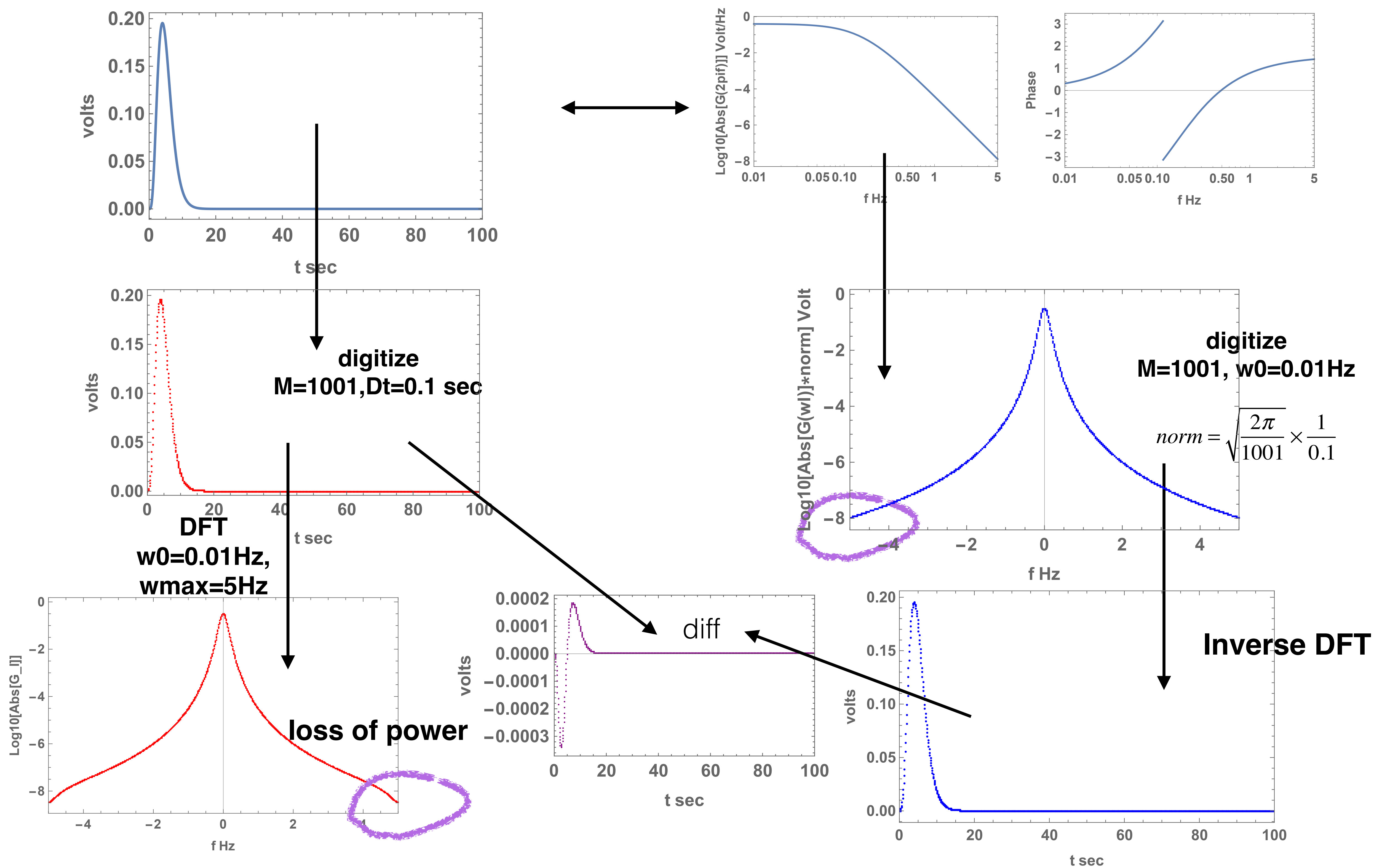
X_1 to $X_{(M-1)/2}$ are complex and X_{M-1} to $X_{(M+1)/2}$ are conjugate.

This means there are only $(M-1)/2\times 2+1 = M$ independent numbers.

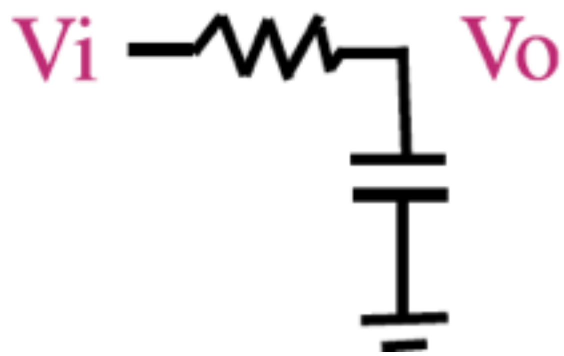
The DFT is often implemented with fast algorithms called "Fast Fourier Transforms".
The FFT uses symmetries in clever ways to cut down the number of complex summations that must be performed. Reference: Numerical Recipes textbook.

non-trivial example (I used symmetric version)

$$g(t)=\frac{e^{-t}t^4}{4!}u(t)\Leftrightarrow G(\omega)=\frac{1}{\sqrt{2\pi}}\frac{1}{(i\omega+1)^5}\quad \text{here } u(t)=\begin{cases}1 \ \mathit{if} \ t\geq 0\\ 0 \ \mathit{otherwise}\end{cases}$$



Example 1



Low pass, integrator.

Input is connected with ideal voltage source with zero impedance and output is measured with infinite impedance.

$$V_o(t) = Q/C, \quad \text{and} \quad V_i(t) - V_o(t) = R \cdot I(t)$$

For any V_i, we can calculate the frequency components (or Fourier transform), multiply by the filter function, and invert.

$$\frac{dV_o(t)}{dt} = \frac{V_i - V_o}{\tau} \text{ where } \tau\text{=RC}$$

Use $\text{V}_i(t) = v_i e^{i\omega t}$ for the input and $\text{V}_o(t) = v_o e^{i\omega t}$ as output.

$$i\omega v_o + v_o / \tau = v_i / \tau$$

$$v_o = v_i \times \frac{1/\tau}{i\omega + 1/\tau} = v_i \frac{1}{\left[1 + (\tau\omega)^2\right]^{1/2}} e^{i\theta} \text{ where } \theta = \text{Arctan}(\tau\omega)$$

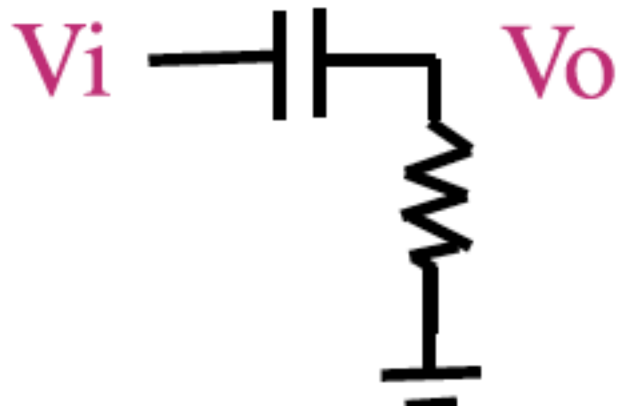
Notice that for $\omega\tau \ll 1$ the filter just passes the input through

Take inverse Fourier transform of the transfer function to get inpulse response

$$h(t) = \frac{e^{-t/\tau}}{\tau} \text{ for } t > 0 \text{ and } 0 \text{ for } t < 0$$

If V_i is a pulse with duration less than τ then V_o becomes an integral of V_i / τ.

Example 1



High pass, differentiator

Input is connected with ideal voltage source with zero impedance and output is measured with infinite impedance.

$$V_o(t) + Q/C = V_i(t)$$

$$\frac{dV_o(t)}{dt} + I(t)/C = \frac{dV_i(t)}{dt} \text{ where } I(t) \text{ is the current.}$$

$$\mathrm{V}_o(t) = I(t) \times R \text{ therefore}$$

$$\frac{dV_o(t)}{dt} + \frac{V_o(t)}{\tau} = \frac{dV_i(t)}{dt} \quad \text{where } \tau = RC \text{ is called the time constant.}$$

Use $\mathrm{V}_i(t) = v_i(\omega)e^{i\omega t}$ for the input and $\mathrm{V}_o(t) = v_o(\omega)e^{i\omega t}$ as output.

$$v_o = v_i \times \frac{i\omega}{i\omega + 1/\tau} = v_i \frac{1}{\left[1 + 1/(\tau\omega)^2\right]^{1/2}} e^{i\theta} \text{ where } \theta = \mathrm{Arctan}(1/\tau\omega)$$

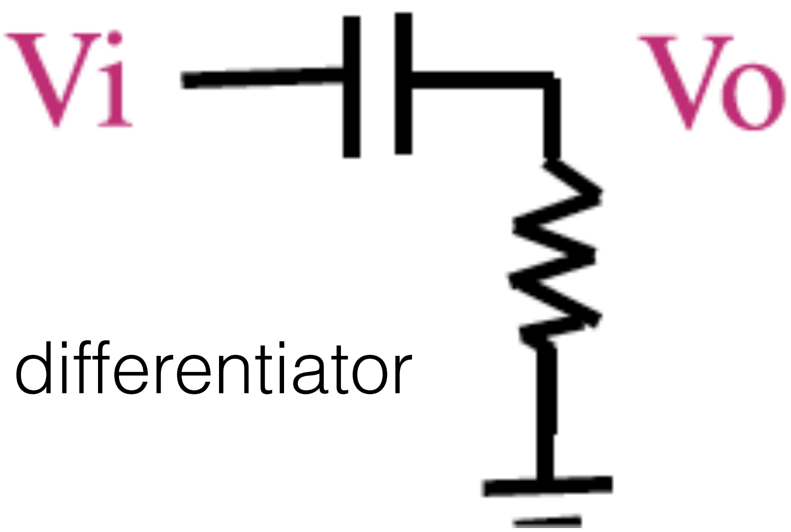
Inverse Fourier transform of the transfer function.

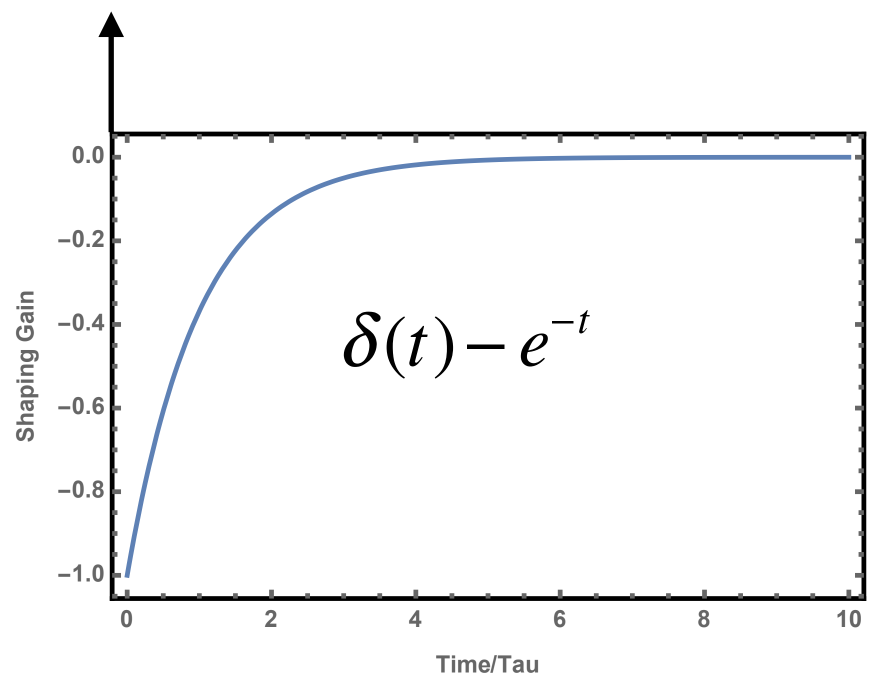
$$h(t) = \delta(t) - \frac{e^{-t/\tau}}{\tau} \quad \text{for } t > 0 \text{ and } \delta(\mathrm{t}) \text{ for } t < 0$$

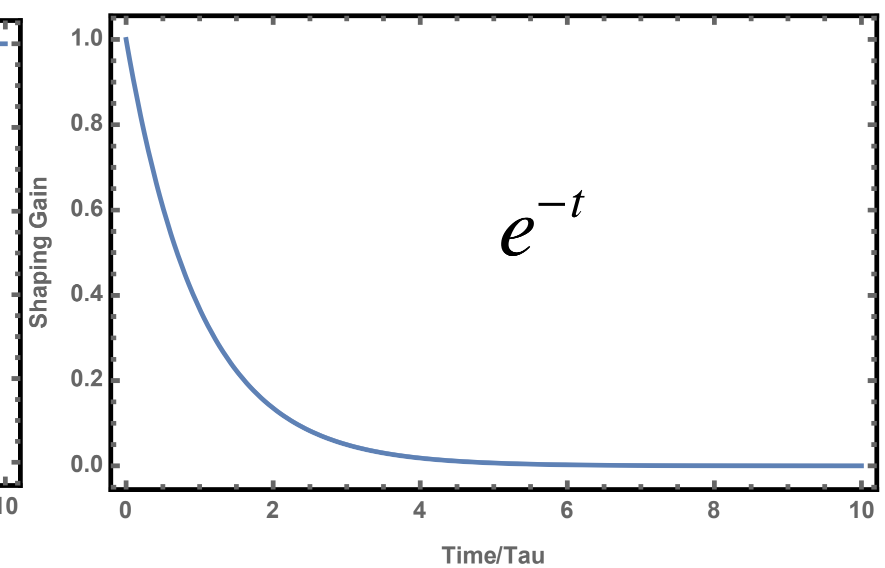
Set RC = 1

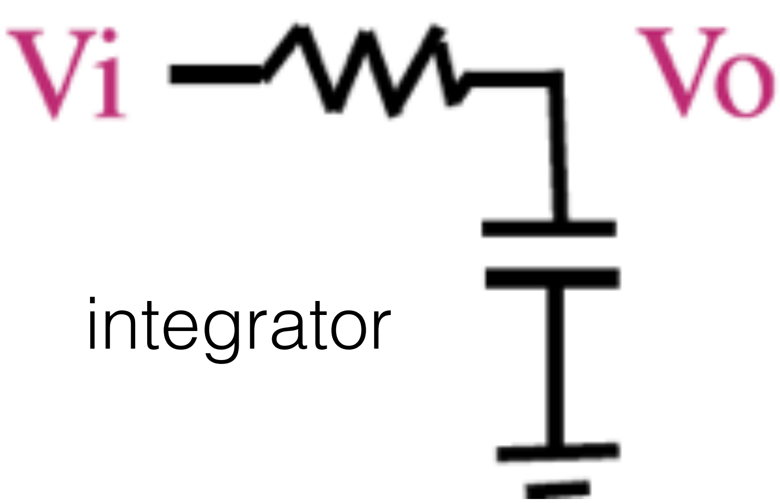
Output if Input is delta(t)

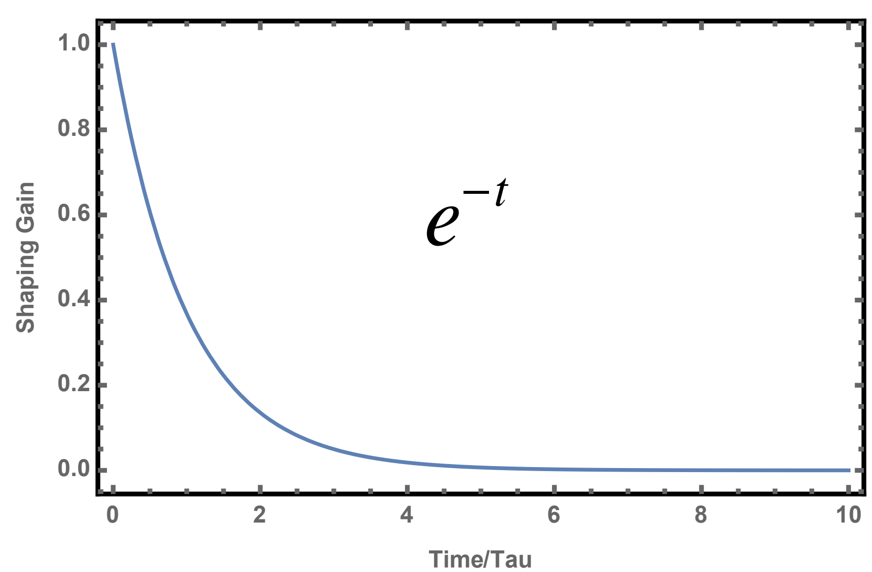
Output if Input is Step function u(t) = 1 for t>0

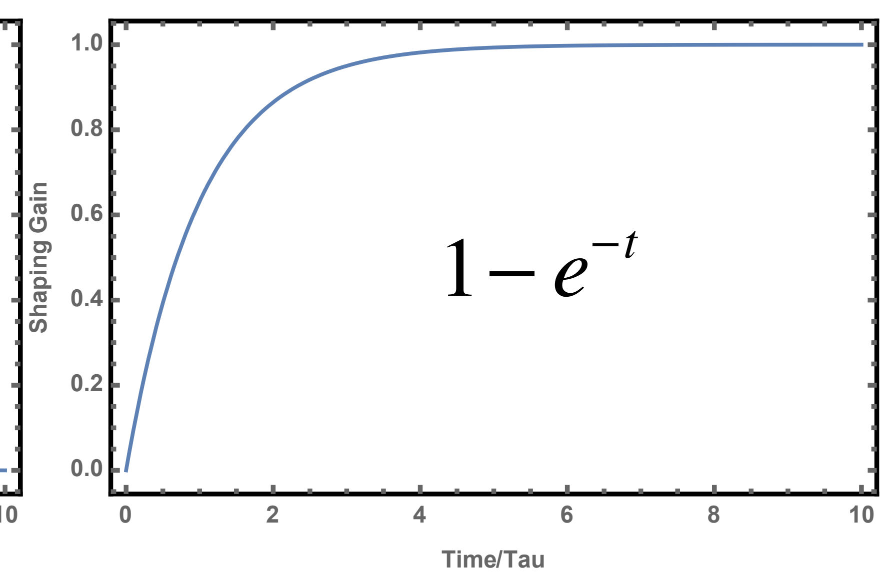




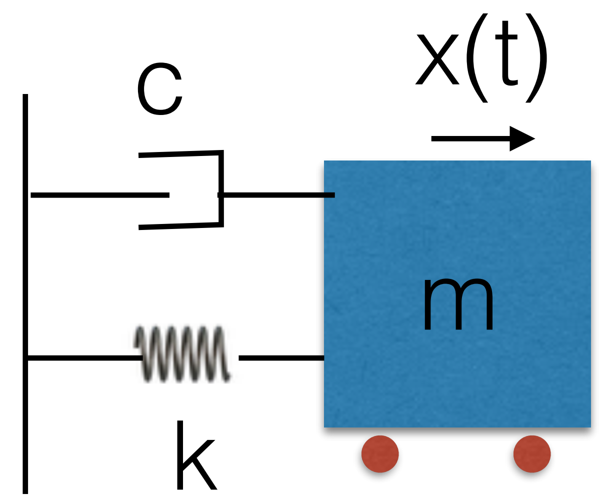








Simple 1 DOF system(this has electrical analog when inductors are present)



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

Set $x(t)=Xe^{i\omega t}$

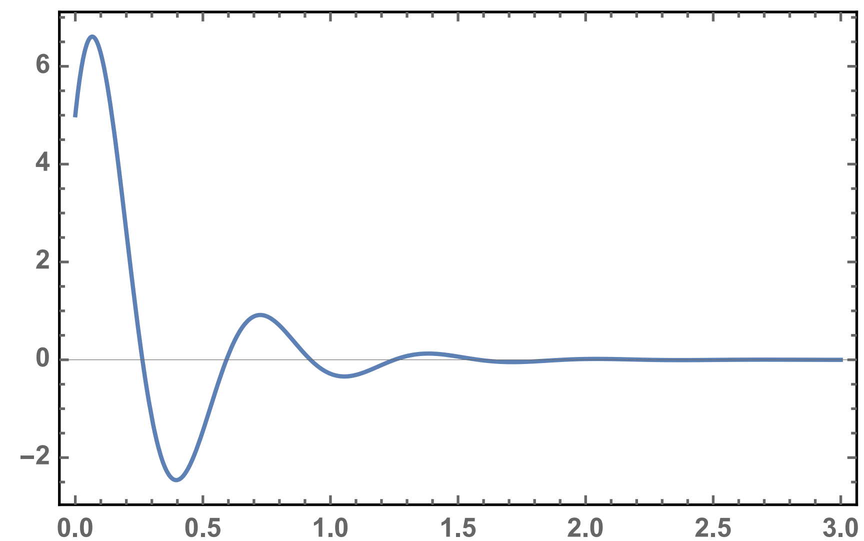
$-\omega^2 mX+i\omega cX+kX=0$ This yields a solution for ω

$$\omega=-i\frac{c}{2m}\pm\sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}$$

Set $\zeta=\dfrac{c}{2m\omega_n}$, $\omega_n^2=\dfrac{k}{m}$ is the natural frequency

$\omega=-i\zeta\omega_n\pm\omega_n\sqrt{1-\zeta^2}$, the complete solution is then

$$x(t)=e^{-\xi\omega_n t}\left(x_0\,\mathrm{Cos}(\omega_d t)+\frac{v_0+\xi\omega_n x_0}{\omega_d}\mathrm{Sin}(\omega_d t)\right)$$

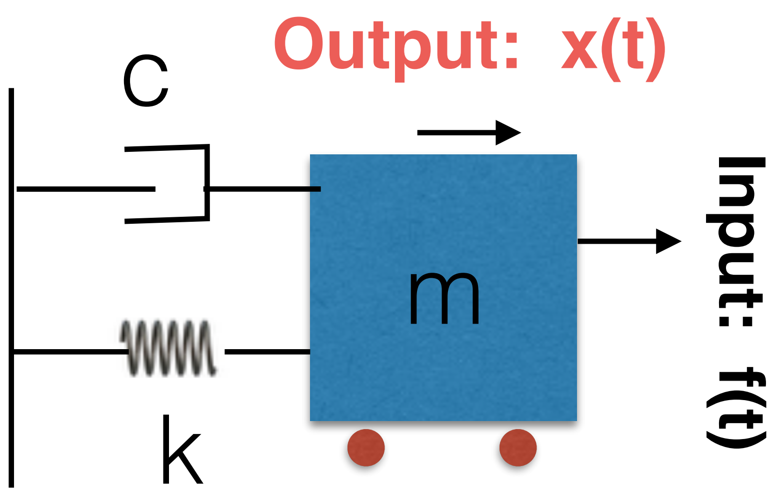


where $\omega_d=\omega_n\sqrt{1-\zeta^2}$ is called the damped frequency. At ζ=1 (critical damping), there is no oscillatio

If we assume small damping, then the intercept of this motion is the initial displacement x_0 and the initial slope corresponds to $\sim v_0$

This is not a forced system and it will respond in case of an initial condition that is non-zero.

Example 3 forced mechanical system



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Fourier: $x(t)\Leftrightarrow X(\omega);\ f(t)\Leftrightarrow F(\omega)$

$$-\omega^2 mX+i\omega cX+kX=F(\omega)$$

$$\frac{X}{F}=\frac{1}{k}\left[\frac{1}{(1-\omega^2/\omega_n^2)+2i(\omega/\omega_n)\zeta}\right]$$

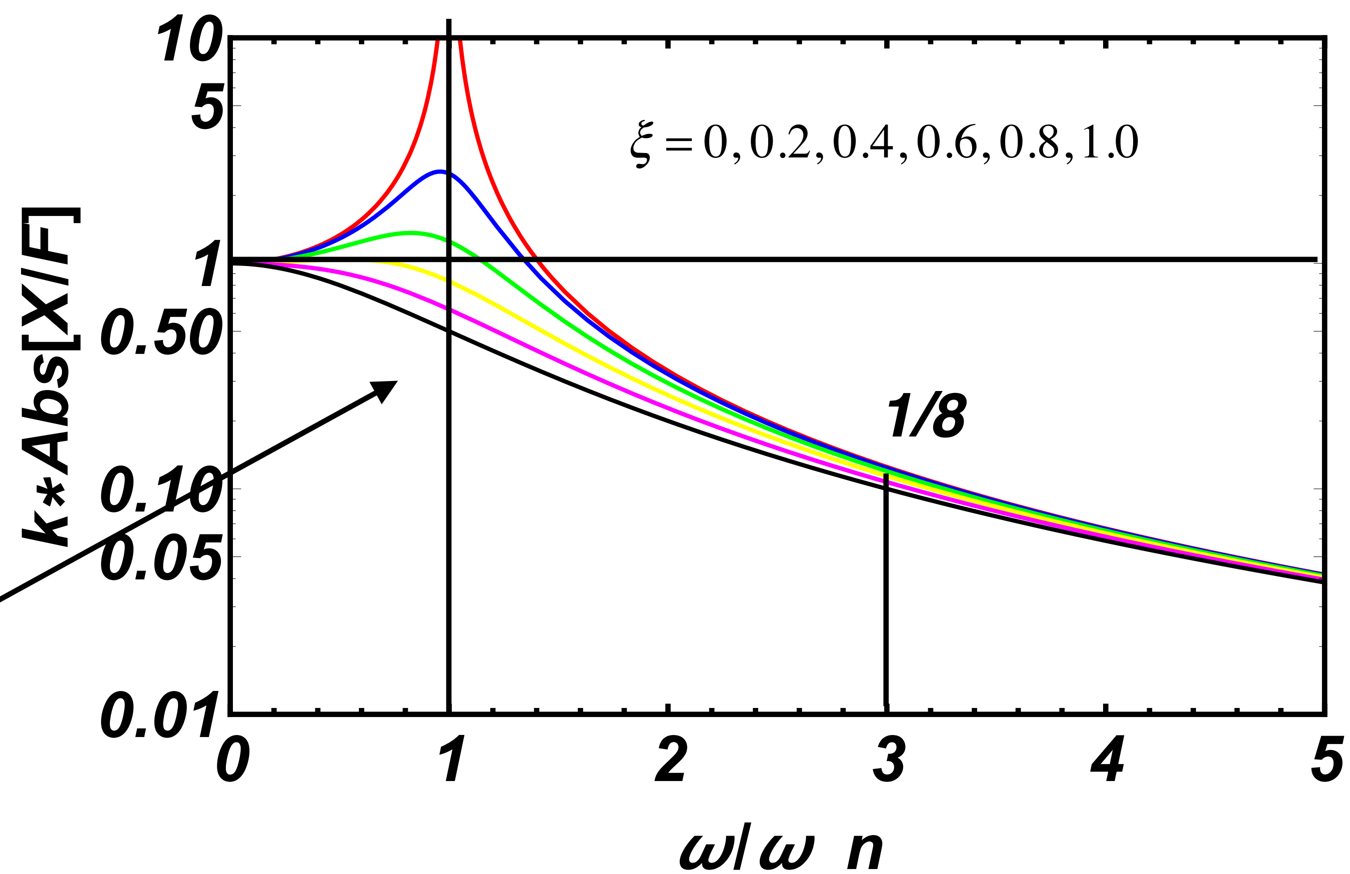
Units k: N/m, c: N/m/s
f(t): applied force in N

$$\omega_n^2=\frac{k}{m}\quad \zeta=\frac{c}{2\omega_n m}$$

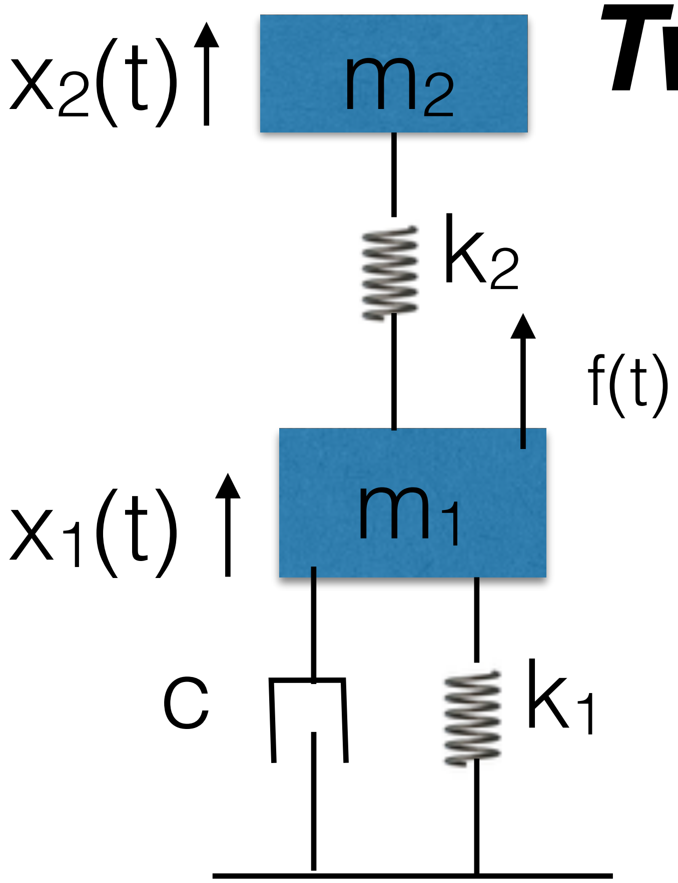
$$\text{Natural Freq}=\nu_n=\frac{\omega_n}{2\pi}\quad \text{Hz}$$

Damping $=\zeta$ is unitless

This is called the transfer function. The height of the response at the natural frequency depends on the damping.



If f(t) is a delta function: This is like an initial condition solution on the previous page.



Two degrees of freedom with f(t)

$$m_1 \frac{d^2 x_1}{dt^2} + c\frac{dx_1}{dt} + k_1 x_1 + k_2(x_1 - x_2) = f(t)$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_2 x_2 - k_2 x_1 = 0$$

In Fourier space

$$\begin{pmatrix} -\omega^2 m_1 + i\omega c + (k_1 + k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F(\omega) \\ 0 \end{pmatrix}$$

First assume $c = 0$ to get the normal modes

solving for ω gives

$$\omega^2 = \frac{1}{2}\left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \pm \frac{(-4k_1k_2m_1m_2 + (k_2m_1 + k_1m_2 + k_2m_2)^2)^{1/2}}{m_1m_2}\right]$$

Ratio of amplitudes for the two modes are

$$\frac{X_2}{X_1} = \frac{-\omega^2 m_1 + k_1 + k_2}{-k_2}$$

Here one plugs in the two eigenvalues of ω_+ and ω_-

We also define $\omega_1^2 = \frac{k_1}{m_1}$; $\omega_2^2 = \frac{k_2}{m_2}$

Do this as a homework problem. Also think of how to do this where f(t) is replaced by floor vibration.

Transfer Function

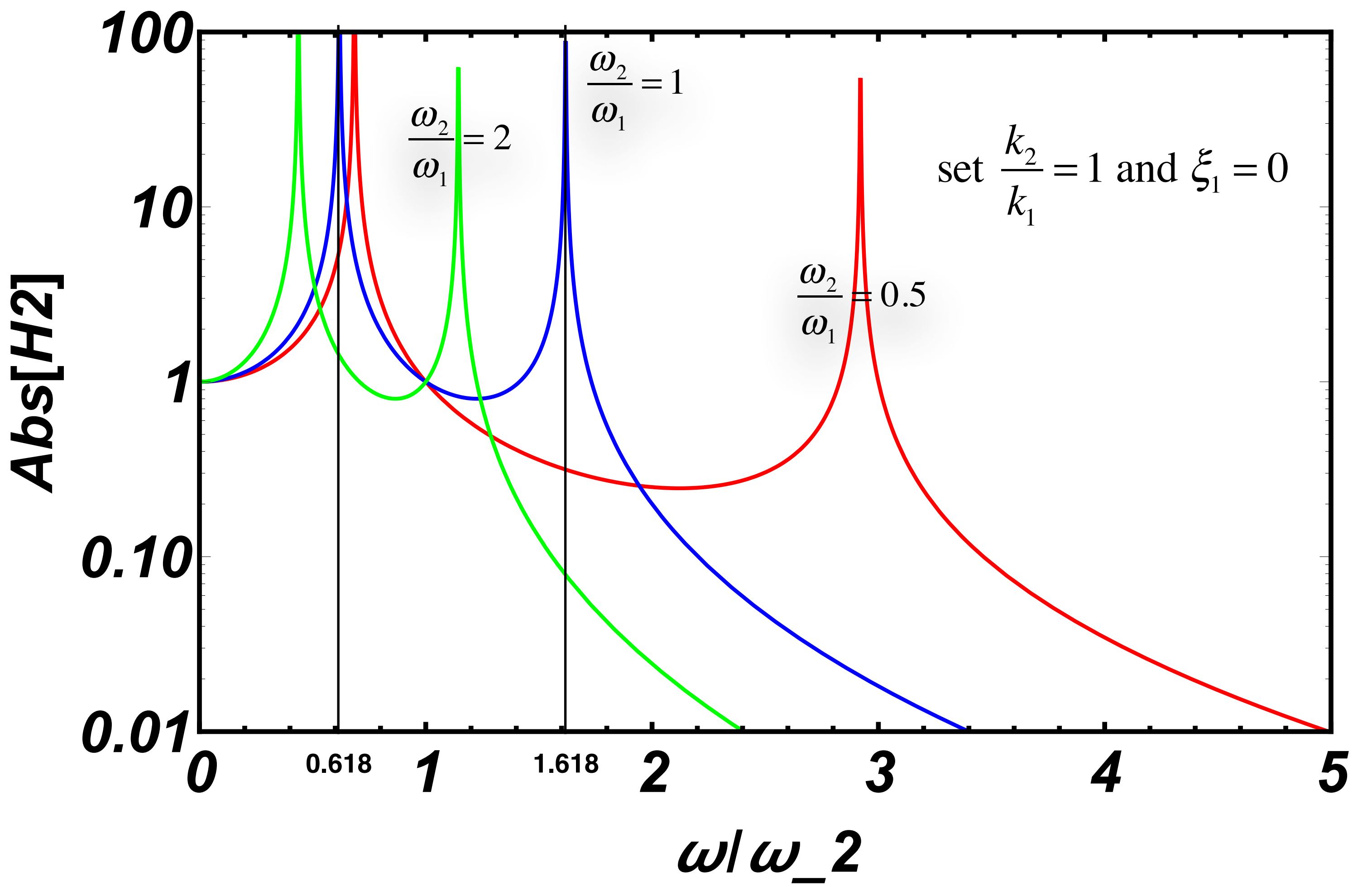
$$\frac{X_1}{F} = \frac{-(k_2 - m_2\omega^2)}{k_2^2 - (k_1 + k_2 - m_1\omega^2 + ic\omega)(k_2 - m_2\omega^2)}$$

$$\omega_1^2 = \frac{k_1}{m_1} \qquad \xi_1 = \frac{c}{2\omega_1 m_1}$$

Damping = ξ_1 is unitless

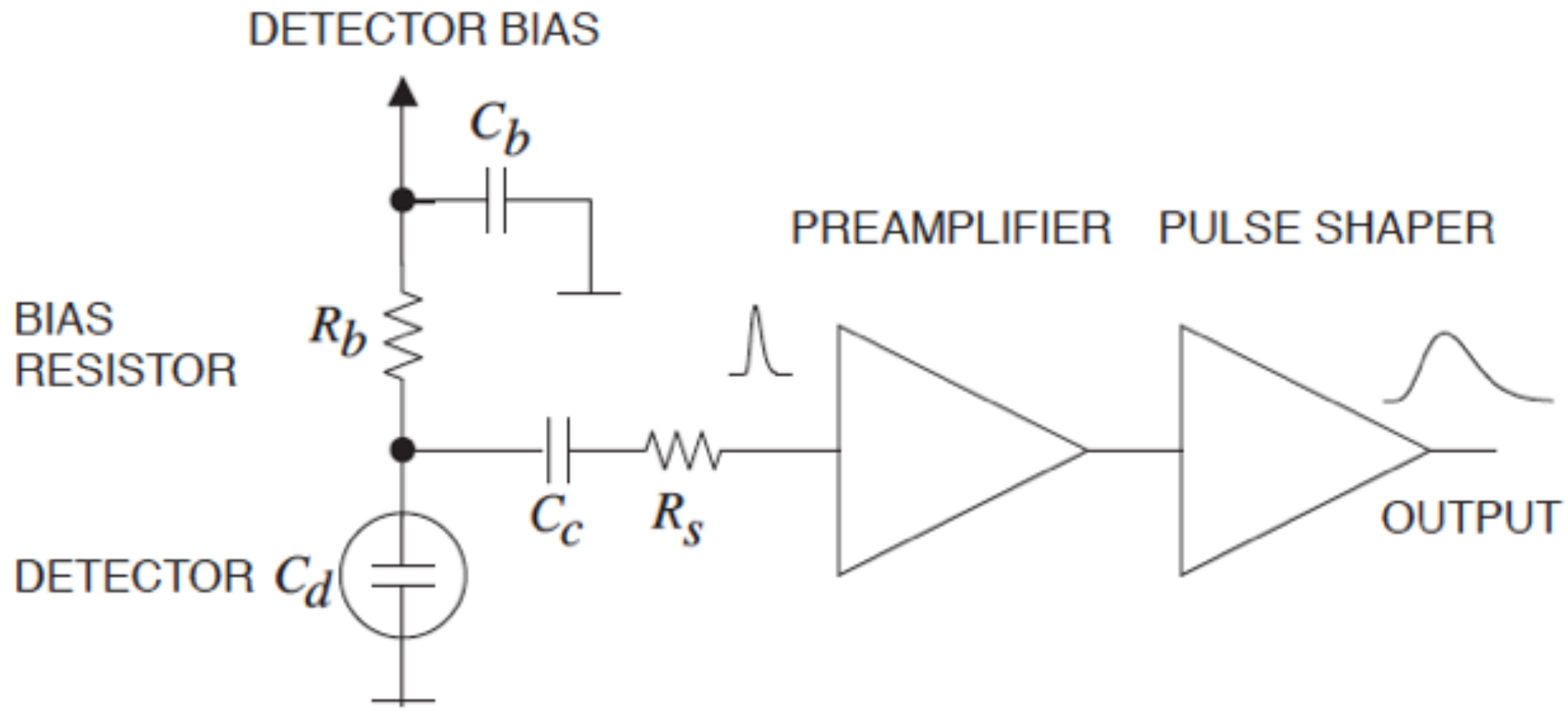
$$\frac{X_2}{F} = \frac{k_2}{k_1k_2 + ic\omega(k_2 - m_2\omega^2) - (k_2m_1 + k_1m_2 + k_2m_2)\omega^2 + m_1m_2\omega^4}$$

$$H_2(\omega) = \frac{X_2}{F/k_1} = \frac{1}{1 + i2\frac{\omega_2}{\omega_1}\frac{\omega}{\omega_2}\xi_1\left[1 - \frac{\omega^2}{\omega_2^2}\right] - \left[1 + \frac{k_2}{k_1} + \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega^2}{\omega_2^2} + \frac{\omega_2^2}{\omega_1^2}\frac{\omega^4}{\omega_2^4}}$$



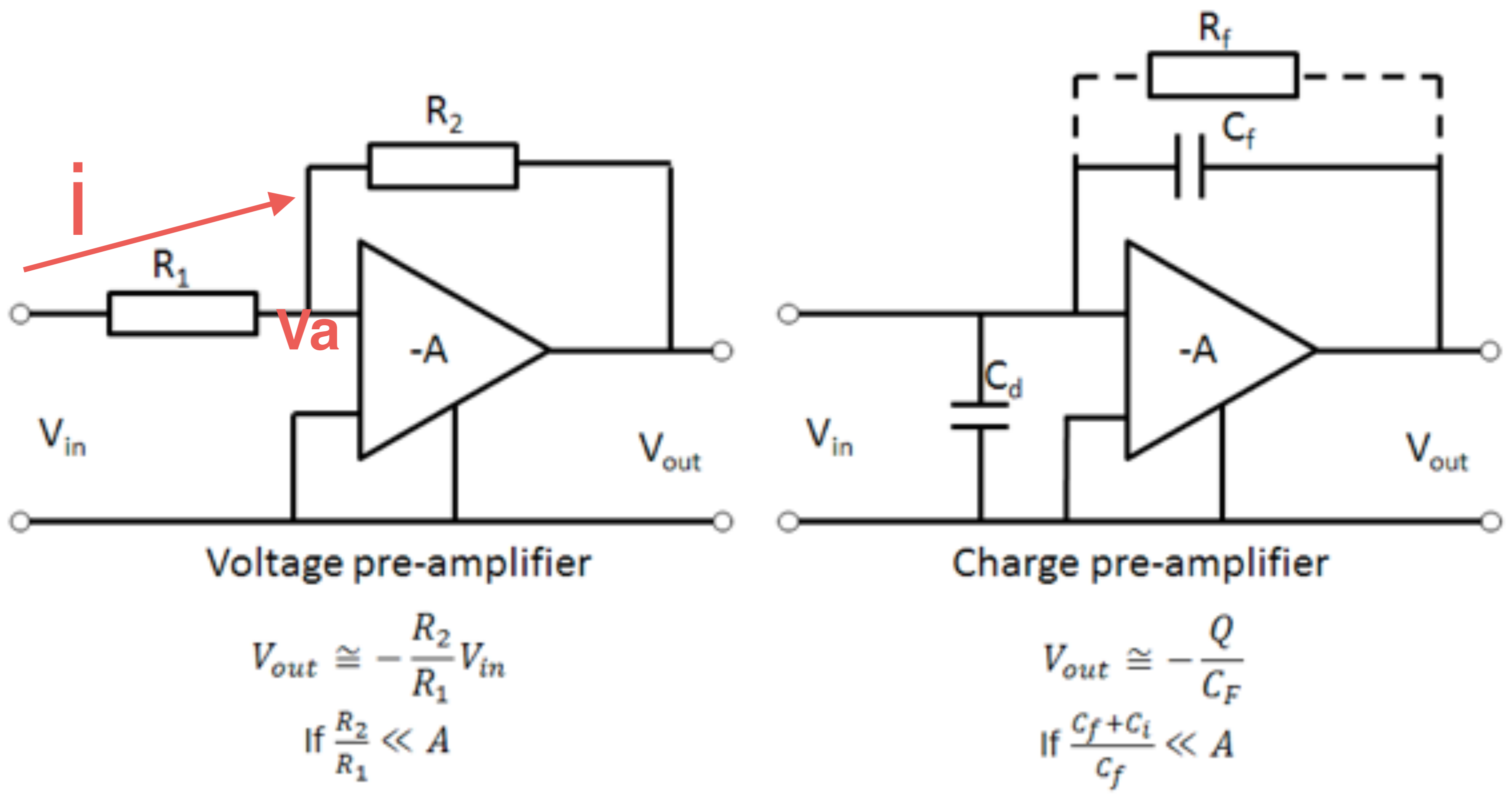
It is better to have $\omega_2 > \omega_1$

Front end electronics (General Principles) for radiation detection



- Detector is assumed to produce a current pulse i(t)
- Detector is modeled by capacitance C_d
- Any device the produces current can be modeled as an ideal current source with an impedance parallel to the source. (Norton's theorem)
- There has to be a bias high voltage to create the current. This is blocked from the amplifier by a capacitor C_c. The current will go through a path of resistance R_s to the preamp and then a shaper will eliminate unwanted signal structure.

Amplifiers(how to control output)



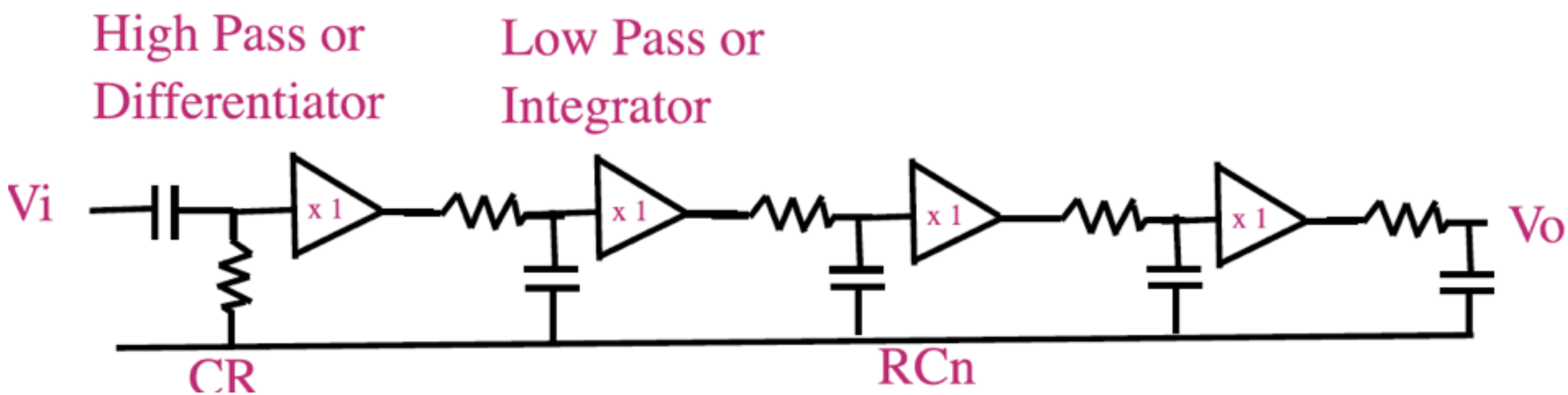
$$V_o = -AV_a$$

$$i = \frac{V_a - V_{in}}{R_1} = \frac{V_o - V_a}{R_2}$$

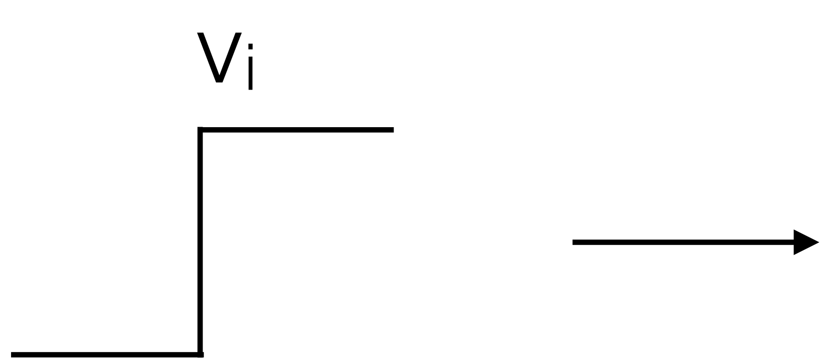
$$V_o = -V_{in} \frac{R_2}{R_1} \left[1 + \frac{(R_1 + R_2)}{R_1 A} \right]^{-1}$$

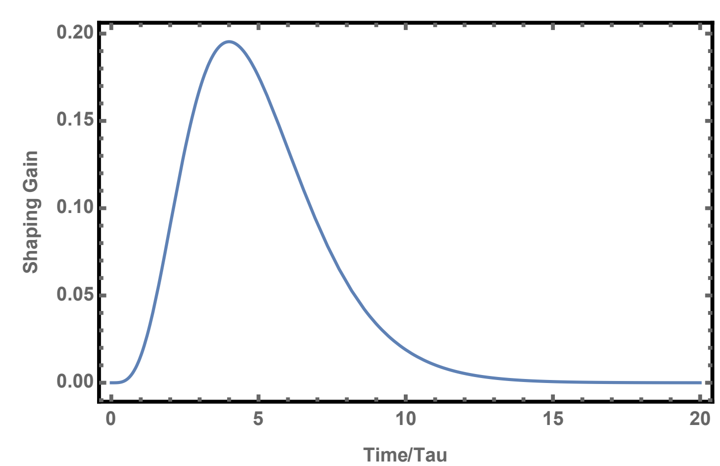
- Analysis of such circuits can be done using the ideal Op-amp in which A is infinite, and the input has infinite impedance.
- Amplifier inverts signal, and small amount is fed back to control the output.
- Voltage Preamp amplifies the voltage at the input if the detector capacitance is constant.
- It is usual in particle physics to have a charge sensitive preamp since detector capacitance can vary and it is not good to have noisy resistances at input.
- The chain of amp/shaper has a transfer function. The pulse is shaped for optimum S/N.

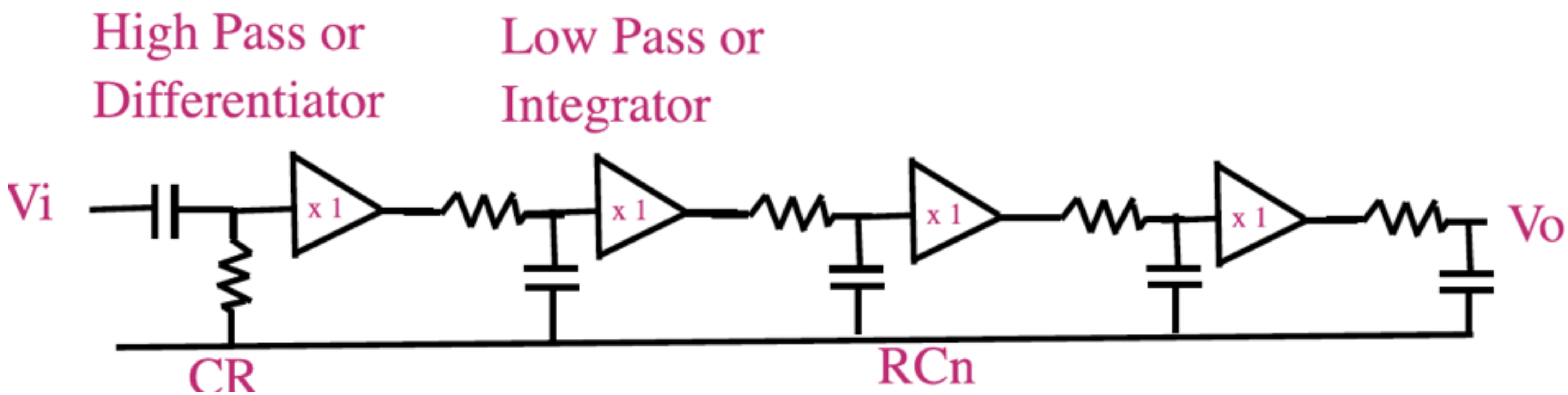
Example 4: CR-RC⁴ filter



Purpose is to create a Gaussian shaped pulse from an initial step voltage. The height of the pulse should be the voltage step. The peak of the pulse is given by the peaking time which is n τ=RC = 1/γ







CR-RC⁴ we can just multiply the transfer functions to get the complete transfer function. Using Laplace transforms we get. (in this case s -> iω to get Fourier)

$$V_o(s) = V_i(s) \times \frac{s}{s + 1/\tau} \times \frac{1}{(s + 1/\tau)^4}$$

This can be inverted to obtain time domain pulse for some V_i. For a unit step pulse →

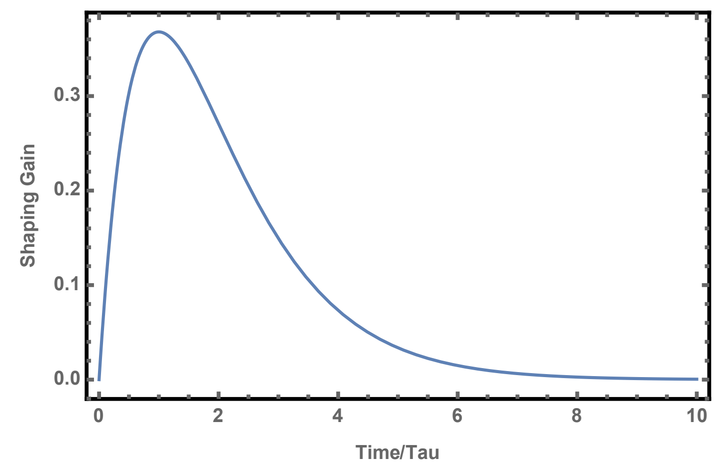
$$V_i(s) = 1/s$$

$$V_o(t) = \frac{t^4}{4!} e^{-t/\tau}$$

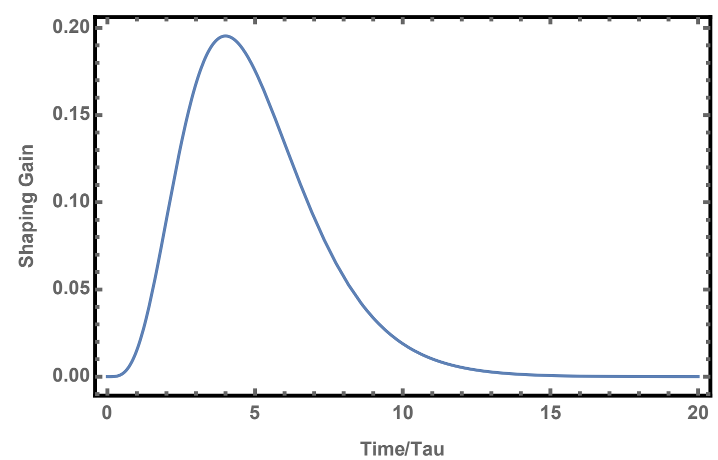
There are ways of making this more symmetric by introducing complex "poles" in the transfer function.

The ideal preamp produces a step function called a "tail pulse". This step must be shaped.

Input is step function with a CR-RC filter. Peak is at time $=1*\tau$

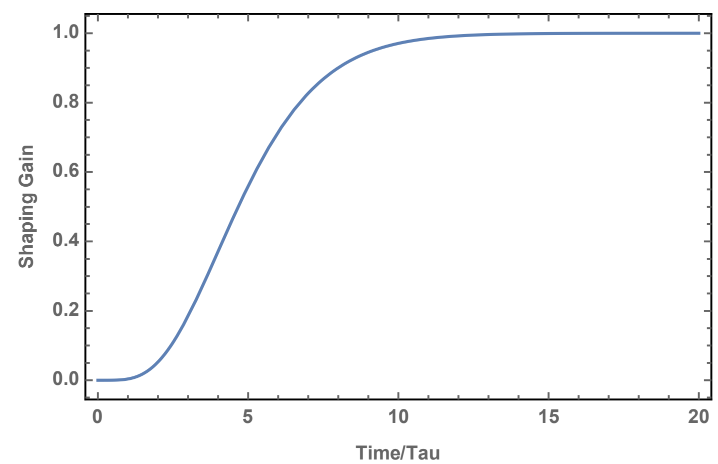


Input is step function with a CR-RC^4 filter. Peak is at time $=4*\tau$



delta response will have an undershoot

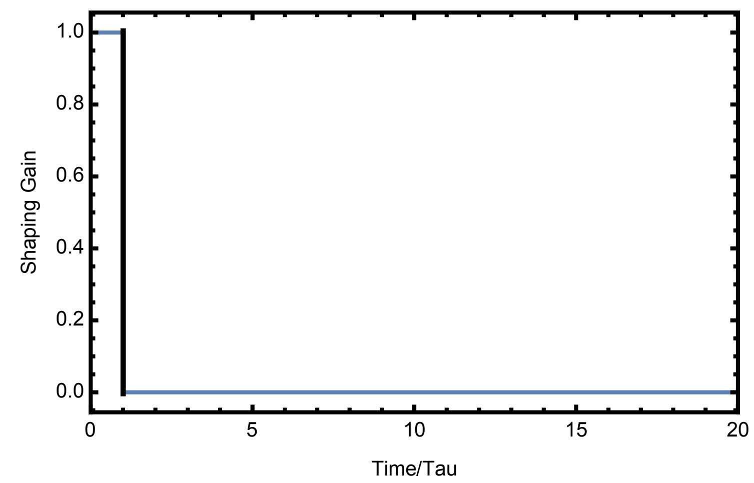
Input is step function with a RC^5 integrator.



work out the response to a square pulse

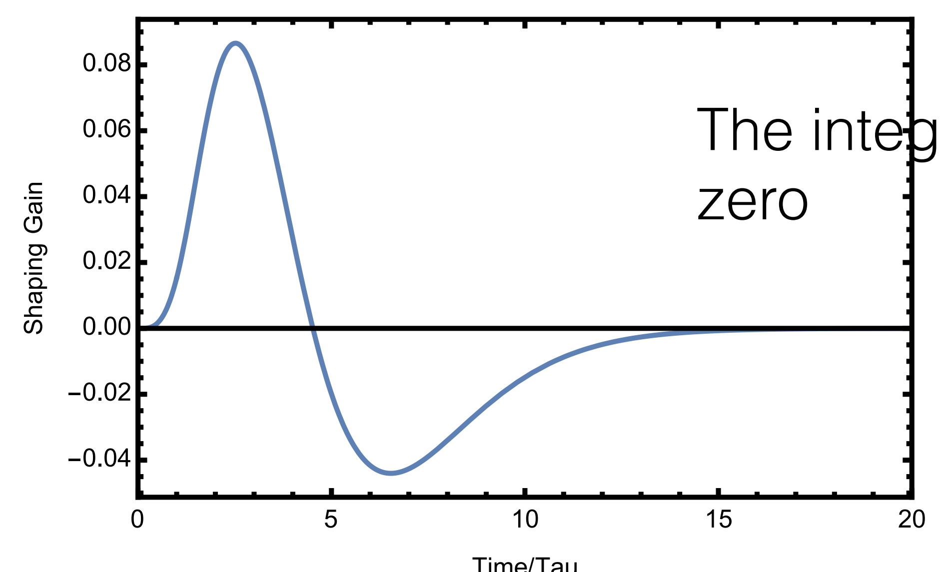
The real input pulses are pulses with some widths. Or they have a long shaping time to bring them back to baseline.

Input



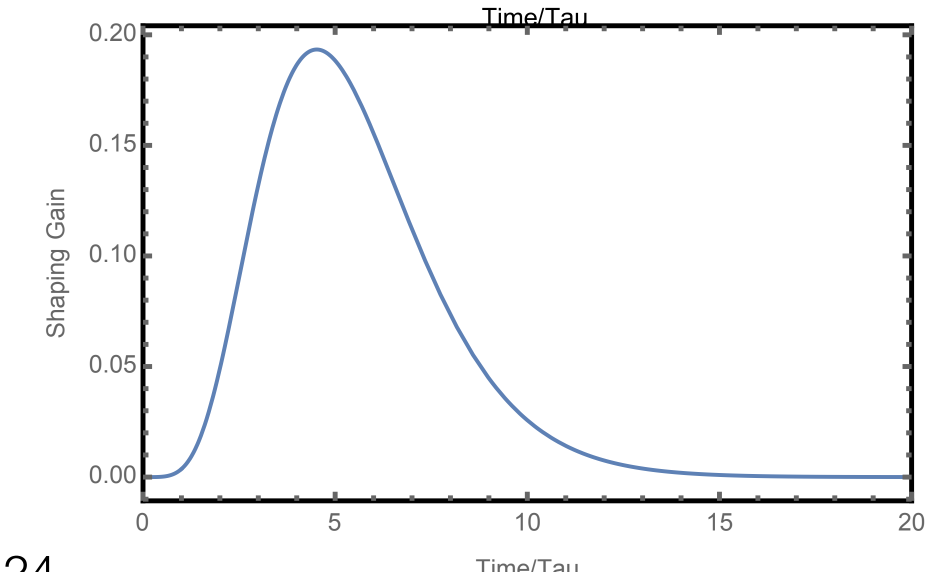
1 mu-sec square pulse

Output after CR-RC4



The integral will be zero

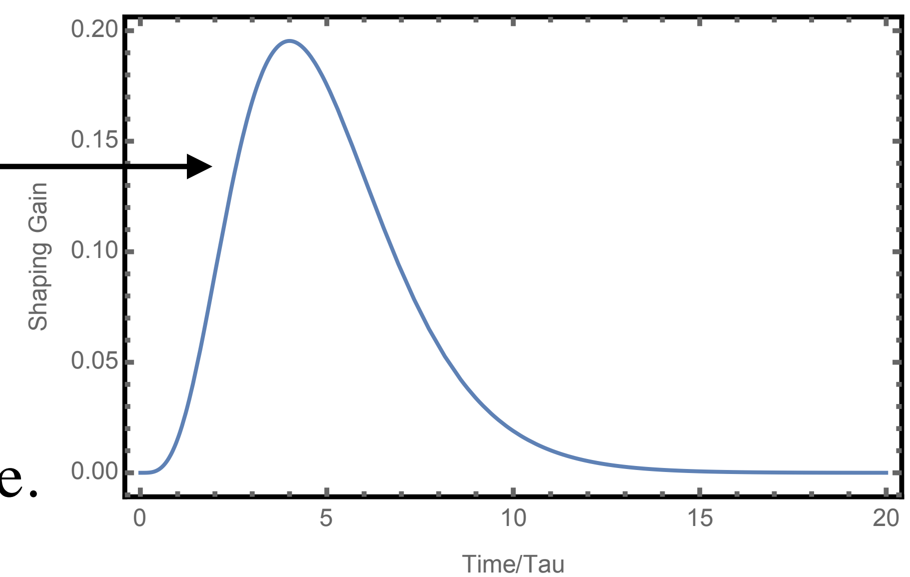
Output after RC5



More about shaping, a 5th order shaper

$$F(s) = \frac{1}{(s+a)^5} \rightarrow f(t) = \frac{t^4}{4!} e^{-at}$$

This has a maximum at $t = 4/a$



$$f(4/a) = \frac{32}{3} \frac{e^{-4}}{a^4} = \frac{0.195}{a^4}$$

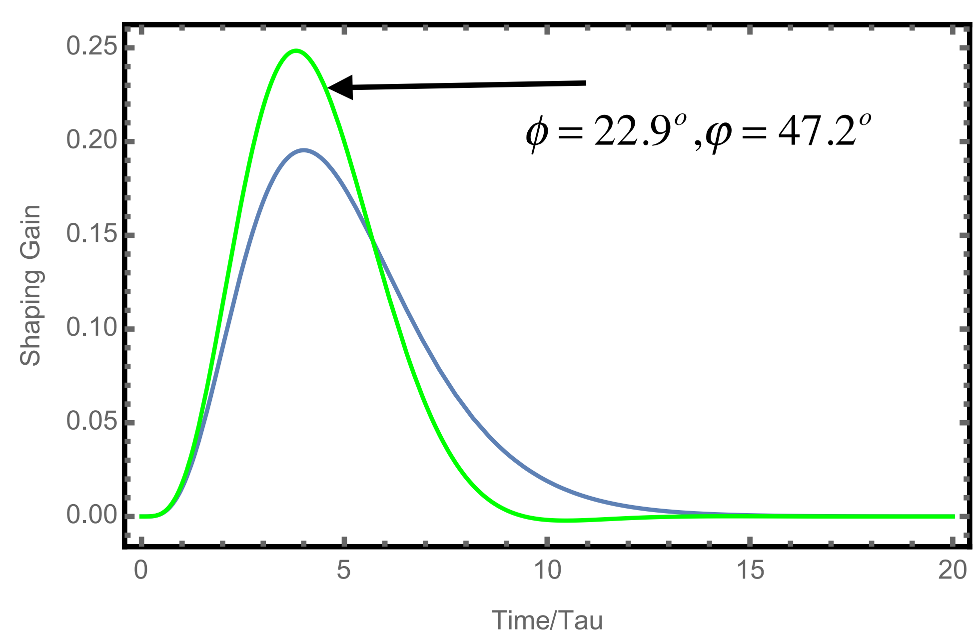
$f(15/a) \approx 0.6 * 10^{-3} \cdots$ It takes 15 times to restore baseline.

Complex poles allow the baseline to be restored much faster.

$$F(s) = \frac{1}{(s+a)((s+a\cos(\phi))^2 + a^2\sin^2(\phi))((s+a\cos(\varphi))^2 + a^2\sin^2(\varphi))}$$

$$\rightarrow f(t) = Ae^{-at} + \sum_{i=2,3} B_i e^{-r_i t} \cos(c_i t + \gamma_i)$$

In addition, we can adjust the amplitude to obtain the same peak for any value of shaping time.



Ohkawa,Yoshizawa,Husimi, NIM 138, 85-92, 1976

If you want to explore more….(also look at control theory)

There is an extensive theory behind optimum shaping of analog pulses using either analog or digital methods. Digital shaping can be done by gated integration. Shaping using analog electronics is called time-invariant shaping. It uses a suitable configuration of poles to create a semi-Gaussian output. Examples of unipolar shaping are

$$T(s) = \frac{1}{(s+p)^n} \quad \text{r-shapers } n = 2,3,4...$$

$$T(s) = \frac{1}{\prod_{i=1}^{n/2} [(s+r_i)^2 + c_i^2]} \quad \text{c-shapers } n = 2,4,6...$$

r_i and c_i are real and imaginary parts of complex-conjugate poles.

Time domain pulses can be obtained by using partial fraction expansion and inverse Laplace

$$f(t) = \frac{1}{(n-1)!} t^{n-1} e^{-tp} \quad \text{this is for r-shapers } n = 2,3,4,...$$

$$f(t) = \sum_{i=1}^{n/2} 2\left|K_i\right| e^{-r_i t} \operatorname{Cos}(-c_i t + Arg(K_i)) \quad \text{for c-shapers } n = 2,4,6...$$

K_i are obtained from partial fraction expansion. As n increases this becomes more Gaussian.

The peaking time (τ_p) characterizes the frequencies that are filtered and the noise performance.

The width (τ_w) or time to baseline defines the rate capability. For a given τ_p, the higher the order, the shorter the τ_w

Noise waveforms at the output

- ***Systems will produce random outputs in response to random fluctuations at the input or in internal components. These fluctuations could be due to thermal motions, statistical fluctuations in electrical currents, or environmental disturbances.***

- ***The waveforms can be thought of as continuous variables of time or they could be digitized at discrete intervals.***

- ***How do we categorize and analyze these noise waveforms ? Can it be done generally for all systems ?***

- ***We will do this using some mathematical devices. Most important is the delta function.***

- ***The problem of the random walk arises in many situations.***

- ***Brownian motion - Einstein (1905)***

- ***Stock markets - Original formulation by Louis Bachelier (1900)***

- ***It is the basis of the advanced financial mathematics that is currently taught in business schools.***

Partial bibliography

- *S.O. Rice Bell Syst. Tech. J 23:283-332 (1944) and 24:46-156 (1945).*
- *W. Schottky Ann. Phys. 57:451-567 (1918). W. Schottky Phys. Rev. 28 (1926).*
- *Edoardo Millotti, ArXiv: physics/0204033 (has large bibliography)*
- *W. H. Press Comments, Astrophys 7 (1978) 103.*
- *statlect.com digital textbook on probability theory*
- *G.F.Knoll's text book on radiation detectors.*
- *For an introduction to the method of characteristic functions see: any good textbook on probability and statistics. Introduction to Probability Theory … William Feller (1950).*

Analysis of noise is an old subject that is still evolving. For deeper understanding, need familiarity with: 1) real and complex analysis, 2) probability theory, 3) calculus of randomness.

There is a lot of other good material. But some of it is confusing. Confusion has origins in either the material being too dated with old definitions or having sloppy mistakes in units of Fourier transforms and power spectra (as pointed out by Millotti).

White noise

Take time interval $0 \to T$

Assume there are N elementary noise excitations in this time interval.

Each characterized by q_i

$$e(t) = \sum_{i=1}^{N} q_i \delta(t - t_i)$$

If each elementary excitation produces a response $g(t)$ (with Fourier transform G(ω))

then the noise waveform is given by

$$eg(t) = \sum_{i=1}^{N} q_i g(t - t_i)$$

Fourier transform of this is given by

$$EG(\omega) = \sum_{i=1}^{N} q_i G(\omega) e^{-i\omega t_i}$$

We have to examine this function as $N \to \infty$, and q correspondingly goes to 0.

Well-known characteristics of the noise waveform

Let $eg(t)$ be the noise output (voltage) at time t. This is a real random number at time t.

$$eg(t) = \sum_{i=1}^{N} q_i g(t - t_i) \Rightarrow EG(\omega) = \sum_{i=1}^{N} q_i G(\omega) e^{-i\omega t_i}$$

1) Probability Density Function $f(eg(t))$ is independent of t. (stationarity).

2) The joint probability density of $f(eg(t),\ eg(t'))$ only depends on $(t - t')$

3) The mean $\langle eg(t) \rangle = 0$ (just for convenience).

It is well known that $f(eg(t))$ has a Gaussian distribution (Find out how to prove this)

These items are needed to specify all characteristics of $eg(t)$.

The variance is given by $e_{rms} = \langle eg(t)^2 \rangle = P$

$ac(\tau) = \langle eg(t).eg(t+\tau) \rangle$...Autocorrelation.

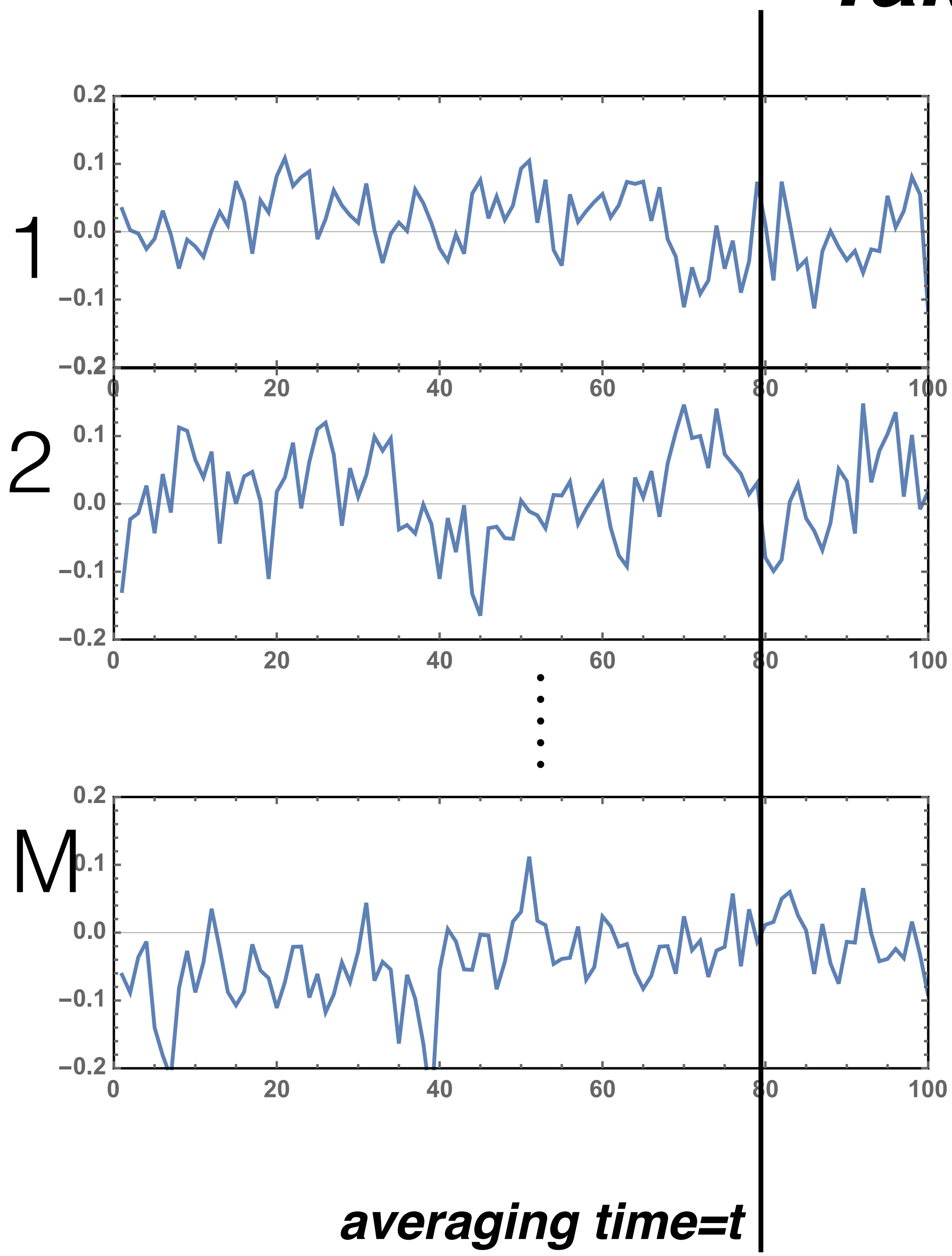
We will show that the $ac(t)$ is simply the inverse Fourier transform of the power spectrum.

In actuality, one can write down multi-point autocorrelation

$\mathrm{ac}_n(\tau_1, \tau_2, ..., \tau_n) = \langle eg(t) eg(t+\tau_1) ... eg(t+\tau_n) \rangle$

For a stationary process all of these must not depend on t.

Taking the average



The average value of $eg(t)$ is not an average over time $(0 \rightarrow T)$, but an average over great many intervals of length T with t held fixed. Imagine that g(t) is a pulse much shorter than T, and rate r =N/T constant.

Then let the number of intervals go to infinity.

$$\langle eg(t) \rangle = r \langle q \rangle \int_{-\infty}^{\infty} g(t)\,dt$$

This is the average value of the pulse multiplied by the rate.

$$\langle eg(t)^2 \rangle - \langle eg(t) \rangle^2 = r \langle q^2 \rangle \int_{-\infty}^{\infty} g(t)^2\,dt$$

This proof takes a bit of work, ...

This is called Campbell's theorem. In its most general form, the averages of powers of g(t) form the coefficients of expansion of the logarithm of the charateristic function. We leave this for you to investigate.

Autocorrelation and the power spectrum

$$ac(\tau) = \int_{-\infty}^{\infty} eg(t)\cdot eg(t+\tau)dt$$ is the autocorrelation function for the noise.

Convert to Fourier transform $EG(\omega) = Fourier[eg(t)]$

$$ac(\tau) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, EG(\omega')e^{i\omega' t} EG(\omega)e^{i\omega t} e^{i\omega\tau}$$

First integrate over t

$$ac(\tau) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega (2\pi\delta(\omega+\omega'))EG(\omega)EG(\omega')e^{i\omega\tau}$$

Switch $\omega' \to -\omega$, recall that EG is Hermitian $EG(-\omega) = EG^*(\omega)$

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|EG(\omega)\right|^2 e^{i\omega\tau}$$

This is called the Weiner-Khinchin theorem. It states that the autocorrelation is the inverse Fourier transform of the power spectrum of the noise.

Therefore only the power spectrum is needed to specify the characteristics of the noise. When τ=0, we end up with Parseval's theorem.

$$\int_{-\infty}^{\infty} eg(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|EG(\omega)\right|^2$$ We now need to figure this in terms of g(t) or G(ω)

This is the simple version of this

We now work on the previous result for white noise and relate it to the impulse response function of the system.

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|EG(\omega)\right|^2 e^{i\omega\tau}$$

Goes away for $t_i - t_j \neq 0$

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{i=1}^{N}\sum_{j=1}^{N} q_i q_j G(\omega) G^*(\omega) e^{-i\omega(t_i - t_j)} e^{i\omega\tau}$$

as $N \to \infty$ and if all q_i have the same magnitude

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G(\omega) G^*(\omega) \bullet N \bullet \left|q\right|^2 \bullet e^{i\omega\tau}$$

If we set $q = \pm Q/\sqrt{N}$ where Q is the elementary noise charge then

$$ac(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left|G(\omega)\right|^2 \bullet Q^2 \bullet e^{i\omega\tau}$$

This shows that the noise characteristics for white noise after a filter are only dependent on the power spectrum (or transfer function) of the filter.

I have played a small trick to make this simple.

And so we finally have

$$\left\langle eg(t)^2 \right\rangle - \left\langle eg(t) \right\rangle^2 = r \times Q^2 \int_{-\infty}^{\infty} g(t)^2\, dt = r \times \frac{Q^2}{2\pi} \int_{-\infty}^{\infty} \left| G(\omega) \right|^2 d\omega$$

This states that variance (RMS) of the noise depends on the filter power spectrum. The argument was done starting with white noise, but is general. The noise has a finite variance only if the integral converges. This depends on the nature of the filter function and its poles.

What do we do if the noise at the input is not white ?

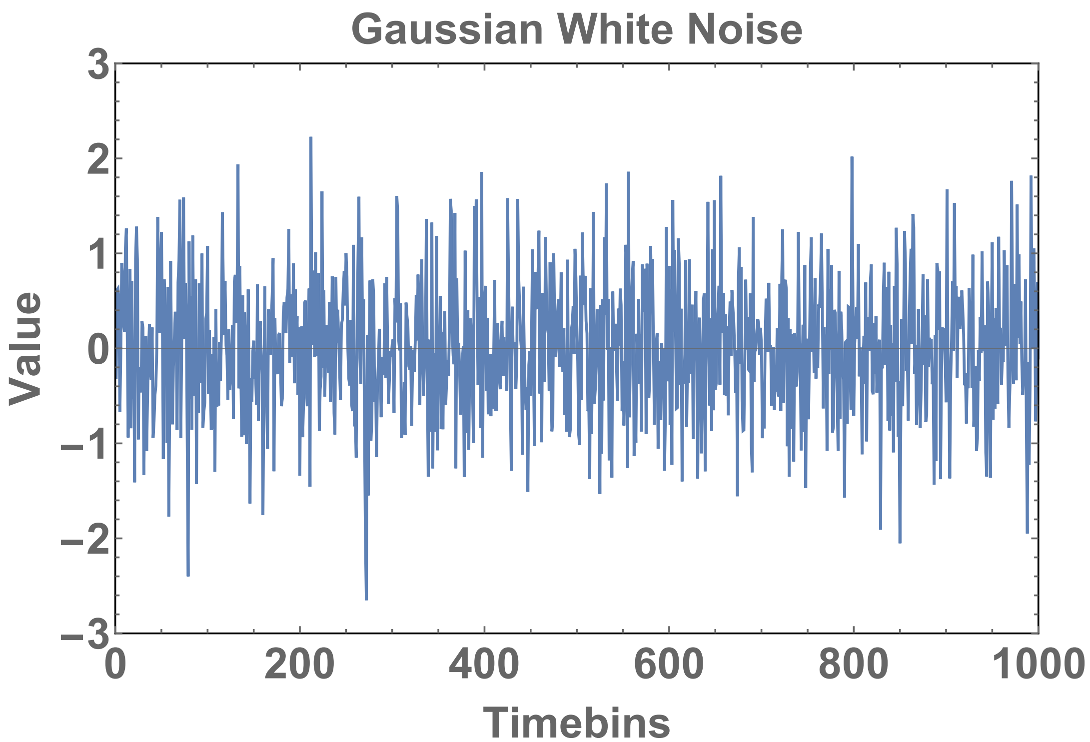
a) Numerically, we can absorb the additional frequency dependence into $G(\omega)$
b) Clearly the native input noise must also be limited in frequency domain and so additional terms may be needed in the model.
c) The best might be to use an empirical model using measurements of the system.
d) In practice there should be no divergence, but there are interesting cases where noise will wander off to very high values.

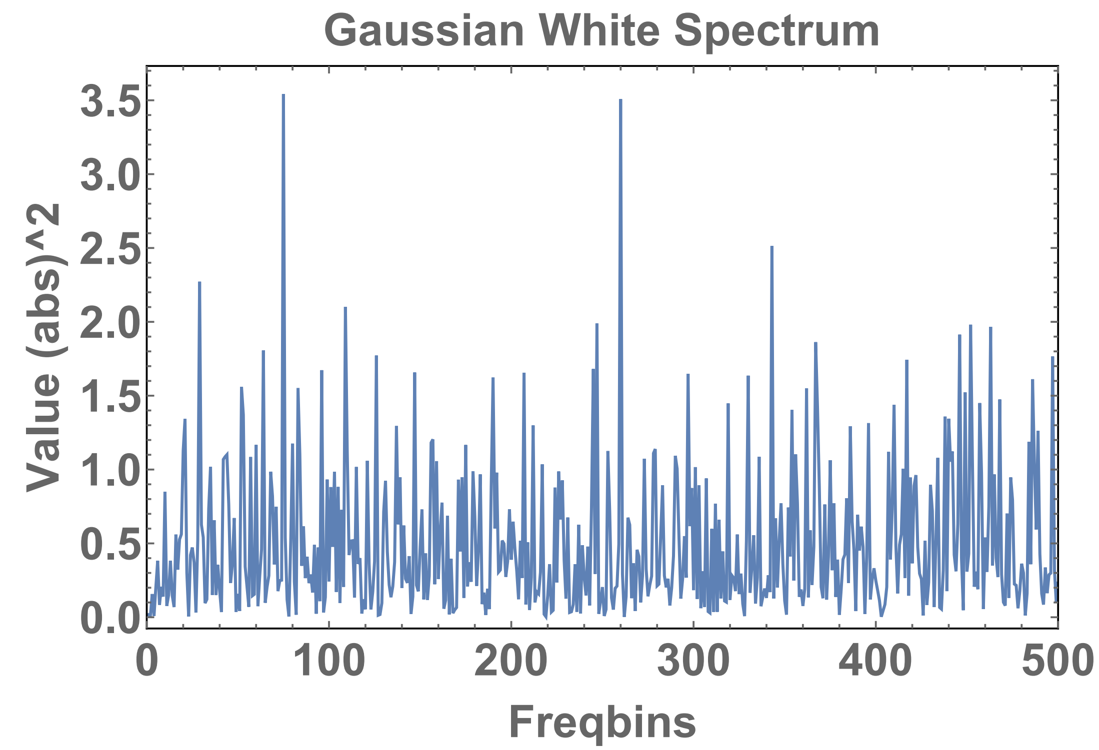
Power spectrum of noise from detector systems

There is a vast literature on this subject

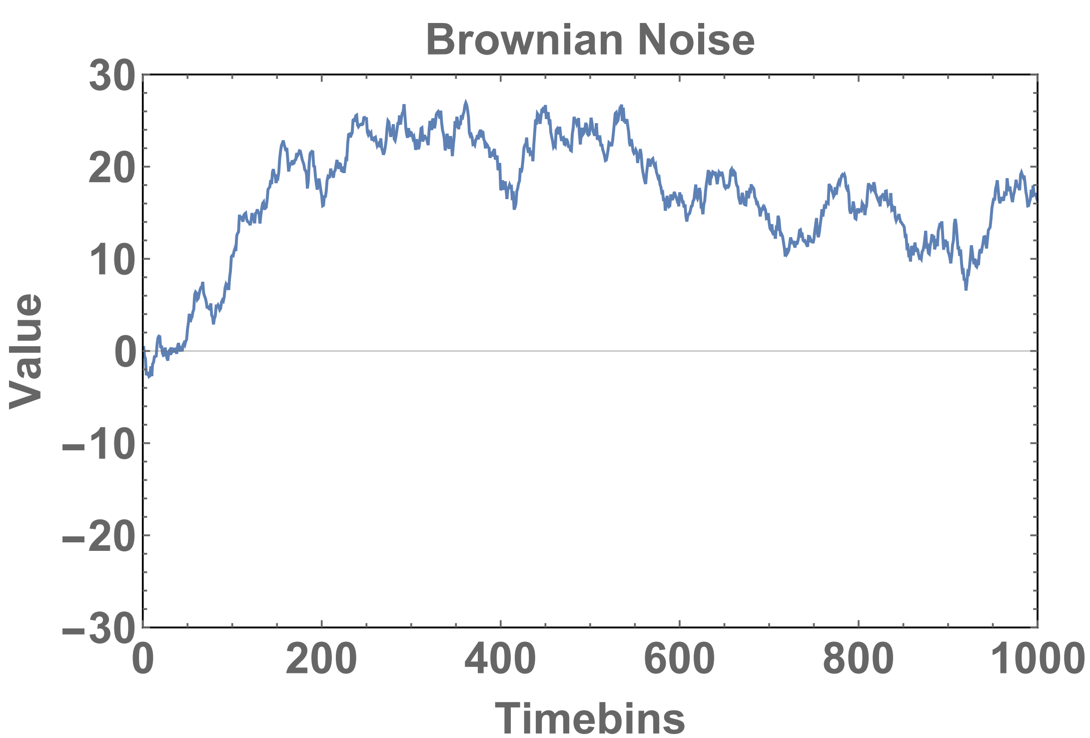
Noise type	Pulse Model	Power spectrum	Divergence	Practical devices
White	train of $\delta(t)$	~ 1	Total noise diverges at infinite frequency	High frequency cutoff
Brownian motion	Integral of train of $\delta(t)$	$\sim 1/f^2$	Total noise power diverges at 0 frequency	Low frequency cutoff
Flicker or 1/f	Train of pulses that have a $1/\sqrt{(t-t_0)}$ fall	$\sim 1/f$	Total noise diverge at both 0 and infinity	Both high and low cutoff

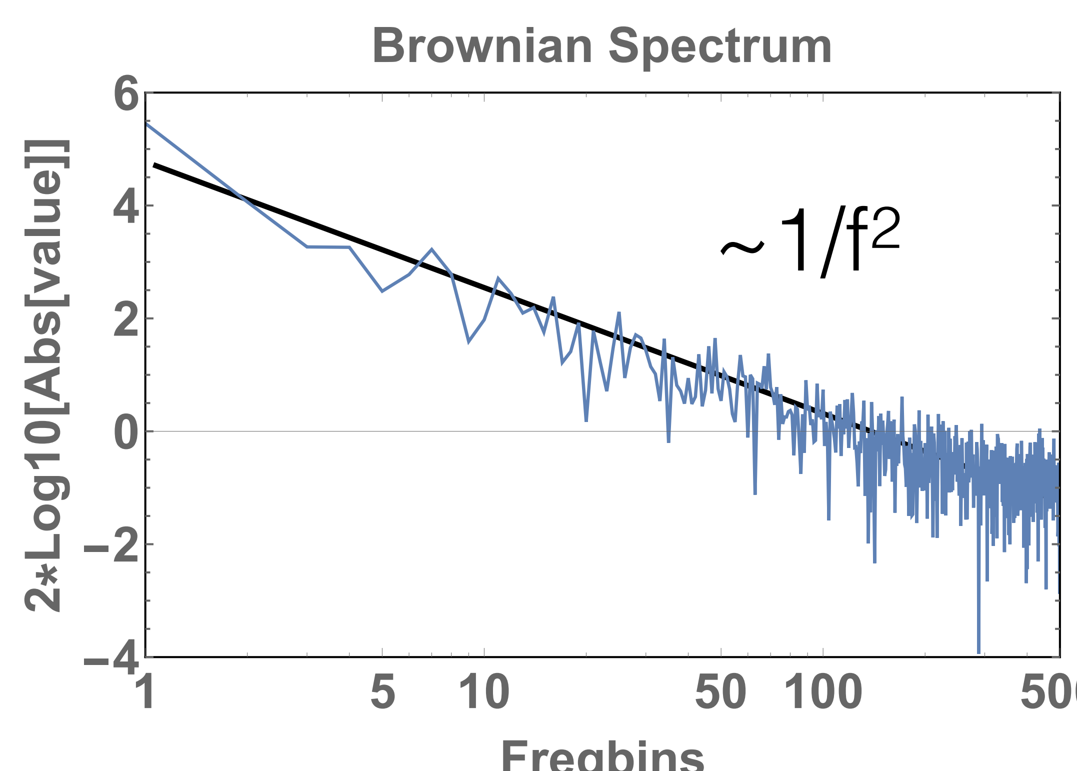
These infinities do not happen because practical devices (including analysis processing) have cutoffs at low and high frequencies. These cutoffs need to be understood to evaluate the result.





Integration of white noise is Brownian noise





eventually this looks the same forward or backwards in time. It is not possible to figure our the direction of time from a snippet of data.

closer look at Brownian motion noise

Part of the system noise could be due to integration of white noise.

What is the spectral power of such noise ?

Recall $Fourier[\int_{-\infty}^{t} f(x)dx] = \frac{1}{\sqrt{2\pi}}(\frac{F(\omega)}{i\omega} + \pi F(0)\delta(\omega))$

Start with white noise $e(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} s_i\delta(t-t_i)$ $\quad s_i = \pm 1$ and t_i are random.

Fourier transform of the integral of $e(t)$ is

$$E(\omega) = \frac{1}{\sqrt{2\pi N}}\sum_{i=1}^{N} s_i(\frac{e^{-i\omega t_i}}{i\omega} + \pi\delta(\omega))$$ ...second term is the dc component

$$E(\omega)\cdot E^*(\omega) = \frac{1}{2\pi N}\left[\frac{N}{\omega^2} + \pi^2\sum s_i s_j \delta^2(\omega)\right]$$ second term zero for no dc component

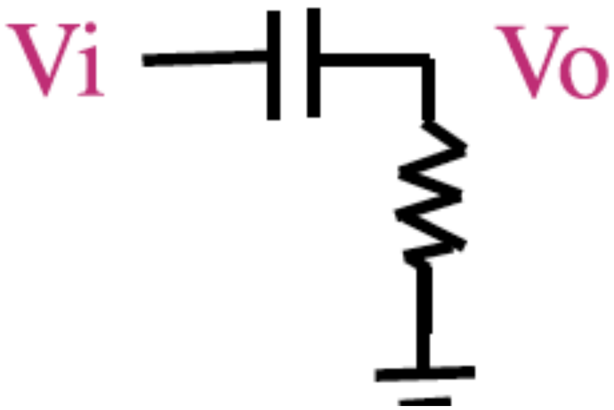
This converges at high ω, but diverges at zero.

If there is an additional shaping filter $G(\omega)$ then the spectral power is given by

$$S_{brownian}(\omega) = \frac{1}{2\pi}\left[\frac{G(\omega)\cdot G^*(\omega)}{\omega^2}\right]$$

To make sure this converges there must be a filter that cancels out the denominator.

Or in another words differentiates the waveform.



High pass, differentiator

With the application of a high pass filter with time constant τ the Brownian noise can be relaxed so that it does not diverge at low frequency.

$$S_{brownian-relaxed}(\omega)=\left[\frac{|G(\omega)|^2}{\omega^2}\times\frac{\omega^2}{(\omega^2+1/\tau^2)}\right]$$

$$S_{brownian-relaxed}(\omega)=\left[\frac{|G(\omega)|^2}{(\omega^2+1/\tau^2)}\right]$$

If G(ω)=1 then the filter simply is flat for ω<<1/τ and falls as $1/\omega^2$ for ω>>1/τ

$$V_o(t)+Q/C=V_i(t)$$

$$\frac{dV_o(t)}{dt}+I/C=\frac{dV_i(t)}{dt}$$

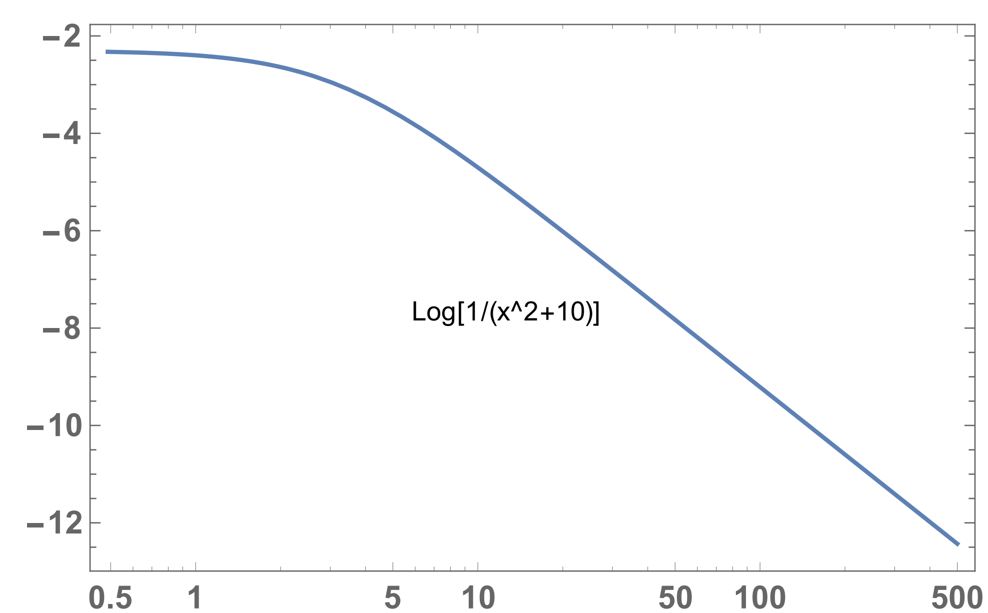
Use Fourier Transforms to solve

$$F[V(t)]=V(\omega),\ F[V'(t)]=i\omega V(\omega)$$

$$i\omega V_o(\omega)+V_o(\omega)/RC=i\omega V_i(\omega)$$

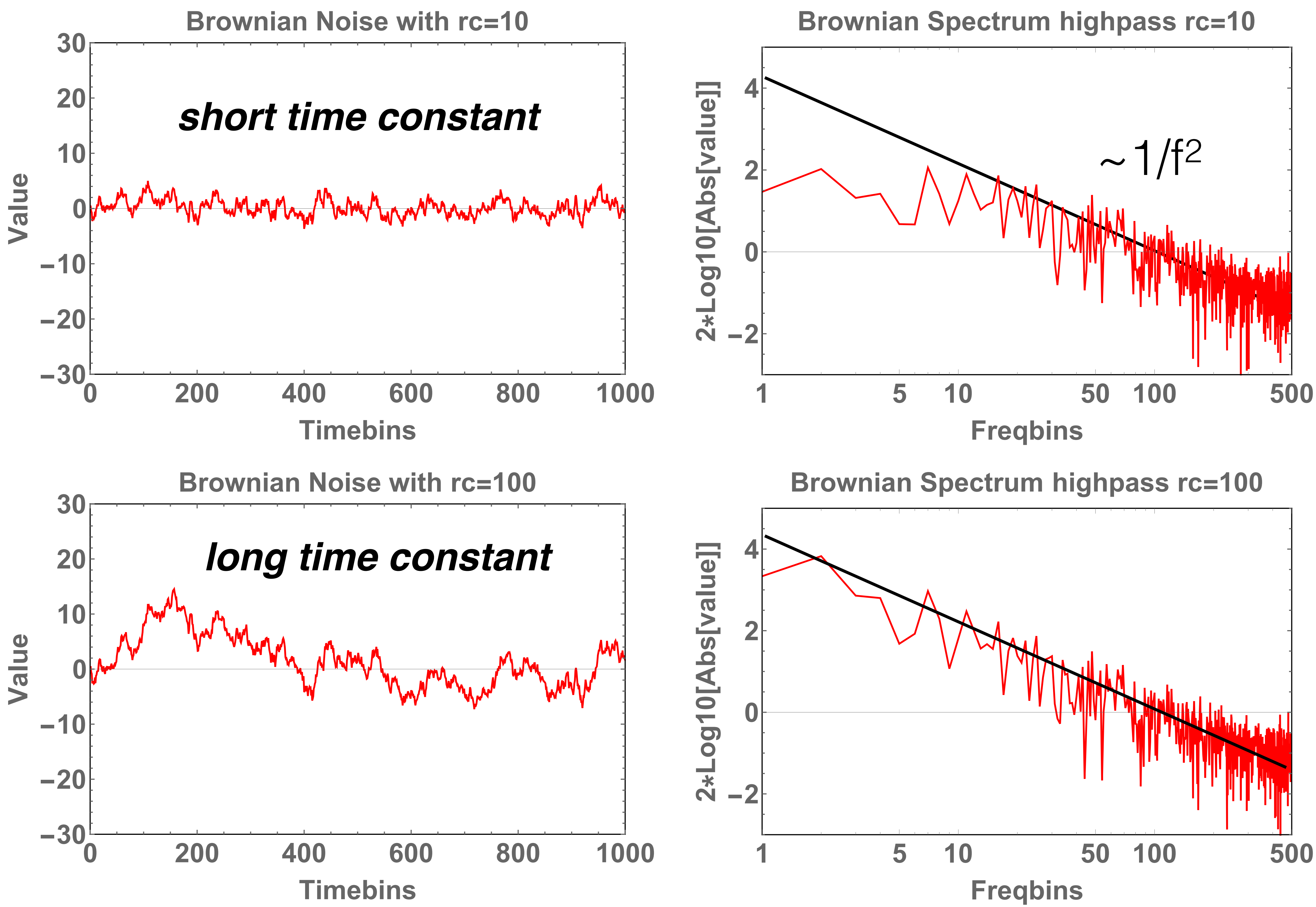
$$V_o(\omega)=V_i(\omega)\times\frac{i\omega}{i\omega+1/\tau}$$

$$\left|\frac{V_o}{V_i}\right|^2=\frac{\omega^2}{(\omega^2+1/\tau^2)}$$

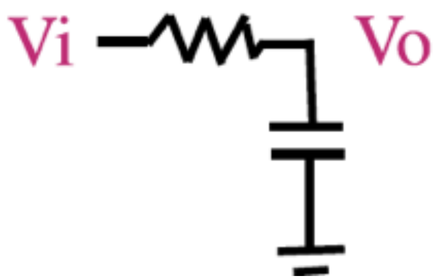


Large capacitors charge up spontaneously and need to have a discharging resistor even while being stored.

Brownian noise after application of high pass filter



Origin of flicker noise

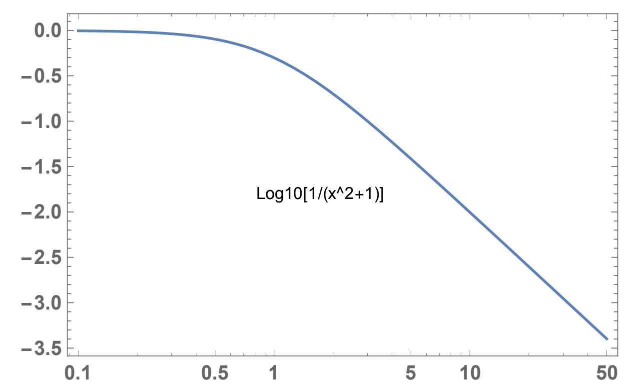


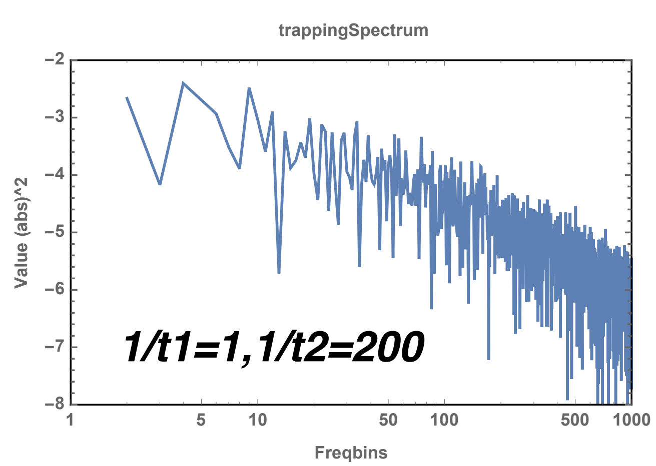
There does not seem to be an obvious mechanism that produces noise with spectrum that varies as $\sim 1/\omega$.

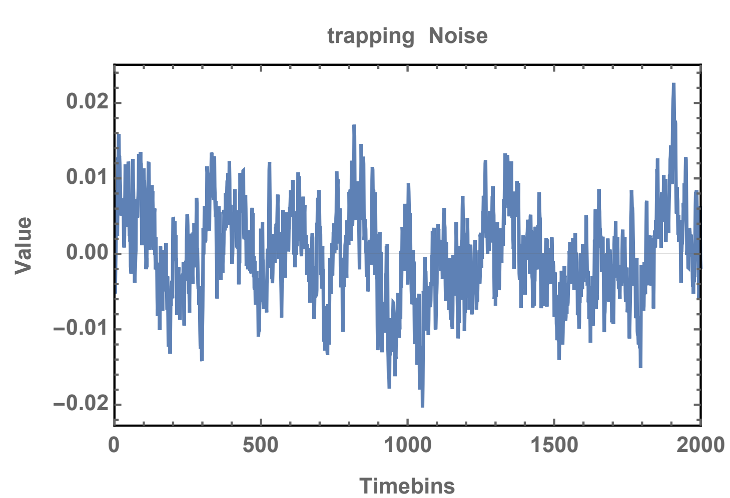
If the elementary pulses go through an exponential filter (τ=RC) then

$$EG(\omega) = \sum_{i=1}^{N} q_i e^{-i\omega t_i} G(\omega) = \sum_{i=1}^{N} q_i e^{-i\omega t_i} \times \frac{1}{1+i\omega\tau}$$

Power spectrum $S(\omega) = Q^2 \frac{1}{1+(\omega\tau)^2}$ falls as $\sim 1/\omega^2$







What if the time constant itself is uniform randomly changing between $1/\tau_1$ to $1/\tau_2$ then the spectrum has to be averaged over this

$$S(\omega) = \frac{1}{1/\tau_1 - 1/\tau_2} \int_{1/\tau_2}^{1/\tau_1} Q^2 \frac{1}{1+(\omega/\lambda)^2} d\lambda$$

The solution to this has a region where

$$S(\omega) \approx \frac{\pi}{2(1/\tau_1 - 1/\tau_2)} \times \frac{1}{\omega} \quad \text{when } 1/\tau_2 < \omega < 1/\tau_1$$

We can do this by Monte Carlo.

Flicker noise. What kind of pulse in time domain ?

White noise has a flat frequency spectrum $S_w(\omega) \propto 1$

Brownian motion noise is the integral of white and $S_B(\omega) \propto \frac{1}{\omega^2}$

What kind of noise has a power law $S_F \propto \frac{1}{\omega^{1+\alpha}}$ with $\alpha < 1$?

Construct Fourier transform of $(1/\omega)$ noise: $F_F(\omega) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{-i\omega t_i} \times \frac{1}{\sqrt{\omega}}$

(In mathematics this represents a fractional integral in time domain.)

Perform Inverse Fourier of $F_F(\omega)$. Just take one pulse at t_0

$$f_F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega + \int_{0}^{\infty} \frac{1}{\sqrt{\omega}} e^{-i\omega t_0} e^{i\omega t} d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \frac{1}{\sqrt{-\omega}} e^{+i\omega(t_0 - t)} d\omega + \int_{0}^{\infty} \frac{1}{\sqrt{\omega}} e^{+i\omega(t - t_0)} d\omega \right]$$

This is a crazy integral

This integral has a branch cut because of the $\sqrt{\omega}$. The answer is

$$f_F(t) = \begin{cases} 2/\sqrt{t - t_0} & t > t_0 \quad = \quad J_+ \\ 0 & t \le t_0 \\ 0 & t \ge t_0 \\ 2/\sqrt{t_0 - t} & t < t_0 \quad = \quad J_- \end{cases}$$

A linear combination of J_+, J_-

(advanced and retarded)
This is why this noise is not time reversible.
In retarded case, the noise dies off as $1/t^{0.5}$ after a pulse.

Such a pulse is very problematic because it gives very long time correlations. The past never goes away.

Examples of 1/f spectra

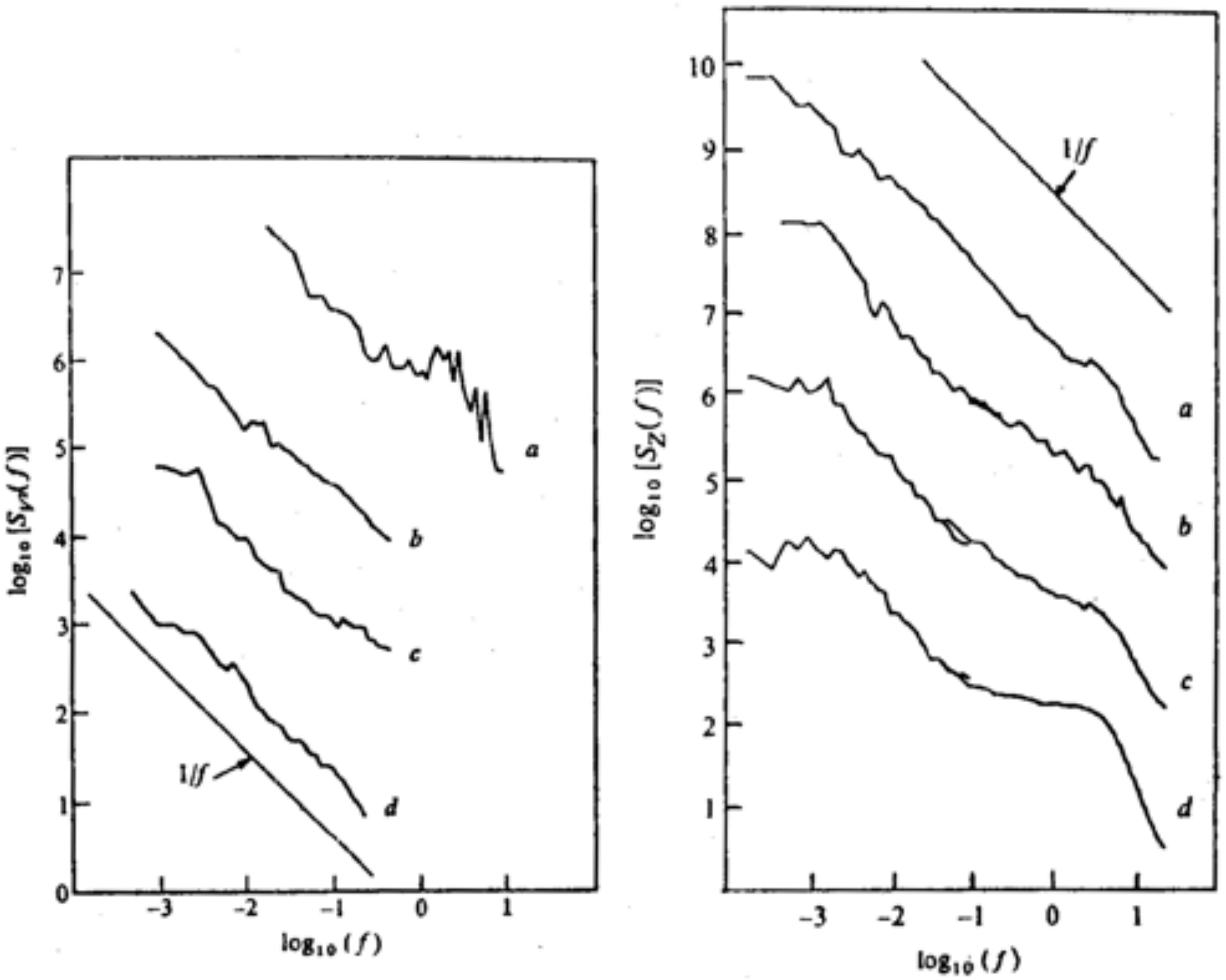
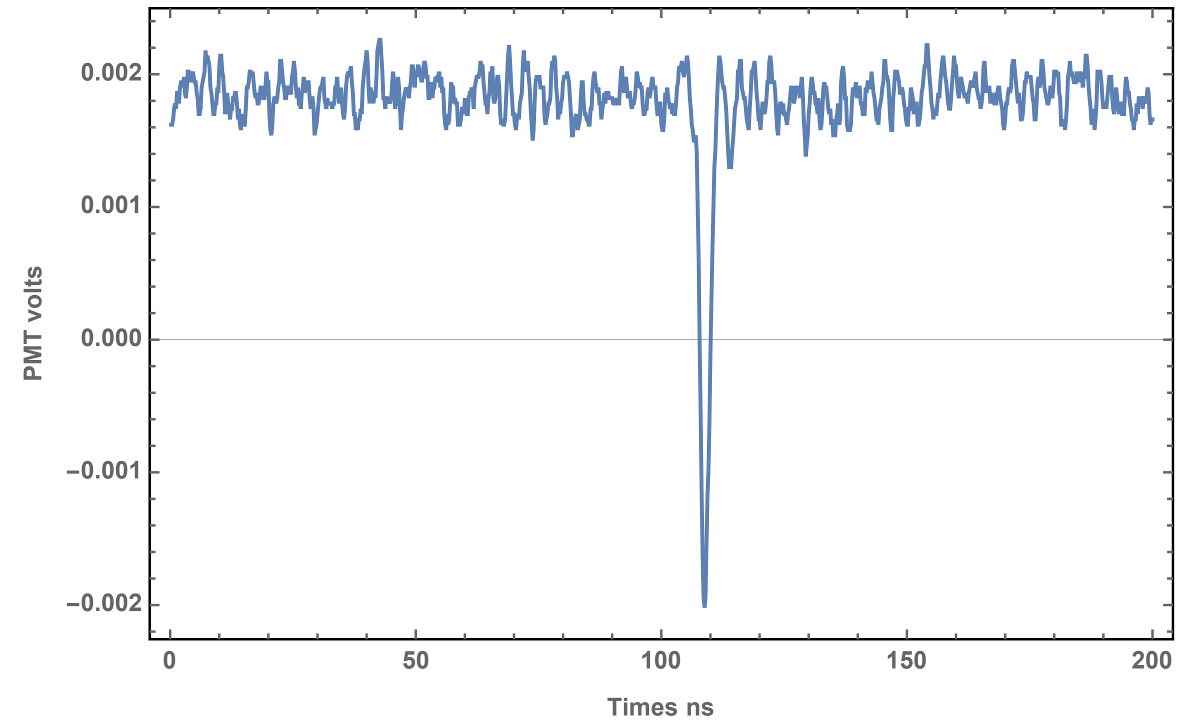
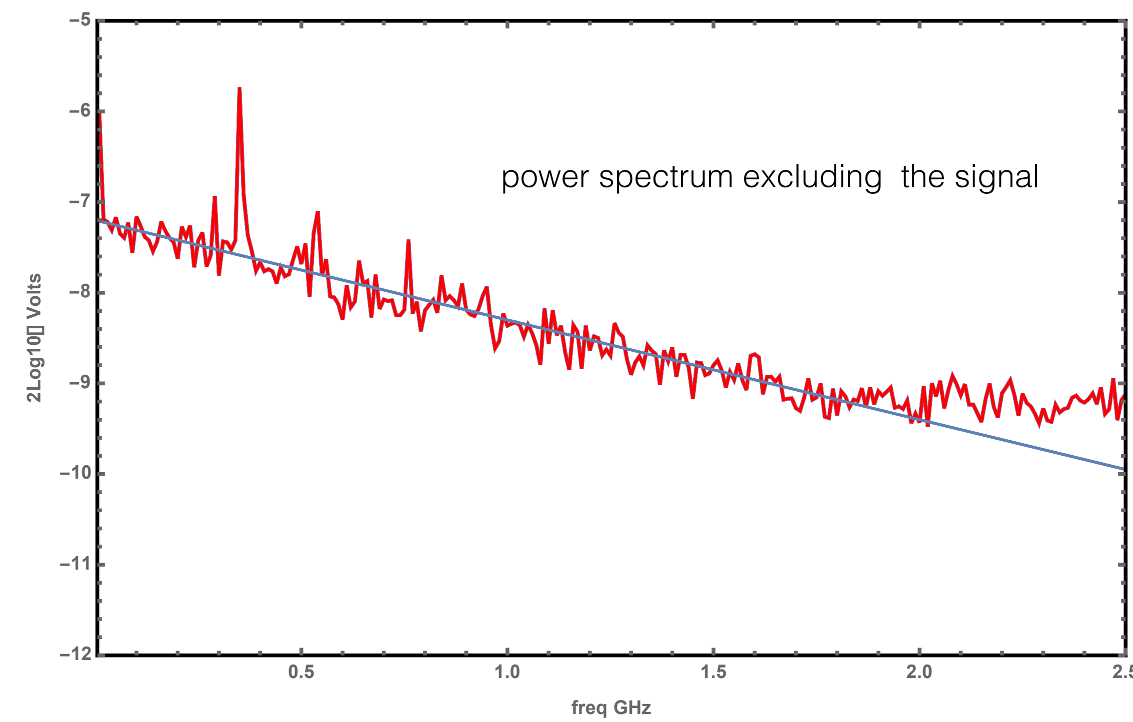


FIGURE 9 (*left*) Loudness fluctuation spectra as a function of frequency f (in Hz) for: *a*, Scott Joplin Piano Rags; *b*, classical radio station; *c*, rock station; *d*, news-and-talk station. (Reproduced from Voss and Clarke.[23]) (*right*) Power spectra of pitch fluctuations for four radio stations: *a*, classical; *b*, jazz and blues; *c*, rock; *d*, news-and-talk. (Reproduced from Voss and Clarke (1975).)

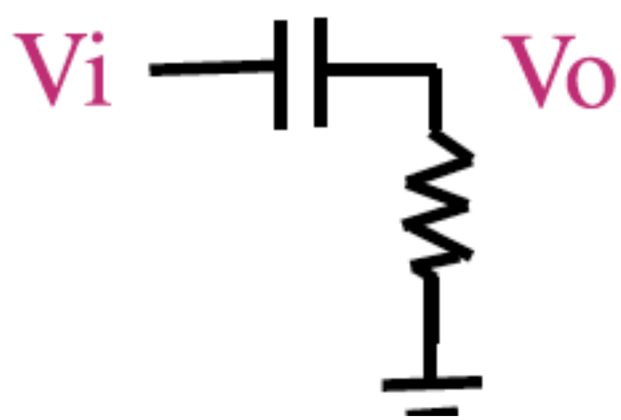




This noise is deeply related to the structure of the device and therefore could be sensitive to failure modes.

Measurement of light pulses from a PMT by Aiwu Zhang

High pass, differentiator

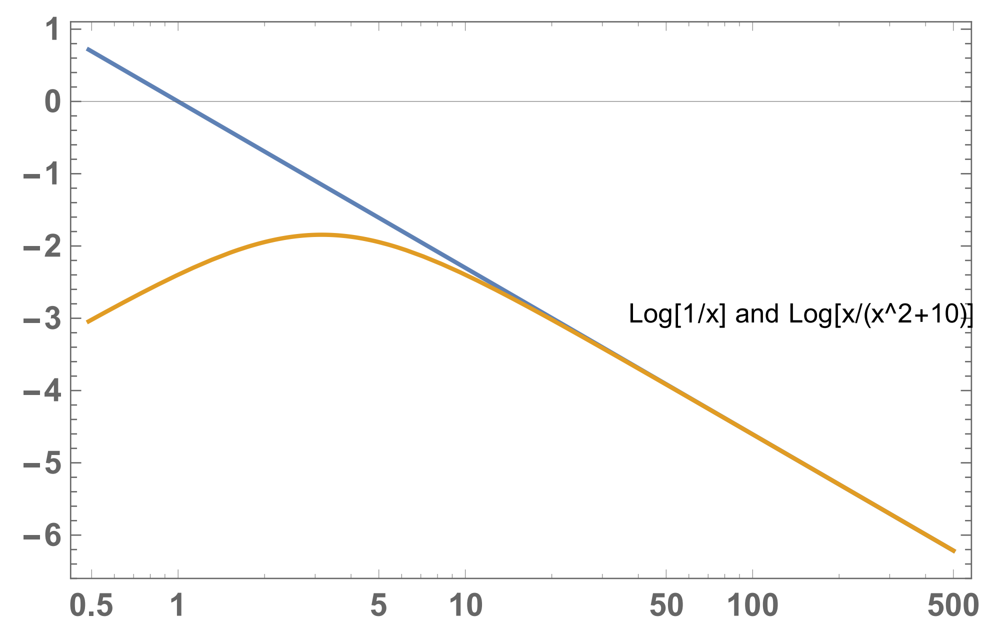


With the application of a high pass filter with time constant τ the 1/f noise can be relaxed so that it does not diverge at low frequency.

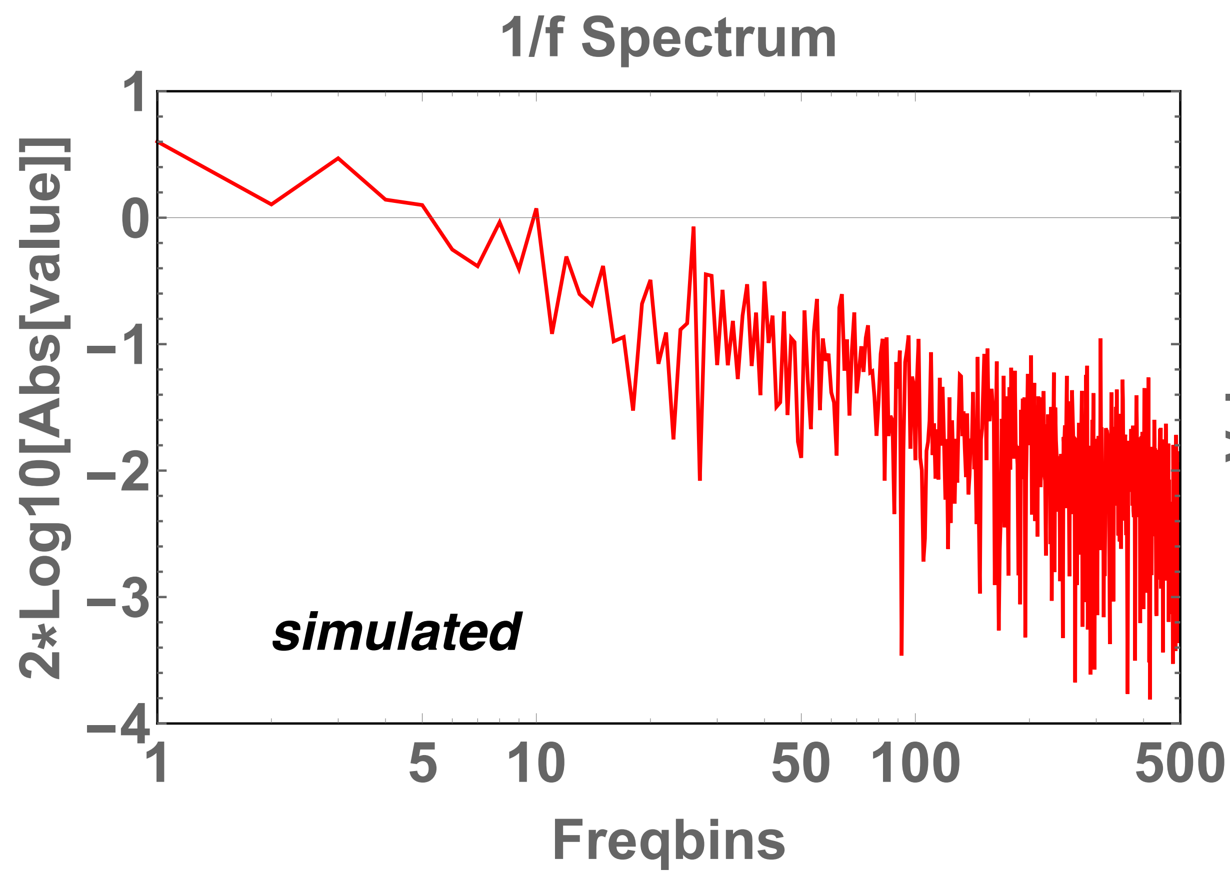
$$S_{Flick}(\omega) = \frac{1}{2\pi}\left[\frac{|G(\omega)|^2}{\omega} \times \frac{\omega^2}{(\omega^2 + 1/\tau^2)}\right]$$

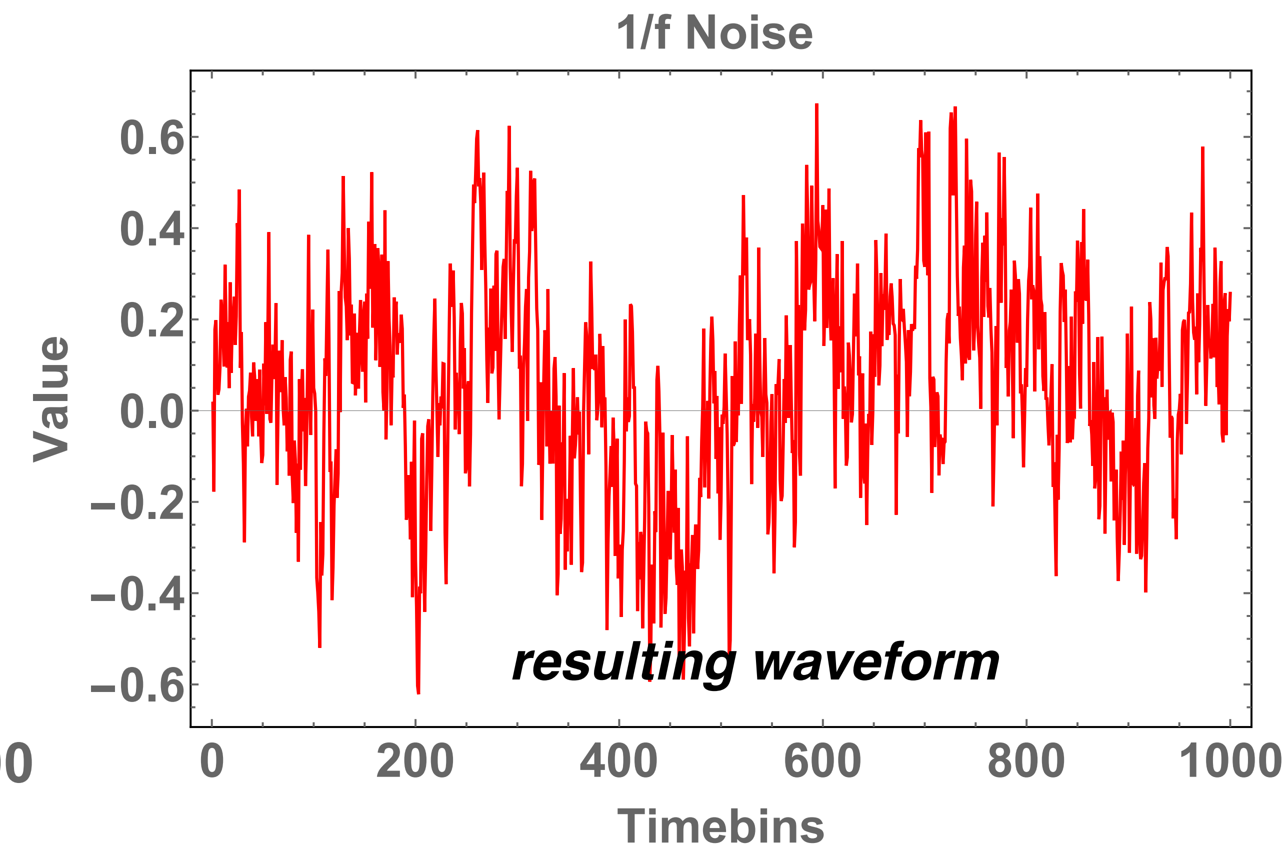
$$S_{Flick}(\omega) = \frac{1}{2\pi}\left[\frac{|G(\omega)|^2 \times \omega}{(\omega^2 + 1/\tau^2)}\right]$$

If $G(\omega)=1$ then the filter gives 0 for DC and rises for $\omega << 1/\tau$ and falls as $1/\omega$ for $\omega >> 1/\tau$

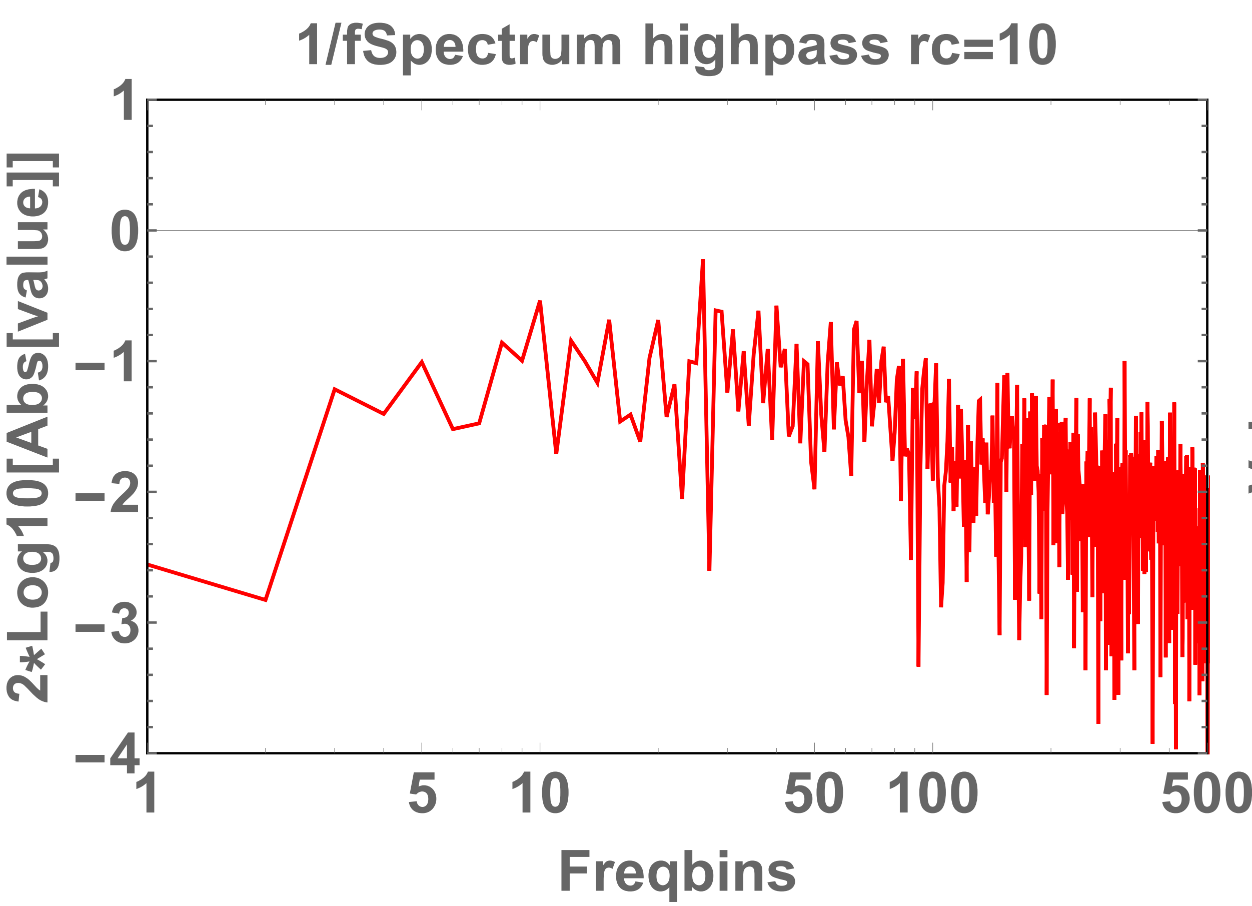


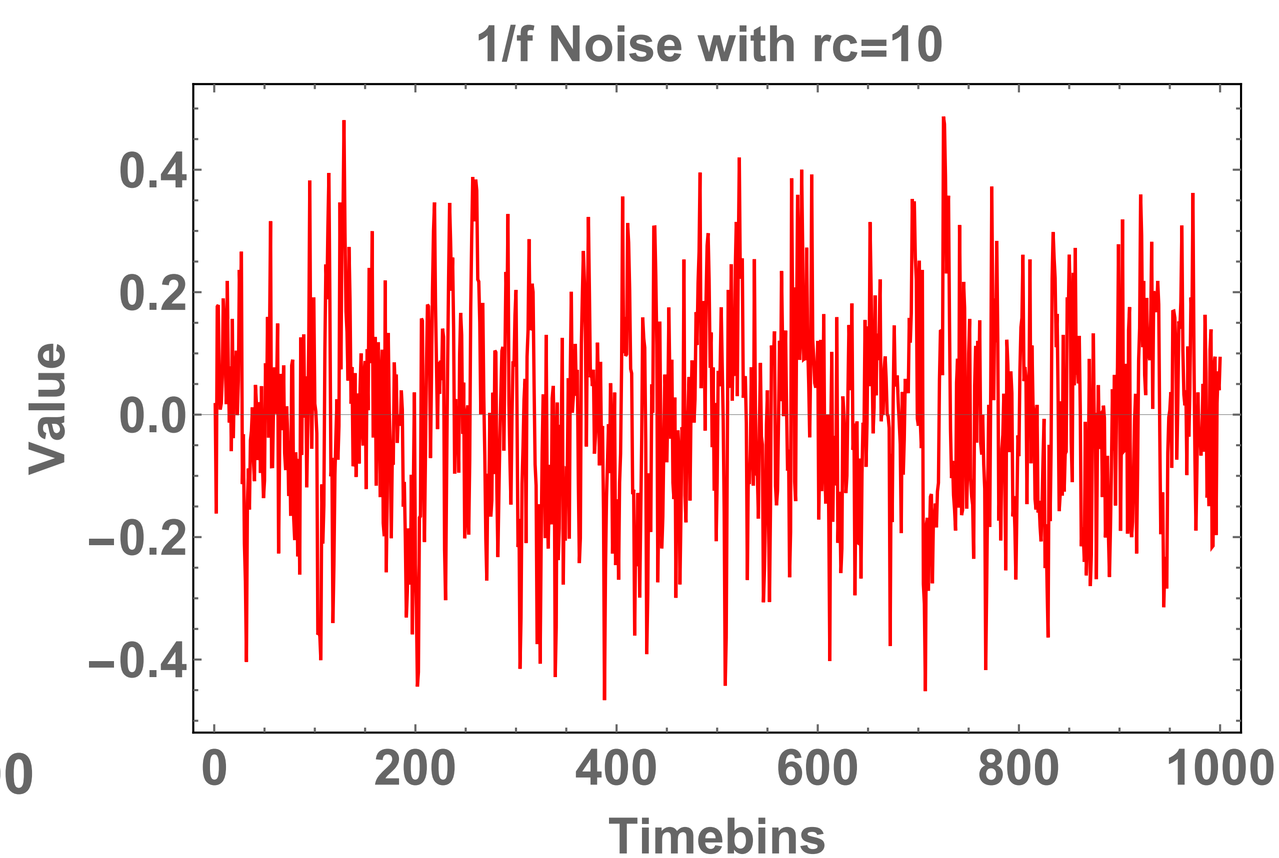
Notice that the power is zero for at f=0 regardless of the value of the time constant.





Sounds like a water fall





Summary

- ***We learned or reviewed the use of Fourier/Laplace transforms for linear systems.***
- ***We learned how they are used to understand electronics and mechanical systems. (There are sophisticated codes for doing this).***
- ***The tools of using delta functions can be easily extended to perform Monte Carlo calculations.***
- ***We learned about classification of noise spectra and their origin.***
- ***I have left a lot of important details out. It is easy to follow up using many papers on the web or textbooks. Try to derive the relations yourself.***