

$$\cancel{d\phi} \{ \} = \{ GM + O(GM)^2 \}$$

$$= (GM) \left\{ 1 + \dots GM + \dots \right\}$$

↓
this will cancel the GM in K

$$\left[\text{Recall } \Rightarrow d\phi = \frac{K}{r^2} \sqrt{\lambda(r)} dr \times \dots \right]$$

$$\left[\right] = \text{constant} - \frac{1}{r^2} \left(\frac{1}{r_+^2} - \frac{1}{r_-^2} \right)^{-1} \left\{ \frac{1}{4(r_+)} - \frac{1}{4(r_-)} \right\} \\ + 1 + \frac{2GM}{r} + \frac{4G^2 M^2}{r^2}$$

Quadratic in $\frac{1}{r}$

So it must be of the form

$$C \left(\frac{1}{r} - \alpha \right) \left(\frac{1}{r} - \beta \right) = C \left(\frac{1}{r} - \frac{1}{r_+} \right) \left(\frac{1}{r} - \frac{1}{r_-} \right)$$

Coefficient
of $\frac{1}{r^2}$

$\frac{1}{r_+}$

$\frac{1}{r_-}$

(since r_+ & r_- are roots of $\frac{d\phi}{dr} = 0$)

$$\text{Also, } C = - \left(\frac{1}{r_+^2} - \frac{1}{r_-^2} \right)^{-1} \times \left\{ 1 + \frac{2GM}{r_+} + \frac{4G^2 M^2}{r_+^2} \right. \\ \left. - \frac{2GM}{r_-} - \frac{4G^2 M^2}{r_-^2} \right\}$$

$$+ 4G^2 M^2$$

$$\Rightarrow C = GM \left[- \left(\frac{1}{r_+^2} - \frac{1}{r_-^2} \right)^{-1} \left\{ \frac{2}{r_+} + \frac{4GM}{r_+^2} - \frac{2}{r_-} - \frac{4GM}{r_-^2} \right\} \right. \\ \left. + 4GM \right]$$

∴ we have,

Ex →
$$d\phi = \frac{ds}{r} \left(1 + \frac{MG}{r}\right) \left[\left(\frac{1}{r} - \frac{1}{r_+}\right) \left(\frac{1}{r} - \frac{1}{r_-}\right) \right]^{-1/2}$$

$\times \left\{ 1 + \frac{MG}{r} \left(\frac{1}{r_+} + \frac{1}{r_-}\right) \right\}$

↓
this should be there to ensure that for $r_- < r < r_+$ the square root term is real

• Change of variables :-

① $x = \frac{1}{r}$ $-\int_{1/r_-}^{1/r_+} dx \dots$

↓ ↓ gives half the shift in ϕ — multiply by 2 to get the complete ans.

② $x - \frac{1}{2} \left(\frac{1}{r_+} + \frac{1}{r_-}\right) = \frac{1}{2} \left(\frac{1}{r_-} - \frac{1}{r_+}\right) \sin \alpha$

$-\int_{\pi/2}^{-\pi/2} d\alpha \dots$

Ex. Show that
$$\phi(r_+) - \phi(r_-) = \pi \left\{ 1 + \frac{3}{2} \frac{MG}{r} \left(\frac{1}{r_+} + \frac{1}{r_-}\right) \right\}$$

$$\Delta\phi \Big|_{\text{total}} = 2\pi \left\{ 1 + \frac{3}{2} \frac{MG}{r} \left(\frac{1}{r_+} + \frac{1}{r_-}\right) \right\}$$

↓ [change bet. 2 max. (or 2 min.)]

∴ the orbit precesses by an angle of

$3\pi \frac{MG}{r} \left(\frac{1}{r_+} + \frac{1}{r_-}\right)$ per revolution.

• (It is apparent from this that the larger the r is, the smaller the precession is. Closer the orbit is to the sun, larger is the precession.)

For Mercury :- 43 arcsecond / 100 years.

