\[ f_1(x', F(x', x')) = f_1(x', x') \]

For \( x' \neq x' \),

\[ f_{3ij}(x', F) \frac{\partial F^i}{\partial x'^k} \frac{\partial F^j}{\partial x'^l} = f_{3kl}(x', x') \]

---

**Homogeneity**:

Given any two points \((x, x')\) & \((x, x')\), there is an isometry \( F \) such that

\[ F(x, x') = x' \]

This implies \( f_1(x', y') = f_1(x', x') \) for any pair of points in space.

[follows from (3)]

\[ f_1(x', x') = f_1(x', x') \] (indep. of \( x' \))

(Isotropy is not going to give any further restriction on \( f_1 \) being away it is independent of \( x' \)).

\[ f_{3ij}(x', F) \frac{\partial F^i}{\partial x'^k} \frac{\partial F^j}{\partial x'^l} = f_{3kl}(x', x') \]

---

Fix \( x' \) (think of it as a parameter without thinking of spacetime) & think of \( f_{3ij} \) as a 3-dimensional metric.

Then, \( x' = F^i(x') \) is an isometry of the 3-dimensional metric \( f_{3ij}(x', x') \) for every \( x' \).
\[ dS^2 = f_3 (x', x) \, dx' \, dx \]

& make a coord. trs. \( x' = F(x) \)
then this is an isometry of the 3-d metric

\[ \text{if} \quad f_3 (x', F) \frac{\partial F^1}{\partial x^1} \frac{\partial F^3}{\partial x^3} = f_3 (x, x') \]

is satisfied.

**List of homogeneous and isotropic metrics**
in \( D = 3 \)

1. **Flat space:**

\[ f_3 (x') \, dx^1 \, dx^2 \]

\[ = \int (x) \left\{ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right\} \]

arbitrary for of time

\[ x \rightarrow x' + \alpha' : \text{establishes homogeneity} \]

\[ \text{SO(3) rotation around origin} \rightarrow \text{isometry} \]

2. **Surface of a 3-dimensional sphere \( S^3 \):**

\[ \sum_{i=1}^{4} (x_i')^2 = R^2 \]

\[ ds^2 = d\Omega^2 + dz_i^2 \]

\[ \text{SO(3)} \]

\[ \text{Given any 2 pts. on the 2D sphere \( S^2 \), there is a rot. about an origin which takes first pt. to the other pt.} \]
Constant -ve curvature metric:
\[ z^2 - (z_1^2 + z_2^2 + z_3^2) = R^2 \]
\[ ds^2 = -(dt)^2 + \sum_{i=1}^{3} (dx_i)^2 \]

A homogeneous metric should have a constant 3-d curvature.
But a constant 3-d Ricci scalar doesn't imply a homogeneous metric, because it only gives one property of the space.

Isometries:
- \( x = t' \)
- \( x_i = \Phi_i(x', t') \)

\[ ds^2 = f_{3ij}(x', t') \, dx_i \, dx_j + \sum_{i=1}^{3} (dx_i)^2 \]

List of homogeneous & isotropic metrics in \( D = 3 \):

1. Flat space:
\[ f_{3ij} = f_{ij} = \delta_{ij} \]
\[ ds^2 = \sum_{i=1}^{3} (dx_i)^2 + r^2 \sin^2 \theta \, dr^2 + r^2 \, d\theta^2 \]

2. 3-dimensional sphere \( S^3 \):
\[ (z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 = R^2 \]
\[ ds^2 = \sum_{i=1}^{3} (dx_i)^2 + d^2 z_4^2 - \theta^2 \]

Embedding space metric.
\[ z_4 = R \cos \Psi \]
\[ z_3 = R \sin \Psi \cos \Phi \]
\[ z_1 = R \sin \Psi \sin \Phi \cos \Theta \]
\[ z_2 = R \sin \Psi \sin \Phi \sin \Theta \]

Ex. Check that
\[ ds^2 = ds^2_{S^3} = R^2 \left( \frac{dr^2}{1-r^2} + r^2 d\Omega^2 + r^2 \sin^2 \Phi \cos \Theta \sin \Phi \right) \]

\[ r = \sin \Psi \]
\[ dr = \cos \Psi \ d\Psi + \sqrt{1-x^2} \ d\Psi \]
\[ ds^2_{S^3} = R^2 \left( \frac{dr^2}{1-r^2} + r^2 \frac{d\Theta^2}{1+r^4} + r^4 \sin^2 \Phi \cos \Theta \sin \Phi \right) \]

Now, \( ds^2 \) is invariant under
\[
\begin{pmatrix}
    z_1 \\
    z_2 \\
    z_3 \\
    z_4
\end{pmatrix} \rightarrow
\begin{pmatrix}
    u_{z_1} \\
    u_{z_2} \\
    u_{z_3} \\
    u_{z_4}
\end{pmatrix}
\]

Where \( u^T u = I \) \( \rightarrow 4 \times 4 \) orthogonal matrix

\( (r, \theta, \phi) \) satisfies \( r^2 + \theta^2 + \phi^2 = R^2 \),
then \( (r', \theta', \phi') \) also satisfies \( \frac{r'^2}{1 + r'^2} = R^2 \)

Any \( \Phi \rightarrow \Theta \) is an isometry, analysing the cases \( \Theta \) & \( \Phi \)

→ induces an isometry on the sphere
Take 2 pts. on the sphere $\rightarrow$ rotate it by $U$. Distance is preserved (in distance there is compared with metric of the embedding space $\Rightarrow S^3$ is isotropic)

\[
R' = \begin{pmatrix}
\sin \theta' & \sin \phi' \\
\cos \theta' & \cos \phi'
\end{pmatrix} \cdot R' = \begin{pmatrix}
\sin \theta & \sin \phi \\
\cos \theta & \cos \phi
\end{pmatrix}
\]

(Bcos of orthogonality, one eqn. is trivial)

(Applying the above eqn., we will see that $U_0$ is an isometry of $\mathbb{R}^3$)

1. (Given any 2 pts.) we can find an isometry which takes us from one to another -- this is by $SO(4)$

$\rightarrow$ Homogeneity is satisfied

2. $\{x_0 + 8x_v^2\}$

\[f(x_0 + 8x_v^2) = f(x_0 + 8x_v) - f(x_0) - f(0) \cdot 8x_v\cdot \sin (\theta) \cdot \sin (\phi)\]

So $x_0$ is a symmetry that isometry must be such that even for $\{x_0 + 8x_v^2\}$, $\{x_0 + 8x_v\}$

the same isometry should take from $\{x_0 + 8x_v^2\}$ to $\{x_0 + 8x_v\}$ for isotropy

Note: $\{x_0 + 8x_v^2\}$
Choose a point: $4 \neq 0$

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{pmatrix}
\]

pt. on the sphere & on the z-axis

\[
U = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]

$U^3 \colon 3 \times 3$ orthogonal matrix

This proves the space is also isotropic

(3) **Hyperboloid**

\[
z_4^2 - \left( z_1^2 + z_2^2 + z_3^2 \right) = R^2
\]

\[
ds^2 = - (dz_4)^2 + \sum_{i=1}^{3} (dz_i)^2
\]

$|z_4| > R \overset{\text{this means}}{\Rightarrow} z_4 > R$

or $z_4 < -R$

(We choose one particular branch because our 3-d space is connected & we will never know about the disconnected part)

\[
\begin{array}{c}
A \rightarrow B \\
\rightarrow A \quad B
\end{array}
\]

(These 2 paths aren't deformable to one another if $A \& B$

are identified --- our universe may be like this (not simply connected))

Choose the branch: $z_4 > R \rightarrow$ Three dim space
Coordinates:

\[ \begin{align*}
Z_1 &= R \cosh \gamma \\
Z_2 &= R \sinh \gamma \cos \theta \\
Z_3 &= R \sinh \gamma \sin \theta \\
Z_4 &= R \sinh \gamma \sin \theta \sin \phi \\
\end{align*} \]

Ex:

\[ d\ell^2_3 = d\ell^2_4 \bigg|_{\gamma = 0} = R^2 \left( d\gamma^2 + \sinh^2(\gamma) (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

The particular surface on which the induced metric is

\[ \text{hyperboloid} \]

\[ \text{has an induced Euclidean metric (it need not have been the case for other} \]

\[ \text{surfaces).} \]

\[ \text{at the end whatever metric we get} \]

\[ \text{should describe an Euclidean space} \]

\[ \text{This is imp.} \]

Take:

\[ \gamma = \sinh \gamma \]

Then:

\[ d\ell^2_3 = R^2 \left( \frac{d\gamma^2}{1+\gamma^2} + \sin^2 \theta d\phi^2 \right) \]

4-dim. Lorentz tos. maps a point

on the hyperboloid to another point

on the hyperboloid and preserves the

form of the metric \( \text{SO}(3,1) \) group of

isometries

(The 3d metric is derived from the 4d metric. So if

The 3d metric preserves the 4d metric & the equ

\[ Z_1^2 - Z_2^2 = R^2 \]

it should preserve the 3d metric

as well)
Homogeneous \rightarrow (by \, Lcr. \, Trs. \, we \can \, take \, any \, Lcr. \, vector \, to \, another \rightarrow \, one \, from \the \, surface \, of \, hyperboloid \, to \, another)\)

Isotropy \rightarrow \text{choose} \, y=0, \, z= R \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}

(Find \, a \, set \, of \, isometries \, which \, preserves \this \, pt. \, & \, take \, us \, from \, one \, to \, another \, \textit{of} \, any \, 2 \, pts., \, equivalent \, from \, z.) \, \text{Isometry \, follows \, from:} \quad U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix}

Homogeneous \, and \, isotropic \, \cosmological \, metric: \quad ds^2 = -f_1(t) \, dt^2 + f(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}

k = \begin{cases} 0 & \rightarrow \text{flat space (zero curvature)} \\ 1 & \rightarrow \text{sphere (positive \, \textit{}} \) \\ -1 & \rightarrow \text{hyperboloid (negative \, \textit{}} \) \\
\end{cases}

Under \, t \rightarrow F(t) \, (we \, don't \, violate \, any \, of \, the \, assumptions \, we \, made \, earlier - \, we \, don't \, violate \, the \, idea \, of \, cosmic \, time \, \& \, hence \, the \, cosmological \, principle - \, of \, course \, \, t \rightarrow F(t; \vec{x}) \, isn't \, allowed) \quad -f_1(F(t)) \left( \frac{dF(t)}{dt} \right)^2 \, dt^2 = dt^2

if \, we \, \text{choose} \, F^0 \, \text{such that} \quad \frac{dF^0}{dt} = \left( \sqrt{-f_1(F(t))} \right)^{-1}
\[
\text{\( ds^2 = -dt^2 + \frac{f (F^0 (U'))}{\left( \frac{\delta}{\delta x^2} \right)^2} \left\{ \frac{dr^2}{1-b^2} + r^2 d\theta^2 \right\} \) }
\]

\[
\Rightarrow \text{\( ds^2 = -dt^2 + (\lambda (t))^2 \left\{ \frac{dr^2}{1-b^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\} \) }
\]

Here, \( \lambda (t) \) is known as the Scale factor.

(\text{Homogeneity & isotropy don't restrict } \lambda (t))

---

If we know the avg. matter dist. in universe, we can find how \( \lambda \) evolves in time from Einstein's eqns.

If data about \( \lambda \) is available, matter content can be found from Einstein's eqns.

---

\text{Ex. Consider the eqn. for geodesic in the 4-d space.}

Show that if we project it onto the 3-dim. space (forget \( x^0 = t \)), then it is a geodesic in the 3-dimensional space.

\text{\( e.g., \) 2-d motion}

\text{projection of the full trajectory in the space part}

---

To get the time embedding, we need to draw it in 3-d, with time \( t \) along z-axis.
\[ ds^2 = -dt^2 + (\lambda(t))^2 \sum_{i} \frac{dx^i}{dx^i} dx^i dx^i \]

\( L \) for any metric of this form the statement holds - only in this case we can talk about the 3-d geodesic without reference to \( t \).

We have to show the \( \lambda^i \)-can are the ones in which terms like \( T^i_{00} \) or \( T^i_{0j} \) are absent in

\[ \frac{d^2 x^m}{ds^2} + \Gamma^m_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 \]

Take 2 points \( P \) & \( Q \) in this 3-d space:

\( P \) \hspace{2cm} \( Q \)

Consider a light ray that starts at \( P \) at \( t = t_1 \) & ends at \( Q \) at \( t = t_2 \).

Given \( P \), \( Q \) and \( t_1 \), we want to calculate \( t_2 \).

Firstly, \( ds^2 = 0 \)

\[ \int_{t_1}^{t_2} \frac{dt}{\lambda(t)} = \int_{P}^{Q} \sqrt{g^{ij}(x^i) \frac{dx^i}{dn} \frac{dx^j}{dn}} dn \]
where \( u \) is the parameter labelling the path.

The path connecting \( P \) & \( Q \) must be a geodesic connecting \( P, Q \).

(\(*\): sphere is homogeneous, we choose our coord. such that \( P \) is at the North pole — then the \( \theta \)-coord. of \( Q \) is the answer for the geodesic length from \( P \) to \( Q \).)

\( P \) emits another signal at \( t_1 + \Delta t_2 \)
\( Q \) receives it at \( t_2 + \Delta t_2 \).

\[
\int_{t_1 + \Delta t_1}^{t_2 + \Delta t_2} dt = \int_{t_1}^{t_2} \frac{\sqrt{g_{ij}(x')} \frac{dx^i}{du} \frac{dx^j}{du}}{\sqrt{g_{ij}(x')}} \, du
\]

\[
\Rightarrow \frac{\Delta t_2}{\Delta t_1} = \frac{\Delta t_1}{\Delta t_2}
\]

for small \( \Delta t_1, \Delta t_2 \)

\[
\Rightarrow \Delta s^2 = \Delta t_1 \frac{\lambda(t_2)}{\lambda(t_1)}
\]

Comoving coordinates
\( \Rightarrow \) when we define the cosmic time & the 3-d metric takes the form \( g_{ij}(\bar{x}) \frac{d\bar{x}^i}{du} \frac{d\bar{x}^j}{du} \)

written where \( g_{ij} \) is indep. of \( t \)
\[ \Delta t_1 = \frac{1}{\nu} \]

intrinsic frequency of the source

\[
\frac{1}{\Delta t_2} = \frac{1}{\Delta t_1} \frac{\lambda(t_1)}{\lambda(t_2)} = \nu \frac{\lambda(t_1)}{\lambda(t_2)}
\]

Same source placed at \( P \) will have frequency \( \Delta t = \frac{1}{\nu} \)

\[ \nu_\text{obs} = \nu \text{ intrinsic} \frac{\lambda(t_1)}{\lambda(t_2)} \quad (A) \]

\( t_1 \) : current time to (of the observer)

We can change \( t_2 \) by looking at diff. stars at diff. distances.

One observes a series of lines & calculating the ratio of the lines, which is an invariant for a particular intrinsic spectrum, we can do the comparison.

\( (A) \) is a linear relation – ratio is same...
\[ v' = \frac{\lambda(t)}{\lambda(t_0)} v \]

To denote the present time

\[ \lambda(t) \]

One observes several spectral lines & calculating their ratios, which are the same always, finds the red shift.

Expt.:

\[ v' < v \] for faraway objects

\[ \lambda(t) \leq \lambda(t_0) \]

\[ d \leq t_0 \]

\[ \Rightarrow \lambda(t) \text{ is increasing with time} \]

\[ \Rightarrow \text{Universe is expanding} \]

\[ ds^2 = -dt^2 + \left( \lambda(t) \right)^2 g_{ij}(x') dx'^i dx'^j \]

\[ \text{physical distance} \Rightarrow \text{distance measured in the metric} \ g_{ij}(x') \times \lambda(t) \]

\[ (\text{i.e., as } \lambda \uparrow, \text{ phys. dist. } \uparrow \text{ for any any 2 points in the comoving coord.)} \]

Hubble constant \[ H_0 = \frac{1}{d(t_0)} \frac{d\lambda}{dt} \lambda(t) \mid_{t=t_0} \]

\[ \text{measures rate of change of } \lambda \text{ at present time} \]

we normalise it by \( \lambda(t_0) \)
\[ \frac{\lambda(t)}{\lambda(t_0)} = 1 + \frac{1}{2} \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{2} \frac{\dot{\lambda}(t_0)}{\lambda(t_0)} \left( \frac{d\lambda}{dt} \right)^2 \]

\[ \ddot{\lambda}(t_0) = -\frac{1}{H_0(t_0)^2 \lambda(t_0) \frac{\partial \lambda}{\partial t}} \]

\[ \lambda(t) \text{ deceleration parameter} \]

Red shift parameter

\[ z(t_1) = \frac{\lambda(t_0)}{\lambda(t_1)} - 1 \]

(\text{This is due to expansion of universe. For far away objects, gravitational red shift is small compared to cosmological red shift.})

\[ \frac{d^2 \lambda}{dt^2} \]

9\(t\) is the gravitational wave - 9\(t\) is directly measurable - 9\(t\) can be converted to scale factor, which can be converted to time

For consistency of our space-time energy-momentum tensor should have same isometries as 9\(t\)

\[ R_{\mu
\nu} - \frac{1}{2} G_{\mu\nu} = -8\pi G T_{\mu\nu} \]

\( R_{\mu\nu}, R \) are form-invariant but 9\(t\) is form-invariant - \( R_{\mu\nu} \) & \( R \) are constructed from 9\(t\) \( \Rightarrow \) \( T_{\mu\nu} \) must also have similar properties

General Coord. Trans.: \( T_{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} T_{\lambda\sigma}(x) \)

Isometry \( \Rightarrow \)

\[ T_{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} = T_{\lambda\sigma}(x) \]
\[ ds^2 = -dt^2 + (\mathbf{k}(x))^2 \gamma_{ij}(x') dx^i dx^j \]

If the tensor in consideration is related to \( \mathbf{J} \), it should have the same isometries — e.g., for EM field, the energy-momentum \( \mathbf{J} \) should have the same isometry — of course, \( \mathbf{J} \) can have diff. It may not obey the isometry, while \( \mathbf{q} \) obeys it, it is very diff. to have such a situation — so better \( \mathbf{J} \) also obeys it or say the pot. \( \mathbf{J} \) obeys it also.

\[
\begin{align*}
  t' &= t \\
  \mathbf{x}' &= F(x) \tag{1}
\end{align*}
\]

\[ T_{00}(x) = T_{00}(x') \tag{1} \]

\[ T_{00}(t, x') = T_{00}(t, \mathbf{x}') \tag{1} \]

\[ P_{0i}(t, x') = T_{0i}(t, \mathbf{x}') \frac{dx_i}{dt} \tag{2} \]

\[ T_{ij}(t, x') = T_{ij}(t, \mathbf{x}') \frac{dx_i}{dt} \frac{dx_j}{dt} \tag{3} \]

(1) implies \( T_{00}(t, x') = g(t) \)

(2) implies

Regarding \( t \) as a fixed parameter.

\( T_{0i}(t, x') \) is a form-invariant covariant rank 1 tensor in 3-dimension.

\( T_{ij}(t, x') \) is a form invariant covariant rank 2 tensor in 3-dimensions.

(form-inv. tensors are very hard to find)
$k = 0$ case: (flat space)

\[ V(x') = V(x + a') \]

$\Rightarrow V$ is constant

$\Rightarrow V = 0$ (isotropy)

\[ V_{ij}(x) = V_{ij} = \text{constant} \]

\[ x'_{i} = R_{ij} x_{j} \]

\[ V_{ij} = R_{ik} R_{lj} V_{kl} \Rightarrow \]

\[ V = C \delta_{ij} \]

\[ V = C F_{ij} \]

(Also in flat space $F_{ij} = \delta_{ij}$)

(Analysis for $k = \pm 1$ cases also give the same result, though the calculations are more involved)

Solution:

\[ T_{00}(t, x') = f(t) \]

\[ T_{0i}(t, x') = 0 \]

\[ T_{ij}(t, x') = f(t) \frac{\partial f}{\partial x^i} = f(t) \lambda(t)^{-1} \frac{\partial f}{\partial x^i} \]

\[ D_{ij} = 0 \]

Show that this gives:

\[ \frac{\partial}{\partial t} (f x^3) = -3 \phi \lambda^2 \]
Note: No one gives us \( f(t) \) - this has to be determined from dynamics - of use \( \Phi \) may be able to find \( \Phi(t) \) by solving.

\[
\text{Ex.} \quad ds^2 = -dt^2 + \lambda(x)^2 \left( \frac{dx^2}{1-kx^2 + 2^2 x^2 + 2 \sin^2 \theta \, dx^2} \right)
\]

Check that:

\[
R_{00} = \frac{3 \dot{x}^2}{\lambda}, \quad R_{ij} = -(\lambda \dot{x}^2 + 2 \ddot{x}^2 + 2 \dot{x} \ddot{x})g_{ij}
\]

\[
R = g^{00}R_{00} + g^{iy}R_{iy} = -6 \frac{\dot{x}^2}{\lambda} - 6 \left( \frac{\ddot{x}}{\lambda} \right)^2 - 6 \theta^2 / \lambda^2
\]

\[
R_{uv} - \frac{1}{2} R g_{uv} = -8 \pi T g_{uv}
\]

\[
\text{Ex.} \quad \text{00 - component gives}
-8 \pi G \rho = -3 \left( \frac{\dot{x}^2}{\lambda} \right)^2 - 3 \dot{x}^2 / \lambda^2 \quad \text{--- 1}
\]

\[
\text{& } \text{i}j \text{- component gives}
-8 \pi G \phi = 2 \ddot{\phi} - \frac{\dot{\phi}^2}{\lambda} + \frac{\dot{x}^2}{\lambda^2} + \frac{\ddot{x}^2}{\lambda^2} \quad \text{--- 2}
\]

(only one eqn, lec 6 both LHS & RHS are proportional to \( \dot{x}^2 \))

\[
\text{1 & 2 aren't independent} \quad \text{becos} \quad Du \left( R_{uv} - \frac{1}{2} R g_{uv} \right) = 0 \text{ is an identity}
\]

\[
\text{Ex.} \quad \text{Take} \frac{\partial}{\partial t} \text{\( \Phi \)} \text{and substitute } \dot{x} \text{ from 2}
\]

\[
\left( \frac{\partial}{\partial t} \right) f(x^3) = -3 \dot{x} \ddot{x} \quad \text{gives}
\]

want indisp. lec cos 2 \text{ can be seen to follow from 1}

\[
& \frac{\partial}{\partial t} (\Phi x^2) = -3 \dot{x} \ddot{x} \quad \text{also as we have the Bianchi identity}
\]
So, we can work with the equations:
\[
\frac{d}{dt}(3x^3) = -3p x^2 \\
3\left(\frac{a}{2}\right)^4 + 3k/x^2 = 8\pi G \rho
\]

\((x, p, f) \rightarrow 3\) unknowns, so 2 first order diff. eqns for 3 unknowns — so time evolution isn’t done completely.

Unknown functions \(f, p, \lambda(t)\) — need some more information.

(This is given by the \(\Phi\) of energy-momentum tensor — not that we need to know it as a function of \(t\).)

Additional information needed: Equation of State.

Functional relationship between \(p, \rho\).

\((\text{This eqn. of state is independent of background.})\)

\(T_{00} = -\rho(t) \Box 00\)
\(T_{ij} = p(t) \Box_{ij}\)

(\(T_{00}\) transforms exactly as \(\rho\);
\(T_{ij}\) transforms as \(p\).

So, relation between \(\rho\) & \(p\) doesn’t depend on the coordinate frame we have chosen.

We can show that relation \(\rho \neq p\) won’t change under \(x \rightarrow f(x)\).

\(T_{ij} \rightarrow \tilde{T}_{ij}(\tilde{x})\)
Possible sources in cosmology

1. Non relativistic matter, \( \rho \ll c_s \)

2. Radiation

3. Cosmological Constant
   (can be either thought of a source in T\(\mu\nu\), or as a term modifying Einstein's eqn)

\[
R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} = -8\pi G \rho + \Lambda g_{\mu\nu}
\]

\( \Rightarrow \) Cosmological Constant

1. \( \Rightarrow \rho = 0 \) (Dark matter is included here)

2. \( \Rightarrow \Phi = 8/3 \)

3. \( \Rightarrow \nabla^2 \rho_{\mu\nu} = -8\pi G (\rho_{\mu\nu} - \frac{\Lambda}{8\pi} \delta_{\mu\nu}) \)

\( \therefore \cos \Theta = -\frac{\Lambda}{8\pi G} \rho_{\mu\nu} \Rightarrow \Phi = -\frac{\Lambda}{8\pi G} \)

Recall:

\( T_{\mu\nu} = \rho \delta_{\mu\nu} \)

\( T_{ij} = \rho \delta_{ij} \)

(This is sometimes called dark energy)

When we talk about NR matter, we essentially mean temp. \( T = 0 \)

Imagine int. dark matter & radiation is so small that they aren't equilibrating ---)
Now,

\[ T^{\mu \nu} = \sum \int \sum_{n} \int_{0}^{\infty} \frac{d\omega}{\omega} \left( \frac{d}{dt} \right)^{-1} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \delta(x - X(n)) \]

Non-relativistic matter:

\[ \left| \frac{dx^\mu}{dt} \right| << \frac{dx^0}{dt} \]

\[ |v| \ll c \]

So, \( T_{ij} = 0 \) for non-relativistic matter

\[ \implies p = 0 \]