What kind of trajectory does a description in 3d subspace?

→ Forget about time. The geodesic in 4d is also a trajectory geodesic in 3d. That geodesic is a curve/line for which \(\theta = 0, \phi = 0\) at every \(p^t\).

→ The light travels along the spacetime curve

\[-dt^2 + \lambda(t)^2 \frac{dr^2}{1-kr^2} = 0\]

\[\int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = \int_{r_0}^{r_0} \frac{dr}{\sqrt{1-kr^2}}\]

\(t_1\) = initial time (when the light was emitted from the source)

\[= \begin{cases} 
2\lambda_0 & \text{for } k = 0 \\
\sin^{-1} \lambda_0 & \text{for } k = 1 \\
\sinh^{-1} \lambda_0 & \text{for } k = -1
\end{cases}\]

\(\sqrt{\lambda_1}\) can be found in terms of \(\lambda_0, H_0, -2m, \lambda_1, -2\).

Suppose the source emits a total energy \(E\) per unit time.

Energy/Unit area measured by the observer: \(F = \frac{E}{4\pi r^2} \frac{2H(t)}{\lambda_0}\)

\(\sqrt{F}\) is observed.

The energy of each photon decreases by \(\frac{dH(t)}{H(t)}\).
$F$ is known for a Standard Candle

Relation between $x_0$, $t_1$:

$H_0, \lambda, x_0, \beta$

We do integral over $\Omega$ & $\phi$ for a fixed $r$ ($= x_0$)

so we get $\Upsilon$

for $k = 0, 1, -1 = 1^2 \int_{0}^{2\pi} \sin^2 \phi d\phi = \frac{2\pi}{3}$

Using this, we can determine $t_1$ for a given source as a function of the cosmological parameters.

$\Upsilon = \frac{E}{H_0 \lambda^2 \alpha \lambda}$

$\frac{N(t)}{N(\infty)} = 1 - H_0 (t - t_0) - \frac{1}{2} \frac{9}{H_0^2} H_0^2 (t_0 - t_1)^2$

known as a fit of the cosmological parameters & luminosity observation.

[But we don't know the cosmological parameters.

But every observation of cosmological parameters can be det. from these obs.]

For $H_0$ the analysis can be simplified:

$\frac{N(t)}{N(\infty)} = 1 - H_0 (t_0 - t_1) + O \left \{ (t_0 - t_1)^2 \right \}$

To calculate $H_0$, we need to study the relations for small $(t_0 - t_1)$.
\[
\int_{t_0}^{t_1} \frac{dt}{s(t)} = \int_{0}^{\tau_0} \frac{dr}{\sqrt{1-k^2r^2}}
\]

gives us
\[
\frac{1}{\tau_0} (t_0-t_1) \approx \tau_0
\]

(Similarly, we get)\[
F \approx F \frac{4\pi \tau_0 \lambda_0}{4\pi \tau_0 \lambda_0}(\text{for } t_0 \text{ close to } t_0)
\]

\[
\therefore \ t_0-t_1 = \lambda_0 \tau_0 = \sqrt{\frac{E}{4\pi F}}
\]

\((t_0-t_1)\) is a dimensional quantity so the question is whether the dimensionless quantity \(\tau_0 (t_0-t_1)\) is small or large.

\(\tau_0\) can be calculated by looking at nearby objects, as it is related to first deriv.

\[
H(0-t_1) \text{ is the relevant quantity which determines the expansion is good or not.}
\]

\((t_0-t_1)\) should be small w.r.t. some unit of measurement of time, say the time in which universe doubles its site.
**Other measurements**

\( \text{1) Age of the universe} \)

\[ \frac{\dot{a}}{a} = \sqrt{\frac{8 \pi G}{3} \rho - \frac{k}{a^2}} \]

\[ = \sqrt{\frac{8 \pi G}{3} \rho_c} \left( \frac{\rho}{\rho_c} - \frac{k}{a^2} \right)^{\frac{3}{2}} \]

where \( \rho_c = \frac{3 H_0^2}{8 \pi G} \)

\[ \Rightarrow \frac{\dot{a}}{a} = H_0 \left( \frac{\rho}{\rho_c} - \frac{k}{a^2 H_0^2} \right)^{\frac{3}{2}} \]

where \( \rho = \rho_m + \rho_\Lambda + \rho_\gamma \)

\[ \Rightarrow \frac{\dot{a}}{a} = H_0 \left\{ \frac{\rho_m}{\rho_c} \frac{H_0^2}{a^3} + \frac{\rho_\Lambda}{\rho_c} + \frac{\rho_\gamma}{\rho_c} \right\}^{\frac{1}{2}} \]

\[ \Rightarrow \frac{\dot{a}}{a} = H_0 \left\{ \Omega_m \frac{H_0^2}{a^3} + \Omega_\Lambda + \Omega_\gamma \right\}^{\frac{1}{2}} \]

\[ \Rightarrow \frac{\dot{a}}{a} = H_0 \left\{ -\Omega_m \frac{H_0^2}{a^3} + \Omega_\Lambda + \Omega_\gamma \right\}^{\frac{1}{2}} \]

\[ \Rightarrow \frac{\dot{a}}{a} = H_0 \left\{ -\Omega_m \frac{H_0^2}{a^3} + \Omega_\Lambda + \Omega_\gamma \right\}^{\frac{1}{2}} \]

**Define:** \( \xi = \lambda u \)

\[ \Rightarrow \frac{\dot{u}}{u} = H_0 \left\{ -\Omega_m u^3 + \Omega_\Lambda + \Omega_\gamma u^4 \right\}^{\frac{1}{2}} \]

**Origin of time:** \( u = 0 \) at \( t = 0 \)
Integrating, we get
\[ \int_0^t \frac{1}{H_0} \int \frac{a}{u^3 (1 + \frac{\Omega_m}{3} \frac{u^2}{u^4} + \frac{\Omega_k}{2} \frac{u^4}{u^4})} du \]

Age of the universe

If we knew an independent way of determining $t$, to, that will give info about $a(t)$, $H_0$.

But unfortunately, we don't have any such a way.

But we can put a bound on $t_0$ by knowing the oldest object.

E.g., globular clusters -- we know how they evolve & so can find their age.

Measurement of Cosmic Microwave background radiation

$T' = 2.73 K$ (Energy content of CMBR is very small)

$V(t) = V(t_0) \frac{a(t_0)}{a(t)}$

Previously, it wasn't detected properly.

There wasn't much problem about not considering dark energy dominated by $\Omega_k$ dominated over $\Omega_m$ at earlier times.

Most of contribution to the integral comes from $u$ close to 1 so age of oldest star or galaxy is close to the age of the universe.
\[ a(t_0) > a(t), \quad \therefore \eta(t) > \eta(t_0) \]

\[ T(t) = T_0 \frac{a(t_0)}{a(t)} \]

\[ \text{At } \frac{a(t_0)}{a(t)} \sim 10^3 \]

\[ T(t) \sim \text{ionization temperature} \]

\[ \text{Hydrogen} \]

Above this temperature, matter was ionized.

\[ \text{has strong interaction with radiation} \]

\[ \text{(as a result)} \]

\[ \text{matter & radiation were in thermal equilibrium} \]

\[ \text{(radiation stopped interacting with matter, except occasionally, when universe cooled & positrons & electrons combined to form H-atom)} \]

Microwave radiation we see today consists of the photons emitted from matter at the ionization temp.

\[ \text{(So if we observe CMBR, fluctuations in CMBR will give us an idea about the } \eta \text{ matter at the time of decoupling)} \]

\[ \text{Fluctuation in microwave background over angular coordinate } \Rightarrow \text{fluctuation in spatial distribution of matter at the time of matter-radiation decoupling} \]
(we have a thing if matter fluctuation at the time earlier times decreeing if it tells us)

\[ \frac{a(t)}{a(t)} \]

\[ \frac{a(t_{1})}{a(t_{1})} \]

Decoupling

\[ a(t) \quad t \]

How will these fluctuations look today?

Recall:
\[ f(t) \]
\[ \frac{\dot{a}}{a} \]
\[ \Delta \phi \]
\[ \Delta \phi \quad \text{angular scale of fluctuations in CMBR} \]
\[ \Delta \phi \quad \text{can be observed} \]
\[ \text{an equation involving the cosmological parameters} \]

Evolves the geodesic backward to \( r_0 \).

The main quantity that enters into the calculation is \(-2\, \text{tot} = 2m + \Delta a + \Delta r\)

\[ \Delta \phi \]
\[ \Delta \phi \quad \text{known from observation} \]
\[ r_0 \quad \text{known from the Eq.} \]
\[ \text{We know distance back the big bang extrapolating back in time} \]

This CMBR measurement is very accurate and very reliable.
Phase 2 together gives us:

\[ \lambda_m = 0.26 \]

\[ a = 0.79 \]

\[ \lambda = 10^{-5} \]

Direct measurements of \( \lambda_m \) isn't very reliable.

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Gravitational action principle

In a constrained dynamical system we can get EOMs by writing an action. To look for an action \( S \) such that extremizing \( S \) we get Einstein's eqns. + other possible field eqns.

\[ \delta S = 0 \Rightarrow \mathcal{R} \mu \nu - \frac{1}{2} R g \mu \nu = - \frac{8}{c^4} \pi G T \mu \nu \]

\[ \delta S \propto \int d^4x \sqrt{-g} \left[ \frac{1}{2} R \mu \nu (g) + \mathcal{F}^\mu \nu (g) \mathcal{F}_\mu \nu (g) \right] \]

gets contributions from metric, Maxwell field, constants

\[ \mathcal{R} \mu \nu \] + other eqns of motion

First concentrate on the other eqns of motion:

Particle motion

Eqns of motion \( \rightarrow \) geodesic eqn.

\[ \mathcal{F}_{\mu \nu} = \mathcal{L} \int \frac{1}{d^4x} \sqrt{g} \mu \nu (g) \frac{d x^m}{d \lambda} \frac{d x^n}{d \lambda} \]  

This is not defined in Minkowski space. Since \( d s^2 \) isn't +ve-definite, we take a variant in Euclidean space & calculate to Minkowski:

\[ \delta S = \int \frac{1}{d^4x} \sqrt{g} \mu \nu (g) \frac{d x^m}{d \lambda} \frac{d x^n}{d \lambda} \]

How about variation of \( S \) with respect to \( g_{\mu \nu} \)?

Change \( g_{\mu \nu} (x) \rightarrow g_{\mu \nu} (x) + \delta g_{\mu \nu} (x) \)

Given that particle is a part of \( S \), we can ask what is the variation of \( S \) under a variation of the metric?
\[ S_{\text{particle}} + S_{\text{gravity}} \]
\[ = c \int_0^1 ds \frac{d}{ds} \left( \frac{0}{0} + 0 \frac{d}{ds} \right) \]
\[ = c \int_0^1 ds \sqrt{g_{\mu\nu}(x)} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \]
\[ \{ 1 + 2 \delta g_{\mu\nu}(x) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} \} \]
\[ \frac{g_{\kappa\sigma}(x)}{dX^\kappa} \frac{dX^\sigma}{ds} \]

\[ S_{\text{kinetic}} = \frac{c}{2} \int d^4x \sqrt{-\det g(x)} \delta g_{\mu\nu}(x) \int_0^1 ds \left( \frac{1}{d^2 d^2 g(X(\xi))} \right)^{\frac{1}{2}} \]
\[ \delta(4) (x - X(\xi)) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} \]

\[ = \frac{c}{2} \int d^4x \delta g_{\mu\nu}(x) \sqrt{-\det g(x)} T^{\mu\nu}(x) \]

\[ \text{Take } S_{\text{particle}} = m \int_0^1 ds \sqrt{g_{\mu\nu}(x)} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \]

\[ S_{\text{particle}} = \frac{1}{2} \int d^4x \delta g_{\mu\nu}(x) T^{\mu\nu}(x) \]

In a sense, the particle action already gives the correct contribution to \( T_{\mu\nu} \) under variation of the metric.
Consider Maxwell's equation:

\[ \text{D}_\mu F^{\mu\nu} = 0 \]

Now let us do the same thing for the Maxwell's action.

In free space, it is obtained by varying the action:

\[
S_{\text{Maxwell}} = -\frac{1}{4} \int d^4x \, \text{Tr} \left[ F_{\mu\nu} F^{\mu\nu} \right]
\]

Differentiate with respect to a variation of the field:

\[
\delta S_{\text{Maxwell}} = -\frac{1}{2} \int d^4x \sqrt{-g} \, g^{\mu\nu} \delta A^\mu \partial_\nu \delta A^\nu
\]

What action gives the curved space Maxwell's equation?

\[
S_{\text{Maxwell}} = -\frac{1}{4} \int d^4x \sqrt{-\text{det} g} \, g^{\mu\nu} F_{\mu\nu}
\]

This action correctly reproduces the EOM... we should not modify its dependence on \( A^\mu \).

Check that:

\[
\delta S_{\text{Maxwell}} = \int d^4x \, \delta A^\nu \left( \frac{\partial F_{\mu\nu}}{\partial A^\mu} - \frac{\partial F_{\mu\nu}}{\partial A^\nu} \right)
\]

Under \( A^\mu \to A^\mu + \delta A^\mu \):

\[
\text{D}_\mu F^{\mu\nu} = 0
\]

Now let us study \( S_{\text{Maxwell}} \) under \( g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \):

\[ S_{\text{Maxwell}} = -\frac{1}{4} \int d^4x \sqrt{-g} \, g^{\mu\nu} \partial_\mu A^\nu \]

\[ \delta \text{g}^{\mu\nu} = -g^{-1} \delta g \quad \text{g}^{-1}
\]

\[ \delta g_{\mu\nu} = -g_{\mu\nu} \delta g
\]

\[ \delta \text{det} g = \frac{1}{2} (\sqrt{-\text{det} g})^{-1} \delta (\sqrt{-\text{det} g})
\]

After:

\[ \text{det} \left( g + \delta g \right) = \text{det} g \left( 1 + g^{-1} \delta g \right)
\]

\[ = \text{det} g \left[ 1 + \text{tr} \left( g^{-1} \delta g \right) \right]
\]

\[ = \text{det} g \left( 1 + g_{\mu\nu} \delta g_{\mu\nu} \right)
\]

\[ \delta (\text{det} g) = -(\text{det} g) \quad g_{\mu\nu} \delta g_{\mu\nu}
\]

\[ \Rightarrow \delta \sqrt{-\text{det} g} = \frac{1}{2} \sqrt{-\text{det} g} \quad g_{\mu\nu} \delta g_{\mu\nu} \]
\[ 8S_{\text{Maxwell}} = \frac{1}{2} \int d^4x \sqrt{-\det g} \left( \nabla_m \nabla_n \varepsilon_{mn} - \varepsilon_{mn} \frac{\partial}{\partial \varepsilon_{mn}} g_{\mu
u}(x) \right) \]

\[ 8S_{\text{matter}} = \frac{1}{2} \int d^4x \sqrt{-\det g} \left( \partial_m \varepsilon_{mn} \right) \]

Now consider the gravitational field equation of motion:

\[ (R_{\mu
u} - \frac{1}{2} R g_{\mu
u}) + 8\pi G T_{\mu
u} = 0 \]

We need \( S_{\text{grav}} \) such that

\[ 8S_{\text{grav}} = \frac{1}{16\pi G} \left( R_{\mu
u} - \frac{1}{2} R g_{\mu
u} \right) \]

Then

\[ 8S_{\text{grav}} + 8S_{\text{matter}} = \frac{1}{16\pi G} \left\{ R_{\mu
u} - \frac{1}{2} R g_{\mu
u} + 8\pi G T_{\mu
u} \right\} \]

\[ \text{Ex.} \quad \text{Show that} \]

\[ 8S_{\sqrt{-\det g}} R = - \int d^4x \sqrt{-\det g} \left( R_{\mu
u} - \frac{1}{2} R g_{\mu
u} \right) \varepsilon_{\mu
u}(x) \]

\[ \Rightarrow S_{\text{grav}} = - \frac{1}{16\pi G} \int d^4x \sqrt{-\det g} R \]

All the equations of motion can be derived by varying \( S_{\text{grav}} + S_{\text{matter}} \) with respect to the dynamical variables.
Cosmological horizon: Maximum distance light would have travelled since $t=0$, $a \rightarrow 0$

Two points separated by more than this distance could not have communicated ever.

Assume that light travels radially outward from $r=0$ at $t=0$.

Then $ds^2 = \frac{dr^2}{1-kr^2} - dt^2 \Rightarrow dt^2 = \frac{dr^2}{\lambda(t)^2 \frac{1}{1-kr^2}}$

$$\therefore \int \frac{dt}{\lambda(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$

```
\text{Comoving horizon}
```

Physical distance as seen today

$$= \lambda(t) \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = \lambda(t) \int_0^{t} \frac{dt'}{\lambda(t')} = \Delta x(t)$$

If we know the current cosmological parameters, we can extrapolate $\lambda(t')$ backwards using 

$$\left(\frac{\Delta x}{\lambda}\right)^2 + kx^2 = \frac{8\pi G}{3} f$$

where $f = c^{2-n}$

with $n = 3$ marks

$n = 4$ redshift

$n = 0$ cosmological constant

Assuming that only one component dominates at any epoch, we have

$$f = c^{2-n}$$

$$\Rightarrow \left(\frac{\Delta x}{\lambda}\right)^2 + kx^2 = \frac{8\pi G}{3} c^{2-n}$$
Also assume $\lambda = 0$ (since current observation is $\lambda'/\lambda < 0.5 \text{GHz}$ \(= \frac{\Omega_{m, 0}}{\Lambda^{1/3}}\))

\[ \therefore \lambda^2 = \frac{8\pi G}{3} \epsilon \lambda^{-\alpha + 2} \]

\[ \Rightarrow \lambda^{(\alpha-2)/2} \lambda' = \sqrt{\frac{8\pi G}{3}} \]

\[ \Rightarrow \frac{2}{\alpha} \lambda^{1/2} = \sqrt{\frac{8\pi G}{3}} (t-K)^{2/\alpha} \]

\[ \therefore \lambda = \lambda_0 \left(\frac{\alpha}{2} \left(\frac{8\pi G}{3}\right)^{1/\alpha}\right)^{2/\alpha} \]

\[ \Rightarrow \lambda = \lambda_0 \left(t-K\right)^{2/\alpha} \]

Suppose this epoch begins at $t_i$ & ends at $t_f$. For a matter-dominated universe, $t_f \sim$ today.

\[ dH(t_i) = \lambda(t_i) \int_{0}^{t_i} \frac{dt'}{X(t')} \]

\[ \Rightarrow \int_{0}^{t_i} \frac{dt'}{X(t')} = \frac{dH(t_i)}{\lambda(t_i)} \]

\[ dH(t_f) = \lambda(t_f) \left\{ \int_{0}^{t_i} \frac{dt'}{X(t')} + \int_{t_i}^{t_f} \frac{dt'}{X(t')} \right\} \]

\[ = \frac{\lambda(t_f)}{\lambda(t_i)} \left[ dH(t_i) + \int_{t_i}^{t_f} \frac{dt'}{X(t-K)^{2/\alpha}} \right] \times \lambda(t_f) \]

\[ = \frac{\lambda(t_f)}{\lambda(t_i)} dH(t_i) + \frac{\lambda(t_f) (t-f)^{-2/\alpha + 1}}{\lambda_0 (-2/\alpha + 1)} \]
\[ d_H(t_f) = \frac{\lambda(t_f)}{\lambda(t_i)} d_H(t_i) + \left(\alpha \frac{t_f}{\alpha - 2}\right) \frac{1}{t_i} \left(\frac{t_f}{t_i - K}\right)^{1 - \frac{2}{\alpha}} \] 

\[ = \frac{x_0}{(t_i - K)^{2/\alpha}} d_H(t_i) \] 

\[ + \left(\frac{t_f}{t_i - K}\right)^{1 - \frac{2}{\alpha}} \frac{x_0}{(t_f - K)^{2/\alpha}} \] 

\[ \Rightarrow \text{No dependence on } x_0 \] 

\[ d_H(t_f) = (t_f - K) \int \frac{x}{x^2} \left(1 - \left(\frac{t_i - K}{t_f - K}\right)^{1 - \frac{2}{\alpha}}\right) \] 

\[ + \frac{d_H(t_i)}{t_i - K} \left(\frac{t_i - K}{t_f - K}\right)^{1 - \frac{2}{\alpha}} \] 

\[ \alpha = 3 \Rightarrow 1 - \frac{2}{\alpha} = \frac{1}{3} \] 

\[ \alpha = 4 \Rightarrow 1 - \frac{2}{\alpha} = \frac{1}{2} \] 

Typically an epoch runs for a long time, i.e. \( t_i \ll t_f \) 

\[ \Rightarrow \frac{t_i - K}{t_f - K} < 1 \] 

\[ \therefore \text{if } \frac{d_H(t_i)}{t_i - K} \text{ is finite,} \] 

\[ d_H(t_f) \approx (t_f - K) \left(\frac{x}{x^{\alpha-2}}\right) \] 

(If \( d_H(t_i) \) is very large, the second term might also contribute)
Standard Big-Bang Cosmology

Recombination (time of CMB)

Before recombination
photons scattered
by charged matter
\Rightarrow coupled

\text{CMB is "untouched"}

\text{since after recombination} \text{CMB corresponds to time of recombination.}

\Rightarrow we work in time after CMB was emitted for matter-dominated universe.

From \( t = 0 \) to \( t_{\text{recomb}} \), \( \alpha = 4 \)

\( \alpha(0) = 0, \; \; \alpha = \alpha_0 t^{1/2} \)

\( \Delta H(t_i) = 0 \)

\Rightarrow \Delta H(t_f) = (t_f) \frac{4}{(4-2)} = 2 t_f \Rightarrow \text{Radiation dominated universe dominated for past.}

\text{Take this as the initial condition for matter dominated past.}

\begin{align*}
\Delta H(t_i) &= 2 t_{\text{eq}} \\
\frac{\Delta H(t_i)}{t_i - t} &\sim 1
\end{align*}

\Rightarrow \text{our approximation is valid.}

\Rightarrow \text{only first term dominates.}
At the boundary:
\[ n_0 (t = t_{eq})^{1/2} = n_0 (t = t_{eq} - K)^{1/2} \]
\[ \Rightarrow t_{eq} - K = (t_{eq})^{3/2} \]
\[ \Rightarrow K = t_{eq} \left[ 1 - (t_{eq})^{1/2} \right] \]

\[ \frac{dH(t_{eq})}{dH(\text{now})} = \frac{4 \times 10^5}{10^{14}} \rightarrow \text{small number} \]

\[ dH(\text{now}) = (t_{eq} - K) \times 3 \]

" horizon today \approx 10^{14} \text{ light yrs,}"

" if we consider light from two most distant stars at opposite sides of the sky, their separation \approx 2 \times \text{horizon.}"

So at the time of recombination, what would have been the horizon?

- Regions at A \& B can never communicate.

So temperature at A need not be the same as temperature at B.

But observed \( \Rightarrow \) CMB is isotropic to order \( 10^{-5} \)

\[ \Rightarrow \text{HORIZON PROBLEM!} \]
Only solution is the two points have been in the horizon since time zero.

Our estimate of the horizon is a horrible underestimate.

No solution in standard big bang cosmology.

\[ d_H(t) \] is large at same time.

\[ \Rightarrow \] second term dominates.

EX: How does \( d_H \) in the cosmological constant dominated universe depend on \( \rho \)?

\[ \rho_{\text{vac}}, \text{cosm. cont. could at some pt. be dominated by } K/N^2 \]

\[ \Rightarrow \] \( d_H \) (const. cont.) \( \propto e^t \)

\[ \Rightarrow \] perhaps there was an inflationary phase when a phase transition causes the cosmological constant to dominate.

Observer at rest w.r.t. CMB \( \equiv \) observer has fixed comoving coordinates.

(If not at rest, CMB will be anisotropic!)

Also, in comoving coordinates, the average over velocities of all objects will be zero.
For cosmological constant dominated universe, $\frac{1}{a^2}$ term may dominate over $\frac{1}{t}$ in the past.

**DESCRIPTION OF THE HORIZON PROBLEM IN COMOVING COORDINATES**

Horizon = $d(t) = \int \frac{dt'}{a(t')} = \text{comoving coordinates}$

For a given epoch dominated by only one component of $\rho$, we have

$$\rho = \rho_0 a^{-3}$$

$$\Rightarrow \left(\frac{a'}{a}\right)^2 = \frac{8\pi \rho G}{3} \rho_0 a^{-3}$$

$$\Rightarrow \left(\frac{a'}{a}\right)^2 = \left(\frac{8\pi \rho G}{3}\right)$$

$$\Rightarrow x = \int \left(\frac{8\pi \rho G}{3}\right)$$

$$\Rightarrow \frac{2a'}{a} = \sqrt{\frac{8\pi \rho G}{3} (t-t')}$$

$$\Rightarrow \lambda = \left\{ \sqrt{\frac{8\pi \rho G}{3}} \left(\frac{a'}{a}\right) (t-t')^{1-2/a} \right\}^{2/a}$$

$$\Rightarrow (t-t')^{1-2/a} = \left[ \frac{2}{a} \left( \frac{8\pi \rho G}{3} \right) \right]^{1-2/a} \lambda^{(1-2/a)}$$

$$\Rightarrow (t-t') = 2\lambda \left( \frac{8\pi \rho G}{3} \right)^{1-2/a} \lambda^{x/2}$$

If $t_{in}$ is the beginning of the epoch, then

$$d(t) - d(t_{in}) = \int_{t_{in}}^{t} \frac{dt'}{a(t')} = \int_{t_{in}}^{t} \frac{dt'}{(t'-t)^{1-2/a}}$$

$$= \left\{ \frac{2}{a} \left( \frac{8\pi \rho G}{3} \right)^{1-2/a} \lambda^{x/2} \right\}$$
\[
= \left\{ \frac{\alpha}{2} \left( \frac{8 \pi G \rho}{3} \right)^{1/4} \right\}^{-2/\alpha} \int_{\text{tin}}^{t} \frac{dt}{(t'-K)^{2/\alpha}}
\]
\[
= \left\{ \frac{\alpha}{2} \left( \frac{8 \pi G \rho}{3} \right)^{1/4} \right\}^{-2/\alpha} \left[ (t'-K)^{1-2/\alpha} \right]_{\text{tin}}^{t}
\]
\[
= \left( \frac{\alpha}{\alpha-2} \right) \left\{ \frac{\alpha}{2} \sqrt{\frac{8 \pi G \rho}{3}} \right\}^{-2/\alpha} \left[ \lambda(t)^{\frac{\alpha}{2}-1} - \lambda(t_{\text{in}})^{\frac{\alpha}{2}-1} \right]
\]
\[
\approx \frac{2}{\alpha-2} \sqrt{\frac{3}{8 \pi G \rho}} \lambda(t)^{\frac{\alpha-2}{2}}
\]

On the radiation dominated era, \( \lambda(t_{\text{in}}) \approx 0 \)
\[ t_{\text{in}} = t_{\text{eq}} \quad (t_{\text{naive}} = \text{redshift}) \]
\[
\lambda(t_{\text{er}}) = \frac{2}{4-2} \sqrt{\frac{3}{8 \pi G \rho}} \lambda(t_{\text{er}})^{\frac{4-2}{2}}
\]
\[
\approx \sqrt{\frac{3}{8 \pi G \rho} \lambda(\text{eq})}
\]
\[
\dot{\rho}_{\text{eq}} = \sqrt{\frac{3}{8 \pi G \rho}} \lambda_{\text{eq}}
\]
\[ S_n = \left( \frac{c}{\lambda_n} \right)^n \Rightarrow \left( \frac{S_n}{S_m} \right)_0 = \left( \frac{S_n}{S_m} \right)_0 \times \frac{\lambda_{eq}}{\lambda_0} \]

\[ \frac{\lambda_{eq}}{\lambda_0} = \left( \frac{S_n}{S_m} \right)_0 \Rightarrow \lambda_{eq} \approx 10^9 \]

In the matter-dominated era,

\[ d(t) - d(t_{eq}) = 2 \sqrt{\frac{3}{8\pi Gc}} \lambda(t)^{1/2} \]

Temperature:

\[ T(x) \propto \lambda^{-1/4} \Rightarrow T \propto \lambda^{-1/2} \]

\[ \Rightarrow T_{eq} \approx 3 \times 10^9 \text{ K} \]

For recombination:

\[ kT < \text{ionisation energy of } H_2 \]

\[ \Rightarrow T \approx 3000K \]

\[ \text{recombination took place in matter-dominated universe,} \]

\[ d(t_{eq}) = 2 \sqrt{\frac{3}{8\pi Gc}} \left( \frac{\lambda(t_{eq})}{\lambda_0} \right)^{1/2} \]

\[ = 2 \sqrt{\frac{3}{8\pi Gc}} \left( \frac{\lambda(t_{eq})}{\lambda_0} \right)^{1/2} + d(t_{eq}) \]

\[ \frac{G}{\lambda_{eq}^4} = \frac{C_m}{\lambda_{eq}^3} \Rightarrow G = C_m \lambda_{eq} \]

\[ \sqrt{\frac{3}{8\pi Gc}} \left( \frac{\lambda_{eq}}{\lambda_0} \right)^{1/2} \frac{L_{eq}}{L_0} \]

\[ \Rightarrow d(t_{eq}) = \left( \frac{3}{8\pi Gc} \right)^{1/2} \left( 2\lambda(t_{eq}) \right)^{1/2} - \lambda_{eq}^{1/2} \]
\[ d(t_{\text{rec}}) \propto 2 \sqrt{\frac{3}{8\pi G \rho}} \sqrt{\frac{1}{A(t_{\text{rec}})^{1/2}}} \]

Horizon of light emitted at \( t_{\text{rec}} \)

\[ = d(t_{\text{now}}) - d(t_{\text{rec}}) \]

\[ = 2 \sqrt{\frac{3}{8\pi G \rho}} \sqrt{\left[ \frac{3}{A(t_{0})^{1/2}} - A(t_{\text{rec}})^{1/2} \right]} \]

\[ = 2 \sqrt{\frac{3}{8\pi G \rho}} (A(t_{0})^{1/2}) \]

Distance between diametrically opposite located sources of CMB photons

\[ d = 4 \sqrt{\frac{3}{8\pi G \rho}} A(t_{0})^{1/2} \]

\[ \frac{A(t_{0})}{A(t_{\text{rec}})} \ll 1 \]

Isotropy of the CMB cannot be a result of equilibrium due to interaction.

Unless there is a LARGE additive correction to \( A(t_{\text{rec}}) \). Since the matter dominated universe is well understood, this cannot occur till at least ten, in fact, this can be pushed back to ten (beginning of radiation dominated era: will not have \( \rho = 0 \) anymore).
PROPOSED SOLUTION: INFLATION

Before the radiation dominated era, the universe was dominated by a large cosmological constant. By a phase transition, the energy density of this cosmological constant got completely converted to radiation.

\[ (\frac{\Lambda}{\lambda})^2 = \frac{8\pi G_0}{3} c_0 \Rightarrow \lambda = \kappa_0 \exp \left[ \sqrt{\frac{8\pi G_0}{3}} t \right] \]

Evolving in cosmological constant dominated phase:

\[ d(t_3) - d(t_0) = \int_{t_0}^{t_3} \frac{dt'}{K_0 e^{\frac{\sqrt{3}}{8\pi G_0} (t_3 - t')}} \]

\[ = -\frac{1}{K_0 \sqrt{\frac{3}{8\pi G_0}}} \frac{1}{\lambda(t_3)} \left\{ e^{\sqrt{\frac{3}{8\pi G_0}} (t_3 - t_0)} - 1 \right\} \]

\[ = \frac{\sqrt{3}}{8\pi G_0} \frac{1}{\lambda(t_3)} \left( e^N - 1 \right) \]

[where \( N = 1 + \frac{8\pi G_0}{3} (t_3 - t_0) \)]

Take this as the initial condition for the radiation dominated era:

\[ \lambda_{in} = \lambda(t_3) \]

\[ d_{in} = d(t_3) = \sqrt{\frac{3}{8\pi G_0}} \left( e^N - 1 \right) \frac{1}{\lambda(t_0)} + d(t_0) \]
How large does $N$ have to be to solve the problem?

i.e.

\[ \frac{A}{d(\text{tree})} < 1 \]

\[ \sin = C_0 \]

Using current knowledge of matter spread & knowledge about $\sin (= C_0 \sim (10^{15} \text{ GeV})^4)$, calculate the min. value of $N$. [1 keV = $10^3$ K]