14.09.2011

A_{l_1, m_1} B_{l_2, m_2} $(2l_1+1)(2l_2+1)$

row column

$$C_{lm} = \sum_{m_1, m_2} W(l_1, m_1, l_2, m_2; l, m)\, A_{l_1, m_1} B_{l_2, m_2}$$

(form irreducible representation)

$$A_{l_1, m_1} B_{l_2, m_2} = \sum V(l, m\,;\, l_1, m_1, l_2, m_2)\, C_{lm}$$

(normalization fixed)

$$V = W^{-1} = W^{\dagger}.$$

These V's are the same as the Clebsch-Gordon coefficients. $\left(C^{l\, l_1\, l_2}_{m\, m_1\, m_2}\right)$

$$\langle n, l, m \mid A_{l_1, m_1} \mid n_2, l_2, m_2 \rangle$$

$$= C^{l\, l_1\, l_2}_{m\, m_1\, m_2} \langle n, l \| A_{l_1} \| n_2, l_2 \rangle$$

(normalization not fixed)

If $A = A^{\dagger}$, then $(WAW^{\dagger})^{\dagger} = (WAW^{\dagger})$

$$(WAW^{-1})^{\dagger} = (W^{-1})^{\dagger} A^{\dagger} W^{\dagger}$$

$$\overset{?}{=} WAW^{-1} \Longrightarrow W^{-1} = W^{\dagger}$$

Preserve the hermiticity

$$[A_i, P_\alpha] = \sum_{\alpha'} R_{\alpha'\alpha} P_{\alpha'}$$

$$Q^T = P^T M$$

$$P^T = Q^T M^{-1}$$

$$Q_\alpha = \sum_\beta M_{\beta\alpha} P_\beta$$

$M \to$ unitary

$$[A_i, Q_\alpha] = \sum_\beta M_{\beta\alpha} [A_i, P_\beta]$$

$$= \sum_{\beta, \beta'} M_{\beta\alpha} R_{\beta'\beta} P_{\beta'}$$

$$= \sum_{\beta, \beta', \gamma} M_{\beta\alpha} R_{\beta'\beta} (M^{-1})_{\gamma\beta'} Q_\gamma$$

$$= \sum (M^{-1} R M)_{\gamma\alpha} Q_\gamma$$

$$[L_i, A_{l_1 m_1} B_{l_2 m_2}]$$

$$= \sum_{m_1', m_2'} \left(D^{l_1, i}_{m_1' m_1} \delta_{m_2' m_2} + D^{l_2, i}_{m_2' m_2} \delta_{m_1' m_1} \right) A_{l_1 m_1'} B_{l_2 m_2'}$$

$$\begin{pmatrix} D^{l_1+l_2, i}_{m'm} & & \\ & D^{l_1+l_2-1, i}_{m'm} & \\ & & \ddots \end{pmatrix}$$

Addition of angular momenta :-

Suppose, in a given QM, we have two sets of operators (M_x, M_y, M_z) and (N_x, N_y, N_z) such that $\longrightarrow$

$$[M_x, M_y] = i\hbar M_z \qquad [N_x, N_y] = i\hbar N_z$$
$$[M_y, M_z] = i\hbar M_x \qquad [N_y, N_z] = i\hbar N_x$$
$$[M_z, M_x] = i\hbar M_y \qquad [N_z, N_x] = i\hbar N_y$$

$$[M_i, N_j] = 0 \text{ for } i, j = x, y, z.$$

(*) Two particles at (x_1, y_1, z_1) , (x_2, y_2, z_2)

$$H = -\frac{\hbar^2}{2m_1} \vec{\nabla}_1^2 + V_1(r_1) - \frac{\hbar^2}{2m_2} \vec{\nabla}_2^2 + V_2(r_2).$$

$$r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Ex Check that $\hat{H}, \hat{M}^2, \hat{N}^2, \hat{M}_z, \hat{N}_z$ are mutually commuting

States can be labelled as

$$|n, l_1, m_1, l_2, m_2\rangle$$

$$M_i |n, l_1, m_1, l_2, m_2\rangle = \hbar \sum_{m_1'} D^{l_1, i}_{m_1' m_1} |n, l_1, m_1', l_2, m_2\rangle$$

$$\vec{M}^2 |n, l_1, m_1, l_2, m_2\rangle = \hbar^2 l_1(l_1+1) |n, l_1, m_1, l_2, m_2\rangle$$

$$N^2 |n, l_1, m_1, l_2, m_2\rangle = \hbar^2 l_2(l_2+1) |n, l_1, m_1, l_2, m_2\rangle$$

$$M_z |n, l_1, m_1, l_2, m_2\rangle = \hbar m_1 |n, l_1, m_1, l_2, m_2\rangle$$

$$N_z |n, l_1, m_1, l_2, m_2\rangle = \hbar m_2 |n, l_1, m_1, l_2, m_2\rangle$$

$$N_i |n, l_1, m_1, l_2, m_2\rangle = \hbar \sum_{m_2'} D^{l_2, i}_{m_2', m_2} |n, l_1, m_1, l_2, m_2'\rangle$$

Define $L_i = M_i + N_i$.

$$\left.\begin{aligned} [M_i, M_j] &= i\hbar \sum_k f_{ijk} M_k \\ [N_i, N_j] &= i\hbar \sum_k f_{ijk} N_k \end{aligned}\right\} \quad [M_i, N_j] = 0$$

$$[L_i, L_j] = i\hbar \sum_{k=1}^{3} f_{ijk} L_k .$$

$\hat{H}, M^2, N^2, L^2, L_z$ are mutually commuting

$|n, l_1, l_2, l, m\rangle' \longrightarrow$ another labelling

$$M^2 |n, l_1, l_2, l, m\rangle' = \hbar^2 l_1(l_1+1) |n, l_1, l_2, l, m\rangle'$$

$$N^2 |n, l_1, l_2, l, m\rangle' = \hbar^2 l_2(l_2+1) |n, l_1, l_2, l, m\rangle$$

$$L^2 |n, l_1, l_2, l, m\rangle' = \hbar^2 l(l+1) |n, l_1, l_2, l, m\rangle'$$

$$L_z |n, l_1, l_2, l, m\rangle' = m\hbar |n, l_1, l_2, l, m\rangle'$$

$\rightsquigarrow$ We must find ~~the~~ a transformation between these bases.

$$L_i |n, l_1, m_1, l_2, m_2\rangle$$
$$= (M_i + N_i) |n, l_1, m_1, l_2, m_2\rangle$$

$$= \hbar \sum_{m_1', m_2'} \left(D^{l_1, i}_{m_1', m_1} \delta_{m_2' m_2} + D^{l_2, i}_{m_2', m_2} \delta_{m_1' m_1} \right) |n, l, m_1', l_2, m_2'\rangle$$

Define $|n, l_1, l_2, l, m\rangle'' = \sum_{m_1, m_2} W(l_1, m_1, l_2, m_2; l, m) \, |n, l_1, m_1, l_2 m_2\rangle.$

$$L_i |n, l_1, l_2, l, m\rangle''$$

$$= \sum_{m'} D^{l, i}_{m'm} |n, l_1, l_2, l, m'\rangle''$$

$$L^2 |n, l_1, l_2, l, m\rangle'' = \sum_{m'} \sum_i \left(D^{l,i} D^{l,i} \right)_{m'm} |n, l_1, l_2, l, m'\rangle''$$

$$\underline{l(l+1)\delta_{m'm}}$$

$\sum_i L_i L_i$

$$\therefore \begin{cases} L^2 |n, l_1, l_2, l, m\rangle'' = \hbar^2 \, l(l+1) |n, l_1, l_2, l, m\rangle'' \\ L_z |n, l_1, l_2, l, m\rangle'' = m\hbar \, |n, l_1, l_2, l, m\rangle''. \end{cases}$$

So, $\boxed{|n, l_1, l_2, l, m\rangle'' \equiv |n, l_1, l_2, l, m\rangle'}$

Conclusion $= \sum_{m_1, m_2} W(l_1, m_1, l_2, m_2; l, m) |n, l_1, m_1, l_2, m_2\rangle$

Add to the earlier Hamiltonian a term

$$\lambda \sum_{i=1}^{3} M_i N_i \longrightarrow \lambda \text{ some no.}$$

$$[L_i, \sum_k M_k N_k]$$

$$= \sum_k \left([L_i, M_k] N_k + M_k [L_i, N_k] \right)$$

$$= \sum_k \left([M_i, M_k] N_k + M_k [N_i, N_k] \right)$$

$$= i\hbar \sum_k \sum_l \left(f_{ikl} M_l N_k + M_k f_{ikl} N_l \right)$$

$$= i\hbar \sum_k \sum_l f_{ikl} (M_l N_k + M_k N_l)$$

sum symmetric

antisymmetric

$$= 0.$$

$$[L^2, \sum_k M_k N_k] = 0 . \quad \text{(also holds!)}$$

We can take $|n, l_1, l_2, l, m\rangle$. as basis

In the first case, we may have taken any of the bases.
Here, we have to take $|n, l_1, l_2, l, m\rangle$.

$$\sum_i M_i N_i = \frac{1}{2}\left\{(M+N)^2 - M^2 - N^2\right\}$$

$$= \frac{1}{2}\left\{L^2 - M^2 - N^2\right\}$$

$$|n, l_1, l_2, l, m\rangle \longrightarrow \frac{\hbar^2}{2}\left\{l(l+1) - l_1(l_1+1) - l_2(l_2+1)\right\}$$

15. 09. 2011

Spin

$$\psi = \begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \end{pmatrix} \quad \text{Direct sum of two vector space.}$$

$$\psi + \psi' = \begin{pmatrix} \psi_1 + \psi_1' \\ \psi_2 + \psi_2' \end{pmatrix} \qquad \langle \psi, \psi' \rangle$$

$$= \int dx\, dy\, dz \left(\psi_1^* \psi_1' + \psi_2^* \psi_2'\right)$$

$$= \int dx\, dy\, dz\, \psi^\dagger \psi .$$

General linear operator (acting on this vector space)

$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

(*) $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are usual operators acting on single functions.

$$= \begin{pmatrix} \hat{A}\psi_1 + \hat{B}\psi_2 \\ \hat{C}\psi_1 + \hat{D}\psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \overset{3\times 3}{A} & \overset{3\times 3}{B} \\ \overset{3\times 3}{C} & \overset{3\times 3}{D} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\hat{H} = \begin{pmatrix} \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right) & 0 \\ 0 & \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right) \end{pmatrix}$$

(Diagonal).

→ special class of Hamiltonians

$$L_x = -i\hbar \begin{pmatrix} y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} & 0 \\ 0 & y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \end{pmatrix}$$

Define

$$[L_i, L_j] = i\hbar \sum_k f_{ijk} L_k.$$

$$\underbrace{[L_i, H] = 0.}_{(\checkmark)}$$

Eigenvalue equation

$$\hat{H}\Psi = E\Psi$$

$$\left\{ \begin{aligned} \left(-\frac{\hbar^2}{2m}\nabla^2\Psi_1 + V(r)\right)\Psi_1 &= E\Psi_1 \\ \left(-\frac{\hbar^2}{2m}\nabla^2\Psi_2 + V(r)\right)\Psi_2 &= E\Psi_2 \end{aligned} \right\}$$

Conclusion :- Ψ_1 and Ψ_2 have the same energy.

$$\Psi_1 = a\,\overleftrightarrow{f_{nl}(r)\,Y_{lm}(\theta,\phi)}^{\text{normalized}}$$

$$E_{nl}$$

$$\Psi_2 = b\,\overleftrightarrow{f_{nl}(r)\,Y_{lm'}(\theta,\phi)}^{\text{normalized}}$$

$$\sum_{m=-l}^{l} a_m \begin{pmatrix} f_{nl}(r)\,Y_{lm}(\theta,\phi) \\ 0 \end{pmatrix}$$

$$\sum_{m'=-l}^{l} b_{m'} \begin{pmatrix} 0 \\ f_{nl}(r)\,Y_{lm'}(\theta,\phi) \end{pmatrix}$$

$$\Rightarrow |a|^2 + |b|^2 = 1.$$

degeneracy :- $(2l+1) + (2l+1)$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pauli matrices

$[S_x, S_y] = i\hbar S_z$ etc... are satisfied.

$[S_i, L_j] = 0. \quad (\forall\, i, j).$

$[S_i, H] = 0.$

$J_i = L_i + S_i$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$= \frac{3}{4}\hbar^2 \cdot \mathbb{1} \quad \left(s = \frac{1}{2} \text{ representation}\right)$$

origin of twofold $(2l+1)$ degeneracy

$$\left| n, l, m_l, s = \frac{1}{2}, m_s = \frac{1}{2} \right\rangle$$

$$= f_{nl}(r)\, Y_{l,m_l}(\theta, \phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left| n, l, m_l, s = \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle$$

$$= f_{nl}(r)\, Y_{l,m_l}(\theta, \phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenstates of $\hat{H}, L^2, S^2 = \frac{3}{4}\hbar^2, L_z, S_z$

<u>Another set of operators</u>

$\hat{H}, L^2, S^2 = \frac{3}{4}\hbar^2, J^2, J_z$

Call the eigenstates $\left| l, s = \frac{1}{2}, j, m_j \right\rangle$

$$\left\{ \begin{aligned} L^2 |\ \rangle &= \hbar^2 l(l+1)\, |\ \rangle \\ S^2 |\ \rangle &= \frac{3}{4}\hbar^2\, |\ \rangle \\ J^2 |\ \rangle &= \hbar^2 j(j+1)\, |\ \rangle \\ J_z |\ \rangle &= m_j \hbar\, |\ \rangle \end{aligned} \right\}$$

$$\sum_{m_l, m_s} W \cancel{(l, m_l, s = \tfrac{1}{2})} \left| l, m_l, s = \frac{1}{2}, m_s \right\rangle$$

$$\left(l, m_l, s = \frac{1}{2}, m_s \,;\, j, m_j\right)$$

$$j = \left(l + \frac{1}{2}\right) \text{ or } \left(l - \frac{1}{2}\right).$$

$$\underset{\Downarrow}{} \qquad \underset{\Downarrow}{}$$

$$2l+2 \; + \; 2l \longrightarrow 2(2l+1).$$

The second set (of eigenkets) is useful where there is spin-orbit coupling.

Suppose, we add to the Hamiltonian an extra term.

$$f(r)\, \vec{L} \cdot \vec{S} = f(r)\left[L_x S_x + L_y S_y + L_z S_z\right]$$

$$= -i\hbar f(r) \left[\begin{pmatrix} 0 & y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) & 0 \end{pmatrix} + \begin{pmatrix} x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} & 0 \\ 0 & -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \end{pmatrix} \right]$$

$$\hat{H}\Psi = E\Psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_1 + V(r)\Psi_1 - i\hbar f(r)\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Psi_2 - i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\Psi_2 + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\Psi_1\right] = E\Psi_1.$$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(J^2 - L^2 - S^2\right).$$

$$f(r)\left| l, s = \frac{1}{2}, j, m_j \right\rangle \longrightarrow \text{Basis}$$

$$= f(r)\sum_{m_l}\left[W\left(l, m_l, s, \tfrac{1}{2}; j, m\right) Y_{lm_l}(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} + W\left(l, m_l, s, -\tfrac{1}{2}; j, m\right) Y_{lm_l}(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\left[\begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2+V & 0 \\ 0 & -\frac{\hbar^2}{2m}\nabla^2+V \end{pmatrix} + f(r)\vec{L}\cdot\vec{S}\right] \left| l, s=\frac{1}{2}, j, m_j \right\rangle$$

So,

$$-\frac{\hbar^2}{2m r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - \frac{1}{2mr^2} l(l+1)\hbar^2 f + V(r) f$$

$$+ F(r)\ \frac{\hbar^2}{2}\left\{ j(j+1) - l(l+1) - \frac{3}{4} \right\} f(r)$$

$$= E f(r)$$

$$E = E_{n,l,j}$$

16.08.2011

$$\psi(x,y,z) = \begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \end{pmatrix}$$

$|\psi(x,y,z)|^2\, dx\, dy\, dz \longrightarrow$ Born interpretation

$(|\psi_1|^2 + |\psi_2|^2)\, dx dy\, dz \longrightarrow$ Analog.

$|\psi_1|^2\, dx\, dy\, dz \rightarrow$ Probability of finding an $m_s = 1/2$ particle in $dx\, dy\, dz$

$|\psi_2|^2\, dx\, dy\, dz \rightarrow$ Probability of finding an $m_s = -1/2$ particle in $dx\, dy\, dz$

$$\langle n, l, m \,|\, A^{l_1}_{m_1} \,|\, n_2, l_2, m_2 \rangle$$

$$= C^{l\ \ l_1\ \ l_2}_{m\ \ m_1\ \ m_2} \langle n, l \,\|\, A^{l_1} \,\|\, n_2, l_2 \rangle$$

(universal numbers)

CG coeff^n.

What about $\langle n, l, m \mid A^{l_1}_{m_1} \; B^{l_2}_{m_2} \mid n_3, l_3, m_3 \rangle$

$$\sum_{l', m'} C^{l' \; l_1 \; l_2}_{m' \; m_1 \; m_2} \; P^{l'}_{m'}$$

$$l' \rightarrow |l_1 - l_2| \text{ to } (l_1 + l_2)$$

$$= \sum_{l'} \sum_{m'} \langle n, l, m \mid P^{l'}_{m'} \mid n_3, l_3, m_3 \rangle \; C^{l' \; l_1 \; l_2}_{m' \; m_1 \; m_2}$$

$$= \sum_{l'} \sum_{m'} C^{l' \; l_1 \; l_2}_{m' \; m_1 \; m_2} \; C^{l \; l' \; l_3}_{m \; m' \; m_3} \; \langle n, l \| P^{l'} \|$$

$$= \underbrace{\sum_{l' = |l_1 - l_2|}^{|l_1 + l_2|} C^{l' \; l_1 \; l_2}_{(m_1+m_2), \, m_1, \, m_2} \; C^{l \; l' \; l_3}_{m \; \cancel{m'} (m_1+m_2) \; m_3}}_{\text{known}} \; \langle n, l \| P^{l'}$$

$$\underline{(l_1 + l_2) - |l_1 - l_2| + 1}$$

Both the C's have to be ~~zer~~ non-zero for giving nonzero contribution.

$$|l' - l_3| \leq l \leq |l' + l_3|$$

$$\langle n, \tfrac{1}{2}, m \mid A^{\frac{1}{2}}_{m_1} \; B^{\frac{1}{2}}_{m_2} \mid n_3, \tfrac{1}{2}, m_3 \rangle$$

$$C^{l' \; \frac{1}{2} \; \frac{1}{2}} \quad C^{\frac{1}{2} \; l' \; \frac{1}{2}} \Rightarrow \underline{l' = 0, 1.}$$

One needs two numbers.

$$\sum_{n',l',m'} \langle n, l, m \,|\, A^{l_1}_{m_1} \,|\, n' l' m' \rangle \langle n' l' m' \,|\, B^{l_2}_{m_2} \,|\, n_3, l_3, m_3 \rangle$$

$$= \sum_{n',l',m'} C^{l\ l_1\ l'}_{m\ m_1\ m'} C^{l'\ l_2\ l_3}_{m'\ m_2\ m_3} \langle n, l \| A^{l_1} \| n', l' \rangle \langle n', l' \| B^{l_2} \| n_3, l_3 \rangle$$

$$= \underbrace{\sum_{l'} C^{l\ l_1\ l'}_{m\ m_1\ m_2+m_3} C^{l'\ l_2\ l_3}_{m_2+m_3\ m_2\ m_3}}_{\text{The } \cancel{\text{conditions}} \text{ number of } \cancel{l, l_1, l'}\ l' \text{ are the same actually}} \underbrace{\sum_{n'} \langle n, l \| A^{l_1} \| n', l' \rangle \langle n', l' \| B^{l_2} \| n_3, l_3 \rangle}_{\text{unknown no. } K_{l'}}$$

$\frac{1}{2}$ (l) $\quad \frac{1}{2}$ (l_1) $\quad \frac{3}{2}$ (l_2) $\quad \frac{3}{2}$ (l_3) $\quad$ 2 constants

$1, 2 \qquad \underbrace{0, 1, 2, 3}$ (x) (x) $\quad$ 2 constants

Given $\langle n, \frac{7}{2}, -\frac{7}{2} \,|\, A^{\frac{3}{2}}_{-\frac{3}{2}} \,|\, n', 2, -2 \rangle = a$

Calculate $\langle n, \frac{7}{2}, -\frac{5}{2} \,|\, A^{3/2}_{-3/2} \,|\, n', 2, -1 \rangle$

~~We have to calculate CG coefficients for~~

$\cancel{\in}$ $\quad l = \frac{7}{2}, \ m = -\frac{7}{2}$

$\cancel{\langle n, \frac{7}{2}, \frac{7}{2}}$ $\quad l_1 = 2, \ l_2 = -2$

$C^{l\ l_1\ l_2}_{m\ m_1\ m_2} \quad C^{l'\ l\ l_1}_{m'\ m\ m_1}$

$$L_+ |2\ -2\rangle = \sqrt{2 \cdot 3 - (-2)(-1)}\ |2\ -1\rangle$$

$$= 2\ |2\ -1\rangle$$

$$L_- |2 \; -1\rangle = \sqrt{2\cdot 3 - (-1)(-2)} \; |2 \; -2\rangle$$

$$= 2\,|2 \; -2\rangle \Rightarrow |2 \; -2\rangle = \frac{1}{2} L_- |2 \; -1\rangle$$

$$\left\langle \frac{7}{2} \; -\frac{7}{2} \right| A^{3/2} \left| 2 \; -2 \right\rangle = \frac{1}{2} \left\langle \frac{7}{2} \; -\frac{7}{2} \right| A^{3/2}_{-3/2} L_- |2 \; -1\rangle$$

$$= \frac{1}{2} \left\langle \frac{7}{2} \; -\frac{7}{2} \right| L_- A^{3/2}_{-3/2} \left| 2 \; -1 \right\rangle$$

$$L_+ \left| \frac{7}{2} \; -\frac{7}{2} \right\rangle$$

$$= \sqrt{\frac{7}{2}\cdot\frac{9}{2} - \frac{(-7)(-5)}{4}} = \sqrt{7} \left| \frac{7}{2} , -\frac{5}{2} \right\rangle$$

$$= \frac{\sqrt{7}}{2} \left\langle \frac{7}{2} \; -\frac{5}{2} \right| A^{3/2}_{-3/2} \left| 2 \; -1 \right\rangle = \frac{2}{\sqrt{7}} \alpha$$

$$= \alpha$$

$m = \frac{3}{2}, \quad m_1 = \frac{3}{2}, \quad m_2 = \frac{-1}{2} \quad : \alpha$

$m = \frac{3}{2}, \quad m_1 = -\frac{1}{2}, \quad m_2 = \frac{5}{2}, \quad m_3 = -\frac{1}{2} \quad : \beta$

Calculate $\left\langle \frac{3}{2}, m \right| A^{\frac{5}{2}}_{m_1} \; B^{\frac{5}{2}}_{m_2} \left| \frac{1}{2}, m_3 \right\rangle$

3, 2

Try explicitly.
