e.

$$E = mc^2 \left(1 + \frac{2E_{NR}}{mc^2}\right)^{1/2}$$

$$c \to \infty \quad \text{Lt.} \longrightarrow \quad mc^2\left(1 + \frac{E_{NR}}{mc^2} + \cdots\right)$$

$$= mc^2 + E_{NR}. \quad (\checkmark)$$

(Agrees in NR limit)

29.02.2012

(1) Sudden approximation

(2) Adiabatic "

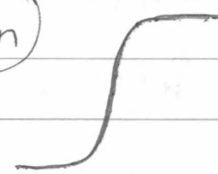
Suppose $\hat{H}(t) = \hat{H}$ for $t < t_0$.

$= \hat{\tilde{H}}$ for $t > t_0$.

Suppose, $\hat{H} u_n = E_n u_n$

& $\psi(t) = u_n(t) \exp\left(-iE_n t/\hbar\right)$ for $t < t_0$.

(Sudden)



What happens for $t > t_0$?

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \quad \text{for } t < t_0$$

$$= \hat{\tilde{H}}\psi \quad \text{for } t > t_0$$

$$\Rightarrow \psi(t_0 - \epsilon) = \psi(t_0 + \epsilon) \text{ as } \epsilon \to 0.$$

$$\lim_{\epsilon \to 0_+} \psi(\vec{x}, t_0 + \epsilon) = u_n(\vec{x}, \cancel{t_0})\, e^{-iE_n t_0/\hbar}.$$

(We assume ψ & $\frac{\partial \psi}{\partial t}$ are continuous at $t = t_0$ in time domain)

If $\hat{\tilde{H}} \tilde{u}_n = \tilde{E}_n \tilde{u}_n$, Expand $u_n(\vec{x})$ in terms of $\tilde{u}_m(\vec{x})$

$$\text{So, } \lim_{\epsilon \to 0_+} \psi(t_0 + \epsilon) = \psi(t) = \sum_m \tilde{u}_m(\vec{x})\, e^{-i\tilde{E}_m(t - t_0)/\hbar}\, e^{-iE_n t_0/\hbar} \langle \tilde{u}_m | u_n \rangle$$

for $t > t_0$

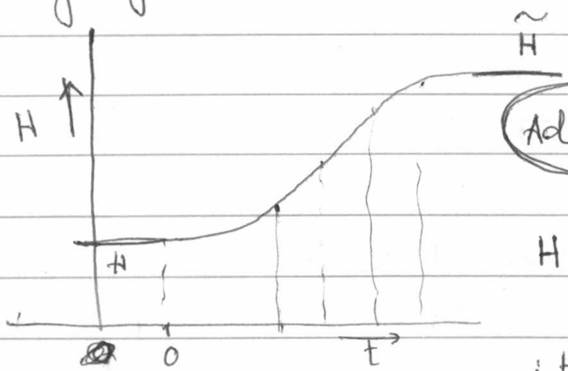
(We have done $|u_n\rangle = \sum_m |\tilde{u}_m\rangle \langle \tilde{u}_m | u_n\rangle$. at $t = t_0$)

Suppose, we have Hamiltonian

$$H \quad \text{for} \quad t \leq 0$$
$$\tilde{H} \quad \text{for} \quad t \geq \tau$$

$\frac{\partial H}{\partial t}$ is small (order $(1/\tau)$) for $0 \leq t \leq \tau$.

We are going to take $\tau \to \infty$ limit.



$H(t) \to H(t/T)$

Adiabatic

$T \to \infty$.

$H(t) \to$ interpolating Hamiltonian

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H(t)\,\Psi(t)$$

$u_n(\vec{x};t)$: n th eigenstate of $\hat{H}(t)$

$E_n(t)$: n^{th} eigenvalue

$$\hat{H}(t)\, u_n(\vec{x};t) = E_n(t)\, u_n(\vec{x};t).$$

$$\Psi(\vec{x},t) = \sum_n a_n(t) \exp\left[-i\int_0^t E_n(t')\,dt'/\hbar\right] u_n(\vec{x},t)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \Rightarrow \sum_n \left\{ i\hbar \frac{\partial a_n}{\partial t} u_n(\vec{x},t) + E_n a_n(t)\, u_n(\vec{x},t) + i\hbar\, a_n(t) \frac{\partial u_n(\vec{x},t)}{\partial t} \right\}^{*} \exp\left(-i\int_0^t E_n(t')\,dt'/\hbar\right)$$

$$= \sum_n E_n a_n(t)\, u_n(\vec{x},t) \exp\left(-i\int_0^t E_n(t')\,dt'/\hbar\right)$$

(*) Middle term cancels

Act on by $\int d^3x\; u_m^*(\vec{x},t)$ ....

$$i\hbar \frac{da_m(t)}{dt} \exp\left(-i\int_0^t E_m(t')\,dt'/\hbar\right) + i\hbar\sum_n a_n(t) \exp\left(-i\int_0^t E_n\right.$$
$$\times \int d^3x\; u_m(x,t)^* \frac{\partial u_n(\vec{x},t)}{\partial t} = 0.$$

$$\therefore \frac{da_m(t)}{dt} = -\sum_n a_n(t)\, \exp\left[i \int_0^t \frac{E_m(t') - E_n(t')}{\hbar} dt' \right] \int d^3x\, u_m^*(\vec{x}, t) \frac{\partial}{\partial t} u_n(\vec{x}, t)$$

$$\hat{H}(t)\, u_n(\vec{x}, t) = E_n(t)\, u_n(\vec{x}, t)$$

$$\frac{\partial}{\partial t} : \quad \frac{\partial \hat{H}}{\partial t} u_n(\vec{x}, t) + \hat{H}(t) \frac{\partial u_n(\vec{x}, t)}{\partial t} = \frac{\partial E_n}{\partial t} u_n(\vec{x}, t) + E_n(t) \frac{\partial u_n(\vec{x}, t)}{\partial t}.$$

Calculate $\int d^3x\, u_m^*(\vec{x}, t) \times (\cdots)$ on both sides

$$\int d^3x\, u_m^*(\vec{x}, t) \frac{\partial \hat{H}(t)}{\partial t} u_n(\vec{x}, t) + E_m(t) \int u_m^*(\vec{x}, t) \frac{\partial u_n(\vec{x}, t)}{\partial t} d^3x$$

$$= \frac{\partial E_n}{\partial t} \delta_{mn} + E_n(t) \int d^3x\, u_m^*(\vec{x}, t) \frac{\partial u_n(\vec{x}, t)}{\partial t}$$

- Try to do calculation for $m \neq n$.

$$\int d^3x\, u_m^*(\vec{x}, t) \frac{\partial u_n(\vec{x}, t)}{\partial t} = \frac{1}{[E_n(t) - E_m(t)]} \int d^3x\, u_m^*(\vec{x}, t) \frac{\partial \hat{H}}{\partial t} u_n(\vec{x}, t)$$

(non-degenerate)

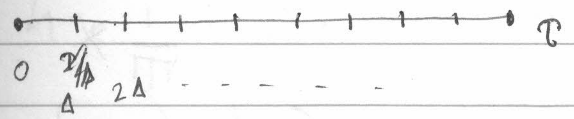
$$\frac{da_m}{dt} = -\sum_{n \neq m} a_n(t)\, \exp\left[i \int_0^t \frac{E_m(t') - E_n(t')}{\hbar} \right] \cdot \frac{1}{\{E_n(t) - E_m(t)\}} \times \int d^3x\, u_m^*(\vec{x}, t) \frac{\partial \hat{H}(t)}{\partial t} u_n(\vec{x}, t)$$

for $m \neq n$

$+$ extra term $(n = m)$.

Suppose, initially $\Psi(t) = u_{n_0} e^{-iE_{n_0} t/\hbar}$ for $t < 0$.

$$N\Delta = \tau$$



Assume

$\Delta \to$ large $\quad \tau \longrightarrow \infty \quad \Delta/\tau \to$ small

$N \to$ large $\quad$ Over Δ, no appreciable change

$\Delta \to$ large wrt $\sim \dfrac{1 \cdot \hbar}{E_n(t) - E_m(t)} \ldots$

For $k\Delta \leq t \leq (k+1)\Delta$

Among $a_n(t)$'s, only a_{n_0} contributes apprecia…

(*) Assume that $a_m(k\Delta) = 0$ for $m \neq n_0$. We shall prove that $a_m((k+1)\Delta) \simeq 0$ for $m \neq n_0$.

$$\frac{da_m}{dt} \simeq -a_{n_0}(k\Delta) \cdot \frac{1}{E_{n_0}(k\Delta) - E_m(k\Delta)} \times \langle m | \frac{\partial \hat{H}}{\partial t}(k\Delta) | n \rangle \Big|_{t=k\Delta} \times$$

$$\times\ \text{~~exp} \left\{ \frac{i}{\hbar} t \left(E_m(k\Delta) - E_n(k\Delta)\right) \right\}\text{~~}$$

If $a_m(k\Delta) \sim \frac{k\Delta}{T}$ then for $k = N$, $a_m(k\Delta) \sim 1$ (not good enough)

$$\left[\exp\left(i \int_0^{k\Delta} dt' \frac{E_m(t') - E_n(t')}{\hbar} \right) \times \exp\left(i \frac{E_m(k\Delta) - E_{n_0}(k\Delta)}{\hbar} t \right) \right.$$

(?) highly oscillatory

So, $a_m((k+1)\Delta) - a_m(k\Delta)$

$$\cong -a_{n_0}(k\Delta) \frac{1}{E_{n_0}(k\Delta) - E_m(k\Delta)} \langle m | \frac{\partial \hat{H}}{\partial t} | n \rangle \Big|_{t=k\Delta} \times$$

$$\exp\left(i \int_0^{k\Delta} dt' \left(\frac{E_m(t') - E_n(t')}{\hbar} \right) \right) \times \frac{\hbar}{i\left(E_m(k\Delta) - E_n(k\Delta)\right)}$$

$$\times \underbrace{\left\{ \exp\left(\frac{i}{\hbar} \Delta \left(E_m(k\Delta) - E_n(k\Delta) \right) \right) - 1 \right\}}$$

$O\left(\frac{1}{T}\right)$

So, small change (!) (✓) ? how big?

$$\frac{1}{T} \times N = \frac{1}{\Delta}$$

$$a_m(t) \text{~~}\frac{da_m}{dt}\text{~~} = \frac{\hbar}{E_m(k\Delta) - E_n(k\Delta)} \exp(\quad)$$

01.03.2012

If we begin with $u_{n_0}(\vec{x})$ at $t=0$, then we end with $a_{n_0}(\tau)\, \tilde{u}_{n_0}(\vec{x})$ at $t=\tau$.

$$\left.\begin{aligned} H u_{n_0} &= E_{n_0} u_{n_0} \\ \tilde{H} \tilde{u}_{n_0} &= \tilde{E}_{n_0} \tilde{u}_{n_0} \end{aligned}\right\} \qquad \frac{da_{n_0}}{dt} = -a_{n_0}(t) \int d^3x\, u^*_{n_0}(\vec{x};t) \times \partial/\partial t\, u_{n_0}(\vec{x};t)$$

$$= -a_{n_0}(t) \left\langle n_0, t \left| \frac{\partial}{\partial t} \right| n_0, t \right\rangle$$

$$\rightsquigarrow a_{n_0}(t) = \exp\left[- \int_0^t dt' \langle n_0, t' | \partial/\partial t' | n_0, t' \rangle \right]$$

$$\therefore a_{n_0}(\tau) = \exp(i\phi) \quad \text{where } \phi = i \int_0^\tau dt' \left\langle n_0, t' \left| \frac{\partial}{\partial t'} \right| n_0, t' \right\rangle$$

$\phi \to$ has to real if $a_{n_0}(\tau) \longrightarrow$ (pure phase)

$$\frac{\partial}{\partial t} \langle n, t | n, t \rangle = 0$$

$$= \left(\frac{\partial}{\partial t} \langle n, t | \right) | n, t \rangle + \left\langle n, t \left| \frac{\partial}{\partial t} \right| n, t \right\rangle$$

$$= \left\langle n, t \left| \frac{\partial}{\partial t} \right| n, t \right\rangle + \left\langle n, t \left| \frac{\partial}{\partial t} \right| n, t \right\rangle^*$$

$$= 0.$$

So, $\text{Re} \left\langle n, t \left| \frac{\partial}{\partial t} \right| n, t \right\rangle = 0$.

- If we choose our wavefunctions to be real, $\phi = 0$.

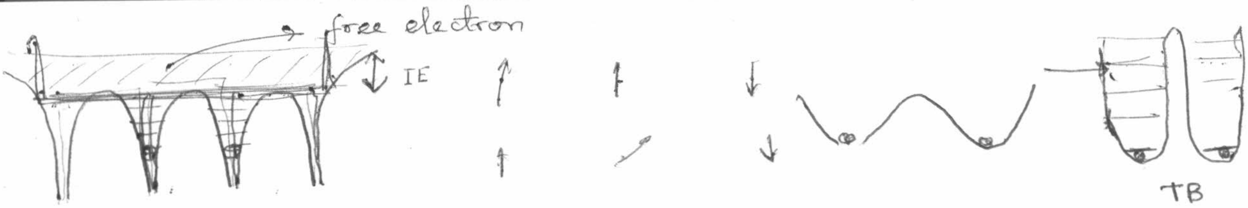
If $H(t) = H(0)$, then $\phi(t)$ does not depend on the choice of basis (!) $\Rightarrow$ something non-trivial (!)

$H(R_1, R_2, \ldots, R_n)$ $\longrightarrow$ parameters

Time dependence comes from time dependence of $R_1, R_2, \ldots R_k$.

$$|n, R_1, R_2, \ldots R_k\rangle \equiv |n, \vec{R}\rangle$$

$\vec{R}$ ↳ k-dim. vector

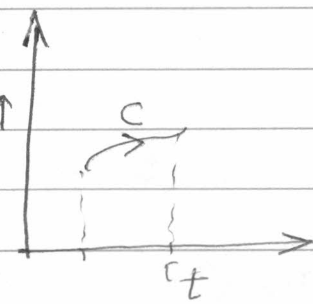


$$\phi = i \int_0^{\tau} dt \langle n_0, \vec{R}(t) | \frac{\partial}{\partial t} | n_0, \vec{R}(t) \rangle$$

$$= i \int_0^{\tau} dt' \sum_{i=1}^{k} \langle n_0, \vec{R}(t) | \frac{\partial}{\partial R_i} | n_0, \vec{R}(t) \rangle \frac{dR_i}{dt'}$$

$$= i \int_C \sum_i dR_i \langle n_0, \vec{R} | \frac{\partial}{\partial R_i} | n_0, \vec{R} \rangle$$

$\longrightarrow$ $\frac{dR_i}{dt}$'s may be anything ?



$H(f(t)) \longrightarrow f(t)$ is monotonic

$\left. \begin{matrix} f(0) = 0 \\ f(t) - \tau \end{matrix} \right\}$

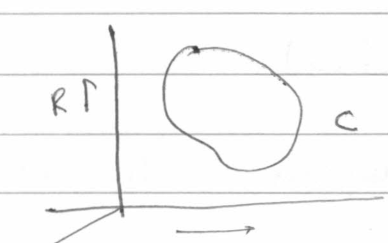
What happens in this case ?

$$\tilde{\phi} = i \int_0^{\tau} \langle n_0, f(t') | \frac{\partial}{\partial t'} | n_0, f(t') \rangle dt'$$

$$= i \int_0^{\tau} d\tau \langle n_0, \tau | \frac{\partial}{\partial \tau'} | n_0, \tau \rangle = \phi \; (!)$$

<u>Closed contour ?</u> ← (flux) (EM-field)

$$\phi = i \oint_C \sum_i dR_i \langle n_0, \vec{R} | \frac{\partial}{\partial R_i} | n_0, \vec{R} \rangle$$



{independent of choice of basis} → vector field in abstract field

$$\oint A_i^{(\vec{R})} dR_i$$

$$A_i^{(R)} = i \langle n_0, \vec{R} | \frac{\partial}{\partial R_i} | n_0, \vec{R} \rangle$$

- Pancharatnam - Berry phase

<u>Example</u> :- Two - state system

- Hilbert space → 2 D.

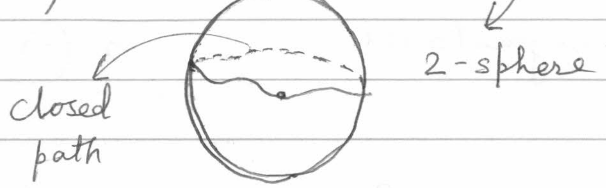
$H = \hat{n} \cdot \vec{\sigma}$

↘ unit vector

$$\hat{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$H = \sin\theta\cos\phi\ \sigma_1 + \sin\theta\sin\phi\ \sigma_2 + \cos\theta\ \sigma_3$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \qquad \theta, \phi \rightarrow \text{parameters}$$



Eigenvalue 1

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\cos\theta + b\sin\theta\, e^{-i\phi} = a$$

$$\Rightarrow a = \frac{b\sin\theta\, e^{-i\phi}}{1-\cos\theta}$$

$$|a|^2 + |b|^2 = 1. \quad \text{(normalisation)}$$

$$\Rightarrow |b|^2\left(1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right) = |b|^2\left(1 + \frac{1+\cos\theta}{1-\cos\theta}\right) = |b|^2\frac{2}{(1-\cos\theta)}$$

$$= \frac{|b|^2}{\sin^2\frac{\theta}{2}}.$$

$$b = \sin\frac{\theta}{2}, \quad a = e^{-i\phi}\cos\frac{\theta}{2}$$

So, $$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$\therefore A_\theta = i\left(e^{i\phi}\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\right)\begin{pmatrix} -\frac{1}{2}e^{-i\phi}\sin\frac{\theta}{2} \\ \frac{1}{2}\cos\frac{\theta}{2} \end{pmatrix}$$

$$= 0.$$

$$A_\phi = i\left(e^{i\phi}\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\right)\begin{pmatrix} -ie^{-i\phi}\cos\frac{\theta}{2} \\ 0 \end{pmatrix}$$

$$= \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$$

- Connection with magnetic monopole

Exercise $\rightarrow$ degenerate case

$$\frac{da_m}{dt} = -\sum_n a_n(t) \oint \langle m,t | \frac{\partial}{\partial t} | n,t \rangle$$

$$= -\sum_n M_{mn}(t)\, a_n(t)$$

matrix

Sum runs over the degenerate eigenstates

$$a(t) = e^{-Mt}\, a(t)$$

for const. M

$$\cong T\left\{ \exp\left(-\int_0^t dt'\, M(t') \right) \right\} a(0)$$

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!}(-1)^k \int_0^T dt_1\, \oint M(t_1) \int_0^T dt_2\, M(t_2) \cdots \int_0^T dt_k$$

(not the solution)

$$= \int_0^T dt_1 \cdots \int_0^T dt_k\, T\left(M(t_1)\, M(t_2) \cdots M(t_k) \right)$$

M $\rightarrow$ (berry matrix) $\rightsquigarrow$ $(k \times k)$ unitary matrix.

02.03.2012

Supersymmetric Quantum Mechanics :-

(*) Trick to solve 1-D QM problems

$$H_1 = -\frac{\hbar^2}{2m} d^2/dx^2 + V_1(x).$$

- Assume, by adding constant to $V_1(x)$, that the ground state Ψ_0 has 0 energy.

$$\left(\frac{1}{2} m\omega^2 x^2 - \frac{1}{2}\hbar\omega \right)$$

$\rightarrow$ If one gives you Ψ_0, you can solve for $V_1(x)$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_0}{dx^2} + V_1(x)\Psi_0 = 0.$$

$$\boxed{V_1(x) = \frac{\hbar^2}{2m}\frac{\Psi_0''}{\Psi_0}} = W^2 - \frac{\hbar}{\sqrt{2m}} W'$$

$$W \xrightarrow{\text{def.}} -\frac{\hbar}{\sqrt{2m}}\frac{\Psi_0'}{\Psi_0}.$$

$$\frac{\hbar^2}{2m}\left(\frac{\psi_0'}{\psi_0}\right)^2 - \frac{\hbar}{\sqrt{2m}}\left(-\frac{\hbar}{\sqrt{2m}}\right)\frac{\psi_0''}{\psi_0} - \frac{\hbar^2}{2m}\left(\frac{\psi_0'}{\psi_0}\right)^2$$

$$= +\left(\frac{\hbar}{\sqrt{2m}}\right)^2 \frac{\psi_0''}{\psi_0}.$$

$W \rightarrow$ superpotential

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2 - \frac{\hbar}{\sqrt{2m}} W'$$

$$= A^\dagger A$$

$$A = +\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

$$A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

($W(x)$ is real.)

$$A\psi_0 = 0$$

(from definition of $W(x)$.)

Let's define a new Hamiltonian

$$H_2 = AA^\dagger$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x)$$

where

$$V_2(x) = W^2 + \frac{\hbar}{\sqrt{2m}} W'$$

$$= V_1 + \frac{\sqrt{2}}{\sqrt{m}}\hbar W'$$

Suppose,

$$H_1\psi_n = E_n\psi_n$$

$$\Rightarrow A^\dagger A\psi_n = E_n\psi_n$$

$$H_2 A\psi_n = AA^\dagger A\psi_n = E_n A\psi_n.$$

$\Rightarrow$ If ψ_n is an eigenstate of H_1 with eigenvalue, E_n, $A\psi_n$ is an eigenstate of H_2 with the same eigenvalue.

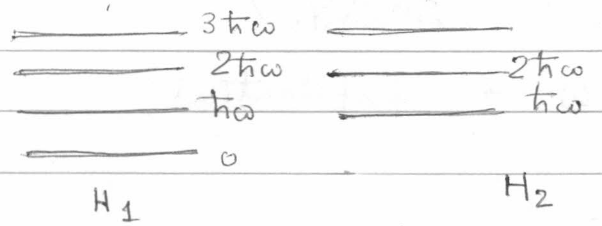
- Except for $n = 0$
- Ground state is bound

$\psi_0 \rightarrow$ unique

$$A\psi_0 = 0.$$

$$V_1 = \frac{1}{2} m \omega^2 x^2 - \frac{1}{2}\hbar\omega$$

(Concrete example)



} spectra

$H_1 A^\dagger \cancel{\psi_n}\, u_n = E_n A^\dagger \psi_n$

If $H_2 u_n = E_n u_n$.

$$V_2 = \frac{1}{2} m \omega^2 x^2 + \frac{1}{2}\hbar\omega$$

→ Check this by calculating W.

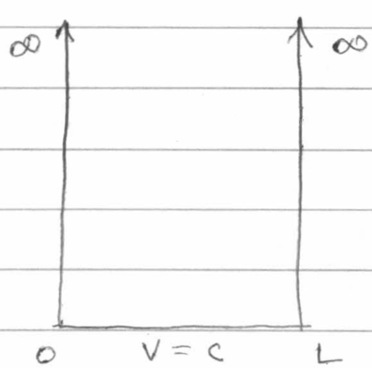
A → ~~takes a p~~ takes you from $\hbar\omega$ to 0.

$A^\dagger$ → ,, $\hbar\omega$ to $2\hbar\omega$.

} In H_1

(Spl. for HO Hamiltonian)

Take infinite square well potential.



$V = C$ for $0 \le x \le L$

$= \infty$,, x outside this range

$$\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad L = \cancel{0}1, 2, \ldots$$

Energy → $E_n = \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L^2} + C$.

- We can take $C = -\frac{\hbar^2}{2m} \pi^2 / L^2$.

$$W = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'}{\psi_0} \qquad V_1, V_2 = W^2 \mp \frac{\hbar}{\sqrt{2m}} W'$$

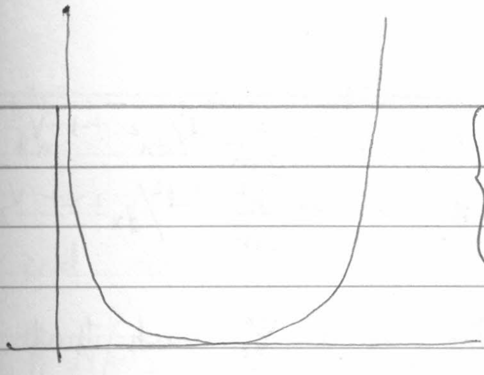
$$= -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{\pi x}{L} \qquad \cancel{W}^2 = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} \cot^2 \frac{\pi x}{L}$$

$$V_1, V_2 = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} \left(\cot^2 \frac{\pi x}{L} \mp \operatorname{cosec}^2 \frac{\pi x}{L} \right) \quad \pm \frac{\pi^2}{L^2} \frac{\hbar^2}{2m} (-) \operatorname{cosec}^2 \frac{\pi x}{L}$$

-1 for V_1, $\left(1 + 2\cot^2 \frac{\pi x}{L}\right)$ for V_2.

$$V_2 = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} \left(1 + 2\cot^2 \frac{\pi x}{L} \right)$$

V_2



$\left\{ \begin{array}{l} \text{eigenstates are} \\ \quad \text{completely known} \end{array} \right\}$ (!)

Eigenfunctions

$$A\psi_n = \left(+ \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W \right) \psi_n$$

$W = -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{x}{L}$

$H_2 \rightsquigarrow$ superpartner

→ Not a systematic procedure. (✓)

Origins :-

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \qquad \psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad \text{(two component)}$$

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^{\dagger} = \begin{pmatrix} 0 & A^{\dagger} \\ 0 & 0 \end{pmatrix}$$

$$Q \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi_n \end{pmatrix}.$$

→ $[Q, H] = 0$. (symmetry !) Same is true for $Q^{\dagger}$.

Exercise Check that

$$[Q, H] = 0, \quad \{Q, Q^{\dagger}\} = H \quad Q^2 = (Q^{\dagger})^2 = 0.$$

(anti-commutator) (anti-commutator)

[Symmetry algebra]

- SUPERSYMMETRIC ALGEBRA ⟶ condition
- Anticommutator of two generators ⟶ Hamiltonian

If $H = \{Q, Q^{\dagger}\}$

$\Rightarrow \langle \psi | H | \psi \rangle = \langle Q^{\dagger}\psi | Q^{\dagger}\psi \rangle + \langle Q\psi | Q\psi \rangle$

⟹ positive $\qquad H \rightarrow$ all positive eigenvalues

⇝ No guarantee of zero eigenvalue

$$A = + \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W \qquad H_1 = -\frac{\hbar^2}{2m} d^2/dx^2 + V_1 = A^\dagger A$$

$$A^\dagger = - \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W \qquad H_2 = -\frac{\hbar^2}{2m} d^2/dx^2 + V_2 = AA^\dagger$$

If there is a Ψ_0 such that $A\Psi_0 = 0$ then we have 0-energy eigenstate.

$$\begin{pmatrix} \Psi_0 \\ 0 \end{pmatrix} \qquad A\Psi_0 = 0$$

$$\Rightarrow \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \Psi_0 = 0$$

$$\Rightarrow \frac{\hbar}{\sqrt{2m}} \frac{d\Psi_0}{dx} = -W(x)\Psi_0$$

$$\Rightarrow d\Psi_0 / \Psi_0 = -\frac{\sqrt{2m}}{\hbar} W(x)$$

$$\Rightarrow \Psi_0(x) = C \exp\left(-\frac{\sqrt{2m}}{\hbar} \int_0^x W(x')\, dx' \right)$$

[This should be normalizable] — positive & grow…

$$\frac{\sqrt{2m}}{\hbar} \int_x^0 W(x')\, dx' \leftarrow$$ negative & falling

$\Psi_0 \to$ { supersymmetric ground state }

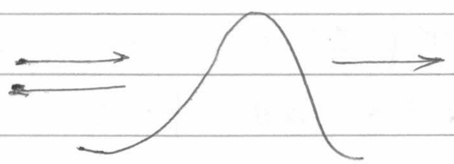
Scattering :-

$$V_{1,2} = W^2 \pm \hbar \frac{W'}{\sqrt{2m}}$$

~~x → W~~ If $W \to W_+$ as $x \to \infty$

$W \to W_-$ as $x \to -\infty$

$$V_1, V_2 \to \left\{ \begin{matrix} W_+^2 & \text{as } x \to \infty \\ W_-^2 & \text{as } x \to -\infty \end{matrix} \right\}$$



$$\psi^{(1,2)} \longrightarrow e^{ikx} + R_{1,2}\, e^{-ikx} \quad \text{as } x \to -\infty$$

$$\longrightarrow T_{12}\, e^{ik'x} \quad \text{as } x \to \infty.$$

$= A^\dagger A$

$= A A^\dagger$

$\Psi_0 = 0$

& growing

$$E = \frac{\hbar^2 k^2}{2m} + W_-^2 = \frac{\hbar^2 k'^2}{2m} + W_+^2$$

$$\psi^{(2)} = N A \psi^{(1)}$$

$x \to -\infty$ end $\Rightarrow$

$$e^{ikx} + R_2 e^{-ikx} = N\left(\frac{\hbar}{2m}\frac{d}{dx} + W_-\right)\left(e^{ikx} + R_1 e^{-ikx}\right)$$

$$= N\left(W_- + \frac{ik\hbar}{\sqrt{2m}}\right) e^{ikx} + N R_1 \left(W_- - \frac{ik\hbar}{\sqrt{2m}}\right) e^{-ikx}$$

$x \to \infty$

$$T_2 e^{ik'x} = N\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_+\right) T_1 e^{ik'x}$$

$$= N T_1 \left(W_+ \frac{ik'\hbar}{\sqrt{2m}}\right) e^{ik'x}$$

Now, compare

$$N = \frac{1}{W_- + \frac{ik\hbar}{\sqrt{2m}}} \qquad R_2 = \frac{W_- - \frac{ik\hbar}{\sqrt{2m}}}{W_- + \frac{ik\hbar}{\sqrt{2m}}} R_1 .$$

$$T_2 = \frac{W_+ + \frac{ik'\hbar}{\sqrt{2m}}}{W_- + \frac{ik\hbar}{\sqrt{2m}}} T_1$$

If ($R_1 = 0$, $R_2 = 0$.)

Take a constant potential.

Take $W^2 - \frac{\hbar}{\sqrt{2m}} W' =$ constant { get non-trivial reflectionless potential }

$W = A \tanh(\alpha x)$ (take as an ansatz)

→ Adjust A & α appropriately

↓

constant pot. ($V_1 = C$)