What is string theory and why should we study it?

Current understanding of elementary constituents of matter
- Standard Model (Glashow, Weinberg, Salam)
- Based on Quantum Field Theory
- Generalization of Quantum Electrodynamics (QED)

In QED we have
- electron \( \psi \) and photon \( A \)
- Dirac field \( \psi \)
- Gauge field \( A \)

Action:
\[ S = \int d^4x \left( \overline{\psi} i \gamma^\mu D_\mu \psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

Generalization:
- Quarks & Leptons:
  - W, Z
  - \( u, c, t; \overline{u}, \overline{c}, \overline{t} \)
  - \( d, s, b; \overline{d}, \overline{s}, \overline{b} \)
  - \((1+i\gamma^5)\psi_{\text{weak}}\)

Higgs Scalar \( \Phi \)

Strong \& \text{Weak} Interactions (violated)
Problem 2: Inclusion of gravity.

- An important force at long distances (astrophysical scale).

Non-relativistic limit + weak gravitational field \Rightarrow \text{Newtonian gravity.}

\text{Otherwise General Theory of Relativity.}

Every known object gravitates \Rightarrow \text{Their constituent elementary particles must also gravitate.}

\text{But the interactions among elementary particles described by the standard model does not include gravity.}

Q. Why then is standard model so successful in explaining observed experimental results?

Compare gravitational and electromagnetic force between two protons:

\[ \frac{G_N M_p^2 / r^2}{e^2 / r^2} = \frac{G_N M_p^2}{e_p^2} \approx 10^{-36} \]

Thus grav. force \ll electromagnetic & other forces.
But however small it may be, gravitational interaction is certainly present in nature and must be included in a complete theory.

Also, gravitational interaction does not always remain small.

Consider two protons moving opposite to each other with large energy and colliding.

\[ \begin{array}{c}
\frac{\text{p}}{\text{E}} \quad \frac{\text{p}}{\text{E}} \\
\end{array} \]

Effective mass of each proton: \( \frac{E}{c^2} \cdot m_p \)

Gravitational interaction between two protons get magnified by a factor of:

\[ \left( \frac{E}{m_p c^2} \right)^2 \]

Electromagnetic interaction between two protons also grows with \( E \) but only logarithmically:

\[ \sim \ln \left( \frac{E}{m_p c^2} \right) \]

If \( E/m_p c^2 \approx 10^{12} \), then the gravitational and electromagnetic interactions become comparable. Result of such an experiment cannot be predicted by the standard model.
Note: At present we cannot carry out this experiment due to lack of powerful enough accelerators.

But a complete theory must be able to predict the results of all experiments, including those which cannot be performed due to practical limitations.

Conclusion: A complete theory must include the effect of gravitational interaction between elementary particles.

What do we know about gravity?

Classical gravity is described by general theory of relativity.

Dynamical variables:

Metric $g_{\mu\nu}(\vec{x}, t)$ describing geometry of space-time

$$\mu, \nu: 0, 1, 2, 3.$$ 

$g_{\mu\nu}$ satisfies Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

$R_{\mu\nu}$: Ricci tensor. $T_{\mu\nu}$: Energy momentum tensor.
General Relativity is a classical field theory with $g_{\mu \nu} (x, t)$ as the classical fields.

In contrast, the standard model is a quantum field theory with quark and lepton fields, gauge fields, etc. as dynamical fields.

At present there is no direct experimental evidence that gravity satisfies the rules of quantum mechanics.

Could it be that the complete theory is

Standard Model

+ Classical General Relativity?

Not possible since elementary particles like electrons, which are known to *not* obey the laws of quantum mechanics, act as the source of gravity.

**Note:**
1. An electron produces a gravitational field.
2. If we could measure all components of the metric and their time derivatives exactly, then we could reconstruct...
and momentum to arbitrary accuracy.

This will violate the rules of quantum mechanics for the electron.

Gravity must also satisfy the rules of quantum mechanics, i.e., there should be an uncertainty principle involving the measurement of the metric and its time derivative.

An aside: The same argument may be used to show that the electromagnetic field must be quantized.

Electric field of an electron depends on its coordinate.

Magnetic field of an electron depends on its velocity.

If we could measure the electric and the magnetic fields at every point to arbitrary accuracy, then we could use this information to find the coordinate and momentum of an electron to arbitrary accuracy.

Violation of Uncertainty Principle.

The electric and magnetic fields must satisfy an uncertainty principle among themselves.
Conclusion: A complete theory must contain:

1. Standard Model.
2. Quantum General Relativity.

Why can't we simply quantize general relativity and append it to the standard model?

Follow the same path as QED.

Dynamical variables: \( g_{\mu\nu}(x, t) \).

Alternatives:

1. Operator approach: \( g_{\mu\nu}(x, t) \) are operators satisfying appropriate commutation relations.
2. Path integral approach:

\[
S\left[g_{\mu\nu}\right] < S\left[g\right]
\]

Action

In either approach, quantization of \( g_{\mu\nu} \)

→ A particle mediating gravitational interaction (Analog of photon).

⇒ Graviton
Photon comes from a vector field $A_\mu(x,t)$.

- has spin 1.

Graviton comes from a tensor field $g_{\mu\nu}(x,t)$

- has spin 2.

Graviton propagator

Graviton interaction vertex

The form of the propagator as well as the vertices can be derived from knowing the Lagrangian of classical general relativity.

Graviton scattering can be computed by putting together the vertices and propagators in the usual fashion.

Tree diagrams (no momentum integration)
Loop diagrams:

\[
\begin{array}{cccc}
1 & \quad & 3 \\
\quad & \quad & \quad \\
2 & \quad & 4 \\
\end{array}
\]

\[\rightarrow \text{Infinite answer.} \]

(Ultraviolet divergence).

**Origin:**

\[\rightarrow \text{Divergent for large } k.\]

1) In momentum space:

\[k_1 \rightarrow k_1 - k \rightarrow k_3\]

\[k_1 \rightarrow k_1 - k - k_3\]

\[k_2 \rightarrow k_2 + k \rightarrow k_4 + k - k_3\]

\[\int d^4k \times \text{Propagators} \times \text{Vertex factors} \]

2) In the position space:

\[X_1 \quad X_2 \quad X_3 \quad X_4 \]

\[\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \times \text{Propagators} \times \text{Vertex factors} \]

\[\rightarrow \text{Divergent from the region where all } x_i \text{'s coincide.} \]
Note: such divergences also exist in QED and the standard model.

But they can be removed by the procedure of renormalization.

(Final answer is not divergent although in the intermediate stages there is divergence.)

But the renormalization procedure does not work for gravity. Naive quantization of general relativity does not give a consistent theory.

String theory is an attempt to resolve this problem.
Basic idea in string theory:

Different elementary particles are all single type of vibrational modes of a string.

- Closed strings
- Open strings

Shouldn't we have seen this in the experiment involving elementary particles?

Not if the typical size of a string is smaller than the resolution that an accelerator can provide.

\( L_s \) : string length

To see the string nature of elementary particles we need "light" of wavelength \( \lambda < L_s \).

\[ \Rightarrow \text{Energy} \geq \frac{h}{\lambda} > \frac{h}{L_s} \]

For small \( L_s \) we need very high energy particle.
Standard scenario: \( l_s = 10^{-33} \) cm.

\[ \frac{hc}{\lambda_s} \approx 10^{19} \text{ GeV} \gg 10^3 \text{ GeV} \]

(Present accelerator energy)

\( \Rightarrow \) At present day experiments, strings will appear to be point-like (particle-like) objects.

\( \Rightarrow \) A given string is

\( \Rightarrow \) collection of \( \infty \) no. of harmonic oscillators (normal modes of oscillation).

\( \Rightarrow \) \( \infty \) no. of energy levels.

Each such energy level

\( \Rightarrow \) a type of elementary particle.

Thus it would appear that a string theory contains infinite number of elementary particles.

\( \Rightarrow \) seem to contradict what we know about nature.

Resolution: Most of these elementary particles have mass \( \frac{hc}{\lambda_s} \approx 10^{19} \) GeV

\( \Rightarrow \) cannot be produced in present day accelerator.
Only those states which have mass \( \ll \frac{\hbar c}{\kappa} \) can be observable at current accelerators.

One particular quantum state of the string represents a massless spin 2 particle.

- Recall that graviton is a massless spin 2 particle.
- String theory automatically contains a graviton.

But before we can say that string theory contains gravity, we should see if this massless spin 2 particle interacts in the same way as a graviton.

**String interaction**

Interaction between point particles is determined by Feynmann diagrams.

\[
\begin{array}{c}
\text{Propagator: } \Delta(x, z) = \text{obtained by summing over all paths between } x \text{ and } z, \\
\end{array}
\]
Full Feynman diagram obtained by summing over all paths with this topology.

We follow the same philosophy for describing string interactions.

Sum over all surfaces describing the process \( A + B \to C + D \).

\( \Rightarrow \) Scattering amplitude for strings \( A \& B \) \( \to \) strings \( C \& D \).

But if we want scattering of strings in specific energy eigenstates (e.g., the graviton state) we need to take the convolution of the space amplitude with the corresponding wave function.
This way we can calculate the graviton scattering amplitude.

Result from the diagram:

When the energies of the external states are small compared to \( hc/l_s \), the scattering amplitude calculated from string theory agrees exactly with that calculated from tree level graphs in general relativity.

The spin 2 particle in string theory has interactions exactly like that of the graviton.
This is also true for higher joint functions.

Note: In quantum general relativity the form of the vertex is fixed by the action giving the Einstein's equation.

In string theory there is no vertex as the surface is smooth.

We have no freedom in specifying how the strings should interact.

Yet we get out an interaction that contains a quantum general relativity.

Thus in string theory:

1. Quantum mechanics and principles of special relativity are inputs.
2. General Relativity is an output.
What about the problem of ultraviolet divergences?

The divergence comes when all four vertices come close to each other.

What will be the corresponding diagram in string theory?

This diagram has no vertices.

Thus when we sum over all surfaces (rather than paths) we do not encounter configuration analogous to collapsed vertices.

⇒ String theory amplitudes have no ultraviolet (short distance) divergence!
Result: string theory provides a finite quantum theory of gravity!

Note: This is even better than renormalizable quantum field theories since here there is no divergence in the first place.

Problems:

1. This theory is consistent only when the number of space-time dimensions is 26 (instead of 4)!

2. Besides the graviton and other massless states, this theory contains a particle with negative mass². A "tachyon"

3. This theory contains gravity but not other known interactions in nature (strong, weak, electro-magnetic).

Furthermore, this does not have fermions.

Conclusions: This does not look like the theory we observe in nature.
The problem of not having gauge interactions can be resolved by including open strings in the theory.

Just like the states of a closed string represent a massless spin 2 particle and resembles a graviton, some of the modes of open string correspond to massless spin 1 particles and resembles non-abelian gauge bosons in their interaction.

But in order to resolve the other problems we need to turn to superstring theory.

(The theory described here is called the bosonic string theory.)
Superstring theory:

Strings, instead of vibrating in usual space-time, also has some internal degree of freedom.

These internal degrees of freedom are fermionic.

Then a superstring could be regarded as a collection of infinite no of bosonic and fermionic harmonic oscillators.

- Can be quantized in a manner similar to that for bosonic strings.

- The good features (e.g. emergence of gravity and UV finiteness) are preserved, but some bad features go away.

1. The theory is consistent in 10 space-time dimensions instead of 4.
2. The tachyonic mode is absent.
3. There are five fully consistent string theories in $D=10$

Type IIA, Type IIB, Type I, $E_8 \times E_8$ heterotic, $SO(32)$ heterotic

Open and closed
String

4. The last 3 theories have non-abelian gauge interaction besides gravity.
5. Except for type IIA, the other theories also contain chiral fermions.
Recall that nature contains chiral fermions like the neutrinos.

All of these theories have a special kind of symmetry, known as supersymmetry, which relates bosonic states to fermionic states and vice versa.

This is closer to nature compared to the bosonic string theory described earlier, but we are still far from a realistic model.

How do we reduce the number of dimensions from 10 to 4?

Kaluza-Klein mechanism / Compactification

As a toy example we shall start from a $(k+1)$ dimensional theory and show how one can get a $(1+1)$ dimensional theory from there.

Consider the two dimensional space to describe the surface of an infinite cylinder rather than an infinite plane.

If $R \gg$ the range of the most powerful telescope, then the scientists living in the space will not know that they are living on a cylinder, and not on a plane.
If R is not so large then scientists with powerful telescopes will see it. As R becomes smaller and smaller and more and more people can feel the effect.

But now suppose R is so small that it is smaller than the resolution of the most powerful microscope.

The space will look one dimensional. Thus the same space in one limit may look 2-dimensional, and in another limit may look 1-dimensional.

This is the same principle that can be used to make a 4-dimensional theory look (3+1)-dimensional.

Take the ten dimensional space of the form:

\[ M^4 \times K \]

A 4-dimensional Minkowski space \( A 6 \)-dimensional compact space.

Size (K) \( \ll \) Resolution of most powerful microscopes (accelerator)

\( \sim 10^{-17} \) cm.
When \( k \) is more than one dimensional then there are many different possible choices for \( k \).

For 2-dimensional manifold we can have sphere, torus, double torus etc.

Similarly there are many different six dimensional manifolds.

Not all of them are consistent (e.g. the metric does not satisfy Einstein's equation).

A certain subset of six dimensional manifolds are known as Calabi-Yau manifolds.

Taking \( k \) to be a Calabi-Yau manifold gives a four dimensional theories with all the qualitative features of the standard model.
For example, for a suitable choice of $K$, we can get four dimensional theories containing

1. Gravity.
2. $SU(3) \times SU(2) \times U(1)$ gauge interactions.
3. Weyl fermions.
4. 3 generations of quarks and leptons including neutrinos which are Weyl fermions.

etc.

However, so far nobody has found a suitable manifold $K$ which gives exactly the standard model with all the masses and couplings correctly.

More importantly, there is as yet no principle for deciding which $K$ is chosen by nature.
Developments during the last 5 years.

1. All the five apparently different string theories have been found to be different limits of the same underlying theory.

   \[ \Rightarrow \text{M-theory.} \]

2. String theories have been unified.
Our aim will be to quantize a string consistent with the principles of quantum mechanics and special relativity.

We shall do this in analogy with the quantization of relativistic point particle.

Set \( k = 1, \ c = 1 \)

Starting point: Classical dynamics of relativistic point particle.

Eq. of motion: \( \frac{d^2 x^k}{d\tau^2} = 0 \).

\( \tau \): proper time.

\[
\gamma^2 = -\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}
\]

\[
\gamma = \begin{pmatrix}
-1 \\
\vdots \\
-1
\end{pmatrix}
\]

Action:

\[
S = \int_a^b L \, ds = \frac{1}{2} \int_a^b \left( -\gamma_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{1/2} ds.
\]

\[
\delta S = \frac{1}{2} \int_a^b \left( -\gamma_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{-1/2} \left( -\eta_{\mu
u} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right) \delta x^\eta \, ds
\]

\[-\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{d(\delta x^\nu)}{ds} \]
\[ \mathcal{L} = e \left( \frac{d}{ds} \left( \gamma^{\mu} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} \right)^{1/2} + \frac{1}{2} \gamma_{\mu\nu} \frac{dx_{\mu}}{ds} \right) \delta x^{\nu}(s) \]

Eqs. of motion:
\[ \frac{d}{ds} \left( \gamma^{\mu} \frac{dx_{\mu}}{ds} \frac{dx^{\nu}}{ds} \right)^{1/2} \frac{dx^{\nu}}{ds} = 0 \]

\[ \frac{dx}{ds} = \left( -\gamma_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \right)^{1/2} \frac{dx}{ds} \]

\[ \frac{d}{ds} = \left( -\gamma_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \right)^{1/2} \frac{dx}{ds} \]

\[ \left( -\gamma_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \right)^{1/2} \frac{d^2 x^{\nu}}{ds^2} = 0 \]

Either \[ \gamma_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = 0 \] (massless particle)

or \[ \frac{d^2 x^{\nu}}{ds^2} = 0 \] (massive particle)

Thus the action that we started with correctly reproduces the equations of motion.

Our next task is to quantize it.

This is made difficult by the square root in the action.

So we first need to simplify it.
Strategy: Replace the action \( S dt \) by another action which is classically equivalent to the action \( S dt \), i.e. which gives the same eqs. of motion.

Then quantize it.

Consider the action

\[
S = \frac{1}{2} \int ds \left[ - \frac{dx^m}{ds} \frac{dx^n}{ds} \eta_{mn} \right] \lambda(s) + \lambda(s)^{-1} \]

\[
\frac{\delta S}{\delta \lambda(s)} = 0,
\]

\[
\Rightarrow \lambda(s)^{-2} = \left( - \eta_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} \right).
\]

\[
\Rightarrow S = c \int ds \left( - \frac{dx^m}{ds} \frac{dx^n}{ds} \eta_{mn} \right)^{1/2}
\]

\( S \) same action.

Alternatively look at \( X^m \) eqs. of motion.

\[
\frac{\delta S}{\delta X^m(s)} = 0, \quad \frac{d}{ds} \left( \frac{\lambda(s)}{\xi(s)} \eta_{mn} \frac{dx^n}{ds} \right) = 0
\]

\[
\Rightarrow \frac{d}{ds} \left( \left( - \eta_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} \right)^{1/2} \eta_{mn} \frac{dx^n}{ds} \right) = 0.
\]

\( S \) same eqs. of motion.
A re-interpretation of the action:

Define:

\[ g_{ss} = \lambda(s)^{-2} \]

- a 1x1 matrix

\[ \det g = \lambda(s)^{-2} \quad g^{ss} = (g_{ss})^{-1} = \lambda(s)^2 \]

\[ S = \frac{1}{2} C \int ds \sqrt{\det g_{ss}} \left[ -g^{ss} \eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 1 \right] \]

\[ \Rightarrow \text{A one dimensional general covariant action} \]

\[ g_{ss} : \text{Metric in the space labelled by s.} \]

Reparametrization invariance:

\[ s \rightarrow f(s), \quad g_{ss} \rightarrow g_{ss} (f'(s))^2 \]

Choosing \( f'(s) = (g_{ss})^{-1/2} \) we can fix the gauge

\[ g_{ss} = 1. \]

\[ S = \frac{1}{2} C \int ds \left[ -\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 1 \right]. \]

\[ g_{ss} \text{ eqs of motion:} \]

\[ g_{ss} = -\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \Rightarrow -\eta_{\mu\nu} \frac{ds}{ds} \frac{ds}{ds} = 1 \Rightarrow s = \tau \]
5: Free field action

Quantization is straightforward.

\[ P_\mu = \frac{\partial S}{\partial (\partial_\mu X^n)} = -c \eta_{\mu \nu} \frac{dx^\nu}{ds} \]

\[ \mathcal{H} = P_\mu \frac{dx^\mu}{ds} - L \]

\[ = -\frac{1}{2c} P_\mu P_\nu \eta^{\mu \nu} - \frac{1}{2} c^2 \]

**Basis:** Momentum eigenstates

\[ P_\mu |k\rangle = k_\mu |k\rangle \]

\[ k_\mu: \quad 4\text{-momentum eigenvalue} \]

\[ P_\mu: \quad 4\text{-momentum operator} \]

**Constraint:** \[ \frac{1}{c^2} \eta^{\mu \nu} P_\mu P_\nu = -1 \]

\[ \Rightarrow \eta^{\mu \nu} k_\mu k_\nu = -c^2 \]

Compare with \( E = mc \). This agrees with the expected property of a relativistic point particle if \( c = m \).
Light cone gauge

- A different gauge fixing which destroys manifest covariance, but the constraints are easier to solve at the operator level.

\[ S = \frac{i}{2} m \int ds \left[ \left( \frac{dx^+}{ds} \frac{dx^-}{ds} \eta_{\mu\nu} \right) \lambda(s) + (\lambda(s))^{-1} \right] \]

Choose \( s \) such that

\[ s = \sigma^0 X^+ \equiv X^0 + X^ \]

\[ s = \frac{i}{2} m \int ds \left[ \left( \frac{dx^-}{ds} - \frac{dx^0}{ds} \right) \frac{dx^0}{ds} \right] \lambda(s) + (\lambda(s))^{-1} \]

\[ x^+ \] eqn. of motion:

\[ \frac{d}{ds} \left( \lambda(s) \frac{dx^-}{ds} \right) = 0 \]

- Construct the Hamiltonian:

\[ P^- = \frac{\partial S}{\partial (\partial^- \lambda)} = \frac{1}{2} m \lambda \]

\[ P_\lambda = \frac{\partial S}{\partial (\partial_\lambda \lambda)} = 0 \]

\[ P_i = \frac{\partial S}{\partial (\partial_i \lambda)} = -\lambda m \frac{dx^i}{ds} \]

\[ \mathcal{H} = P^- \partial^- \lambda + P_\lambda \partial_\lambda \lambda + P_i \partial_i \lambda \]

\[ = -\frac{i}{2m\lambda} P^- P_\lambda - \frac{m}{2X^i} P_i P_i - \frac{m^2}{4\lambda\lambda P^-} \]
Commutation relations can be worked out using the constrained Hamiltonian formalism. In this case they are the same as the original usual ones:

\[ [X^j, P^i_\pm] = i\hbar, \quad [X^j(t), P^i(t)] = i\hbar \delta_{ij}. \]

Other commutators vanish.

Interpretation of \( H \): Generator of \( \Lambda \) evolution.

But \( \Lambda = X^+ \).

\( \Rightarrow \) \( H \) should generate \( X^+ \) evolution.

\( \Rightarrow \) \( H = P^+ \).

Eqs. of motion:

\( \frac{\partial}{\partial x_i} (p_i^\pm) = 0, \quad \frac{\partial}{\partial p_i} (p_i^\pm) = 0. \)

\( X^+ \) eqn:

\[ \frac{d}{ds} \left( \frac{p_+}{m} \frac{dx^-}{ds} \right) = 0. \]

\( \Rightarrow \) \( \frac{dx^-}{ds} = \text{constant} \).

\[ \frac{dx^-}{ds} = \left[ H, X^- \right] = \mp \frac{p_-}{2m} p_+ + \frac{m^2}{4m^2} \frac{p_+}{p_-}. \]

Since \( p_+ \) & \( p_- \) are constants, this eqn is automatically satisfied.
Hilbert space state:
\[ \psi(k^- x + k^+ x^+ \hat{\sigma}) = (k_-, \hat{k}) \]

\[ P_- | k_-, \hat{k} \rangle = k_- | k_-, \hat{k} \rangle \]
\[ P^+ | k_-, \hat{k} \rangle = k^+ | k_-, \hat{k} \rangle \]

Note: \( k_- \) & \( k^+ \) are unconstrained variables.

Interpretation of \( \hat{H} \):
\[ \hat{H} | \psi \rangle = -\hbar^2 \frac{\partial^2}{\partial x^2} | \psi \rangle = \hbar^2 \frac{\partial^2}{\partial x^2} | \psi \rangle \]

On the other hand
\[ P_+ | \psi \rangle = -\hbar \frac{\partial}{\partial x^+} | \psi \rangle \]

\[ \Rightarrow \quad \hat{H} = -P_+ \]

\( \hat{H} \) eigenvalue \( = k^+ \)

\[ \Rightarrow \quad k^+ = +\sqrt{\frac{\hbar^2}{4\theta k_-} + \frac{m^2}{4\theta k_-}} \]

\[ \Rightarrow \quad 4\theta k_+ k_- \sqrt{\hat{k}^2} = m^2 \]

\[ k_+ = \frac{1}{2} (k^0 \mp \vec{k}') \quad \Rightarrow \quad (k_0)^2 - (\vec{k}')^2 = \frac{\vec{k}^2}{6} = m^2 \]

\[ \Rightarrow \quad \text{usual relation} \]
Quantization of a relativistic string:

Action $s = T \int d^2 \xi \sqrt{-\text{det} \left( h_{\alpha \beta} \right)}$

Tension $T$ area element

Parametrize the surface by two parameters $(\xi^0, \tau)$

$\xi^0 = \tau \rightarrow \xi^1 = \xi^0 = \tau$

$S_{\text{action}} = - T \int d^2 \xi \sqrt{-\text{det} \left( h_{\alpha \beta} \right)} \rightarrow$ Reparametrization invariant

$\xi^0 \rightarrow \xi^0 \left( \xi^1 \right)$

$h_{\alpha \beta} = \begin{pmatrix} 2 \dot{\xi} \cdot \dot{x} & \xi' \cdot \dot{x} \\ \dot{x} \cdot \dot{x} & \xi'' \end{pmatrix}$, 2-d field theory

First we need to remove $\sqrt{}$

Alternative action:

$S = - \frac{T}{2} \int d^2 \xi \sqrt{-\text{det} \left( g_{\alpha \beta} \right)} \gamma_{\alpha \beta} \dot{\xi}^\alpha \dot{\xi}^\beta \eta_{\mu \nu}$

$g_{\alpha \beta}$ symmetric

$\gamma_{\alpha \beta}$ $2 \times 2$ matrix valued field

$\gamma_{\alpha \beta}$ inverse of $\gamma_{\alpha \beta}$
\[ \delta S \text{ of action for } \gamma : \]
\[ \delta S = -\frac{\pi}{2} \int d^2 \sigma \left[ \delta \sqrt{-\text{det } \gamma} \gamma^{\alpha \beta} + \sqrt{-\text{det } \gamma} \delta \gamma^{\alpha \beta} \right] \]
\[ = -\frac{\pi}{2} \int d^2 \sigma \left[ \left( -\text{det } \gamma \right)^{1/2} \gamma^{\alpha \beta} \delta \gamma^{\alpha \beta} \right] \]
\[ = -\frac{\pi}{2} \int d^2 \sigma \gamma^{\alpha \beta} \delta \gamma^{\alpha \beta} \]
\[ = -\frac{\pi}{2} \int d^2 \sigma \gamma^{\alpha \beta} \delta \gamma^{\alpha \beta} \]
\[ = -\frac{\pi}{2} \int d^2 \sigma \gamma^{\alpha \beta} \delta \gamma^{\alpha \beta} \]
\[ = \frac{\pi}{2} \int d^2 \sigma \gamma^{\alpha \beta} \delta \gamma^{\alpha \beta} \]

\[ \Rightarrow \gamma^{\alpha \beta} = F(\gamma^{\alpha \beta}) \]

So \( \gamma^{\alpha \beta} = 0 \) for any \( F(\gamma^{\alpha \beta}) \).
Substitute into the expression for $\tilde{S}$:

$$\tilde{S} = \frac{F^2}{8} \left( - \frac{1}{2} \right) \sqrt{\text{det} \gamma} \frac{1}{2 \gamma^{\alpha \beta}} \frac{1}{2 \gamma^{\alpha \beta}}$$

Thus $\tilde{S}$ and $S$ describes the same action after we eliminate $\gamma_{\alpha \beta}$ in $\tilde{S}$ by its equation of motion.

Symmetries of $\tilde{S}$.

1. Re-parametrization invariance:

$$X^\mu' \left( \frac{\bar{x}'}{\bar{x}} \right) = X^\mu \left( \frac{\bar{x}}{\bar{x}} \right) \Rightarrow \frac{\bar{x}}{\bar{x}} \frac{\partial}{\partial \bar{x}^\mu} \phi \left( \frac{\bar{x}}{\bar{x}} \right)$$

$$\tilde{S} \left( X^\mu, \gamma_{\alpha \beta} \right) = \tilde{S} \left( X^\mu', \gamma_{\alpha \beta} \right)$$

Two parameter family of gauge symmetry.

2. Weyl invariance:

$$X'^\mu \left( \frac{\bar{x}}{\bar{x}} \right) = X^\mu \left( \frac{\bar{x}}{\bar{x}} \right)$$

$$\gamma'_{\alpha \beta} \left( \frac{\bar{x}}{\bar{x}} \right) = \phi \left( \frac{\bar{x}}{\bar{x}} \right) \gamma_{\alpha \beta} \left( \frac{\bar{x}}{\bar{x}} \right)$$

$$\tilde{S} \left( X^\mu, \gamma'_{\alpha \beta} \right) = \tilde{S} \left( X^\mu, \gamma_{\alpha \beta} \right)$$

$\phi \left( \frac{\bar{x}}{\bar{x}} \right)$ arbitrary function.
3. A one parameter family of local symmetries.

\[ X' \, ^\mu (\tilde{\xi}) = \Lambda ^\mu_\nu \, X ^\nu (\tilde{\xi}) \]
\[ \gamma' _{\alpha \beta} (\tilde{\xi}) = \Omega \gamma _{\alpha \beta} (\tilde{\xi}) \]

\[ \Lambda : \text{ Lorentz transformation} \]
\[ \Lambda \eta \Lambda ^T = \eta \]

\[ S (X' \, ^\mu , \gamma' _{\alpha \beta}) = S (X ^\mu , \gamma _{\alpha \beta}) \]


3. Gauge symmetries.

\[ \text{We can fix three of the D+3 fields} \, X ^\mu , \gamma _{\alpha \beta} \]

\[ \text{We have different possible choices.} \]
1. Covariant gauge fixing:

\[ \gamma_{\alpha\beta} = \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ S = -\frac{1}{2} \int d^2 x \, \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \]

\( \Rightarrow \) A free field theory.

Constraints: \( \gamma_{\alpha\beta} \) equations of motion:

\[ T_{\alpha\beta} = 0 \]

\[ \eta_{\mu\nu} \partial_\mu x^\nu \partial_\nu x^\mu - \frac{i}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\mu x^\nu \partial_\nu x^\nu \gamma_{\alpha\beta} \]

Note: \( \eta^{\alpha\beta} T_{\alpha\beta} = 0 \) identically.

\( \Rightarrow \) One of the three constraints is redundant.

Define: \( U \equiv 8 \Theta \Theta \xi^\nu \xi_{\nu}, \quad V = \xi^\nu \Theta \Theta \xi_{\nu} \equiv 8 \Theta \Theta ^+ \xi_{\nu} \)

Independent constraints:

\[ T_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu} \partial_\mu x^\rho \partial_\nu x^\rho = 0 \quad \text{check} \]

\[ T_{\nu\nu} = \frac{1}{2} \eta_{\mu\nu} \partial_\nu x^\mu \partial_\nu x^\nu = 0 \]
Strategy:
1. Quantize the free scalar field theory of the $X^i$'s.
2. Impose the physical state condition

$$\langle \text{phys} \mid T_{\mu\nu} \mid \text{phys}' \rangle = 0$$

$$\langle \text{phys} \mid T_{15} \mid \text{phys}' \rangle = 0.$$

Step 1 for closed string theory

$$X^\mu (\sigma + 2\pi R) = X^\mu (\sigma)$$

$$X^\mu (\sigma, \tau) = \sum_{n \neq 0} (2\pi R) \epsilon^{\mu \nu} + X^\mu$$

Substitute into the action:

$$\tilde{S} = -\frac{\alpha'}{2\pi^2} \sum_{n \neq 0} \left( \partial_\sigma \phi^+_n \partial_\tau \phi^+_n + \partial_\tau \phi^-_n \partial_\tau \phi^-_n \right) + \frac{\alpha'}{2\pi} \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu}$$

$$T = \frac{1}{2\pi \alpha'} (\phi^+_n)^*$$

$$\tilde{S} = \frac{1}{2\alpha'} \left[ \sum_{n \neq 0} \left( \partial_\tau \phi^+_n \right)^* \partial_\tau \phi^+_n - n^2 \phi^+_n \phi^-_n \right] + \eta_{\mu\nu} \partial_\tau \phi^+_n \partial_\tau X^\nu$$
A collection of harmonic oscillators.

Conjugate variable to $\phi^k$:

$$\Pi^\nu_{\mu n} = \frac{\delta S}{\delta (\partial_\nu \phi^\mu_n)} = \frac{i}{\alpha'} (\phi^k \phi^k_{-n})^* \eta_{\mu \nu}$$

$$[\phi^k_n, \phi^l_m \Pi^\nu_{\mu n}] = 2 \delta^k_\nu \delta^l_\mu \delta^\nu_m \delta^\mu_n = \gamma^\mu_\mu \alpha'$$

$$H = \sum_{n \neq 0} \Pi^\nu_{\mu n} \partial_\nu \phi^\mu_n - \frac{\delta S}{\delta (\partial_\nu \alpha')} + \sum_{n \neq 0} (\phi^k_n)^* \phi^{-n}_k$$

$$= \sum_{n > 0} \Pi^\nu_{\mu (\mu - n)} \Pi^\nu_{\mu n} \eta^{\nu \mu} + \frac{i}{\alpha'} \sum_{n > 0} (\phi^k_n)^* \phi^{-n}_k \eta^{\nu \mu}$$

$$+ \frac{\alpha'}{2} \eta^{\mu \nu} p_\mu p_\nu$$

$$\Pi_{\mu n} = \Pi_{\mu (-n)} \ , \ (\phi^k_n)^* = \phi^{-n}_k$$

$$H = \sum_{n > 0} \Pi^\nu_{\mu (\mu - n)} \Pi^\nu_{\mu n} \eta^{\nu \mu} + \frac{i}{\alpha'} \sum_{n > 0} \phi^k_{-n} \phi^{-n}_k \eta^{\nu \mu}$$

$$+ \frac{\alpha'}{2} \eta^{\mu \nu} p_\mu p_\nu$$

$$\alpha^\mu_{-n} = \phi^k_{-n} + \frac{\alpha'}{2} (\eta^{\mu \nu} \Pi^\nu_{\mu n} + 2 i n \phi^{-n}_n)$$

$$(\alpha^k_{-n})^* = \frac{1}{\alpha'} (\eta^{\nu \mu} \Pi^\nu_{\mu (-n)} - 2 i n \phi^k_n) + \alpha^k_{+n}$$

$$(\alpha^k_{+n})^* = \frac{1}{\alpha'} (\eta^{\nu \mu} \Pi^\nu_{\mu n} - 2 i n \phi^k_{-n})$$
\[ [\alpha^\mu_n, \alpha^\nu_m] = \frac{i}{2} [\gamma^\mu \Phi_n + i n \Phi_{-n}, \gamma^\nu \Phi_m + i n \Phi_{-m}] \]

\[ = \frac{i}{2} (i \delta m + n, 0) + i n \eta^{\mu\nu} (i \delta m + n, 0) \]

\[ = 8 m \delta_{m+n,0} \eta^{\mu\nu} \]

\[ [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = m \delta_{m+n,0} \eta^{\mu\nu} \]

\[ [\alpha^\mu_n, \tilde{\alpha}^\nu_n] = 0 \]

\[ H = \sum_{n>0} (\alpha^\mu_n \tilde{\alpha}^\nu_n + \tilde{\alpha}^\mu_n \alpha^\nu_n) \eta^{\mu\nu} \]

\[ + \frac{\alpha'}{2} \eta^{\mu\nu} p^\mu k^\nu \]

\[ p^\mu (k\beta) = k^\mu (k\beta) \quad \alpha^\mu_n \rightarrow \alpha^\mu_n, \alpha^\nu_m \rightarrow \alpha^\nu_m \quad \{k\beta\} \]

Constraint eqs:

\[ \nabla_{\mu} = 0 \]

\[ \exists \frac{1}{2} \eta^{\mu\nu} (\partial_\nu - \partial_{-\nu}) X^\mu (\partial_\nu - \partial_{-\nu}) X^\nu = 0 \]

\[ \Theta (\partial_\nu - \partial_{-\nu}) X^\mu \]

\[ = \sum (\tilde{\alpha}^\mu_n - i \eta \Phi_n) e^{in\sigma} + \partial_\nu \tilde{\alpha}^\mu_n \]

\[ = \sum (\tilde{\alpha}^\mu_n \eta_{\mu\nu} - i \eta \Phi_n) e^{in\sigma} + \alpha' \eta^{\mu\nu} k_{\nu} \]

\[ = \sum \frac{\alpha'}{\sqrt{2\alpha'}} \tilde{\alpha}^\mu_n e^{in\sigma} + \alpha' \eta^{\mu\nu} k_{\nu} \]
\[ T_{\mu \nu} = \alpha' \sum_{n} L_n e^{-i n \sigma} \]

\[ L_n = \frac{\alpha'}{2} \sum_{m-n} \alpha^m \alpha^\nu \eta_{\mu \nu} + \frac{1}{2 \sqrt{2 \alpha'}} \sum_{m} \alpha^m \eta_{\mu \nu} b^\mu b^\nu \]

\[ L_0 = \sum_{n>0} \sum_{m} \frac{\alpha^m}{2} \alpha^n \eta_{\mu \nu} + \frac{\alpha'}{2} \eta_{\mu \nu} b^\mu b^\nu + \tilde{c} \]

(same as above)

\[ T_{\nu \theta} = \alpha' \sum_{n} L_n e^{-i n \sigma} \]

\[ T_n = \frac{\alpha'}{2} \sum_{m-n} \alpha^m \alpha^n \eta_{\mu \nu} + \frac{1}{2 \sqrt{2 \alpha'}} \sum_{m} \alpha^m \eta_{\mu \nu} b^\mu b^\nu \]

\[ T_0 = \sum_{n>0} \sum_{m} \frac{\alpha^m}{2} \alpha^n \eta_{\mu \nu} + \frac{\alpha'}{2} \eta_{\mu \nu} b^\mu b^\nu + \tilde{c} \]

(normal ordering)

constant
Procedure for implementing the constraint:

\[
\begin{align*}
\mathcal{L}_n |\text{phys}\rangle &= 0 \quad \text{for } n \geq 0 \\
\mathcal{\tilde{L}}_n |\text{phys}\rangle &= 0 \quad \text{for } n \geq 0
\end{align*}
\]

\[ \exists \quad < \text{phys} | \mathcal{L}_n^+ = 0 \quad \implies \quad < \text{phys} | \mathcal{L}_{-n} = 0 \quad \text{for } n \geq 0 \]

\[ < \text{phys}^\prime | \mathcal{\tilde{L}}_n^+ = 0 \quad \implies \quad < \text{phys}^\prime | \mathcal{\tilde{L}}_{-n} = 0 \]

\[
\text{Then} \quad < \text{phys}^\prime | \mathcal{L}_n | \text{phys} \rangle = 0 \quad \forall \quad n \geq 0
\]

Summary

Hilbert space states are given by

\[
\alpha^{n_1}_{-m_{1}} \cdots \alpha^{n_k}_{-m_k} \hat{\Phi}^{n_1}_{-n_1} \cdots \hat{\Phi}^{n_k}_{-n_k} |\text{Fock vac}\rangle
\]

Physical states satisfy

\[
\mathcal{L}_n |\text{phys}\rangle = 0, \quad \mathcal{\tilde{L}}_n |\text{phys}\rangle = 0 \quad \text{for } n \geq 0
\]

Example

\[
|\text{Fock vac}\rangle
\]

\[
\mathcal{L}_m |\text{Fock vac}\rangle = \sum_{n \geq 0} \left[ \frac{1}{2} 2 \alpha^{m-n} \hat{\Phi}^{n}_{-n} \hat{\Phi}^{n}_{-n} \eta_{\mu} |\text{Fock vac}\rangle \\
+ \frac{1}{2} \sqrt{2} \alpha^{m} \hat{\Phi}^{m}_{-n} \hat{\Phi}^{m}_{-n} |\text{Fock vac}\rangle \right] \quad \text{for } m \geq 0
\]

Either \( m = n \) or \( n > 0 \) in first term.

\[ \exists \quad \mathcal{L}_m |\text{Fock vac}\rangle = 0 \quad \text{for } m \geq 0 \]
\[ L_0 = \sum_{n \geq 0} \alpha' \eta^{\mu \nu} k^\mu k^\nu + c \langle k^3 \rangle \]

\[ = \langle k^3 \rangle \cdot 0 \quad \text{for } n > 0 \]

\[ \Rightarrow L_0 \langle k^3 \rangle = \left( \frac{\alpha'}{4} \eta^{\mu \nu} k^\mu k^\nu + c \right) \langle k^3 \rangle \]

**Physical state condition:**

\[ \frac{\alpha'}{2} \eta^{\mu \nu} k^\mu k^\nu + c = 0 \]

\[ \Rightarrow \left( -k^3 \right)^2 + (k^3)^2 \]

\[ \Rightarrow (k^3)^2 - (k^3)^2 = \frac{c}{\alpha'} \]

\[ \Rightarrow c = \tilde{c} \]

\[ \Rightarrow \exists \quad \text{a single } \tilde{c} \quad \text{describes a state} \]

\[ \Rightarrow m^2 = \frac{4}{\alpha'} \tilde{c} \]

Thus, the state \( \langle k^3 \rangle \) of the string represents a particle with mass \( m^2 = \frac{4}{\alpha'} \tilde{c} \).
We shall now examine the physical state condition on other states.

Check:

\[ [L_m, \hat{x}^\nu_m] = n \hat{x}^\nu_{m+n} \quad \text{for} \quad m \neq n. \]

\[ = \frac{1}{2} \sqrt{2} \alpha' \cdot n \cdot p^\nu \quad \text{for} \quad m = n \]

\[ [L_m, p_\nu] = 0. \]

\[ [\hat{L}_m, \hat{x}^\nu_m] = n \hat{x}^\nu_{m+n} \quad \text{for} \quad m \neq n \]

\[ = \frac{1}{2} \sqrt{2} \alpha' \cdot n \cdot p^\nu \quad \text{for} \quad m = n. \]

\[ L_0 \langle \hat{x}^\nu_{-\nu_1} \cdots \hat{x}^\nu_{-\nu_k} \hat{x}^\nu_{-\nu_i} \cdots \hat{x}^\nu_{-\nu_k} \rangle (123) \]

\[ = \left( \sum_{i=1}^{k} m_i + c + \frac{\alpha'}{4} \eta^\nu_\nu k_k k_i \right) \hat{x}^\nu_{-\nu_1} \cdots \hat{x}^\nu_{-\nu_k} (123) \]

\[ \implies \frac{\alpha'}{4} \sum_{i=1}^{k} m_i + c + \frac{\alpha'}{4} \eta^\nu_\nu k_k k_i = 0. \]

\[ \implies -\eta^\nu_\nu k_k k_i = \frac{1}{\alpha'} \left( c + \sum_{i=1}^{k} m_i \right) \]

\[ (k^0)^2 - \vec{k}^2 \]

\[ (\text{mass})^2 \]

\[ L_0 = 0 \quad \text{constraint} \]

\[ -\eta^\nu_\nu k_k k_i \Phi = \frac{1}{\alpha'} \left( c + \sum_{i=1}^{k} m_i \right) \]

\[ \implies \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} n_i \implies \text{level.} \]
Theorem:
\[ L_m \alpha_{-m} \cdot \alpha_{mk} \cdot \alpha_{r} \cdot \alpha_{-nk} \cdot \alpha_{-mk} \cdot \alpha_{-r} \cdot \alpha_{nk} \ |_{\{k\}} = 0 \]
if \( m > \sum_{x=1}^{k} m_x \).

Proof:
\[ L_m \alpha_{-m} = m \cdot \alpha_{m} + \alpha_{-m} \cdot L_m. \]

\[ \Rightarrow \ \text{we get} \]
\[ m \cdot \alpha_{m} \cdot \alpha_{-m} - m \cdot \alpha_{m} \cdot \alpha_{-m} + \alpha_{m} \cdot L_m \cdot \alpha_{-m} - \alpha_{-m} \cdot L_m \cdot \alpha_{m} \]
\[ + \alpha_{m} \cdot \alpha_{-m} \cdot L_m \cdot \alpha_{m} - \alpha_{-m} \cdot \alpha_{m} \cdot L_m \cdot \alpha_{-m} \]
\[ \text{In the first term:} \]
\[ m = \sum_{x=1}^{k} m_x \]
\[ \Rightarrow \ m - m_x \geq \sum_{x=1}^{k} m_x \]
\[ \Rightarrow \ m - m_x > m_x \text{ for } x = 2, \ldots, k. \]
\[ \alpha_{m} \text{ can be taken to the left and right.} \]
\[ \alpha_{-m} \text{ can be taken to the left.} \]
\[ \alpha_{m} \text{ can be taken to the right.} \]

Second term can be analyzed as before.

Continuing this way we can bring \( L_m \) to the extreme left and \( \alpha_{m} \) to the extreme right.

Similarly \( \alpha_{-m} \) will annihilate the state for \( m > \sum_{x=1}^{k} m_x \).
Net result: For a level $n$ state we only need to check the condition for $L_0, L_1, \ldots, L_n$ and $\tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_n$.

We shall focus on the level $1$ states.

$$S_{\mu\nu} \alpha_{\mu} \alpha_{\nu} \xi(k^3)$$

$\tilde{L}_0$ condition:

$$- \eta^{\mu\nu} k_\mu k_\nu = \frac{2}{\alpha'} (C' + 1)$$

$$\Rightarrow m_1^2$$

3 possibilities:

1. $C' + 1 < 0 \Rightarrow$ tachyonic state
2. $C' + 1 = 0 \Rightarrow$ massless state
3. $C' + 1 > 0 \Rightarrow$ massive state

Now the expression is written in a frame coordinate where $k_\mu = 0$.

$$L_1 S_{\mu\nu} \alpha_{\mu} \alpha_{\nu} \xi(k^3) = \frac{1}{2} \sqrt{2} \alpha' \eta^{\mu\nu} k_\mu S_{\mu\nu} \alpha_{\nu} \xi(k^3)$$

$$\Rightarrow k_\mu S_{\mu\nu} = 0.$$
Possibility 1

\[ C + 1 < 0 \Rightarrow (k^0)^2 - (\vec{k}^2)^2 = \frac{C + 1}{\alpha^2} < 0. \]

We can choose a frame in which

\[ k^0 = 0, \quad (k^i)^2 = \frac{\tilde{\nu}}{\alpha^2} (C + 1), \quad k^2 = k^3 = \cdots = 0 \]

Condition in \( S^{\mu\nu} \):

\[ S^{\mu\nu} = 0 \quad S^{k\nu} = 0 \quad \forall \mu, \nu. \]

Choose \( \xi \) such that \( S^{00} = \xi \neq 0 \).

All other components of \( \xi \) vanish.

State: \( x_0, x^2, 1 \{ k^3 \} \rightarrow \text{physical} \).

Norm: \[ \langle \{ k^3 \}, x^0, x^2, x_0, x^2, 1 \{ k^3 \} \rangle \]

\[ = (-1)^{\frac{n}{2}} \alpha (C + 1). \quad \langle \{ k^3 \}, \{ k^3 \} \rangle. \]

\[ = (-1)^{\frac{n}{2}} \alpha (C + 1). \]

Negative norm \( \Rightarrow \text{state inconsistent} \).

Conclusion: \( C + 1 \geq 0 \).

One can show that \( C + 3 > 0 \Rightarrow \text{inconsistent} \).

Interacting theory.

\[ C + 3 = 0 \Rightarrow \text{level zero state has} \]

\[ (\text{mass})^2 = \frac{\tilde{\nu}}{\alpha^2} \quad \Rightarrow \quad \text{tachyonic}. \]
Hilbert space of a closed string:

\[ [\alpha^k, \alpha^\nu] = m \delta_{m+n,0} \eta^{k \nu}; \quad [\tilde{\alpha}^k, \tilde{\alpha}^\nu] = m \delta_{m+n,0} \eta^{k \nu} \]

\[ [x^k, p^\nu] = i \eta^{k \nu} \]

Genuine state in the Hilbert space:

\[ \psi_{m, n} = \alpha_{m,k}^{\nu} \tilde{\alpha}_{-m, k}^\nu \phi_m \phi_n \text{ for } m, n > 0 \]

\[ \psi_{m, k}^{R3} = 0, \quad \psi_{m, k}^{L3} = 0 \text{ for } m > 0 \]

\[ \phi_m \phi_n^{R3} = \phi_m \phi_n^{L3} \]

\[ L_m = \frac{1}{2} \sum_{n \neq 0} \alpha_{m-n}^k \alpha_n^\nu \eta_{k \nu} + \frac{i}{2} \sqrt{\alpha'} x^k p_m \text{ for } m > 0 \]

\[ L_0 = \sum_{n > 0} \alpha_{-n}^k \alpha_n^\nu \eta_{k \nu} + \frac{x'}{4} \eta^{k \nu} p_m p_n + C \]

\[ L_m, L_0, \alpha_{m}^k \rightarrow \tilde{\alpha}_{m}^k, C \rightarrow \tilde{C} \]

Physical state constraint:

\[ L_m \lvert \text{phys} \rangle = 0, \quad \tilde{L}_m \lvert \text{phys} \rangle = 0 \text{ for } m \geq 0 \]

Acting on \( \phi_{R3} \), physical state constraint gives:

\[ (k^0)^2 - (\tilde{k}^0)^2 = \frac{4}{\alpha'} C - \frac{4}{\alpha'} \tilde{C} \implies C = \tilde{C} \]

\[ \Rightarrow \text{ A particle of mass } \left(\frac{C}{\alpha'}\right)^2 = \frac{4}{\alpha'} C \]
\[ x_{-m_i}^k, x_{-m_i}^l, x_{-n_i}^v, x_{-n_i}^v \rangle \]

\( L_0 \) and \( \bar{L}_0 \) constraints:

\[-\eta^{k\nu} k_k k_\nu = \frac{i}{\alpha'} (C + \sum m_i) = \frac{i}{\alpha'} (C + \sum n_i) \]

\[ (k^0)^2 - k^2 = \frac{b^2}{\alpha'} m_i = \frac{b^2}{\alpha'} n_i = \mathbb{N} \quad \text{\( L_0 \) level} \]

Then, \( L_m, \bar{L}_m \) acting on a level \( \mathbb{N} \) state automatically gives zero for \( m > \mathbb{N} \)

\( \Rightarrow \) we only need to check the constraints for \( m \leq \mathbb{N} \)

Example: Level 1 state:

\[ S_{mn} x_{-1}^k, x_{-1}^l \rangle \]

\( L_0, \bar{L}_0 \) constraint:

\[ (k^0)^2 - k^2 = \frac{i}{\alpha'} (C + 1) \]

\[ \text{(Mass)}^2 \]

Three cases:

1. \( C + 1 < 0 \)
2. \( C + 1 = 0 \)
3. \( C + 1 > 0 \)
Case 1: Choose a frame in which
\[ k^0 \cdot c_0, \quad k^1 = \frac{1 - \frac{1}{2} (c + 1)}{\alpha^1}, \quad k^2 = \cdots k^{d+1} = 0. \]

Constraint: \( \Sigma_{\mu} \cdot \Sigma_0 = 0 \implies \Sigma_0 \geq 0. \)

Take two states:

\( \alpha \sim \alpha^2 \sim 1 \{ k_3 \}. \)

\[
\text{Norm:} \quad - \langle \{ k_3 \} | \{ k_3 \} \rangle.
\]

\( \alpha \sim \alpha^2 \sim 1 \{ k_3 \}. \)

\[
\text{Norm:} \quad \langle \{ k_3 \} | \{ k_3 \} \rangle.
\]

- Some physical states will have negative norm.
- Unacceptable.

Case 1 \( \Rightarrow \) Massless spin 2 particle.
Case 2 \( \Rightarrow \) Massive spin 2 particle.

In this case one cannot have a consistent interacting string theory.

\( \Rightarrow \quad C + 1 = 0 \implies C = -1. \)

Mass of \( \{ k_3 \} \) = \( \frac{\sqrt{2}}{\sqrt{c}} \), \( C = -\frac{\sqrt{c}}{2} \), \( \Rightarrow \) Tachyonic.
Not result:

1. Level I state is massless.
2. Described by a rank 2 polarization tensor $\mathcal{E}_{\mu\nu}$ satisfying
   \[ k^\mu \mathcal{E}_{\mu\nu} = 0 \]

   \[ \Rightarrow 4 \text{ constraints.} \]

Suppose consider a state

\[ |\Psi\rangle = \sum_{n=0}^{\infty} a_n |X_n\rangle \]

any state $n > 0$

\[ \langle \Psi | \text{phys} \rangle = \frac{1}{2} \sum_{n=0}^{\infty} \langle X_n | \text{phys} \rangle = 0. \]

Similarly if $|\tilde{\Psi}\rangle = \sum_{n=0}^{\infty} a_n |\tilde{X}_n\rangle$ then

\[ \langle \tilde{\Psi} | \text{phys} \rangle = 0. \]

States of the form $|\Psi\rangle, |\tilde{\Psi}\rangle$ are known as spurious states.

A state which is both spurious and physical is called null.

Suppose $|\psi_{\text{phys}}\rangle$ is a physical state

\[ |\psi_{\text{phys}}\rangle = |\psi_{\text{phys}}\rangle + |X\rangle \]

is also a physical state if $|X\rangle$ is null.

\[ \langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle = \langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle \]
We say that $|\mathcal{S}_{\text{phys}}^0\rangle$ and $|\mathcal{S}_{\text{phys}}^{\prime}\rangle$ describe equivalent physical states, as they have the same inner product with all physical states.

$|\mathcal{S}_{\text{phys}}\rangle \sim |\mathcal{S}_{\text{phys}}^{\prime}\rangle$

Example of spinorial state at level 1:

$$
\frac{2}{\sqrt{2\alpha_1}} \left[ (-1)^2 \sum \alpha_{i_1} \alpha_{v_1} \alpha_{-1} \kappa_{3} \right] \alpha_{-1} \kappa_{3} \alpha_{-1} \kappa_{3} \alpha_{-1} \\
= \left[ k_{\mu} \delta_{v_{\mu}} \alpha_{-1} \alpha_{v_{-1}} \alpha_{-1} \kappa_{3} \right] \alpha_{-1} \kappa_{3} \alpha_{-1} \kappa_{3} \alpha_{-1} \\
= (k_{\mu} \delta_{v_{\mu}} + k_{v} \delta_{\mu_{v}}) \alpha_{-1} \alpha_{v_{-1}} \alpha_{-1} \kappa_{3}
$$

This state is physical if

$$k^2 = 0.$$

$k^\mu (k_{\mu} \delta_{v_{\mu}} + k_{v} \delta_{\mu_{v}}) = 0$ (i.e. $k^\mu \delta_{\mu_{v}} = 0$)

$k_v (k_{\mu} \delta_{v_{\mu}} + k_{v} \delta_{\mu_{v}}) = 0$ (i.e. $k_v \delta_{v_{\mu}} = 0$)

$S_{\mu v} \sim S_{\mu} k_v + S_{\nu} k_{\mu} + S_{\mu v}$

If $k^\mu \delta_{\mu_{v}} = 0 = k_v \delta_{v_{\mu}}$

Write $S_{\mu v} = \frac{1}{2}(S^S_{\mu v} + S^A_{\mu v})$

Symmetric $S^S_{\mu v} = S_{\mu v} + S_{\nu \mu}$, Antisymmetric $S^A_{\mu v} = S_{\mu v} - S_{\nu \mu}$
\[ K^S S_{\mu\nu} = 0 \quad , \quad K^A S_{\mu\nu} = 0 \]

\[ S^S_{\mu\nu} = S^S_{\mu\nu} + K^S S_{\mu\nu} + K^S S_{\mu\nu} \quad , \quad K^S S_{\mu\nu} = 0 \]

\[ S^A_{\mu\nu} = S^A_{\mu\nu} + K^A S_{\mu\nu} - K^A S_{\mu\nu} \quad , \quad K^A S_{\mu\nu} = 0 \]

We shall first focus on the states obtained from \( S^S_{\mu\nu} \).

Compare with the polarization tensor of the graviton obtained by quantizing general relativity:

\[ S_{\text{GR}} = K S d^0 x \sqrt{-\det g} R \]

\[ \# \text{ Levi-Civita scalar} \]

\( g_{\mu\nu} \) : metric

Flat Minkowski space : \( g_{\mu\nu} = \eta_{\mu\nu} \)

Fluctuations around the Minkowski space:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

Express \( S_{\text{GR}} \) as a power series in \( h_{\mu\nu} \)

Note: \( g_{\mu\nu} = \eta_{\mu\nu} + O(h) + O(h^2) + \ldots \)

\( \Rightarrow S_{\text{GR}} \) is an infinite order polynomial in \( h_{\mu\nu} \).
Propagator - Only quadratic term is important.

N-point function at tree level:

Contains at most vertex of order $h^n$.

⇒ For any tree level amplitude we can truncate its expansion to order $h^n$.

We can find the spectrum of physical states by examining the quadratic term.

⇒ Massless particles with polarization vector $S_{\mu\nu} = S_{\nu\mu}$ (since $h_{\mu\nu} = h_{\nu\mu}$)

But general coordinate invariance

⇒ Constraints on $S_{\mu\nu}$. 
\[ g'_{\mu\nu}(x') = g_\phi(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}. \]

Take \( x'^\mu = x^\mu + \epsilon^\mu \)

\[ \Delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_\mu \delta_\nu + \nabla_\nu \delta_\mu \]

\[ \delta_\mu = \delta g_{\mu\nu} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ S h_{\mu\nu} = \partial_\mu \delta_\nu + \partial_\nu \delta_\mu + O(h) \]

- Gauge transformation

Effect on polarization tensor:

\[ S S_{\mu\nu}^a = k_\mu \delta_{\nu}^a + k_\nu \delta_{\mu}^a \]

\[ \Rightarrow \delta S_{\mu\nu}^a \equiv \delta S_{\mu\nu}^a + k_\mu \delta_{\nu}^a + k_\nu \delta_{\mu}^a. \]

Gauge fixing: Lorentz covariant gauge

\[ \eta^\mu_\nu \epsilon^\nu_\mu h_{\mu\nu} = 0 \]

\[ \Rightarrow \epsilon^\mu_\nu \delta S_{\mu\nu} = 0 \]

Linearized Einstein's eq. \( \nabla_\mu \frac{R^\mu_\nu - \frac{1}{2} R g^\mu_\nu}{h_{\mu\nu}} = 0 \) gauge:

\[ \partial_\nu \partial_\sigma h^\mu_\nu - \eta^\mu_\nu \partial_\sigma h^\kappa_\mu - \partial h^\kappa_\mu + \Box h_{\mu\nu} = 0. \]

\[ \times \eta^\nu_\sigma (2 - D) \Box h^\kappa_\mu = 0. \Rightarrow \Box h^\kappa_\mu = 0 \] for \( D \geq 2 \)

\[ \partial_\nu \partial_\sigma h^\kappa_\mu + \Box h_{\nu\sigma} = 0. \]
$h^\mu_\nu = \int d^3 k \quad S^a_{\mu \nu}(k) \in i k \cdot x$

$k^\nu k_\sigma = S^k_\mu + k^2 \delta^k_\nu \sigma = 0$

$k^2 = S^k_\mu = 0$.

Either $k^2 = 0$ or $S^k_\mu = 0$.

If $S^k_\mu \neq 0$ then $k^2 = 0$.

$\Rightarrow k^\nu k_\mu S^k_\mu = 0 \Rightarrow \delta^k_\nu \mu = 0$ (unless $k^2 = 0$).

For $S^k_\mu = 0$ we get

$k^2 S^k_\mu = 0$.

$\Rightarrow k^2 = 0$ if $S^k_\mu \neq 0$.

Conclusion: States of a graviton are characterized by momentum $k$ & polarization tensor $S^a_{\mu \nu}$ satisfying:

$k^2 = 0$

$S^a_{\mu \nu} = 0, \quad k^\mu S^a_{\mu \nu} = 0$

$S^a_{\mu \nu} = S^a_{\mu \nu} + (k^\mu S^a_{\mu \nu} + \delta^a_{\mu \nu} k^\mu)$

$
\Rightarrow$ correspondence with states described by $S^a_{\mu \nu}$ except that $S^k_\mu \neq 0$.

$
\Rightarrow$ An extra state.

String theory spectrum has a graviton and a scalar.
$5^A_{\mu \nu} \rightarrow$ describe states of a rank 2 anti-symmetric tensor field $B_{\mu \nu}$.

Gauge invariance:

$$\delta B_{\mu \nu} = \partial_{\mu} \delta_{\nu} \sim \partial_{\nu} \delta_{\mu}.$$ 

Thus the bosonic string theory contains 3 sets of massless fields:

- Graviton: $h_{\mu \nu}$
- Scalar (Dilaton): $\phi$
- Anti-symmetric tensor: $B_{\mu \nu}$. 

Spectrum of closed string theory

Level 0: Scalars with \( m^2 = -\frac{4}{N^2} \)

Level 1: Massless \( S_0, \phi, \pi \) - scalars

Level N:
\[
m^2 = \frac{4}{N^2} (N-1) \rightarrow \text{Massive fields and bosons}.
\]

Analysis for level 2.

General state:
\[
\frac{1}{\sqrt{N^2}} \left( S^0 \otimes \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{i k} \pi_k + \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{i k} \pi_{-k} + \sum_{k=1}^{N} e^{-i k} \pi_k + \sum_{k=1}^{N} e^{-i k} \pi_{-k} \right)
\]

To constrain:
\[
-\frac{1}{N^2} R_{\mu \nu k_0} = \frac{4}{N^2} (2 - \delta_{k_0,0}) = \frac{4}{N^2}, \rightarrow \text{mass}\]

1, 1, 1, 2, 2, 2 + constraints

Possible states on \( S^0, \pi^0, \pi^1, \pi^2, \pi^3, \pi^4 \)

Are there negative norm states satisfying the physical side condition?
Answer: For $D > 26$, there are negative norm states satisfying the physical state conditions.

For $D \leq 26$, there are no negative norm states satisfying these conditions.

⇒ Free string theory is consistent for $D \leq 26$.

But interacting string theory is consistent for only $D = 26$.

⇒ Critical dimension.

Open string theory

Open string theories automatically contain closed strings.

Particle: Open string world-sheet:

Consider a self-energy diagram: 

A closed string.
Thus these theories must also be defined for D=26.

If open string consistency condition gives different conditions in D then there would be a contradiction.

Fortunately this does not happen.

Non-interacting

Open string spectrum

\[ S = \frac{1}{4\pi\alpha'} \int d^2 \xi \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \]

\[ \alpha = 2 \frac{a}{\sqrt{3} \alpha'} \]

Constraint: \( T_{\alpha\beta} = 0 \)

\[ K \left( \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \right) \]

Constant

\[ \eta^{\alpha\beta} T_{\alpha\beta} = 0 \rightarrow \text{Two independent constraints} \]
\[ u = \sqrt{c}, \quad r = x^0 \]

\[ T_{\mu\nu} = \eta_{\mu\nu} \partial_\mu x^\alpha \partial_\nu x^\beta = 0 \]

\[ T_{\mu\nu} = \eta_{\mu\nu} \partial_\mu x^\alpha \partial_\nu x^\beta = 0 \]

In quantum theory

\[ \langle \text{phys} | T_{\mu\nu} | \text{phys} \rangle = 0 = \langle \text{phys} | T_{\mu\nu} | \text{phys} \rangle \]

What are the boundary conditions on \( x^\nu \) at \( \tau = 0, \pi \)?

\[ \delta S = -\frac{i}{2\pi \alpha'} \int d^2 \xi \eta_{\alpha\beta} \partial_\xi (\partial_{\alpha} x^\nu) \partial_{\beta} x^\mu \]

\[ = \frac{i}{2\pi \alpha'} \int d^2 \xi \eta_{\alpha\beta} \partial_{\alpha} x^\mu \partial_{\beta} x^\nu \eta_{\mu\nu} \]

\[ = \frac{i}{2\pi \alpha'} \int d^2 \sigma \partial_{\alpha} x^\mu \partial_{\beta} x^\nu |_{\sigma = 0} \]

In order that \( \delta S = 0 \) reproduces the correct local eqs. of motion we need:

\[ \partial_\tau x^\nu = 0 \quad \text{at} \quad \sigma = 0, \tau = 0 \text{ Neumann br.} \]

or

\[ x^\nu = x_0^\nu \quad \text{at} \quad \sigma = 0, \tau = 0 \text{ Dirichlet br.} \]

Also possible (i) Dirichlet at \( \sigma = 0 \), Neumann at \( \tau = 0 \)

(ii) Vice versa.

(iii) \( y^\nu = x_0^\nu \text{ at } \sigma = 0, \quad x^\nu = x_0^\nu \text{ at } \sigma = \pi \)
If we want to maintain D-dimensional Poincare invariance, then all the $x^\mu$'s must satisfy Neumann b.c. at both ends.

$x^\mu = x_0^\mu$ at $\sigma = 0$ or $\pi$ breaks translational symmetry. $x^\mu \neq x^\mu + a^\mu$.

If we want to maintain D-dimensional Poincare invariance we must put Neumann b.c. on all coordinates.

We first consider this case.

$\partial_\sigma x^\mu = 0$ at $\sigma = 0, \pi$

$x^\mu = \sum_{n \geq 0} \phi_n^\mu \cos n \sigma + x^\mu$

$S = \frac{1}{4\alpha'} \sum_{n \geq 0} \int_0^{\pi} d\sigma \left( 2\partial_\sigma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} - 2\partial_\sigma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} \right)$

$= \frac{1}{8\alpha'} \sum_{n \geq 0} \left( 2\partial_\sigma \phi_n^\mu \partial_\sigma \phi_n^\nu \eta_{\mu\nu} - 2\partial_\sigma \phi_n^\mu \partial_\sigma \phi_n^\nu \eta_{\mu\nu} \right)$

$+ \frac{1}{4\alpha'} \partial_\sigma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu}$

$\Pi^{\mu\nu} = \frac{\delta S}{\delta (2\partial_\sigma \phi_n^\mu)} = \frac{1}{4\alpha'} \partial_\sigma \phi_n^\mu \eta_{\mu\nu}$

$\phi_n = \frac{\delta S}{\delta (\partial_\sigma x^\mu)} = \frac{1}{2\alpha'} \partial_\sigma x^\mu \eta_{\mu\nu}$
In quantum theory:

$$[\phi^k_n, T^m_{\nu \mu}] = 2i \delta_{mn} \delta^k \nu$$

$$[\alpha^k_n, \alpha^k_{\nu}] = 2i \delta^k \nu$$

Define:

$$\alpha^k_n = \eta^{\mu \nu} \sqrt{2\alpha^r} T^r_{\nu \mu} \frac{\phi^k_n}{2\sqrt{2\alpha^r}}$$

$$\alpha^k_{-n} = \eta^{\mu \nu} \sqrt{2\alpha^r} T^r_{\nu \mu} + \frac{2i}{2\sqrt{2\alpha^r}} \phi^k_n$$

for \( n > 0 \)

$$[\alpha^k_m, \alpha^k_n] = 2m \eta^{\mu \nu} \delta_{mn} \delta_{\mu \nu}$$

$$\sqrt{\frac{\alpha^r}{\alpha^k}} \alpha^k_m \rightarrow \text{annihilation operator} \quad (m > 0)$$

$$\sqrt{\frac{\alpha^r}{\alpha^k}} \alpha^k_{-m} \rightarrow \text{creation \ operator}$$

Define \( |k \rangle_3 \) such that:

$$\beta^r_m |k \rangle_3 = 0, \quad \alpha^k_n |k \rangle_3 = 0 \text{ for } m > 0$$

Basis of states:

$$\alpha^k_{-m}, \ldots, \alpha^k_{-m_k} |k \rangle_3$$

Constraints:

$$0 = T^r_{\mu \nu} = \eta^{\mu \rho} \alpha^\rho \mu \eta^{\nu \gamma} \alpha^\gamma \nu \eta_{\mu \nu}$$

$$u = \tau + \sigma$$

$$0 = T^r_{\mu \nu} = \eta^{\mu \rho} \alpha^\rho \mu \eta^{\nu \gamma} \alpha^\gamma \nu \eta_{\mu \nu}$$

$$v = \tau + \sigma$$
Substitute expansion of \( \alpha x^\mu \)

\[
\mathcal{L}_{\text{phys}} = \sum_{m \neq 0} \mathcal{L}_m e^{-im\phi} + \mathcal{L}_0 e^{-i\phi}
\]

\[
\mathcal{L}_m = \frac{1}{2} \sum_{n=1} \alpha_{m-n}^\mu \alpha_n^\nu \eta_{\mu\nu} + \sqrt{2} \alpha' \beta_\mu \alpha^\mu_m
\]

for \( m \neq 0 \).

\[
\mathcal{L}_0 = \sum_{n>0} \alpha_{-n}^\mu \alpha_n^\nu \eta_{\mu\nu} + \alpha' \beta_\mu \beta_\nu \eta_{\mu\nu} + c_0
\]

no normal ordering constant

Physical state condition:

\[
\mathcal{L}_m \mid \text{phys} \rangle = 0 \quad \text{for } m > 0
\]

Spurious state:

\[
\sum_{m>0} \mathcal{L}_m \mid x_m \rangle
\]

A state which is spurious + physical is called null.

\[
\mid \text{phys} \rangle \approx \mid \text{phys} \rangle + \mid \text{null} \rangle
\]

Since \( \langle \text{null} \mid \text{phys} \rangle = 0 \), \( \forall \mid \text{phys} \rangle \).
Level 0 constraint on
\[ \alpha^\mu_{-m_1} \cdots \alpha^\mu_{-m_k} \] 
\[ \alpha^\mu_{-m_1} \cdots \alpha^\mu_{-m_k} \] \( H(k3) \).

\[ \alpha', \eta^{\mu\nu} \kappa^\mu \kappa^\nu + \frac{\kappa}{\sum \frac{m_i}{\alpha'} + C_0} = 0. \]

\[ \eta^{\mu\nu} \kappa^\mu \kappa^\nu = \frac{1}{\alpha'} , \left( \sum \frac{m_i}{\alpha'} + C \right) = \frac{1}{\alpha'} (N + C). \]

Level 0

On a level \( N \) state we need to check \( L_0, L_1, \cdots L_N \) constraints.

Level 0: \( H(k3) \).

\[ \eta^{\mu\nu} \kappa^\mu \kappa^\nu = \frac{1}{\alpha'} , C \]

\( \text{(mass)}^2 \)

Level 1: \( \sum_5 \alpha^\mu_{-1} \) \( H(k3) \).

\( L_0 \) constraint: \[ \eta^{\mu\nu} \kappa^\mu \kappa^\nu = \frac{1}{\alpha'} , (C + 1) \]

\( \text{(mass)}^2 \)

\( L_1 \) constraint: \[ k^\mu 5_\mu = 0. \]

Case 1: \( (C + 1) < 0 \) tachyonic

2. \( (C + 1) = 0 \) massless

3. \( (C + 1) > 0 \) massive
If \((c+1) < 0\) we can choose

\[ k_0 = 0, \quad k_1 = \sqrt{-\frac{1}{c+1}}, \quad k_2 = k_3 = \ldots = 0 \]

1. Constraint: \(\xi' = 0\).

\(\alpha^0_{\pm} |k3\rangle\) & \(\alpha^2_{\pm} |k3\rangle\) are both physical.

\(\alpha^0_{\pm} |k3\rangle\) has norm \(-\langle k3|k3\rangle\).

\(\alpha^2_{\pm} |k3\rangle\) has norm \(\langle k3|k3\rangle\).

Both cannot be the norm - contradiction.

\((c+1) \geq 0\). No negative norm state.

But interacting theory is consistent only for \(c+1 = 0\).

\(\Rightarrow\) Massless particle.

Note:

\(|k3\rangle \rightarrow (\text{mass}) = \sqrt{\frac{1}{c+1}}\)

\(\Rightarrow\) Tachyon.

Spurious state at level 1:

\[ \frac{a}{\sqrt{2x^1}} L_{-1} |k3\rangle = \alpha k^\mu \alpha^{\mu}_{-1} |k3\rangle \]

Physical since \(k^\mu k_\mu = 0\)

Thus \(\xi^\mu = \xi^\mu + \alpha k^\mu\)

Physical state condition: \(k^\mu \xi^\nu = 0\) \(\Rightarrow\) A massless vector gauge particle field.
Absence of negative norm physical states at level 2

\[ \Rightarrow D \leq 26 \]

Consistent interacting theory:

\[ D = 26 \]
Open strings:

\[ S = \frac{1}{4\pi\alpha'} \int d^2 z \eta^\alpha_{\beta} \partial_\alpha X^I \partial_\beta \bar{X}^I \eta_{MN} \left\{ \frac{\partial}{\partial z} \times \frac{\partial}{\partial \bar{z}} \right\} \delta^2 \tau, \]

Possible boundary conditions on \( X^I \):

**Dirichlet:** \( X^I(0) = X^I_0 \) at \( \sigma = 0, \pi \)

**Neumann:** \( \partial_\sigma X^I(0) = 0 \) at \( \sigma = 0, \pi \)

**Neumann b.c. on all \( X^I \)'s is only Poincaré invariant b.c.**

We shall now study the Dirichlet b.c.

Choose b.c.:

**Neumann** for \( \mu = 0, 1, \ldots, 15 \)

**Dirichlet** for \( \mu = 16, 17, \ldots, 25 \)

\[ \partial_\sigma X^I = 0 \] at \( \sigma = 0, \pi \) for \( \mu = 0, \ldots, 15 \)

\[ X^I = X^I_0 \] at \( \sigma = 0, \pi \) for \( \mu = 16, \ldots, 25 \)
### Interpretation

\( x^M = x_0^M \) describes for \( k = k+1, \ldots, 25 \) describes a \((5+1)\) dimensional subspace of the 26 dimensional space time.

**Ex.** If \( k = 24 \), the eqn. \( x^{25} = x_0^{25} \) is a \((2\oplus 4+1)\) dimensional subspace.

\[
\begin{align*}
0 & \leq M \leq k+1 \\
0 & \leq n \leq 25
\end{align*}
\]

Ends of open strings are constrained to lie on this \((k+1)\)-dimensional hypersurface, which breaks translational invariance.

Thus there is something special about this \((k+1)\)-dimensional hypersurface.

- Allows open strings to end there.

### Interpretation: There is a "Soliton" sitting along this hypersurface.

- Dirichlet \( k \)-brane (\( k \)-space, 1 time direction)
p = 0: Point-like object

p = 1: String-like object

p = 2: Membrane-like object etc.

Interpretation of the open strings with ends on the D-brane:

⇒ Oscillation modes of the D-brane.

Expected: A soliton has various modes of vibration:

1. Translational modes.
2. Internal modes

Typically each mode ↔ a (p+1) dimensional field.

\[ \Phi_n(x^M) \text{ or } \Phi_{n+1}(x^M) \]

\[ \Phi_n : A \text{ particular mode.} \]

Value of \( \Phi_n \) may differ from point to point on the brane.

⇒ \( \Phi_n(x^M) \) : A (p+1) dimensional field

E.g. If \( \Phi_n \) ↔ Translational mode

Then \( x^M \) dependent \( \Phi_n \)
If \( \Phi_i \) is one of the solitons in the \( \pm \) direction, then

\[ \Phi_i (x^M) \rightarrow \left[ \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right] \quad \text{etc.} \]

Thus, if the open string's satisfying Dirichlet b.c. describe modes of N0 b.-b. and soliton, then each mode could correspond to a \((p+1)\) dimensional yields.

Try to check if this is correct.

Mode expansion of \( x^M \),

\[ x^M \quad \text{Neumann b.c.} \quad \text{identical mode expansion} \]

\[ x^M = x^M_0 + \sum_n x^M_n \quad \sin n \sigma \]

\[ \text{fixed const.} \]

Note: No zero mode \( \chi^H(\sigma) \)

\[ S = \frac{1}{2 \pi \alpha'} \int \left( \partial_\sigma \phi^i \partial_\sigma \phi^i - \eta^2 \phi^i \phi^i \right) + x^M \text{- contribution} \]

Note: \( \eta_{\pm} = 1 \) as all Dirichlet direction are Euclidean space-like.
Conjugate momentum

\[ \Pi_n^{(i)} = \frac{\delta S}{\delta (\partial_x \phi_n^{(i)})} = \frac{i}{\sqrt{2}} \partial_x \phi_n^{(i)} \]

\[ \alpha_n^{\pi i} = \sqrt{2} \alpha^i \Pi_n^{(i)} - \frac{i n}{\sqrt{2}} \phi_n^{(i)} \]

\[ \alpha_n^{\pi \pi} = \sqrt{2} \alpha^i \Pi_n^{(i)} + \frac{i n}{\sqrt{2}} \phi_n^{(i)} \]

\[ \left[ \phi_n^{(i)}, \Pi_n^{(j)} \right] = \frac{2}{\sqrt{2}} \delta_{ij} \delta_{mn} \]

\[ \left[ \alpha_m^{\pi}, \alpha_n^{\pi} \right] = m \delta_{mn} \left[ \delta_{ij} \right] \]

\[ \left[ \alpha_m^{\pi}, \phi_n^{(i)} \right] = \frac{2}{\sqrt{2}} \delta_{ij} \delta_{mn} \]

\[ 0 \leq M, N \leq b \]

\[ 0 \leq i, j \leq 25 \]

\[ 1(k^3) \]

\[ \left\langle k^3 \right| \rightarrow (k^0, k', \ldots, k_b) \]

\[ \alpha_m^{\pi} \left\langle k^3 \right| = 0, \quad \alpha_m^{\pi} \left\langle k^3 \right| = 0 \text{ for } m > 0 \]

\[ \left[ \delta_M^{\pi}, \left\langle k^3 \right| = k_M \left\langle k^3 \right| \right] \]
General basis state

\[ x^{\prime \mu_1} \ldots x^{\prime \mu_k} x^{2i} \ldots x^{2e} ]_3^R \]

Note: States are labelled by 
\((\mathbb{R}+1)\) dimensional momenta
\(= \) quanta of \((\mathbb{R}+1)\) dimensional field.

We shall now determine what physical kind of states / fields we get.

Constraints:

\[ \langle \text{phys} | T_{uu} | \text{phys} \rangle = 0 = \langle \text{phys} | T_{uu} | \text{phys} \rangle \]

\[ T_{uu} = (\text{const.}) \partial_\mu x^\nu \partial_\nu x^\nu \eta_{\mu\nu} \]

\[ T_{uu} = (\text{const.}) \partial_\mu x^\nu \partial_\nu x^\nu \eta_{\mu\nu} \]

\( u=\bar{\omega}, \quad \bar{u}=\chi+i\sigma \)

Expressed in terms of \(x^\mu_i\)

\[ T_{uu} = (\text{const}) \sum_{m=\infty}^{m=-\infty} \frac{c^m}{e^{im\sigma}} \]

\[ T_{uu} = (\text{const}) \sum_{m=\infty}^{m=-\infty} \frac{c^m}{e^{im\sigma}} \]

\( u \neq \bar{u}, \quad \bar{u} \neq u+i\sigma \)

\( T_{uu} \neq \bar{T}_{uu} \) have been exchanged!

\( T_{uu} \neq \bar{T}_{uu} \) have been exchanged!

\( T_{uu} \neq \bar{T}_{uu} \) have been exchanged!

\( T_{uu} \neq \bar{T}_{uu} \) have been exchanged!
\[ L_m = \frac{1}{2} \sum_n (\alpha^M_{m-n} \alpha^N_n \eta_{MN} + \alpha^L_{m-n} \alpha^L_n) + \sqrt{2} \alpha' p_M \alpha^M_m \]

for \( m \neq 0 \)

\[ L_0 = \sum_n (\alpha^M_{-m} \alpha^N_m \eta_{MN} + \alpha^L_{-m} \alpha^L_m) \]

\[ + \alpha' p_M p_N \eta_{MN} + \mathcal{C}_0 \]

normal ordering constant

**Physical state condition:**

\[ L_m \left| \text{phys} \right. = 0 \quad \text{for} \quad m > 0. \]

**Spurious state**

\[ \sum_n L_{-m} \left| X_m \right. \]

A state is null if it is spurious as well as physical.

\[ \left| \text{phys} \right. \Rightarrow \left| \text{phys} \right. + \left| \text{null} \right. \]

Since \( \left< \text{null} \left| \text{phys} \right. \right> = 0 \quad \forall \quad \text{phys} \)

\[ L_0 = 0 \quad \text{constraint on} \quad \alpha^M_{-m}, \ldots, \alpha^M_{-m_k}, \alpha^L_{-n}, \ldots, \alpha^L_{-n_k} \quad \text{(Kes)} \]

\[ \Rightarrow \quad \eta_{MN} p_M p_N = \frac{1}{\alpha'} (\mathcal{C}_0 + \sum m_x + \sum n_x) \]

\[ \Rightarrow \quad N \text{ (level)} \]

\[ \Rightarrow \quad \alpha' (p+1) \text{ dimensional particle of} \]

\[ (\text{mass})^2 = \frac{1}{\alpha'} (\mathcal{C}_0 + N) \]
Level 0: Scalar with \((\text{mass})^2 = \frac{C_0}{\alpha'}\).

Level 1: \((\text{mass})^2 = \frac{C_0 + 1}{\alpha'}\).

Analysis of Absence of negative norm physical states + Consistent interacting theory

\[ \Rightarrow C_0 + 1 = 0. \]

\[ \Rightarrow \text{Level 0} \Rightarrow \text{Tachyonic State} \]

\[ \text{Level 1} \Rightarrow \text{Massless State} \]

\[ \sum \alpha^M (\mathbf{K}^3)^M + \sum \alpha^i (\mathbf{K}^3)^i. \]

Physical state constraint:

\[ \sum \alpha^M \mathbf{K}^M = 0. \]

Equivalence relation: \( \sum \alpha^M \mathbf{K}^M = \sum \alpha^M + a \mathbf{K}^M. \)

\( \sum \alpha^M \mathbf{K}^M \) = Massless gauge field living on the D-brane.

\[ \Rightarrow \exists \; \phi^i: \text{A set } \phi \text{ of } (25-6) \text{ massless scalars } \phi^i. \]

Translation mode of any D-brane in (25-6) transverse direction \( \phi^i \), D-brane located at \( \mathbf{x}_0 + \mathbf{z} \).
Interpretation: Consider $p$-brane solutions in an ordinary $D$-dimensional field theory.

\[ \eta^{\mu\nu} \partial_{\mu} \phi + V'(\phi) = 0 \]

Classical solution: $\phi = \phi_{ic} (x^{p+1}, \ldots, x^{D-1})$

Independent of $x^0, \ldots, x^p$. $\phi_{ic} \rightarrow \phi_{ic} + \psi$ for some

Suppose energy density is concentrated around

$x^{p+1} \approx 0, x^{p+2} \approx 0, \ldots, x^{D-1} \approx 0$

$\Rightarrow A p$-brane solution.

$\begin{array}{c}
| \begin{array}{ccc}
\varepsilon^1, & \ldots, & \varepsilon^p \\
& \rightarrow & \\
& x^{p+1}, & \ldots, x^{D-1}
\end{array}
\end{array}$

Fluctuations around the classical solution:

$\phi(x^0, \ldots, x^D) = \phi_{ic}(x^{p+1}, \ldots, x^{D-1}) + \delta(x^0, \ldots, x^p)$

Eqs. of motion:

\[ \eta^{\mu\nu} \partial_{\mu} \phi + V''(\phi_{ic}) \delta + \mathcal{O}(x^2) = 0 \]

Define $F_n(x^0)$

Define $F_n(y)$

$\partial_\alpha \partial_\alpha F_n(y) + V''(\phi_{ic}) F_n(y) = -\lambda_n F_n(y)$
\[ \mathcal{X}(x, \dot{x}) = \sum \nabla_n F_n(\dot{x}_i) \psi_n(\dot{x}_i) \]

\[ \Rightarrow \eta^{MN} \partial_n \partial_M \psi_n + \lambda_n \psi_n + G(\psi_n^2) = 0 \]

\[ \Rightarrow \text{A (p+1) dimensional field of mass } \lambda_n \]

Note: Only those modes corresponding to normalizable \( F_n \) \( (\int |F_n(\dot{x})|^2 dx \text{ is finite}) \) correspond to (p+1) dimensional fields localized on the brane.

Excitation modes of the brane.

For non-normalizable mode \( \lambda_n \) is continuous and actually describe D bulk fields. (Scattering states labelled by D-p-1 dimensional transverse momenta)

The open strings living on the D-brane describe the normalizable modes of the soliton.

Note that unlike for ordinary solutions where we first specify the solution and then compute the spectrum of modes living on the brane, D-branes are specified by the spectrum of modes living on the brane.
For ordinary solitons the $O(t^2)$ and higher terms describe interaction of the excitation modes of the soliton with each other and also the bulk fields.

For D-branes the interaction of the excitation modes of the brane among each other and the bulk fields is described by the fundamental rules for string interaction.

\[ \text{sum over all surfaces bounded by external string states} \]

Translation mode:

\[ \phi_{\alpha} \left( \vec{x}, \vec{y} + \vec{a} \right) - \phi_{\alpha} \left( \vec{x}, \vec{y} \right) = a^i \partial_i \phi_{\alpha} \left( \vec{x}, \vec{y} \right) \]

A solv of linearized eqs. of motion.

⇒ Translation modes

Eigenvalue = 0
For conventional solutions in field theory, if we have a pair of solutions, then the spectrum of normalizable modes around the solution representing the pair cannot be easily determined by any simple rules.

We need to solve the eigenvalue equation around the new classical solution representing the pair of branes.

But for D-branes the analysis is simple. Consider a pair of D-branes at $x^k = x_{0}^k$ and $x^k = x_{1}^k$ respectively.

\[
\begin{align*}
(a) & \quad x_{0}^k & (b) & \quad x_{1}^k \\
(c) & \quad x_{1}^\sigma & (d) & \quad x_{0}^\sigma \\
\end{align*}
\]

b.c. $\partial_\sigma x^M = 0 \quad 0 \leq M \leq k$

(a) $x^k = 0 \quad at \quad \sigma = 0, 1$

(b) $x^k = x_1^k \quad at \quad \sigma = 0, 1$

(c) $x^k = x_0^k \quad at \quad \sigma = 0$

(d) $x^k = x_0^k \quad at \quad \sigma = 1$
Spectrum of open strings in sectors (a) and (b) is same as that in the case of a single D-brane.

\[ X^i = (x^i - x_0^i) \frac{\sigma}{\Pi} + x_0^i + \sum \phi_n^* \sin n\sigma \]

\[ S = \frac{1}{4\pi\alpha'} \int d\sigma [\frac{(x^i - x_0^i)^2}{\Pi} + \frac{\Pi}{2} \left( \partial \sigma \phi_n \partial \sigma \phi_n \right) - \eta^2 \phi_n^* \phi_n] \]

\[ \Pi_n^{(i)} = \frac{\delta S}{\delta (\partial \sigma \phi_n^{(i)})} = \frac{1}{\eta \alpha'} \partial \sigma \phi_n^{(i)} \]

\[ \text{Same commutation relation as in the case of sector (a) and same Fock space structure.} \]

Difference comes in the expression for \( L_m \)

\[ T_{uu} = \text{const.} \sum_3 L_m e^{i\alpha} \]

\[ T_{uv} = \text{const.} \sum_3 L_m e^{-i\alpha} \]
\[ L_m = \frac{1}{2} \sum_{m-n} (\alpha_n^m \eta_{m-n} + \alpha_n^m \alpha_n^m) + \sqrt{2} \alpha' \rho_m \alpha_m^m \]

for \( m \neq 0 \)

\[ L_0 = \sum_{n \neq 0} (\alpha_n^m \alpha_n^m \eta_{m-n} + \alpha_n^m \alpha_n^m) \]

\[ + \alpha' \rho_m \rho_m \eta_{m-n} + C_0 + \frac{(\vec{x}_1 - \vec{x}_0)^2}{4 \pi^2 \alpha'} \]

\( C_0 \) takes the same value -1

\( \alpha \) can be seen by analyzing the system at \( \vec{x}_1 = \vec{x}_0 \).

\( \approx \) same open string spectrum except that each state has an additional contribution to \( (\text{mass})^2 \)

\[ m^2_{(e)} = m^2_{(a)} + \frac{(\vec{x}_1 - \vec{x}_0)^2}{4 \pi^2 \alpha'} \]

Similarly \( m^2_{(d)} = m^2_{(a)} + \frac{(\vec{x}_1 - \vec{x}_0)^2}{4 \pi^2 \alpha'} \)

\[ m^2_{(b)} = m^2_{(a)} \]

Now consider the case \( \vec{x}_1 = \vec{x}_0 \).

\( \approx 4 \)-fold degeneracy

\( \approx \) Gauge fields \( \approx U(2) \) gauge fields.

\( \approx \) 4 sets of \( (25-\bar{p}) \) scalar. No \( (25-\bar{p}) \) scalar fields in the adjoint of \( U(2) \).
Each state is 4-fold degenerate

\[ \Lambda \text{ adjoint of } U(2) \]

\[ \Lambda \text{ 2x2 hermitian matrix. } \Lambda \]

\[ \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \text{sector (a)} \]

\[ \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{sector (b)} \]

\[ \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \text{sector (c)} \]

\[ \Lambda = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow \text{sector (d)} \]

\[ \Lambda \text{ Chen-Paton factor} \]

Look at an adjoint scalar associated with a particular scalar (say \( \phi^{25} \)).

\[ \langle \phi^{25} \rangle = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \text{moving brane at } \Sigma_0 + (0,0,\ldots,0,a) \]

\[ \langle \phi^{25} \rangle = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \rightarrow \text{moving brane at } \Sigma_0 + (0,0,\ldots,0,b) \]

\[ \langle \phi^{25} \rangle = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \rightarrow \text{branes at } \Sigma_0 + (0,0,\ldots,0,a) \text{ and at } \Sigma_0 + (0,0,\ldots,0,b) \]

If \( a = b \), the two branes are at the same location \( \Rightarrow \) still 4 massless gauge fields.

\[ \Rightarrow U(2) \text{ is unbroken since all } \Lambda \text{ commute with } (a, b) \]
If $a \neq b$, the only sectors (a) and (b) give massless gauge fields, which commute with only two generators: $(1, 0)$ and $(0, 1)$. Therefore, $U(2) \rightarrow U(1) \times U(1)$ is broken.

Write

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a + b & 0 \\ 0 & a + b \end{pmatrix} + \frac{i}{2} \begin{pmatrix} a - b & 0 \\ 0 & b - a \end{pmatrix}$$

which commutes with all $\lambda$ and does not commute with $\lambda^\dagger$.

Expect the mass of the off-diagonal gauge bosons to be proportional to $|a - b|$. This can be seen from the fact that the mass of the vectors from sectors (a) and (b) are proportional to

$$|\vec{\alpha} + (0, 0, \ldots, 0, a) - \vec{\alpha} - (0, 0, \ldots, 0, b)|$$

$$= |a - b|.$$
Similarly, $N$ coincident $D$-branes

- $U(N)$ gauge theory

$N^2$-fold degenerate

All states are in the adjoint representation of $U(N)$

We can now consider closed string theories with $N$ space-filling $D$-25 branes.

- Open strings with Neumann b.c. in all coordinates and carrying $N \times N$ Chan-Paton factors.

Massless open string spectrum.

- $U(N)$ gauge fields in $(25+11)$ dimensions.

Note: There are no transverse directions.

- No scalar fields.

- $U(N)$ gauge theory + $g_{\mu\nu}$, $B_{\mu\nu}$, $\phi$ fields.

It turns out that once interaction are taken into account this theory becomes inconsistent.

Reason: $D$-branes carry finite energy/unit $p$-volume.

- A $D$-25 -brane $\Rightarrow$ finite energy density in $(25+11)$ dim. space.

- A cosmological constant term in the Einstein-Hilbert action.

$(25+11)$ - Dim. Minkowskian space is no longer a solution to the e.o.s. of motion (otherwise not existent).
Unoriented strings

Consider the transformation:

\[ \Omega : \sigma \rightarrow -\sigma, \quad \tau \rightarrow -\tau \quad \text{for closed strings} \]

A symmetry of the action:

\[ X^\mu (\sigma, \tau) = X^\mu (-\sigma, \tau) \]

\[ X^\mu = \sum \phi^\mu \epsilon^{i m} \]

\[ \phi_n \rightarrow \phi_{-n}. \]

\[ \Rightarrow \quad \alpha_{n} \leftrightarrow \alpha_{-n} \]

We can define a new string theory by restricting only to \( \Omega \) invariant states.

Quotient of the original theory by \( \Omega \).

\( \Rightarrow \) Unoriented string theory.

\( \Omega \) changes orientation:

\( \Omega (k) \) is \( \Omega \) invariant.

\( \Rightarrow \) Tachyon survives.

\( \Sigma_{\mu} \alpha_{n} \alpha_{-n} \) survives for symmetric \( \phi \).

\( \Leftrightarrow \) \( \Sigma_{\mu} \) & \( \phi \) are present.

\( B_{\mu} \) is absent.

\( \Sigma_{\mu} \) & \( \phi \) are present.
It turns out that the process of \$P_2\$ projection generates a negative cosmological constant.

Thus once interactions are taken into account, the unoriented strongly Minkowski space is no longer a solution of the equations of motion.

Can be cancelled by putting in space-filling \$25\$-branes.

Need \$2^{13}\$ 25-branes.

\[2^{13} \times 2^{13}\] matrix \(\Lambda\) for Chan-Paton factors.

A 2 projection on the open string states:

\[\Sigma \to \Pi \sigma\]

\(\phi^\mu \to \sum \phi^\mu_n \sin n\sigma\)

\(\phi^\mu_n \to (-1)^n \phi^\mu_n\)

\(\alpha^\mu_n \to (-1)^n \alpha^\mu_n\)

Action on \(\Lambda\):

\[1, 2 \quad \to \quad 2, 1\]

Consider action on tachyon:

\[\mathcal{Z} (14k_3) \otimes \Lambda = \mathcal{Z}(14k_3) \otimes \Lambda^*\]

\(\Rightarrow \Lambda = \Lambda^*\) for \$Z\$ invariant states.
Level I state:

\[ \mathcal{S}_2 \left( \sum_{\mu=1}^{N} \alpha_{\mu} \left( \sum_{k=1}^{3} \eta_{k} \chi_{k} \right) \right) = -5 \alpha_{\mu} \left( \sum_{k=1}^{3} \eta_{k} \chi_{k} \right) \lambda \]

\[ \lambda = -\lambda^T \quad \text{for SU invariant state} \]

\[ \lambda \quad \text{is purely imaginary} \]

\[ \lambda \leftrightarrow \text{generators of } SO(N) \text{ group} \]

\[ \Rightarrow \quad \lambda = -\lambda^* \quad \Rightarrow \quad \lambda \text{ is purely imaginary} \]

\[ \lambda \leftrightarrow \text{generators of } SO(N) \text{ group} \]

\[ \Rightarrow \quad \text{Gauge symmetry is } SO(N) \]

Even levels: Symmetric \( NXN \) representation of \( SO(N) \)

Odd levels: Anti-symmetric \( NXN \) representation of \( SO(N) \)

Thus we get a theory with gauge group \( SO(2^{13}) \).
Meaning of the $\langle \text{phys} | T_{\mu\nu} | \text{phys}' \rangle = 0$ constant:

$$S = -\frac{i}{2m} \int d^2 \mathcal{S} \sqrt{-\text{det} Y} \gamma_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

- $\gamma_{\alpha\beta}$: Two dimensional metric.

Gauge symmetries:

1. Reparametrization invariance:

$$X'^{\alpha} (\tilde{x}') = X^{\alpha} (\tilde{x})$$

$$\gamma_{\alpha\beta}' (\tilde{x}') = \epsilon_{\alpha\beta} (\tilde{x}) \frac{\partial \tilde{x}^{\gamma}}{\partial \tilde{x}'^{\alpha}} \frac{\partial \tilde{x}^\beta}{\partial \tilde{x}'^{\gamma}}$$

2. Weyl invariance:

$$\gamma_{\alpha\beta}' (\tilde{x}') = \epsilon_{\alpha\beta} (\tilde{x}) \gamma_{\alpha\beta} (\tilde{x})$$

$$X'^{\mu} (\tilde{x}') = X^{\mu} (\tilde{x})$$

Gauge fixing: $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$

$$S = -\frac{i}{2m} \int d^2 \mathcal{S} \eta_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\gamma_{\alpha\beta}$ eqs. of motion:

$$0 = T_{\alpha\beta} = \text{const.} \left( \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{i}{2} \gamma_{\alpha\beta} \eta_{\mu\nu} \partial_\mu X^\rho \partial_\nu X^\sigma \eta_{\rho\sigma} \right)$$

$$\eta_{\alpha\beta} \cdot \gamma_{\alpha\beta} = 0 \Rightarrow \text{Two inequivalent constants}$$

$$0 = T_{\mu\nu} = \left( \text{const.} \right) \partial_\mu X^\rho \partial_\nu X^\sigma \eta_{\rho\sigma}$$

$$0 = T_{\mu\nu} = \left( \text{const.} \right) \partial_\mu X^\mu \partial_\nu X^\nu \eta_{\mu\nu}$$
$S$ has residual gauge invariances.

(1) Reparameterization

$u \rightarrow f(u)$, $v \rightarrow v$

Together with a reparametrization Weyl transformation:

$-du'dv' = -du'dv f'(u)^{-1}$

$\Rightarrow Y'_{\mu'\nu'} = f'(u)^{-1}$

$\delta_{\alpha\beta} \rightarrow f'(u) Y_{\alpha\beta}$, $X'^{\mu} \rightarrow X^{\mu}$

Net result:

$X'^{\mu}(f(u), v) = X^{\mu}(u, v)$

$Y_{\alpha\beta}(f(u), v) = \delta_{\alpha\beta} f'^{2}$

(2) Reparameterization:

$u \rightarrow g(u)$, $v \rightarrow v$, Together with a Weyl transformation:

$Y_{\alpha\beta} \rightarrow g'(u) Y_{\alpha\beta}$

Net result:

$X'^{\mu}(u, g(v)) = X^{\mu}(u, v)$

$Y_{\alpha\beta}(u, g(v)) = Y_{\alpha\beta}(u, v)$

Since these gauge tras are not fixed, we must demand that physical states are invariant under these gauge tras.

Generators for first tras (conformal tras)

$u \rightarrow u + \epsilon \phi(u)$, $v \rightarrow v$

(Do $\phi(u)$ $Tu_{\mu}(u,v)$)

$\delta_{\alpha\beta} \rightarrow \frac{\delta_{\alpha\beta}}{\delta(2\phi)}$

$\delta_{\alpha\beta} \rightarrow \frac{\delta_{\alpha\beta}}{\delta(2\phi)}$

$\delta_{\alpha\beta} \rightarrow \frac{\delta_{\alpha\beta}}{\delta(2\phi)}$
Generator of second symmetry:

\[ U \rightarrow U + \varepsilon \psi(U) \]

\[ \sum_{\nu=0}^{\alpha-1} \psi(U) \rightleftharpoons (U, \varepsilon) \]

Closed string:

\[ \phi(u = \tau - \sigma) = \phi(u = \tau - \sigma - 2\pi i) \]

\[ \Rightarrow \phi(u) = \phi(u - 2\pi) \]

\[ \Rightarrow \psi(u) = \sum_{\nu \in \mathbb{N}_{\nu}} \phi_{\nu} \varepsilon^{\nu u} \]

\[ \Rightarrow \text{Similarly} \]

\[ \psi(u) = \sum_{\nu \in \mathbb{N}_{\nu}} \psi_{\nu} \varepsilon^{\nu u} \]

\[ \Rightarrow \text{Note that generators of conformal trs.:} \]

\[ \sum_{\nu \in \mathbb{N}_{\nu}} \phi_{\nu} \sum_{\nu=0}^{2\pi} \varepsilon^{\nu u} \rightleftharpoons \sum_{\nu \in \mathbb{N}_{\nu}} \phi_{\nu} \varepsilon^{\nu u} \]

\[ \Rightarrow \psi_{\nu} \sum_{\nu=0}^{2\pi} \varepsilon^{\nu u} \rightleftharpoons \sum_{\nu \in \mathbb{N}_{\nu}} \psi_{\nu} \varepsilon^{\nu u} \]

\( \phi_{\nu}, \psi_{\nu} \) arbitrary coefficients.

\[ \Rightarrow \text{Gauge invariance requires:} \]

\[ \langle \text{phys} | \ln | \text{phys} \rangle = 0 = \langle \text{phys} | \tilde{\ln} | \text{phys} \rangle \]

\( \ln, \tilde{\ln} \) - generators of conformal trs.

\[ [\ln, \ln] = (m-n) \ln \rightleftharpoons \frac{e^2}{12} (m^3 - m) \delta_{m+n,0} \]

\[ [\tilde{\ln}, \ln] = (m-n) \tilde{\ln} \rightleftharpoons \frac{e^2}{12} (m^3 - m) \delta_{m+n,0} \]

\[ c = \mathcal{B}D \Rightarrow \text{central charge.} \]
Open strings

\[ \rho \rightarrow \rho + e \phi(u) \]
\[ \omega \rightarrow \omega + \pi \phi(u) \]

\[ \sigma \rightarrow \frac{1}{2}(\omega + \phi(u)) \rightarrow \sigma + \pi (\omega - 2\pi + \phi(u)) \]
\[ \sigma \rightarrow \frac{1}{2}(\omega - \pi - \phi(u)) \rightarrow \sigma - \pi (\omega - \pi - \phi(u)) \]

\[ \sigma = 0 \text{ should map to } \sigma = 0 \]
\[ \sigma = \pi \text{ should map to } \sigma = \pi \]

\[ \Psi(\omega) = \phi(\omega) \]
\[ \Psi(\omega + 2\pi) = \phi(\omega) \]

\[ \psi(\omega) = \sum \psi_n \phi(\omega) = \sum \psi_n e^{i\omega} \]

Note the current:

\[ \sum_{0}^{\pi} d\sigma \sum_{n} \psi_{n}(e^{i\sigma} T_{uu} + e^{-i\sigma} T_{uv}) \]

\[ T_{uu} = \sum L_{m} e^{i\sigma} \]
\[ T_{uv} = \sum L_{m} e^{-i\sigma} \]
\[ \sum_{n} \psi_{n} e^{i\nu n} L_{m} = \sum_{n} \left( e^{i(m-n)\nu} + e^{i(n-m)\nu} \right) L_{m} \]

Thus \( L_{m} \) are the generators of the conformal transformation:

\[ \nu \rightarrow \nu + \epsilon \psi_{n} e^{i\nu n} \]
\[ \nu \rightarrow \nu + \epsilon \psi_{n} e^{i\nu n} \]

\[ [L_{m}, L_{n}] = (m-n) L_{m+n} + \frac{e}{12} (m^{3} - m) \sum_{m+n,0} \]

Note: \[ L_{m} = \sum_{i=0}^{D-1} L_{m}^{(i)}, \quad \sum_{i=0}^{D-1} L_{m}^{(i)} \]

Thus the theory has extra symmetries generated by \( L_{m}^{(i)} \) & \( \sum_{i=0}^{D-1} L_{m}^{(i)} \) for each \( i \), although it is not part of the original gauge invariance.

\[ \langle L_{m}^{(i)} \rangle \neq 0, \quad \langle \sum_{i=0}^{D-1} L_{m}^{(i)} \rangle \neq 0 \] for individual

But we can still use these additional symmetries to our advantage.

We shall see that in the light cone gauge we do not have any constraints, but we still have conformal symmetry which we can use. \[ \gamma = X^{+}, \quad \delta_{\lambda} Y_{\sigma} = 0, \quad d_\lambda = -1 \]
Light cone gauge

$$S = -\frac{1}{\eta_1 n} \int d^2 \xi \sqrt{-\text{det} \gamma} \; \gamma_{\alpha \beta} \partial_\alpha \xi^\mu \partial_\beta \xi^\nu \eta_{\mu \nu}$$

Three gauge transformation parameters:

- 2 from 2-d g.e.t.
- 1 from Weyl trs.

⇒ Need three gauge fixing conditions.

Define:\n\[X^+ = \frac{1}{\sqrt{2}} (X^0 \pm X')\]

Take the 3 gauge fixing conditions to be:

\[X^+ = \gamma = \xi^0 = 0\]
\[\partial_\sigma \gamma_{\sigma \sigma} = 0\]
\[\partial_\nu \gamma = -1\]

\[\gamma_{\tau \tau} \gamma_{\sigma \sigma} = (\gamma_{\tau \tau})^2 = -1\]
⇒ \[\gamma_{\tau \tau} = \frac{(\gamma_{\tau \tau})^2 - 13}{\gamma_{\alpha \alpha}}\]

\[\gamma_{\alpha \beta} = -:\begin{pmatrix} \gamma_{\alpha \sigma} & \gamma_{\alpha \tau} \\ -\gamma_{\sigma \tau} & \gamma_{\tau \tau} \end{pmatrix}\]

⇒ \[S' = +\frac{1}{\eta_1 n} \int d\xi \; d\sigma - \gamma_{\sigma \sigma} \partial_\sigma \xi^\mu \partial_\sigma \xi^\nu \eta_{\mu \nu} - 2\gamma_{\sigma \tau} \partial_\sigma \xi^\mu \partial_\tau \xi^\nu \eta_{\mu \nu}\]
\[+ \gamma_{\tau \tau} \partial_\sigma \xi^\mu \partial_\tau \xi^\nu \eta_{\mu \nu}\]}
\[ a_\mu x^\mu \eta_{\mu
u} - \tilde{a}_\nu x^\nu \tilde{a}_\mu x^\mu = \tilde{a}_\nu x^\nu a_\mu x^\mu \]

\( i, j \) repeated \( \implies \) sum over \( 2, -1 \)

\[ a_\tau x^+ = 1, \quad \tilde{a}_\tau x^+ = 0 \]

\[ S = \frac{1}{4\pi a_\tau} \oint d\alpha \left[ -2 a_\tau x^- a_\tau x^+ \right] \]

\[ + \sum_{\gamma} \phi_{\gamma}^{-1} \left( (\Gamma_{\gamma\tau})^2 - 1 \right) \tilde{a}_{\tau} x^\tau \tilde{a}_{\tau} x^\tau \]

\[ X^- (\tau, \sigma) = X^- (\tau) + \sum_{\Phi} \phi_{\Phi}^{-1} \epsilon_{\Phi}^{\infty \sigma} \]

\[ Y^- (\tau, \sigma) \]

\[ S = \frac{1}{4\pi a_\tau} \oint d\alpha \left[ -2 a_\tau x^- a_\tau x^+ \right] \]

\[ + \sum_{\gamma} \phi_{\gamma}^{-1} \left( (\Gamma_{\gamma\tau})^2 - 1 \right) \tilde{a}_{\tau} x^\tau \tilde{a}_{\tau} x^\tau \]

Note: \( Y^- \) & hence \( \phi_{\gamma}^{-1} (\tau) \) has no time derivative in \( S \).

\( \exists \phi_{\gamma}^{-1} (\tau) \) they of motion should give constraints.

\[ \sum_{\Phi} \phi_{\Phi}^{-1} \epsilon_{\Phi}^{\infty \sigma} \tilde{a}_{\tau} x^\tau = 0 \quad \forall \nu \neq 0 \]

\( \exists \tilde{a}_{\tau} \) constant \( \iff \tilde{a}_{\tau} \gamma_{\tau\tau} = 0 \)
We shall now restrict ourselves to the open string sector with Neumann b.c. but similar analysis can be carried out for other b.c.'s as well.

\[ S = \frac{i}{4\pi\alpha'} \int d^2\xi \sqrt{-\text{det} \gamma} \partial_{\xi} x^\mu \partial_{\xi} x^\nu \eta_{\mu\nu} \]

\[ SS = -\frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{-\text{det} \gamma} \partial_{\xi} x^\mu \partial_{\xi} \left( \delta x^\nu \right) \eta_{\mu\nu} \]

\[ = \frac{1}{2\pi\alpha'} \int d^2\xi \partial_{\xi} \left( \sqrt{-\text{det} \gamma} \partial_{\xi} x^\mu \right) \eta_{\mu\nu} \delta x^\nu \]

\[ = \frac{1}{2\pi\alpha'} \int \partial \sigma \sqrt{-\text{det} \gamma} \partial_{\sigma} x^\mu \delta x^\nu \eta_{\mu\nu} \bigg|_{\sigma=0} \]

\( \exists \) b.c.

\[ \gamma_{\sigma\sigma} \partial_{\sigma} x^\mu = 0 \quad \text{at } \sigma = 0, \pi \]

\( \exists \) \[ \sigma \partial_{\sigma} x^\mu + \gamma_{\sigma\sigma} \partial_{\sigma} x^\mu = 0 \]

\( \exists \) \[ \sigma \partial_{\sigma} x^\mu - \gamma_{\sigma\sigma} \partial_{\sigma} x^\mu = 0 \quad \text{at } \sigma = 0, \pi \]

Take \( \mu = + \).

\( \exists \) \[ \gamma_{\sigma\sigma} = \gamma \sigma_{\sigma} \partial_{\sigma} x^+ = 0 \]

\( \exists \) \[ \gamma \sigma_{\sigma} = 0 \quad \text{everywhere since } \partial_{\sigma} \gamma_{\sigma\sigma} = 0 \]

\[ S = \frac{1}{4\pi\alpha'} \int \partial \sigma \left[ -2 \gamma_{\sigma\sigma} \partial_{\sigma} x^+ \right. \]

\[ + \gamma_{\sigma\sigma} \int \partial \sigma \partial_{\sigma} x^\nu \partial_{\sigma} x^\nu \partial_{\sigma} x^\nu \partial_{\sigma} x^\nu \right] \]
Eqs. of motion:

\[ \gamma_{\sigma\sigma} \text{ eqn.: } -2 T_{\sigma} \partial_{\tau} \gamma_{\sigma} + \int \partial_{\sigma} \partial_{\tau} \gamma_{\sigma} \gamma_{\tau} = 0 \]

\[ + \gamma_{\sigma\sigma}^{-2} \int \partial_{\sigma} \partial_{\tau} \gamma_{\sigma} \partial_{\tau} \gamma_{\sigma} = 0. \]

\[ \gamma_{\sigma} \text{ eqn.: } \partial_{\tau} \gamma_{\sigma} = 0. \]

\[ \gamma_{\tau} \text{ eqn.: } \gamma_{\sigma\sigma} \partial_{\tau} \gamma_{\tau} - \gamma_{\sigma\sigma}^{-1} \partial_{\tau} \gamma_{\tau} = 0. \]

Q. Do these equations contain all the original eqs. of motion?

We have fixed \( X^+, \gamma_{\sigma\sigma} \) & det \( \gamma \).

Check the eqs. of motion of \( X^+ \) & \( \gamma_{\tau\tau} \).

\[ X^+ \text{ eqn.: } \]

\[ \partial_{\tau} \left( \sqrt{\text{det} \gamma} \gamma^{\alpha\beta} \partial_{\beta} X^{-} \right) = 0. \]

In the gauge:

\[ \partial_{\tau} \left( \gamma^{\alpha\tau} \partial_{\tau} X^{-} \right) + \partial_{\tau} (\gamma_{\sigma\sigma} \partial_{\tau} X^{-}) = 0 \]

\[ \gamma_{\sigma\sigma} \partial_{\tau} X^{-} + \gamma_{\sigma\sigma}^{-1} \partial_{\tau} X^{-} = 0 \]

\[ \gamma_{\sigma\sigma} \partial_{\tau} X^{-} = 0. \]

\[ \gamma_{\sigma\sigma} \partial_{\tau} Y^{-} + \gamma_{\sigma\sigma}^{-1} \partial_{\tau} Y^{-} = 0. \]

First eq. a consequence of the eqs.

of motion.

2nd eq. constraint on \( Y^{-} \). But \( Y^{-} \) has disappeared.
from the dynamics.

No extra constraint on the physical degrees of freedom.

\[ \gamma_{\alpha\beta} \text{ eqs.:} \]

\[ T_{\alpha\beta} = \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu} - \frac{1}{2} \gamma_{\alpha\beta} \partial_{\gamma} X^{\mu} \partial_{\gamma} X^{\nu} \eta_{\mu\nu} \]

\[ T_{\gamma\delta \sigma} = \partial_{\gamma} X^{\mu} \partial_{\delta} X^{\nu} \eta_{\mu\nu} - \frac{1}{2} \gamma_{\gamma\delta\sigma} \partial_{\gamma} X^{\mu} \partial_{\delta} X^{\nu} \eta_{\mu\nu} \]

\[ + \left( \gamma_{\gamma\delta\sigma} \right)^{-1} \partial_{\gamma} X^{\mu} \partial_{\delta} X^{\nu} \eta_{\mu\nu} \]

\[ = - \frac{1}{2} \gamma_{\gamma\delta\sigma} \left( \partial_{\gamma} X^{\mu} \partial_{\delta} X^{\nu} \eta_{\mu\nu} + \left( \gamma_{\gamma\delta\sigma} \right)^{-2} \partial_{\gamma} X^{\mu} \partial_{\delta} X^{\mu} \right) \]

\[ \partial_{\gamma} X^{\mu} \text{ eq. is a consequence of the eqs of motion.} \]

\[ \partial_{\gamma} X^{\mu} = \frac{1}{2} \left( \partial_{\gamma} X^{\mu} \partial_{\gamma} X^{\nu} + \left( \gamma_{\gamma\delta\sigma} \right)^{-2} \partial_{\gamma} X^{\mu} \partial_{\gamma} X^{\mu} \right) \]

determines \( X^{-} \) but no constraints on the physical degrees of freedom.

\[ T_{\tau\tau} = \partial_{\tau} X^{\mu} \partial_{\tau} X^{\nu} - 2 \gamma_{\tau\tau} \partial_{\tau} X^{-} \partial_{\tau} X^{\nu} + \frac{1}{2} \gamma_{\tau\tau}^{-1} \]

\[ ( - \gamma_{\tau\sigma} \partial_{\tau} X^{\mu} \partial_{\sigma} X^{\nu} + 2 \gamma_{\tau\sigma} \partial_{\tau} X^{\mu} \partial_{\tau} X^{\nu} \]

\[ + \gamma_{\tau\sigma}^{-1} \partial_{\tau} X^{\mu} \partial_{\tau} X^{\mu} \eta_{\mu\nu} \]

\[ = \frac{1}{2} \left( - \partial_{\tau} X^{\mu} \partial_{\tau} X^{\mu} + \partial_{\tau} X^{\mu} \partial_{\tau} X^{\mu} + \gamma_{\tau\tau}^{-1} \partial_{\tau} X^{\mu} \partial_{\tau} X^{\mu} \right) \]
Vanishing of \( T_{\sigma r} \) gives the same constraint. Finally:

\[ T_{\sigma r} = \partial_\sigma X^k \partial_\sigma X^\nu \eta^\mu \nu \]

\[ = -\partial_\sigma X^- + \partial_\sigma X^i \partial_\sigma X^i \]

Determines \( \phi_i \) in terms of \( X^i \) - satisfies all other eq.

These two equations are compatible as a consequence of the \( \dot{X}^2 \) equation.

Now back to the gauge fixed action:

\[ S = \frac{1}{4 \pi} \int d\sigma \left[ -2\pi \gamma_\sigma \partial_\sigma X^- + \gamma_\sigma \int d\sigma \partial_\sigma X^i \partial_\sigma X^i \right. \]

\[ \left. - \gamma_\sigma^{-1} \int d\sigma \partial_\sigma X^i \partial_\sigma X^i \right] \]

\[ P_+ = \frac{\delta S}{\delta (\partial_\sigma Y_{\sigma \sigma})} = 0 \quad P_- = \frac{\delta S}{\delta (\partial_\sigma X^i)} = \frac{-\gamma_\sigma}{2\sigma} \quad \bar{P}_+ = \frac{\delta S}{\delta (\partial_\sigma X^i)} \]

\[ = \frac{\gamma_\sigma^2}{2\pi\alpha'} \partial_\sigma X^i \]

\[ H = P_+ \partial_\sigma Y_{\sigma \sigma} + P_- \partial_\sigma X^i + \int d\sigma P_+ (\sigma) \partial_\sigma X^i (\sigma) - \mathcal{L} \]

\[ = - \frac{2}{\pi\alpha'} \int d\sigma \int d\sigma P_+ (\sigma) P_- (\sigma) - \frac{1}{8\pi\alpha'^2} \int d\sigma \int d\sigma \partial_\sigma X^i \partial_\sigma X^i \]

b.c. \( \partial_\sigma X^i = 0 \) at \( \sigma = 0, \pi \)

\( \partial_\sigma X^- = 0 \) at \( \sigma = 0, \pi \) automatically satisfied
Mode expansion

\[ X^\alpha = x^\alpha(\tau) + \sum_{n=1}^{\infty} \phi_n^\alpha(\tau) \cos n \sigma \]

\[
S = \frac{1}{4 \pi \alpha'} \int d\sigma \left[ -2 \pi \gamma_{\sigma \sigma} \partial_x^- x^+ + \pi \gamma_{\sigma \sigma} \partial_x^- x^+ \partial_x^+ x^+ + \frac{\pi}{2} (\gamma_{\sigma \sigma})^{-1} \sum_{n=1}^{\infty} n^2 \phi_n^\alpha \phi_n^\alpha \right]
\]

\[
\Gamma_\sigma = \frac{\delta S}{\delta (\partial_x^- \gamma_{\sigma \sigma})} = 0 , \quad \Gamma^- = \frac{\delta S}{\delta (\partial_x^- x^+)} = -\frac{1}{2 \alpha'} \gamma_{\sigma \sigma} ,
\]

\[
\Pi_n^\alpha = \frac{\delta S}{\delta (\partial_x^- \phi_n^\alpha)} = \frac{1}{2 \alpha'} \gamma_{\sigma \sigma} \partial_x^- \phi_n^\alpha , \quad \Pi_n^\alpha = \frac{\delta S}{\delta (\partial_x^- x^+)} = \frac{1}{2 \alpha'} \gamma_{\sigma \sigma} x^+
\]

\[ \exists H = \Gamma_\sigma \partial_x^- \gamma_{\sigma \sigma} + \Gamma^- \partial_x^- x^+ + \sum_{n=1}^{\infty} \partial_x^- \phi_n^\alpha - L + \sum_{n=1}^{\infty} \Pi_n^\alpha \]

\[ = \frac{2 \alpha'}{\gamma_{\sigma \sigma}} \sum_{n=1}^{\infty} \Pi_n^\alpha \Pi_n^\alpha + \frac{1}{8 \alpha'} \sum_{n=1}^{\infty} n^2 \phi_n^\alpha \phi_n^\alpha + \frac{\alpha'}{\gamma_{\sigma \sigma}} k_+ k^- + \frac{1}{16 \alpha'^2} k_- k_+ \]

Interpretation of $H$:

Generator of $\tau$ translation.

But $\tau = x^+$

\[ \Rightarrow \text{generator of } x^+ \text{ translation} \]

\[ \Rightarrow -p_+ \]

$p_+$: generator of passive $x^+$ transformation

$H$: generator of active $\tau$ transformation

\[ \Rightarrow \epsilon p_+ \epsilon = + \frac{\epsilon}{2} \frac{p_+ k_+}{k_-} + \frac{1}{16 \alpha'^2} \sum_{n=1}^{\infty} n^2 \phi_n^\alpha \phi_n^\alpha + \frac{1}{2} \sum_{n=1}^{\infty} \Pi_n^\alpha \Pi_n^\alpha \]
Now, quantize:

\[
[x^r, p^-] = i, \quad [x^r, p^i] = i\delta^r_i
\]

\[
[i\phi^r_n, \Pi^i_m] = i\delta_m^r \delta_n^i
\]

Define:

\[
\alpha^r_n = \sqrt{2\alpha'} \Pi^r_n + \frac{i\hbar}{2\sqrt{2\alpha'}} \phi^r_n \quad \text{for } n > 0.
\]

\[
\alpha^r_n = \sqrt{2\alpha'} \Pi^r_n + \frac{i\hbar}{2\sqrt{2\alpha'}} \phi^r_n
\]

Then

\[
b^+ = \frac{\mathbf{p}^r \cdot \mathbf{p}^i}{2\mathbf{k}^-} + \frac{1}{2\mathbf{k}^-} \left[ \sum_{n=1}^{\infty} \alpha_{-n}^r \alpha_{n}^r + C_0 \right]
\]

\[
{[\alpha_{-m_1}^r, \alpha_{-m_2}^r]} = m \delta^{i,j} \delta_{m+n,0}
\]

Define \(k^-, \mathbf{k}^-\) s.t. D-2 dim. vec. \((k^2, \ldots, k^{d-1})\)

\[
|k^-, \mathbf{k}^-\rangle = k^- |k^-, \mathbf{k}^-\rangle
\]

\[
\mathbb{R}^2 |k^-, \mathbf{k}^-\rangle = \mathbb{R}^2 |k^-, \mathbf{k}^-\rangle
\]

\[
\alpha_{-n}^r |k^-, \mathbf{k}^-\rangle = 0 \quad \text{for } n > 0.
\]

Generic state:

\[
| \alpha_{-m_1}^r, \alpha_{-m_2}^r, \ldots, \alpha_{-m_k}^r, k^-\rangle
\]

\[
b_+ = \alpha_{-m_1}^r, \ldots, \alpha_{-m_k}^r |k^-, \mathbf{k}^-\rangle
\]

\[
= \left\{ \frac{k^r k^i}{2k^-} + \frac{1}{2k^-} \left( \sum_{k=1}^{k} \mathbb{R} \delta^{i,j} + C_0 \right) \right\} \alpha_{-m_1}^r, \ldots, \alpha_{-m_k}^r, k^+ \]
\[ 2 \mathbf{k}_+ \cdot \mathbf{k}_- - \mathbf{\kappa}_+ \cdot \mathbf{\kappa}_- = \frac{1}{\alpha'} \left( \frac{b}{\Sigma_{x=1}^{N+\infty} m_x + \epsilon_0} \right)^{\frac{1}{2}} \]

\[ \mathbf{\kappa}_+ \cdot \mathbf{\kappa}_- = \frac{1}{\alpha'} \left( N + \epsilon_0 \right) \]

\[ \mathbf{m}_2 \]

\[ \Rightarrow \text{A state with} \]

\[ m^2 = \frac{1}{\alpha'} \left( N + \epsilon_0 \right) \]

Note: Now all states are physical.

But manifest symmetry

\[ \text{SO}(D-2) \subset \text{SO}(D-1,1) \]

Determination of \( \epsilon_0 \):

Note: At level 1, there are \((D-2)\) states.

\[ \chi_{-1} \left( \mathbf{K}_-, \mathbf{\bar{K}} \right) \]

\[ m^2 = \frac{1}{\alpha'} \left( 1 + \epsilon_0 \right) \]

Must form a representation of \( \text{SO}(D-1,1) \)

A massive \( \text{SO}(D-1,1) \) vector has \((D-1)\) helicity states.

A massless \( \text{SO}(D-1,1) \) vector has \((D-2)\) helicity states.

\[ \Rightarrow \text{state must be massless } \Rightarrow \epsilon_0 = -1 \]

\[ \Rightarrow \text{level 0 state is tachyonic.} \]
In order to establish Lorentz invariance of the quantum theory we should be able to explicitly construct the Lorentz generators and show that they satisfy the usual commutation relations.

\[ S = \frac{1}{\sqrt{-g}} \int d^2 \xi \sqrt{-g} \left( \sum_{\mu} \partial_{\mu} X^\mu \right) \left( \sum_{\nu} \partial_{\nu} X^\nu \right) \eta_{\mu\nu} \]

\[ \delta X^\mu = \Lambda^{\mu}_{\nu} X^\nu \]

\[ \delta S = 0. \]

Noether current:

\[ J^{\mu \rho} = \frac{i}{2 \sqrt{-g}} \int d^2 \xi \sqrt{-g} \left( \sum_{\mu} \partial_{\mu} X^\mu \right) \left( \sum_{\nu} \partial_{\nu} X^\nu \right) \eta_{\mu\nu} \]

\[ \eta = \frac{1}{2 \sqrt{-g}} \Lambda^{\mu \rho} J_{\mu \rho} \]

\[ J^{\mu \rho} = \frac{i}{\sqrt{-g}} \int d^2 \xi \sqrt{-g} \left[ \gamma^{\alpha \beta} \left( \partial_{\alpha} X^\mu \right) \partial_{\beta} X^\nu \right] \]

Now we consider the Noether current in the gauge fixed theory.

Note: In the gauge fixed theory the Lorentz trs. need to be accompanied by a compensating gauge trs.
But gauge invariance

\Rightarrow The compensating gauge transformation leaves the physical states invariant.

\Rightarrow We can continue to use the same generators if the gauge symmetry is valid.

\[ \det \gamma = -1, \quad \gamma^{-1} = 0. \]

\[ \gamma_{\tau\tau} = -\gamma_{\sigma\sigma} = 2\chi' \phi. \]

\[ \gamma_{\sigma\tau} = -\gamma_{\tau\sigma} = \left( \frac{\gamma_{\sigma\sigma}}{\gamma_{\tau\tau}} \right)^{-1} = -\left( 2\chi' \phi \right)^{-1} \]

\[ J^{\mu} = \frac{1}{4\pi\alpha'} \int d\sigma \left[ (2\chi' \phi) \partial_\tau x^\mu \cdot x^\rho - (2\chi' \phi) \partial_\rho x^\mu \cdot x^\tau \right] \]

\[ = \frac{2\pi}{2\pi} \int d\sigma \left[ x^\rho \partial_\tau x^\mu - x^\mu \partial_\tau x^\rho \right] \]

\[ J^{++} = \frac{b_-}{2\pi} \int d\sigma \left[ x^+ \partial_\tau x^- - x^- \partial_\tau x^+ \right] \]

\[ J^{2+} = \frac{b_+}{2\pi} \int d\sigma \left[ x^2 \partial_\tau x^+ - x^+ \partial_\tau x^2 \right] \]

\[ J^{2-} = \frac{b_-}{2\pi} \int d\sigma \left[ x^2 \partial_\tau x^- - x^- \partial_\tau x^2 \right] \]

\[ J^{13} = \frac{b_+}{2\pi} \int d\sigma \left[ x^i \partial_\tau x^3 - x^3 \partial_\tau x^i \right] \]
\[
\delta x^+ = \Lambda^+ \cdot x^+ \cdot \Lambda^+ \cdot x^+ + \Lambda^+ \cdot x^+ \cdot \Lambda^+ \cdot x^+
\]

Thus we need compensating gauge for \( \Lambda^+ \) and \( \Lambda^+ \).

Generator:

\[
\Lambda_{\mu \nu}^{+ \pm} = \begin{bmatrix}
\Lambda^+ & \Lambda^+ & \Lambda^+ & \Lambda^+
\end{bmatrix}
\]

\[
\eta_{\mu \nu} = \begin{pmatrix}
+ & - & 2 & 3 \\
0 & -1 & -1 & 0 \\
-1 & 0 & 1 & 1 \\
2 & 3 & 1 & 1
\end{pmatrix}
\]

Thus we expect \( J^+ \), \( J^- \) to be safe since these do not require compensating gauge transformation.

Problematic cases \( J^+ \), \( J^- \).

We need to check that these \( J_{\mu \nu} \)'s satisfy the usual Lorentz algebra.

It turns out that all the \( J_{\mu \nu} \) satisfy the Lorentz algebra except the \( [J^+, J^-] \) commutator.

Expected answer = 0.
\[ J^{\hat{z}} = \frac{1}{2\pi} \int d\sigma \left[ x^2 \frac{d}{dt} x^{-} - x^{-} \frac{d}{dt} x^2 \right] \]

Expressible in terms of \( x, b, x^{-} \).

\[ x^{-} = x^{-}(t) + \sum_{n} \phi_{n}(t) \cos n\sigma \]

\[ \gamma_{\sigma \sigma} \text{ eq. of motion:} \]

\[ \frac{\partial}{\partial \sigma} x^{-} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} x^{-} \frac{d}{dt} x^{-} + (\gamma_{\sigma \sigma})^{-2} a_{\sigma} x^{-} a_{\sigma} x^{-} \right\} \]

\[ \gamma_{\sigma \sigma} = \delta_{\sigma \sigma} - 2a_{\sigma} a_{\sigma} \]

Thus \( \frac{\partial}{\partial \sigma} x^{-} \) can be expressed in terms of \( x^{n}, b^{n}, b^{-} \) etc.

\[ \gamma_{\sigma \nu} \text{ eq. of motion:} \]

\[ \frac{\partial}{\partial \sigma} x^{-} = \frac{\partial}{\partial x} x^{-} \frac{d}{dt} x^{-} \]

\[ \psi \]

\[ -\sum_{n} \phi_{n}(t) \sin(n\sigma) \]

\( \implies \phi_{n}(t) \) can be determined in terms of \( x^{n}, b^{n}, b^{-} \) etc.

\( \implies x^{-} \) can be determined in terms of \( x, x^{n}, b^{n}, b^{-} \) etc.

Net result: \( J^{\hat{z}} \) can be expressed in terms of \( x, x^{n}, x^{-}, b^{n}, b^{-} \).

Compute \( [J^{\hat{z}+}, J^{\hat{z}+}] \).
Net result:

\[ [J^+, J^-] \otimes \mathbb{2} = \text{const.} \sum_{m=1}^{8} \Delta m (x_m^i \alpha_i^j - \alpha_j^i x_m^i) \]

\[ \Delta m = \frac{26-D}{12} + \frac{1}{m} \left( \frac{D-26+C_0}{12} \right) \]

\[ \Rightarrow C_0 = -1, \ D = 26 \]

Light-cone gauge for closed strings

\[ X^+ = \tau \]

\[ \partial_\tau X_{\tau \sigma} = 0 \]

\[ \det \gamma = -1 \]

\[ X^- = \sum_n \phi_n^- e^{i n \sigma} + X^- (\tau) \]

\[ \phi_n^- \text{ eqs. of motion become algebraic.} \]

Gives \[ \partial_\tau X_{\tau \sigma} = 0. \]

For open string we saw that \[ X_{\tau \sigma} = 0 \]

at \( \sigma = 0 \) \& \( \tau \) due to b.c. \[ \Rightarrow X_{\tau \sigma} = 0 \]

This is no longer possible for closed string

Instead here we proceed slightly different manner.

Residual gauge inv.

\[ \sigma' = \sigma + f (\tau) \]
Not possible for open strings since \( \sigma = 0, \pi \) need to be fixed.

\[
\gamma_{\tau\tau} \, d\tau^2 + \gamma_{\sigma\sigma} \, d\sigma^2 + 2 \gamma_{\tau\sigma} \, d\tau \, d\sigma \\
+ \gamma_{\tau\tau} \, d\tau^2 + \gamma_{\sigma\sigma} \left( d\sigma^2 + 2f'(\tau) \, d\tau \, d\sigma + (f'(\tau))^2 \, d\tau^2 \right) \\
+ 2 \gamma_{\tau\sigma} \left( d\sigma + f'(\tau) \, d\tau \right) \, d\tau
\]

\[
= \left( \gamma_{\tau\tau} + 2 \gamma_{\tau\sigma} f'(\tau) + \gamma_{\sigma\sigma} (f'(\tau))^2 \right) \, d\tau^2 \\
+ 2 \left( \gamma_{\tau\sigma} + f'(\tau) \, \gamma_{\sigma\sigma} \right) \, d\tau \, d\sigma \\
+ \gamma_{\sigma\sigma} \, d\sigma^2
\]

\[
\Rightarrow \gamma_{\sigma\sigma} = \gamma_{\tau\tau} + 2 \gamma_{\tau\sigma} f'(\tau) + \gamma_{\sigma\sigma} (f'(\tau))^2 \\
\gamma_{\tau\tau} = \gamma_{\sigma\sigma} + f'(\tau) \, \gamma_{\sigma\sigma} \\
\gamma_{\sigma\sigma} = \gamma_{\sigma\sigma}.
\]

Note: \( \gamma_{\sigma\sigma} \) & \( \text{det } \gamma \) does not change.

\[
\Rightarrow \text{Gauge conditions are preserved.}
\]

Since \( \gamma_{\tau\tau} \) & \( \gamma_{\sigma\sigma} \) are independent of \( \sigma \),
we can make new \( \gamma_{\tau\sigma} \) vanish by

Choosing \( f'(\tau) = -\gamma_{\tau\sigma} / \gamma_{\sigma\sigma} \).

Rest of the analysis proceeds as in the open string case.
Result: Final dynamical variables:
\[ x^-, p^-, \tilde{x}^-, k, \tilde{k}^- \]

\[ -p^+ = H = -\frac{p^+ k^+}{2p^-} + \frac{\bigcirc}{2} \sum_{n=1}^{\infty} (\tilde{x}^+ \tilde{x}_n + \tilde{x}^n \tilde{x}^+ + c + \tilde{c}) \]

Note: There is still a residual gauge freedom \( \sigma \rightarrow \sigma + \text{constant} \).

We set an additional constraint:
\[ \sum_{n=1}^{\infty} \tilde{x}^+ \tilde{x}_n + c = N \]

Symmetry \( x^+ \leftrightarrow \tilde{x}^+ \in c = \tilde{c} \).

Analysis of Lorentz invariance:

\[ \Rightarrow \quad c = \tilde{c} = -1 \]

\[ H = -\frac{p^+ k^+}{2p^-} = \sum_{n=1}^{\infty} (\tilde{x}^+ \tilde{x}_n + \tilde{x}^n \tilde{x}^+) \quad (\text{phys}) = 0 \]

Acting on \( |S\rangle \): \( x^-_{-m_1}, \ldots, x^+_{-m_k} \tilde{x}^-_{-n_1}, \ldots, \tilde{x}^+_{-n_l} (k, \tilde{k}) \)

Constraint: \( \sum_{m} x^+_{-m} = -\sum_{n} \tilde{x}^+_{-n} = N \)

\[ H |S\rangle = \{-\frac{k^+ k^+}{2k^-} - \bigcirc \} (2N-2)^j |S\rangle \]
\[ k_+ = \frac{k^\cdot k^\cdot}{2 k_-} + \frac{4\pi}{\alpha'} k_- (N-1) \]

\[ 2 k_+ k_- - k^\cdot k^\cdot = \frac{\mathcal{g}}{\alpha'} (N-1) \]

\[ m^2 \]

This gives the spectrum of closed strings.

- Same spectrum as in the covariant formulation.
Compactification: Reducing the number of dimensions.

1. Take a $D$ dimensional theory.
2. Take $d$ of the space-like dimensions to be compact & small.

For sufficiently small size, these directions will be invisible to the people and the experiments.

3. The theory will appear $\phi^{(D-d)}$

Field theory example:

1. Start with a scalar field in $D$ dimensions.

\[
(D - m^2) \phi = 0
\]

\[
\sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \partial_\nu \phi - m^2 \phi = 0
\]

$\eta^{\mu \nu} = \text{diag} (-1, 1, 1, \ldots, 1)$

Take the $x^{D-1}$ direction to describe a circle of radius $R$

\[
x^M : 0 \leq M \leq D-2, \quad \text{Remaining coordinates}
\]

\[
\phi (x^M, x^{D-1}) = \sum_n \phi_n (t x^{M}) e^{i n x^{D-1}/R}
\]

\[
\frac{d^2}{d \phi} = \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \partial_\nu \phi + m^2 \phi
\]

\[
= \sum_{\mu, \nu} \eta^{\mu \nu} \partial_\mu \partial_\nu \phi + m^2 \phi
\]
Klein-Gordon Eqn.

\[ 0 = (\Box - m^2) \phi = \sum_n \left( \sum_{M=0}^{D-2} \eta_{MN} \partial_M \partial_N \phi_n - \frac{n^2}{R^2} \phi_n - m^2 \phi_n \right) e^{-i n x^{D-1}/R} \]

\[ \Rightarrow \sum_{n} \eta_{MN} \partial_M \partial_N \phi_n - \left( \frac{n^2}{R^2} + m^2 \right) \phi_n = 0 \]

\( \phi_n \) describes a \((D-1)\) dimensional scalar field of \( (\text{mass})^2 = \left( \frac{m^2 + \frac{n^2}{R^2}}{R^2} \right) \)

Infinite no of such scalar fields since \(-\infty < n < \infty\).

\( \phi_n = \phi_n^* \)

Note: For small \( R \), \( (m^2 + \frac{n^2}{R^2}) \) is large except for \( n = 0 \).

\( \Rightarrow \) As \( R \to \infty \) only the \( n = 0 \) mode will be observable by experimentalists.

\( \Rightarrow \) A single scalar field in \((D-1)\) dimensions.

This way by compactifying \( D \) dimensions we'll get a \((D-1)\) dimensional field theory at energy \( \ll \frac{1}{R} \). (i.e., D-1, D-2, \( \ldots \), D-d)

Gauge field: \( A_n \)

\[ A_M = \sum_n A_M^{(n)} e^{-i n x^{D-1}/R} \]

\[ A_{D-1} = \sum_n A_{D-1}^{(n)} e^{-i n x^{D-1}/R} \]
Substitute in Maxwell's eq:

\[ \partial_t F_{\mu\nu} = 0. \]

\[ A^{(n)}_M : \text{A vector field of (mass)}^2 = \frac{n^2}{R^2} \]

in (0-1) dimension for 0 ≤ M ≤ D-2.

\[ A^{(n)}_{D-1} : \text{A scalar field of (mass)}^2 = \frac{n^2}{R^2} \]

in (0-1) dimension gets absorbed in \( A^{(n)}_M \).

Metric:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ h_{MN} = \sum_n h^{(n)}_{MN} e^{\ln x^{D-1}/R} \quad 0 \leq M, N \leq D-2. \]

\( \to \) spin 2 field of (mass)² = \( \frac{n^1}{R^2} \)

\[ h^{(n)}_{M(D-1)} = \sum_{M(D-1)} h^{(n)}_{M(D-1)} e^{\ln x^{D-1}/R} \]

\( \to \) vector field of (mass)² = \( \frac{n^1}{R^2} \) absorbed in \( h_{MN} \)

\[ h^{(n)}_{(D-1)(D-1)} = \sum_{(D-1)(D-1)} h^{(n)}_{(D-1)(D-1)} e^{\ln x^{D-1}/R} \]

\( \to \) scalar field of (mass)² = \( \frac{n^0}{R^2} \)

As \( R \to 0 \), only \( n=0 \) modes survive.

\( \to \) a massless gauge field, a scalar, and a massless vector.
We now want to implement this procedure in string theory.

String theory has no of states of different masses.

We can apply this procedure to each of these states.

For each state in (26) dimensional string theory we get an infinity of single particle states in the compactified theory, labelled by momenta in the compactified directions.

But besides these states string theory contains extra states associated with "winding modes".

To study these we need to analyze exactly strings theory directly in the compact space-time.

$X^{D-1}$ is compact, say.

Recall mode expansion of $X^k$ in the uncompactified theory:

$X^k = X^k(\tau) + \sum_n \phi^k_n e^{in\sigma}$

(Focus on closed string theory)

New mode expansion:

$X^M = X^M(\tau) + \sum_n \phi^M_n e^{in\sigma}$

$X^{D-1} = \sum_n \phi^{D-1}_n e^{in\sigma}$
As $\sigma \to \sigma + 2\pi$, $x^{D-1} \to x^{D-1} + 2\pi R \nu$

1. The string winds $w$ times along the compact direction.
2. Winding number $w$ does not affect the definition of conjugate momenta.

We can proceed exactly as in the non-compact case.

$$\Pi_{\nu n} = \frac{i}{\alpha^{'}} \eta_{\rho \nu} \partial_\rho \phi_{-n}$$

$$p_\nu = \frac{i}{\alpha^{'}} \eta_{\rho \nu} \partial_\rho x^n$$

$$\alpha^\nu_{-n} = \frac{1}{\sqrt{2}} \left( \eta^{\nu \rho} \sqrt{\alpha^{'}} \Pi_{\rho n} + \frac{2i}{\sqrt{\alpha^{'}}} \phi_{-n} \right)$$

$$\tilde{\alpha}^\nu_{n} = \frac{1}{\sqrt{2}} \left( \eta^{\nu \rho} \sqrt{\alpha^{'}} \Pi_{\rho n} - \frac{2i}{\sqrt{\alpha^{'}}} \phi_{-n} \right)$$

$X^\nu, P^\nu$ hermitian; \((\alpha^\nu_n)^+ = \alpha^{-\nu}_{-n}\), \((\tilde{\alpha}^\nu_n)^+ = \tilde{\alpha}^{-\nu}_{-n}\)

$$[X^\nu, P^\nu] = i \delta^\nu_\nu, \quad [\alpha^\nu_m, \alpha^\nu_n] = m \eta^{\nu \rho} \delta_{mn,0}$$

$$[\tilde{\alpha}^\nu_m, \tilde{\alpha}^\nu_n] = m \eta^{\nu \rho} \delta_{mn,0}$$

Define $|k, \omega\rangle$ such that

$$P_m |k, \omega\rangle = k_m |k, \omega\rangle, \quad \alpha^\nu_n |k, \omega\rangle = 0 - \tilde{\alpha}^\nu_n |k, \omega\rangle$$

for $n > 0$

Note $k^{D-1} = \eta/R; \quad \eta \in \mathbb{Z}$

$$|k, \omega\rangle \rightarrow |\{k^m\}, \eta, \omega\rangle$$
Hilbert space:

\[ \alpha^{m_1, \ldots m_k, n_1, \ldots n_l} \in \{ \mathbb{R}^{M3}, n, \omega \} \]

\[ \text{(D-1) dim. momenta} \quad \frac{n}{R} = \text{momentum along } \alpha^{D-1} \]

(String theory in D-dim with xo compactified on a circle of radius R)

Constraints:

\[ L_m |_{\text{phys}} = 0 \quad m \geq 0 ; \quad \tilde{L}_m |_{\text{phys}} = 0 \quad m \geq 0 \]

\[ L_m = \frac{1}{2} \sum_{n \neq 0, m} \alpha^{k} \alpha^{\nu} \eta_{\mu \nu} + \frac{1}{2} \sqrt{2} \alpha' \alpha^{\mu} \eta_{\mu \nu} \]

\[ + \frac{1}{2} \sqrt{2} \alpha' \alpha^{D-1} \left( \frac{R \omega}{\alpha'} + \frac{n}{R} \right) m \geq 0 \]

\[ L_0 = \sum_{n \geq 0} \alpha^{k} \alpha^{\nu} \eta_{\mu \nu} + \frac{\alpha'}{4} \eta_{\mu \nu} \eta_{\mu \nu} \]

\[ + \frac{\alpha'}{4} \left( \frac{n}{R} - \frac{R \omega}{\alpha'} \right)^2 + c \]

\[ \tilde{L}_m = \frac{1}{2} \sum_{n \neq 0, m} \tilde{\alpha}^{k} \tilde{\alpha}^{\nu} \eta_{\mu \nu} + \frac{1}{2} \sqrt{2} \alpha' \tilde{\alpha}^{\mu} \eta_{\mu \nu} \]

\[ + \frac{1}{2} \sqrt{2} \alpha' \tilde{\alpha}^{D-1} \left( \frac{n}{R} + \frac{R \omega}{\alpha'} \right) m \geq 0 \]

\[ \tilde{L}_0 = \sum_{n \geq 0} \tilde{\alpha}^{k} \tilde{\alpha}^{\nu} \eta_{\mu \nu} + \frac{\alpha'}{4} \eta_{\mu \nu} \eta_{\mu \nu} + \frac{\alpha'}{4} \left( \frac{n}{R} + \frac{R \omega}{\alpha'} \right)^2 + c \]
Generic state:
\[
\alpha_{-m_1} \cdots \alpha_{-m_k} \sim \alpha_{-n_1} \cdots \alpha_{-n_l} \left( \frac{m_1^2}{4\pi^2} \right) \text{ for } n \geq 0
\]

We need to impose constraints:
\[
\langle \text{phys} \mid T_{\mu\nu} \mid \text{phys} \rangle = 0 = \langle \text{phys} \mid T_{\mu\nu} \mid \text{phys} \rangle
\]
\[
\sum n = -\delta_0, \quad \sum n = 0
\]

\[
T_{\mu\nu} = \Phi = (\text{const}) \eta_{\mu\nu} \partial_{\mu} X^k \partial_{\nu} X^k
\]
\[
= (\text{const}) \sum L_m e^{-im\sigma}
\]

\[
T_{\mu\nu} = (\text{const}) \eta_{\mu\nu} \partial_{\mu} X^k \partial_{\nu} X^k
\]
\[
= (\text{const}) \sum \tilde{L}_m e^{-im\sigma}
\]

\[
L_m = \frac{1}{2} \sum_{n \neq 0} \alpha_{-m-n}^k \alpha_n^k \eta_{\mu\nu} + \frac{1}{2} \sqrt{2\alpha^1} \alpha_{-m-n}^k \partial \mu \frac{1}{2} \sqrt{2\alpha^1} \alpha^D_{m-n} \frac{R\omega}{\alpha^1}
\]

\[
L_0 = \sum_{n \geq 0} \alpha_{-n}^k \alpha_n^k \eta_{\mu\nu} + \frac{\alpha'}{\gamma} \eta^{MN} \partial_M \partial_N + \frac{\alpha'}{\gamma} \left( D^{-1} \frac{F^\mu \nu}{\alpha^1} - R\omega \right)^2 + C
\]

\[
\tilde{L}_m = \frac{1}{2} \sum_{n \neq 0} \tilde{\alpha}_{-m-n}^k \tilde{\alpha}_n^k \eta_{\mu\nu}
\]
\[
+ \frac{1}{2} \sqrt{2\alpha^1} \alpha_{-m-n}^k \partial \mu \alpha^D_{m-n} \frac{R\omega}{\alpha^1}
\]

\[
\tilde{L}_0 = \sum_{n \geq 0} \tilde{\alpha}_{-n}^k \tilde{\alpha}_n^k \eta_{\mu\nu} + \frac{\alpha'}{\gamma} \eta^{MN} \partial_M \partial_N + \frac{\alpha'}{\gamma} \left( D^{-1} + \frac{R\omega}{\alpha^1} \right) + \tilde{C}
\]

\[
C = \tilde{C} = -1
\]

Constraint: \[
L_m \mid \text{phys} \rangle = 0 = \tilde{L}_m \mid \text{phys} \rangle \text{ for } m \geq 0
\]
\[ x_{-m_1} \cdots x_{-m_k} x_{n_1} \cdots x_{n_l} \{ k^m_1, \eta, \omega \} \]

**L_0 constraint**

\[ \sum_{k=1}^k \frac{m_k}{\alpha'} + \frac{\alpha'}{\alpha'} \sum_{k=1}^k \left( \frac{n_k}{R} - \frac{R \omega^k}{\alpha'} \right)^2 + \mathcal{O} = 0 \]

**N**

\[- \eta^{MN} k_M k_N = \frac{4}{\alpha'} (N + \bar{c}) + \left( \frac{n}{R} - \frac{R \omega}{\alpha'} \right)^2 \]

**L_0 constraint**

\[- \eta^{MN} k_M k_N = \frac{4}{\alpha'} (N + \bar{c}) + \left( \frac{n}{R} + \frac{R \omega}{\alpha'} \right)^2 \]

\[(\text{Mass})^2 = \frac{4}{\alpha'} (N + \bar{c}) + \left( \frac{n}{R} - \frac{R \omega}{\alpha'} \right)^2\]

\[(\text{Mass})^2 = \frac{4}{\alpha'} (N + \bar{c}) + \left( \frac{n}{R} + \frac{R \omega}{\alpha'} \right)^2\]

**Note**

\[(\text{Mass})^2 = \frac{4}{\alpha'} (N + \bar{c}) + \frac{n^2}{R^2} = \frac{4}{\alpha'} (N + \bar{c}) + \frac{n^2}{R^2} \]

\[c + \bar{c} = \bar{N} + \bar{c} \]

\[(\text{Mass})^2 = \frac{4}{\alpha'} (N + \bar{c}) + \frac{n^2}{R^2} \]

26 - dim \nabla \cdot \phi \rightarrow \text{Contribution from momentum along 25th direction}

Thus, when \( \omega = 0 \) states correspond to the expected Kaluza-Klein states which we get by compactifying an ordinary field theory on a circle. The consistency \( D = 26, c - \bar{c} = -1 \)
W FACTOR STATES ARE NEW STATES COMING FROM "STRINGY" CONSIDERATIONS

2) SPECTRUM IS SYMMETRIC UNDER

\[ R \rightarrow \frac{x'}{R}, \quad n \rightarrow n, \quad \omega \rightarrow \omega. \]

\[ \frac{R}{x'} + \frac{n}{x'} = \frac{x'}{R} + \frac{n}{x'} \]

Not only \( L_0, \tilde{L}_0 \) but also \( L_m, \tilde{L}_m \) are symmetric.

\[ L_m = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n^m \alpha_n^m, \quad \eta_{\mu\nu} + \frac{1}{2} \sqrt{2} \alpha_n \alpha_m \]

\[ + \frac{1}{2} \sqrt{2} \alpha_n \alpha_m (n + \frac{R}{x'}) \]

\[ \tilde{L}_m = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n^m \alpha_n^m, \quad \eta_{\mu\nu} + \frac{1}{2} \sqrt{2} \alpha_n \alpha_m \]

\[ + \frac{1}{2} \sqrt{2} \alpha_n \alpha_m (n + \frac{R}{x'}) \]

\[ L_m \rightarrow L_m, \quad \tilde{L}_m \rightarrow \tilde{L}_m \] under \( R \rightarrow \frac{x'}{R}, \quad n \rightarrow n, \quad \omega \rightarrow \omega. \)

One can show that the interaction rules also obey this symmetry.

**Conclusion:** The 26 dimensional bosonic string theory with one direction compactified on a circle of radius \( R \) is equivalent to 26-dimensional bosonic string theory with one direction compactified on a circle of radius \( x'/R \).

**R = x'/R Duality**

Momentum along \( x^{25} \) ↔ Winding along \( x^{25} \)
Spectrum of massless states

\[ (\text{Mass})^2 = \frac{4}{\alpha'} (N - 1) + \frac{\left(\frac{n^2}{R} + \frac{Rw}{\alpha'}\right)^2}{\alpha'} \]

First consider \( n = 0, w = 0 \)

\[ (\text{Mass})^2 = 0 \Rightarrow \text{Level 1 State} \]

\[ S_{MN} \chi^M_{-1} \chi^N_{-1} \{ k^M, 0, 0 \} \]

\( \rightarrow \) Graviton, dilaton, antisymmetric tensor

\[ (S_{MN} + S_{NM}) \quad (S_{MN} - S_{NM}) \text{ in } D = 25 \]

\[ S_{M} \chi^M_{-1} \chi^25_{-1} \{ k^M, 0, 0 \} \]

\[ S_{M} \chi^25_{-1} \chi^M_{-1} \{ k^M, 0, 0 \} \]

\( \Rightarrow \) Two massless vector fields.

Field theory until pretation.

\[ g_{25 M} \& B_{25 M} \]

Finally:

\[ \chi^25_{-1} \chi^25_{-1} \{ k^M, 0, 0 \} \rightarrow \text{Scalar} \]

Field theory until pretation

\[ g_{25 25} \]

Thus this spectrum coincides with the expected spectrum of a 26-dim. field theory of \( g_{\mu\nu}, \phi \& B_{\mu\nu} \) compactifying one direction on \( S^2 \).
But there are other massless states for $n \neq 0, \omega \neq 0$.

$$\left( \frac{\text{Mass}}{\alpha} \right)^2 = \frac{1}{\alpha} (N-1) + \left( \frac{R}{\alpha} - \omega \frac{R}{\alpha} \right)^2$$

$$= \frac{1}{\alpha} (N-1) + \left( \frac{R}{\alpha} + \omega \frac{R}{\alpha} \right)^2$$

$$\left( \frac{\text{Mass}}{\alpha} \right)^2 = 0$$

$$\Rightarrow \left( \frac{R}{\alpha} - \omega \frac{R}{\alpha} \right)^2 = \frac{1}{\alpha} (1-N)$$

$$\left( \frac{R}{\alpha} + \omega \frac{R}{\alpha} \right)^2 = \frac{1}{\alpha} (1-N)$$

$\Rightarrow N \leq 1, N \leq 1$

1. $N = \tilde{N} = 1 \Rightarrow n = \omega = 0$. → Studied already

2. $N = 1, \tilde{N} = 0$.

$$n = \frac{R}{\alpha} - \omega \frac{R}{\alpha} = 0$$

$$n = \frac{R}{\alpha} + \omega \frac{R}{\alpha} = \pm \frac{2}{\sqrt{\alpha}}$$

$$A \Rightarrow \frac{2n}{R} = \pm \frac{2}{\sqrt{\alpha}}$$

$$R = 1 \text{n} \sqrt{\alpha}$$

$$\omega = \frac{n \alpha'}{R^2} = \pm 1$$

3. $R = \sqrt{\alpha}$

$\Rightarrow$ states:

$\left| \Phi^m \right|_{K^m}, n = \pm 1, \omega = \mp 1$ → Massless vector

$\left| \alpha \right|_{K^m}, n = \pm 1, \omega = \mp 1$ → Massless scalar

satisfy L constraint.

$\Rightarrow$ 2 massless vectors & 2 massless scalar.
\( Z^2 = 1, \quad N = 0. \)

\[
\begin{align*}
\frac{\nabla^2}{R} + \frac{R\omega}{\alpha^2} &= 0, \\
\frac{\nabla^2}{R} - \frac{R\omega}{\alpha^2} &= \frac{2}{\sqrt{\alpha}}.
\end{align*}
\]

Same analysis \( \Rightarrow R = 0 \text{ or } \sqrt{\alpha}, \quad \omega = -n = \pm 1 \)

States:

- \( S_m \chi^m_{-1}, \quad |k| m^3, \quad n = \pm 1, \quad \omega = -n \rightarrow \text{Massless vectors} \)
- \( R^{25} \chi^3_{-1}, \quad |k| m^3, \quad n = \pm 1, \quad \omega = -n \rightarrow \text{Massless scalars} \)

\( \Rightarrow 2 \text{ massless vectors} + 2 \text{ massless scalars} \)

4. \( Z = 0, \quad N = 0. \)

\[
\begin{align*}
\left( \frac{\nabla^2}{R} + \frac{R\omega}{\alpha^2} \right) &= 0, \\
\left( \frac{\nabla^2}{R} - \frac{R\omega}{\alpha^2} \right) &= \frac{4}{\alpha^2}.
\end{align*}
\]

Subtract: \( 4 \frac{R\omega}{\alpha^2} = 0 \Rightarrow n = 0 \text{ or } \omega = 0. \)

If \( n = 0, \quad R^2 \omega^2 = 4 \alpha \)

For \( R = \sqrt{\alpha}, \quad \omega = \pm 2 \)

If \( \omega = 0, \quad \frac{R^2}{\alpha^2} = \frac{4}{\alpha^2}. \)

For \( R = \sqrt{\alpha}, \quad n = \pm 2. \)

3. Four massless zoo states:

- \( \chi^m_{-1}, \quad |k| m^3, \quad n = \pm 2, \quad \omega = 0 \rightarrow \text{Scalar} \)
- \( \chi^m_{-1}, \quad |k| m^3, \quad n = 0, \quad \omega = \pm 2 \rightarrow \text{Scalar} \)
Complete spectrum of massless states at $R = \sqrt{2}a^1$:

1. 25 dim. metric
2. 25 dim. anti-symmetric tensor
3. 25 dim. scalar dilaton
4. 9 scalars:
   1. 1 from $\alpha^{25}_{-1} \tilde{\alpha}^{25}_{-1} \{k^m\}, n = 0, \omega = 0$
   2. 2 from $\alpha^{25}_{-1} \{k^m\}, n = \pm 1, \omega = n$
   2. 2 from $\tilde{\alpha}^{25}_{-1} \{k^m\}, n = \pm 1, \omega = -n$
   2. 24 from $\{k^m\}, n = \pm 2, \omega = 0$
   2. 2 from $\{k^m\}, n = 0, \omega = \pm 2$

5. 6 vectors:
   1. 1 from $\alpha^M_{-1} \tilde{\alpha}^{25}_{-1} \{k^m\}, n = 0, \omega = 0$
   1. 1 from $\alpha^M_{-1} \tilde{\alpha}^{25}_{-1} \{k^m\}, n = 0, \omega = 0$
   1. 2 from $\alpha^M_{-1} \{k^m\}, n = \pm 1, \omega = n$
   2. 2 from $\tilde{\alpha}^{25}_{-1} \{k^m\}, n = \pm 1, \omega = -n$

Study of interaction:

3. The 6 massless vectors correspond to gauge bosons of $SU(2) \times SU(2)$ gauge theory
5. 9 scalars $\to (3,3)$ representation of $SU(2) \times SU(2)$
**Compactification to lower dimensions:**

- Take d of the directions to be circles of radius R₁, ..., Rₙ.

- Momentum & winding modes along each of the d directions.

  1. $R_i \rightarrow \frac{1}{R_i}$ duality for each i.

  2. Larger gauge symmetry groups.

- A theory in $(26-d)$ uncompactified dimensions.

**More general compactification**

Note: 26 dimensional critical string theory has constraints.

$L_{m \text{ (phys)}} = \tilde{L}_{m \text{ (phys)}} = 0 \quad m \geq 0$.

$L_m = \sum_{\mu=0}^{25} L_m^{(\mu)}$ for $m \neq 0$.

$L_0 = \sum_{\mu=0}^{25} L_0^{(\mu)} - 1$ (normal ordering constant).

$\tilde{L}_m = \sum_{\mu=0}^{25} \tilde{L}_m^{(\mu)} \quad m \neq 0$.

$\tilde{L}_0 = \sum_{\mu=0}^{25} \tilde{L}_0^{(\mu)} - 1$. 
\[ L_m = \frac{i}{2} \sum_{n \neq 0, m} \alpha^\mu_m \alpha^\mu_n \eta_{\mu \nu} \eta_{\mu \nu} + \frac{1}{2} \sqrt{2} \alpha' \eta_{\mu \nu} \beta^\mu \alpha^\nu_m \text{ for } m \neq 0. \]

\[ L_0 = \sum_{n > 0} \alpha^\mu_m \alpha^\mu_n \eta_{\mu \nu} + \frac{1}{2} \alpha' \eta_{\mu \nu} \beta^\mu \beta^\mu_0 \eta_{\mu \nu}. \]

\[ \bar{L}_m = \frac{i}{2} \sum_{n \neq 0, m} \bar{\alpha}^\mu_m \bar{\alpha}^\mu_n \eta_{\mu \nu} \eta_{\mu \nu} + \frac{1}{2} \sqrt{2} \alpha' \eta_{\mu \nu} \bar{\beta}^\mu \bar{\alpha}^\nu_m \text{ for } m \neq 0. \]

\[ \bar{L}_0 = \sum_{n > 0} \bar{\alpha}^\mu_m \bar{\alpha}^\mu_n \eta_{\mu \nu} + \frac{1}{2} \alpha' \eta_{\mu \nu} \bar{\beta}^\mu \bar{\beta}^\mu_0 \eta_{\mu \nu}. \]

**Dynamical variables:**

\[ x^\mu, \beta^\mu, \alpha^\mu_m, \bar{\alpha}^\mu_m \]

\[ \alpha^\mu_m - \alpha^\mu_{-m} \overset{\text{even}}{\rightarrow} \bar{\alpha}^\mu_m - \bar{\alpha}^\mu_{-m} \overset{\text{even}}{\rightarrow} | \{ k^\mu_3 \} \rangle \rightarrow \text{A state} \]

\[ p^\mu | \{ k^\mu_3 \} \rangle = k^\mu | \{ k^\mu_3 \} \rangle. \]

\[ \alpha^\mu_n | \{ k^\mu_3 \} \rangle = 0 = \bar{\alpha}^\mu_n | \{ k^\mu_3 \} \rangle \text{ for } n > 0. \]
\[ L_m, \tilde{L}_m : \text{Contribution to } L_m, \tilde{L}_m \text{ due to the modes of } \chi^i. \]

\[ [L_m, L_n] = (m-n) L_{mn} + \frac{e_1}{12} S_{m+n,0} (m^3 - m) \]

\[ [\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{mn} + \frac{1}{12} S_{m+n,0} (m^3 - m) \]

Now consider a string theory based on the following gauge fixed action:

\[ S = -\frac{1}{4\pi \alpha'} \left[ \int d^2 \sqrt{g} \sum_{\mu=0} \eta_{\mu \nu} \partial_\mu x^k \partial_\nu x^l \eta^{kl} \right] + S_{\text{CFT}} \]

\[ S_{\text{CFT}} \text{ describes a conformal field theory of central charge } c \]

\[ (\text{internal degrees of freedom of the string}) \]

\[ \Rightarrow S_{\text{CFT}} \text{ contains symmetry algebra generated by } L_m, \tilde{L}_m : \]

\[ [L_m, L_n] = (m-n) L_{mn} + \frac{e_1}{12} (m^3 - m) S_{m+n,0} \]

\[ [\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{mn} + \frac{1}{12} (m^3 - m) S_{m+n,0} \]
Now quantize the $x^m$'s as usual and define 

$$L_m = \sum_{n=0}^{\infty} L_{m+n} + L_{CFT}^m \quad m \neq 0.$$ 

$$L_0 = \sum_{n=0}^{\infty} L_{m+n} + L_{CFT}$$

$$\tilde{L}_m = \sum_{n=0}^{\infty} \tilde{L}_{m+n} + \tilde{L}_{CFT} \quad m \neq 0.$$ 

$$\tilde{L}_0 = \sum_{n=0}^{\infty} \tilde{L}_{m+n} + \tilde{L}_{CFT}$$

Hilbert space states:

$$\alpha_{-m_1} k_1 \cdots \alpha_{-m_k} k_k \sim \psi_1 \cdots \sim \psi_2 \left| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \left( \sum_{k=1}^{n} \mathcal{O} \right) \right) \right> \times \left| \psi_{CFT} \right>$$

Physical state constraint:

$$L_m |\text{phys}\rangle = 0, \quad \tilde{L}_m |\text{phys}\rangle = 0 \quad \text{for} \quad m \geq 0.$$ 

Q. Does it get rid of the negative states and give a consistent theory?

A. Yes if

$$D + C = 26.$$ 

Thus for a given $n$ we need to choose a conformal field theory with

$$C = 26 - D.$$ 

Stringy "compactification".

Example: for $n = 4$, $C = 22$. Ising model has $C = \frac{1}{2} \implies$ Can use 44 copies of Ising model.
Physical insight:

In the presence of a 2-d metric $\gamma_{ab}$

the action is given by

$$ S = \frac{i}{2 \hbar} \int d^2 x \, \eta_{\mu \nu} \partial_a x^\mu \partial_b x^\nu \sqrt{\det \gamma} \gamma^{ab} 

+ S_{\text{CFT}} (\gamma_{ab}, \ldots) + \text{other fields} $$

$$ \frac{\delta S}{\delta \gamma_{ab}} = \text{const} \left( \sum_{\mu} \gamma^{\mu a} \frac{\delta}{\delta \gamma_{\mu b}} + T_{\text{CFT}} \right) $$

$$ T_{\text{CFT}} (\gamma_{ab}) = \frac{\delta S}{\delta \gamma_{ab} (\gamma)} $$

$\gamma_{ab}$ eqs. of motion

$$ \Rightarrow \Sigma T_{(\mu)}^{(a)} + T_{\text{CFT}} \frac{\delta}{\delta \gamma_{\mu b}} = 0 $$

must be imposed as constraints after
gauge fixing $\gamma_{ab} = \eta_{ab}$

Conformal invariance

$$ \Rightarrow \gamma_{ab} T^{(a)} \frac{\delta}{\delta \gamma_{\mu b}} = 0 $$

$$ \gamma_{ab} T^{(\text{CFT})} \frac{\delta}{\delta \gamma_{\mu b}} = 0 $$

\text{(invariance under $\gamma_{ab} \rightarrow \lambda \gamma_{ab}$)}

\text{two conditions:}

$$ \Sigma T_{\mu \nu} (\gamma_{ab}) + T_{\text{CFT}} (\gamma_{ab}) = 0 \quad \Sigma T_{\mu \nu} (\gamma_{ab}) + T_{\text{CFT}} (\gamma_{ab}) = 0 $$

$$ \Rightarrow \Sigma \delta \gamma_{ab} \text{ the physical state condition:} $$

$$ T_{\text{CFT}} \text{ in eqs.}, \quad T_{\text{CFT}} \gamma_{ab} \text{ in eqs.}$$
Compactification on tori discussed earlier are special cases of the more general compactification.

\[ S = - \frac{1}{4 \pi \alpha'} \int d^2 \sigma \sqrt{\text{det} g} \sum_{\mu \nu = 0}^{D-1} \eta_{\mu \nu} \partial_{\mu} x^\mu \partial_{\nu} x^\nu \eta^{\mu \nu} \]

\[ - \frac{1}{4 \pi \alpha'} \int d\sigma d\alpha' \sum_{i,j=0}^{25} \eta^{ij} \partial_x y_i \partial_{\alpha'} y_j \eta^{i \alpha'} \]

**Internal coordinates**

\[ y^{i \alpha'} = y^{i 2 \pi R_i \alpha'} \]

**Generalize**

\[ S_{\text{eff}} = - \frac{1}{4 \pi \alpha'} \int d\sigma d\alpha' \sum_{i,j=0}^{25} \left( G_{ij} \partial_x y_i \partial_{\alpha'} y_j \eta^{i \alpha'} \right) + B_{ij} \partial_x y_i \partial_{\alpha'} y_j \eta^{i \alpha'} \]

**G, B** : constant matrices

Check invariance under:

\[ y^{i \alpha'}(\sigma, \tau) = y^{i \alpha'}(\sigma, \tau) \]

\[ g_1(\sigma, \tau) + g_2(\sigma, \tau) = f(\sigma + \tau) \]

\[ g_1(\sigma, \tau) - g_2(\sigma, \tau) = g(\sigma - \tau) \]

**Check** invariance under:

\[ S_{\text{eff}} = \frac{1}{4 \pi \alpha'} \int d\sigma d\alpha' \sum_{i,j=0}^{25} \left[ G_{ij} \partial_x y_i \partial_{\alpha'} y_j + B_{ij} \partial_x y_i \partial_{\alpha'} y_j \right] \]

\[ \eta^{i \alpha'} \eta_{i \alpha'} = \delta_{\alpha'} \delta_i \]
Scale invariance:
\[ \nu^0 \rightarrow \lambda \nu^0, \nu \rightarrow \lambda \nu \]
\[ (\sigma \rightarrow \lambda \sigma, \tau \rightarrow \lambda \tau) \]

In general quantum effects may break this symmetry.
- UV divergence \rightarrow regulator
- Broken scale invariance!

But the theory at hand has action quadratic in the fields \( y^i \).

\( \Rightarrow \) free field theory

\( \Rightarrow \) no renormalization necessary.

\( \Rightarrow \) scale invariance is not broken by quantum corrections.

What about the central charge?

For \( g_{ij} = \delta_{ij} \) & \( B_{ij} = 0 \), \( C = 26 - D \).

In the presence of \( G \) & \( B \) one can explicitly construct the \( L_m \) & \( \tilde{L}_m \) & their commutators.

\[ C = 26 - D \] (independent of \( G_{ij} \) & \( B_{ij} \))
A single Ising model

\[ |10\rangle \rightarrow \frac{1}{\sqrt{N}} (|E\rangle + |E\rangle) = |10\rangle \rightarrow \theta \text{ primary} \]

\[ L_{x}^{(\text{Ising})} |10\rangle = 0 \]
\[ |E\rangle = \frac{1}{\sqrt{2}} |10\rangle \]
\[ |10\rangle = \frac{1}{\sqrt{2}} |E\rangle \]

Other states:

\[ \sim |10\rangle \rightarrow L_{-m_{1}} \rightarrow L_{-m_{2}} \rightarrow \sim |10\rangle \rightarrow \text{secondary} \]

Constitutes the full Hilbert space at a single Ising CFT.

A basis state in the Hilbert space of the Ising model.

Direct product of states from each of the $4^4$ sectors.
Light cone gauge compactification

**Non-compact dimension:**

Dynamical variables: $x^\pm, k^\pm, x^m, x^i, \bar{p}_i$

$1 \leq i \leq 25$

**Light cone Hamiltonian:**

$$-\frac{1}{\alpha'} p^+ = \frac{1}{2} \sum_{i=2}^{25} \left( \frac{25}{2} L_i^{(x)} + \frac{25}{2} L_i^{(x)} - 2 \right)$$

$$+ \sum_{i=1}^{25} p_i$$

$$x^n \rightarrow x_i, x^\pm, x_m, x^i, \bar{p}_i \quad (2 \leq i \leq 25)$$

**Light cone Hamiltonian:**

$$-\frac{1}{\alpha'} p^+ = -\frac{1}{\alpha'} (25 \sum_{i=2}^{25} L_i^{(x)} + 25 \sum_{i=2}^{25} L_i^{(x)} - 2)$$

$L_i^{(x)}$ have the same expression in terms of $x_m, x^n, \bar{p}_i$.

**Compactification:**

Take on free bosons $X^0, X^i, \ldots X^{D-1}$ and a CFT of central charge $C$.

**Light cone gauge dynamical variables:**

$x^\pm, k^\pm, x^m, x^i, \bar{p}_i \quad (2 \leq i \leq D-1)$ & the variables of CFT.
Define light-cone Hamiltonian:

\[- \mathbf{p}^\mu = - \frac{\gamma}{\alpha' \beta} \left( \sum_{i=2}^{D-1} \mathcal{L}^{(i)} - \sum_{i=2}^{D-1} \hat{\mathcal{L}}^{(i)} + \mathcal{L}^{(CFT)}(\alpha') + \hat{\mathcal{L}}^{(CFT)}(\alpha') \right) - 2 \]

Does it work?

Need to construct Lorentz generators \( J^{\mu \nu} \) for \( 0 \leq \mu, \nu \leq D-1 \) and check that they satisfy the usual Lorentz algebra

\[ \Delta \xi^\mu \Delta \xi^\nu = \sum_{i=0}^{D-1} \left( \xi^\mu \right)_i \left( \xi^\nu \right)_i + \frac{1}{2} \eta^{\mu \nu} \Delta x^2 \]

Can be done by \( J^{\mu \nu} \) for \( 0 \leq \mu, \nu \leq D-1 \) through

\[ \Delta \xi^\mu \Delta \xi^\nu = \sum_{m=0}^{D-25} \left( \xi^\mu \right)_m \left( \xi^\nu \right)_m + \frac{1}{2} \eta^{\mu \nu} \Delta x^2 \]

(Replace \( \sum_{i=1}^{D-1} \mathcal{L}^{(i)}_m + \sum_{i=1}^{D-1} \hat{\mathcal{L}}^{(i)}_m \) inside the Lorentz generators by \( \mathcal{L}^{(CFT)}_m + \hat{\mathcal{L}}^{(CFT)}_m \) respectively).

The algebra is satisfied if

\[ D + C = 26 \]

Noncompact \( \Rightarrow \) same result as in the covariant gauge.
One loop vacuum energy amplitude function.

Field theory example:

- A free massless scalar field:

\[ S_0 = \frac{1}{2} \int d^4x \left( \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) \quad (\text{Euclidean}) \]

\[ Z = \int d^4\phi \ e^{-S_0} = \int \left[ \prod_k \phi(k) \right] e^{-\frac{1}{2} \sum_k \phi^4 \left( k, k^2 + m^2 \right) \phi \left( -k \right)} \]

\[ = \left( \prod_k \frac{1}{\sqrt{\left( k^2 + m^2 \right)}} \right)^{-\frac{1}{2}} \ e^{-\frac{1}{2} \sum_k \ln \left( k^2 + m^2 \right)} \]

\[ Z = -\frac{1}{2} \left( \frac{\pi m^2}{2} \right)^D \int d^Dk \ \ln \left( k^2 + m^2 \right) \sim \text{UV divergent} \]

\[ \frac{1}{k^2 + m^2} = \int_\infty^0 ds \ e^{-s \left( k^2 + m^2 \right)} \]

\[ \frac{2}{m^2} \ln \left( k^2 + m^2 \right) = \frac{2}{m^2} \int_0^\infty ds \ e^{-s \left( k^2 + m^2 \right)} \]

\[ \ln \left( k^2 + m^2 \right) = -\int_\infty^0 \frac{ds}{s} \ e^{-s \left( k^2 + m^2 \right)} + \text{constant} \]

\[ \sim \ln \left( k^2 + m^2 \right) = -\int_0^\infty \frac{ds}{s} \ e^{-s \left( k^2 + m^2 \right)} \]

\[ \int d^Dk \ \ln \left( k^2 + m^2 \right) = -\int_0^\infty \frac{ds}{s} \ \int d^Dk \ e^{-s \left( k^2 + m^2 \right)} \]

Note: Large \( k \) divergence is regulated.

Large \( k \rightarrow \text{small} \) \( s \)

\[ \rightarrow \ e \sim \text{no UV cut-off}. \]
\[ Z = \frac{V(1/\alpha')}{(2\pi)^{26}} \int \sum_{\text{all states in light cone}} \frac{ds}{2\pi} e^{-\frac{1}{2} \int \frac{ds}{2\pi} \frac{V(\alpha')}{(2\pi)^{26}} \frac{1}{s} } \frac{dV}{dx} \frac{dV}{dx} \]

Note large D value of \( D \)

\[ \approx \text{stronger is the divergence near } \Delta \]

\[ e^{-\frac{1}{2} \frac{1}{\Lambda^2}} \Lambda: \text{UV cut-off mass.} \]

We shall compute one loop vacuum amplitude in string theory and will show that it gives a form identical to the field theory form with two differences:

1. An infinite sum over many terms with different \( m^2 \).

2. Infinite no of particle-like states in string theory.

3. There is a natural lower cut-off to the \( s \)-integral.

- It does not have to be produced by hand.

In light cone gauge: (Naive answer),

\[ m^2 = \frac{1}{\alpha'} (\sum_{m_x} -1) = \frac{1}{\alpha'} (\Sigma_{n_y} -1) \]

\[ \text{for } \alpha - \frac{1}{m_x}, - \frac{1}{m_y}, - \frac{1}{m_z} \]

\[ \Rightarrow Z = \frac{V(1/\alpha')}{(2\pi)^{26}} \int \sum_{\text{all states in light cone}} \frac{ds}{2\pi} e^{-2\pi s (\sum m_x - 1)} - 2\pi s (\sum n_y - 1) - 2\pi s (\sum n_z - 1) \]
\[ \delta_{x_{m_i}, x_{n_j}} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Delta \, e^{2\pi i \theta (\Sigma m_i - \Sigma n_j)} \]

\[ \theta + i \lambda = \tau \quad \text{complex parameter} \]

\[ \Psi = \frac{1}{2} \frac{V(x')^{-13}}{(2\pi)^{26}} \int d^2 \tau \int d^2 \tau' e^{2\pi i \tau (\Sigma m_i - 1)} e^{-2\pi \lambda \tau} (\tau_{n_j} - 1) \]
Details of the computation:

\[ Z = \sum_{\text{gauge}} \int dX^4 \, e^{-S} \] Remove gauge degrees of freedom.

\[ S = \frac{1}{4 \pi^2} \sum \log \det \gamma + \frac{1}{2} \int dX^4 \, \partial_{\mu} X \, \partial_{\nu} X \]

- Euclidean continuation both in space-time and world-sheet.

\( \gamma_{\mu\nu} \) has Euclidean signature.

\( \eta_{\mu\nu} = \delta_{\mu\nu} \)

Note: This computation gives \( Z \) and not \( e^Z \).

\( e^Z \) will require:

\[ 0 + \frac{1}{2} + \frac{1}{3!} \ldots \]
Try to fix a gauge.

Coordinate system: \((\sigma_1, \sigma_2) \equiv (\sigma_1 + 1, \sigma_2) \equiv (\sigma_1, \sigma_2 + 1)\)

(1) A novel parametrization:

\[
\begin{align*}
\sigma_1 & \rightarrow f_1(\sigma_1, \sigma_2) \\
\sigma_2 & \rightarrow f_2(\sigma_1, \sigma_2)
\end{align*}
\]

\[
\begin{align*}
(f_1(\sigma_1 + 1), f_2(\sigma_1 + 1, \sigma_2)) &= (f_1(\sigma_1, \sigma_2) \oplus f_2(\sigma_1, \sigma_2)) + (m, n, l) \\
(f_1(\sigma_1, \sigma_2 + 1), f_2(\sigma_1, \sigma_2 + 1)) &= (f_1(\sigma_1, \sigma_2), f_2(\sigma_1, \sigma_2)) + (m_2, n_2)
\end{align*}
\]

\(f_1, f_2\) are arbitrary functions subject to this restriction.

(2) Weak transformation:

\[
\begin{align*}
\gamma_{\alpha\beta}^{(\sigma_1, \sigma_2)} & \in \Gamma^{(\sigma_1, \sigma_2)} \\
\gamma_{\alpha\beta}^{(\sigma_1, \sigma_2 + 1)} &= \Gamma^{(\sigma_1, \sigma_2)}
\end{align*}
\]

\(\phi(\sigma_1 + 1, \sigma_2) = \phi(\sigma_1, \sigma_2) = \phi(\sigma_1, \sigma_2 + 1)\).
Gauge fixing:

\[ \gamma_{\alpha \beta} = \delta_{\alpha \beta} \]

Not possible.

We can bring an arbitrary \( \gamma_{\alpha \beta} \), via gauge transforms, to the form:

\[ ds^2 = (d\sigma_1 + \tau d\sigma_2)^2 \]

\( \tau \): a complex parameter; \( \text{Im} \tau > 0 \).

\( \tau = i \beta \) \( \Rightarrow \)

\[ ds^2 = (d\sigma_1)^2 + (d\sigma_2)^2 \Rightarrow \gamma_{\alpha \beta} = \delta_{\alpha \beta} \]

But different values of \( \tau \) cannot be related by gauge transforms.

\[ \gamma_{\alpha \beta} = \delta_{\alpha \beta} \text{ cannot be implemented} \]

Residual gauge transforms which preserve the form of the metric:

1. \( \sigma_1 \rightarrow \sigma_1 + a \chi \)
   \( \sigma_2 \rightarrow \sigma_2 + b \)

\[ ds^2 \rightarrow ds^2 \]

The final path integral must be divided by the volume of this gauge:

2. \( \sigma_1 \rightarrow -\sigma_1 \)
   \( \sigma_2 \rightarrow -\sigma_2 \)

3. \( \sigma_1 \rightarrow \sigma_1 + \sigma_2 \)
   \( \sigma_2 \rightarrow \sigma_2 \)

\[ ds^2 \rightarrow |d\sigma_1^2 + (\tau + 1) d\sigma_2|^2 \]

\( \Rightarrow \tau \) and \( \tau + 1 \) describe the same metric.

Restrict \( \tau \) to:

\[ -\frac{1}{2} < \text{Re} \tau < \frac{1}{2} \]
3. \( \sigma_1 \rightarrow \sigma_2 \)
\( \sigma_2 \rightarrow -\sigma_1 \).

\[ ds^2 \rightarrow |d\sigma_2 - \tau d\sigma_1|^2 = |\tau|^2 |d\sigma_1 - \frac{i}{\tau} d\sigma_2|^2 \]

Weyl transform: \( \gamma_{\mu\nu} \rightarrow \gamma^{\prime}_{\mu\nu} = |\tau|^2 \gamma_{\mu\nu} \)

\( \Rightarrow ds^2 \rightarrow |d\sigma_1 - \frac{i}{\tau} d\sigma_2|^2 \)

\( \Rightarrow \tau \) and \( -\frac{i}{\tau} \) describe gauge equivalent metric.

Restrict \( \tau \) to a "fundamental region" unit circle around the origin.

\( \Rightarrow \) Final amplitude:

\[ \int dx_1 dx_2 \times \text{Integrand. } f(\tau, \bar{\tau}) \]

\( \text{Jacobian of } \) trs from \( \exp(\) gauge) to \( d^2 \tau \) (gauge)

\( \Rightarrow \) oX\( ^h \) integral
Thus string theory automatically regularizes the ultraviolet divergences.

Note: For this to work, \( f(x, \tau) \) must be covariant under:

\[ x \rightarrow x + 1, \quad x \rightarrow -\frac{1}{\tau} \]

i.e.

\[ f(x+1, \tau+1) = f(x, \tau) \]

\[ d^2x \cdot f(x, \tau) = d^2(-\frac{x}{\tau}) \cdot f(-\frac{x}{\tau}, \frac{1}{\tau}) \]

\[ \frac{d^2x}{x^4} \]

\( \Rightarrow \) Next \( f(-\frac{x}{\tau}, -\frac{1}{\tau}) = (2\pi)^4 f(x, \tau) \)

- can be explicitly checked.

(Consequence of g.c.i & Weyl invar.)

Now suppose we want to do this for an arbitrary string compactification based on D \( + \) bosonic coordinates fields \( X^k \) and an arbitrary CFT, with \( C = 26 - D \).

\( \Rightarrow \) \( f(x, \tau) \) has to be computed again in light-cone from the spectrum of the theory + gauge.

Q: Is it modular inv.?

If we know for sure that the CFT comes from integral of a local Lagrangian then modular inv. follows from g.c.i & Weyl inv.
Otherwise we need to check it explicitly.

Constraints on which irreps of the Virasoro algebra should be included in the spectrum.
Superstring Theory

Bosonic string theory in covariant gauge:

Action \[ S = -\frac{1}{4\pi\alpha} \int d^2\Sigma \eta^{\alpha\beta} \eta_{\mu\nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu d^2\Sigma. \]

\[ \eta^{\alpha\beta} = \text{diag} (-1, 1), \quad \eta_{\mu\nu} = \text{diag} \begin{pmatrix} -1, 1, 1, \ldots, 1 \end{pmatrix} \]

Constraint: \[ T^{\alpha\beta} = 0 \]

Energy-Momentum tensor

Superstring theory is obtained by supersymmetrizing the action as well as the constraint.

\[ S = -\frac{1}{4\pi\alpha} \int d^2\Sigma \left[ \eta^{\alpha\beta} \eta_{\mu\nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu \right. \]

\[ \left. - i \bar{\psi}^k \gamma^\mu \partial_{\alpha} \psi^k \eta_{\mu\nu} \right]. \]

\( \psi^k \): Two component real fermion:

\[ \psi^k = \begin{pmatrix} \psi_+^k \\ \psi_-^k \end{pmatrix} \]

Exchange \( \pm \) to be consistent with later convention.

\[ P^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad P^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

\( \{ e^0, e^3 \} = -2 \eta^{03} - 2 \sigma \) matrices.
Note: $i \in \mathbb{C}$ is real.

Can take $\psi^{\pm}$ to be real.

Majorana fermion

Supersymmetry:

$\delta x = 0$ under

$\delta \psi^k = \bar{\psi} \gamma^k$

$\delta \psi^k = -i \bar{\psi} \gamma^k x^k \in \mathbb{C}$

$\bar{\psi} = \psi^+ \phi^- \in \mathbb{C}^+$

$\psi^+ = \psi \phi$ anti-commuting

$\rightarrow$ supersymmetry

(Mixes bosons with fermions)

Note:

$[\delta_{e_1}, \delta_{e_2}] \psi^k = \delta_{e_1} \delta_{e_2} x^k - \delta_{e_2} \delta_{e_1} x^k$

$= \alpha^k \bar{\psi} \gamma^k x^k$

$\alpha^k = 2i \bar{\psi} \phi^k \in \mathbb{C}$

$[\delta_{e_1}, \delta_{e_2}] \psi^k = \alpha^k \bar{\psi} \gamma^k$ (using $\psi^a \bar{\psi} \gamma^k = 0$ eqs. of motion)

$\Rightarrow$ commutator of 2 susy brs. gives us translation.
Energy-momentum tensor $T^{ab}$

- Noether current associated with translation invariance.

Super current $j^a \rightarrow 2$ conf. spinor $(\lambda_+^a, \lambda_-^a)$

- Noether current associated with supersymmetry transformation.

A general procedure for computing Noether current:

Suppose there is a continuous global symmetry and let $\lambda$ be the infinitesimal parameter of the symmetry transformation.

$\Rightarrow \delta \lambda S = 0$.

Now, take $\lambda$ to depend on space-time coordinates:

$\lambda = \lambda(\mathcal{X})$

$\Rightarrow \delta_\lambda S = 0$

$= \int d^2 \chi (\partial_a \lambda) \wedge J^a$

- General form:

$J^a$: Noether current
Proof of conservation law of $J^\alpha$:

$\partial_\alpha J^\alpha = 0$

$\delta_{\lambda} S = \int d^2 \delta J^\lambda$

If the field configuration satisfies eqs. of motion then $\delta_{\lambda} S = 0$. to first order for any variation:

$\Rightarrow \delta_{\lambda} S = 0$ for any $\lambda(z)$

$\Rightarrow \partial_\alpha J^\alpha = 0$.

Use this procedure to determine the space-time Noether currents corresponding to translation symmetry and supersymmetry.

Translation symmetry:

$\delta X^l = \alpha^l(\delta) \partial_\alpha X^l$

$\delta \psi^\mu = \alpha^\mu(\delta) \partial_\alpha \psi^\mu$

$\Rightarrow \delta S = -\frac{1}{2i\alpha_l} \int d^2 \delta \left[ \eta^{oo} \eta_{\nu} \partial_\alpha (\alpha^r(\delta) \partial_\alpha X^r) \right]$

$= \int d^2 \delta \partial_\alpha \partial_\lambda \eta^{oo} \eta_{\nu} \partial_\alpha (\alpha^r(\delta) \partial_\alpha \psi^r)$

$= \int d^2 \delta \partial_\alpha \partial_\lambda \eta^{oo} \eta_{\nu} \partial_\alpha (\alpha^r(\delta) \partial_\alpha \psi^r)$

$= \int d^2 \delta \partial_\alpha \partial_\lambda \eta^{oo} \eta_{\nu} \partial_\alpha (\alpha^r(\delta) \partial_\alpha \psi^r)$
\[ T_{\alpha \beta} = C^\mu \left( \partial_\mu \partial_{\nu} \eta_{\mu \nu} + \frac{i}{4} \psi^k \partial_\mu \partial_\nu \psi^k \eta_{\mu \nu} 
 + \frac{2}{3} \overline{\psi}^k \gamma^k \partial_\mu \psi^k \eta_{\mu \nu} \right) - \frac{i}{2} \eta_{\alpha \beta} \mathbf{K} \]

\( \mathbf{K} \neq 0 \text{ is such that } \eta_{\alpha \beta} T_{\alpha \beta} = 0. \)

Similarly we can compute \( J^\alpha \):

\[ J^\alpha = \frac{1}{2} \psi^k \gamma^k \partial_\mu \psi^k \eta_{\mu \nu} \]

We now impose the constraints:

\[ T_{\alpha \beta} = 0, \quad J^\alpha = 0. \]

Note: If we did not impose the constraint \( J^\alpha = 0 \), we would have gotten a compactified bosonic string theory.

\( \psi^k \)'s describe an internal CFT.

\[ C_4 = \frac{D}{2} \quad \text{(can be shown).} \]

We could get

\[ D + \frac{D}{2} = 26 \quad \text{no integer soln. for } D. \]

Even if we had an soln., we would
6. not achieve anything new.

2. The condition \( J^a = 0 \) looks ad hoc.

Proper starting point:

Super symmetrization of gauge invariant action:

\[
S = -\frac{1}{4\pi\alpha'} \int d^2 x \sqrt{-g} \gamma_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}
\]

\( \gamma_{\alpha\beta} \): 2-dim. metric.

Super symmetrization \( \Rightarrow \)

\( \psi^\mu \): Partner of \( X^\mu \).

\( X^\mu \): Partner of \( \gamma_{\alpha\beta} \).

\( \phi \): For each \( \alpha \), \( X^\alpha \) is a 2-component spinor.

\( \Rightarrow S(X^\mu, \psi^\mu, \gamma_{\alpha\beta}, X^\mu) \)

has:

1. General coordinate inv.
2. Weyl.
3. Local supersymmetry
4. Super Weyl
7. Fix gauge

$\gamma_{\alpha\beta} = \delta_{\alpha\beta}, \quad X_\alpha = 0.$

$\gamma_{\alpha\beta}$ eq. $\implies T_{\alpha\beta} = 0$

$X_\alpha$ eq. $\implies J_\alpha = 0$

We shall simply take these constraints as starting points and proceed.

Closed string:

0 \leq \sigma \leq 2\pi

$X^\mu (\sigma, \tau) = X^\mu (\sigma + 2\pi, \tau)$

$\psi_\pm (\sigma, \tau) = \psi_\pm (\sigma + 2\pi, \tau)$

Imagine that the string is moving in a space parametrized by $X^\mu, \psi_\pm$.
We shall take the space to be the quotient of the \( (X, Y) \) space by the brs. \( Y_+ \rightarrow Y_{-1} \) and \( Y_- \rightarrow Y_{-1} \) \((\mathbb{Z}_2 \times \mathbb{Z}_2) \) identify the points \((X, Y^+), (X, -Y^+), (X, Y^-), (X, -Y^-)\).

\[ \Phi \left( (X^{+3}, Y^{+3}) \right) \] must be invariant under \( Y^{+3} \rightarrow -Y^{+3}, Y^{-3} \rightarrow -Y^{-3} \).

\[ \Phi \left( (X^{+3}, Y^{+3}) \right) = \Phi \left( (X^{+3}, \pm Y^{+3}) \right) \]

\[ \Phi \left( (\pm Y^{+3}, Y^{+3}) \right) \]

\[ (-1)^{F_R} \text{ generator of } Y_+ \rightarrow -Y_+ \]

\[ (-1)^{F_L} \text{ generator of } Y_- \rightarrow -Y_- \]

\[ \Phi \left( |\text{phys}\rangle \right) = \Phi \left( |\text{phys}\rangle \right) \rightarrow \text{projection} \]

\[ (-1)^{F_R} \Phi \left( |\text{phys}\rangle \right) = |\text{phys}\rangle \]

We can now consider two / four different kinds of string states:

\[ \psi^\mu \left( \sigma + 2\pi, \tau \right) \rightarrow \psi^\mu \left( \sigma, \tau \right) \rightarrow \{\psi^\mu \} \]

\[ \psi^\mu \left( \sigma + 2\pi, \tau \right) = \psi^\mu \left( \sigma, \tau \right) \rightarrow \{\psi^\mu \} \]

\[ \psi^\mu \left( \sigma + 2\pi, \tau \right) = \psi^\mu \left( \sigma, \tau \right) \rightarrow \{\psi^\mu \} \]

\[ \psi^\mu \left( \sigma, \tau \right) \]
Thus we have two sectors for $\psi^+$

\textbf{NS-sector:} 
\[ \psi^+ (\sigma + 2\pi, \tau) = - \psi^+ (\sigma, \tau) \]

\textbf{R-sector:} 
\[ \psi^+ (\sigma + 2\pi, \tau) = \psi^+ (\sigma, \tau) \]

Similarly, 2 sectors for $\psi^-$

In each sector, we have GSO projection under $(-1)^F_L$ and $(-1)^F_R$.

The full spectrum is obtained by combining the spectrum from these four sectors.

\textbf{Mode expansion:}

\[ \chi^+ (\sigma, \tau) \rightarrow \text{usual} \]

\[ \exists \chi^0, \chi^+, \chi^- \]

\[ [\chi^+, \chi^-] = i \delta^+, \quad [\chi^+, \chi^n] = 8 \sigma m \delta_{m+n,0} \eta^{\mu\nu} \]

\[ [\chi^-, \chi^n] = m \delta_{m+n,0} \eta^{\mu\nu} \]

\[ S_F = \frac{i}{4\pi} \int d\sigma \int d\tau \left[ \psi^+ (\sigma + 2\pi) \psi^- + \psi^+ (\sigma + 2\pi) \psi^- \right] \eta^{\mu\nu} \]

\textbf{Mode expansion:}

\[ \psi^+ = \sqrt{2} b^+ (\tau) \mathbb{1} + i \psi^- \]

\[ \psi^- = \sqrt{2} b^- (\tau) \mathbb{1} - i \psi^+ \]

\[ \gamma \in \mathbb{Z} \text{ in the R sector} \]

\[ \gamma \in \mathbb{Z} + \frac{1}{2} \text{ in the NS sector} \]
\[ S_F = \frac{2 \pi}{\hbar} \sum_{\gamma} \left[ b^\dagger_{\gamma} \left( \partial_\tau + i \kappa \right) b^\gamma + b^\dagger_{-\gamma} \left( \partial_\tau + i \kappa \right) b^\gamma \right] \eta_{\gamma \nu} \]

\[ \Rightarrow \text{Momentum conjugate to } b_{\gamma} \text{ is } \frac{i}{\hbar} \partial_\tau b^\gamma \eta_{\gamma \mu} \text{ for } \kappa \to 0. \]

\[ \frac{i}{\hbar} \partial_\tau b^\gamma \eta_{\gamma \mu} \text{ for } \kappa = 0. \]

Similarly for \( \bar{b}^\gamma \).

Take \( b_{\gamma}^\dagger, b_{\gamma} ; r > 0 \) as coordinates.

\[ b^\dagger_{s-r}, b^{\dagger}_{s+r} ; r > 0 \text{ as momenta.} \]

\[ \{ b_{\gamma}, b_{\gamma}^\dagger \} = \eta_{\gamma \nu} \delta_{\gamma \nu} \delta_{r+s,0} \]

\[ \{ \bar{b}_{\gamma}, \bar{b}_{\gamma}^\dagger \} = \eta_{\gamma \nu} \delta_{r+s,0}. \]

For \( b_{0}, \bar{b}_{0} \), we need special treatment.

\[ \Rightarrow \text{Momentum conjugate to } b_{0} \text{ is } \frac{i}{\hbar} b_{0}, \text{ and } \bar{b}_{0} \text{ is } \frac{i}{\hbar} \bar{b}_{0}. \]

\[ \Rightarrow \text{Coordinates & Momenta are not independent.} \]

\[ \Rightarrow \text{Constrained system.} \]

Compute Dirac bracket of \( b_{0}^\dagger, b_{0} \) with itself.

Replace it by \(-i\) x commutator bracket in quantum theory.
Result:
\[ \{ b^\nu_0, b^\nu_3 \} = \eta^{\nu\nu} \]
\[ \{ \tilde{b}^\nu_0, \tilde{b}^\nu_3 \} = \eta^{\nu\nu} \]
\[ \{ b^\rho_0, b^\rho_3 \} = \eta^{\rho\rho} \]
\[ \{ \tilde{b}^\rho_0, \tilde{b}^\rho_3 \} = \eta^{\rho\rho} \]

\[ \frac{1}{\sqrt{2}} (b^0_0 \pm b^0_3) = x^+ \]
\[ \frac{1}{\sqrt{2}} (b^3_0 \pm i b^3_3) = x^- \]
\[ x^+ = \sum x^+_i, x^- = \sum x^-_i \]

General result:
\[ \{ b^\mu_0, b^\nu_3 \} = \eta^{\mu\nu} S_{\nu+1,0} \]
\[ \{ \tilde{b}^\mu_0, \tilde{b}^\nu_3 \} = \eta^{\mu\nu} S_{\nu+3,0} \]
for all \( \eta \)

\[ H_F = \sum \eta_{\mu\nu} (b^\mu_0 b_3^\nu + \text{constant}) \]

Construction of the Hilbert space:

NS-NS, NS-R, R-NS, RR sectors.

NS-NS sector:

Define \( |k3\rangle \)

\[ P_{k3}(1) |k3\rangle = k_0^\nu |k3\rangle \]

\[ \langle k3 | k_0^\nu = 0 \]

A generic state

\[ | k_1, \ldots, k_3 \rangle \]

GSO projection:

\[ (-1)^{f_2} b^\nu_0 = b^\nu_0 \]
\[ (-1)^{f_3} b^\nu_3 = b^\nu_3 \]
\[ (-1)^{f_4} b^\nu_0 = -b^\nu_0 \]
\[ (-1)^{f_5} b^\nu_3 = -b^\nu_3 \]
\[ (-1) \hat{f}_R |R_3\rangle = 0 - 1 |\bar{R}_3\rangle \quad (-1) \hat{f}_R |R_3\rangle = - |R_3\rangle \]

\[ \Rightarrow \text{NS sector Fock vacuum is projected out by } \hat{a}_0 \text{ projection.} \]

\[ b^\nu \sim b^\mu \quad |\bar{R}_3\rangle \quad \text{is } \hat{a}_0 \text{ allowed} \]

\[ \text{Not } b^\nu |\bar{R}_3\rangle \text{ or } \bar{b}^\nu |R_3\rangle. \]

\[ \text{(At present } (-1) \hat{f}_R, (-1) \hat{f}_R \text{ eigenvalue of } |R_3\rangle \text{ seems ad hoc but it can be derived using BRST formalism).} \]

\[ \text{NS-R sector has some extra complication due to the presence of zero modes.} \]

\[ \Psi_\mu^k : \frac{1}{2} + \text{integer moded} \]

\[ \Psi^\mu : \text{integer moded} \]

\[ \{ b^\mu, b^\nu \} = \eta^\mu_\nu \]

\[ \Rightarrow \text{Clifford algebra.} \]

\[ \text{Quantization } \Rightarrow \text{ a state in the spinor representation of } SO(D-1,1). \]

\[ \text{Covariant group acting on } \Psi. \]
\[\sqrt{2} \left( b_0^1 \pm i \ b_0^2 \right) = \mathcal{O} \ x_1^\pm \]
\[\sqrt{2} \left( b_0^3 \pm i \ b_0^5 \right) = x_2^\pm \text{ etc.} \]

\[\Rightarrow \mathcal{O} \ x_1^\pm \]
\[\{ x_i^-, x_j^+ \} = \delta_{ij} \]
\[\{ x_i^-, x_j^+ \} = \delta_{ij} \]

Define \( 10^\rangle \) s.t. \( x_i^- 10^\rangle = 0 \ \forall \ x_i \)

Other states:

\[ x_i^+ 10^\rangle, \ x_i^- x_j^+ 10^\rangle \]
\[ \{ 10^\rangle, x_i^+ x_j^+ 10^\rangle, x_i^- x_j^- x_k^+ x_l^+ 10^\rangle \} \]

\[ 10^\rangle \rightarrow \text{ (spinor) ref. } \mathcal{R} \text{ of } SO(D-1,1) \]

\[ \{ x_i^+ 10^\rangle, x_i^- x_j^- x_k^+ x_l^+ 10^\rangle \} \]

\[ 10^\rangle \rightarrow \text{ (spinor)' ref. } S' \text{ of } SO(D-1,1) \]

Note:

If \( S \) has \((-1)^F_R = 1\) then \( S' \) has \((-1)^F_R = -1\)

\& \text{ vice versa.}

\Rightarrow \text{ Two choices.}

Also even M.R. ground state can be in \( S \) or \( S' \) representation of \( SO(D-1,1) \).
sector:  

$$ \left( k^3 r \right) \bigotimes \left\{ 1^x_{\alpha} \right\} \bigotimes \left\{ 1^x_{\beta} \right\} \bigotimes \left\{ 1^x_{\gamma} \right\} \bigotimes \left\{ 1^x_{\delta} \right\} \bigotimes \left\{ 1^x_{\epsilon} \right\} .$$

Note: NS-NS, RR: Bosons  
NS-R, R-NS: Fermions

Finally, in each sector we need to impose physical state constraint:

$$\langle \text{phys} | \sum_{n} \left( L^{(o)}_{n} + \tilde{L}^{(o)}_{n} \right) | \text{phys} \rangle' = 0$$  

$$\langle \text{phys} | \sum_{n} \left( \tilde{L}^{(e)}_{n} + \tilde{L}^{(e)}_{n} \right) | \text{phys} \rangle = 0.$$

$$L_{m} = \frac{1}{2} \sum_{n \neq 0, m \neq 0} \alpha_{m-n} \alpha_{n} \eta_{h^2} + \frac{i}{2} \sqrt{2 \alpha_{m-n}} \alpha_{m} \theta_{m-n} \mu$$

$$+ \frac{1}{2} \sum_{n \neq 0} \left( s + \frac{1}{2} m \right) b_{-m} b_{m-n} \eta_{n}$$

$$L_{o} = \sum_{n \neq 0} \alpha_{-n} \alpha_{n} \eta_{h^2} + \frac{1}{4} \eta_{n} \theta_{n} \mu_{n}$$

$$+ \sum_{n \neq 0} \tilde{b}_{-n} \tilde{b}_{n} \eta_{n} \mu_{n}$$

$$L_{o} = \text{normal ordering const.}$$

$$\tilde{L}_{m}, \tilde{L}_{n} : \alpha_{m-n} \rightarrow \alpha_{m-n}, \text{by } b \rightarrow \tilde{b}.$$
\[ G_{\nu} = \sum_{n \neq 0} \alpha_n b_{-n} \eta_{\nu \mu} + \frac{\sqrt{\alpha}}{2} \tilde{b}_{-n} \eta_{\nu \mu} \]

\[ \tilde{G}_{\nu} = \sum_{n \neq 0} \tilde{\alpha}_n \tilde{b}_{-n} \eta_{\nu \mu} + \frac{\sqrt{\alpha'}}{2} \tilde{b}_{-n} \eta_{\nu \mu} \]

**Physical states:**

\[ (\mathcal{L}^n_{\text{phys}}) = 0 = (\mathcal{L}_n_{\text{phys}}) \quad \text{for} \quad n \geq 0 \]

\[ G_{\nu} \mid \text{phys} \rangle = 0 = \tilde{G}_{\nu} \mid \text{phys} \rangle \quad \text{for} \quad n \geq 0 \]

**Acting on**

\[ \alpha^\mu - \alpha_{-n}^\mu \quad - \alpha_{-n}^\nu \quad - \alpha_{-n}^{\nu'} b_{-n} \quad - b_{-n} \]

\[ \tilde{\alpha}_n^\mu - \tilde{\alpha}_{-n}^\mu \quad - \tilde{\alpha}_{-n}^\nu \quad b_{-n} \quad - b_{-n} \]

\[ [k^\mu] \left( \begin{array}{c} 1 \alpha > \\ 0 \end{array} \right) \left( \begin{array}{c} b^n \rangle_R \end{array} \right) \]

---

**1. Constraint:**

\[ \frac{\alpha'}{4} \eta^{\mu \nu} k_n k_\nu + \Sigma \eta m_i + \Sigma \eta \nu_i + C = 0 \]

**2. Constraint:**

\[ \frac{\alpha'}{4} \eta^{\mu \nu} k_n k_\nu + \Sigma \eta m_i + \Sigma \eta \nu_i + \Sigma \lambda_i + \tilde{C} = 0 \]

\[ \Rightarrow \eta^{\mu \nu} k_n k_\nu = \frac{\alpha'}{4} \left( \Sigma \eta m_i + \Sigma \eta \nu_i + \tilde{C} \right) \]

\[ = \frac{\alpha}{\alpha' \eta} \left( \Sigma \eta m_i + \Sigma \eta \nu_i + \tilde{C} \right) \]

\[ \eta_n, \tilde{\eta} : \text{Levels.} \]
Unitarity analysis + consistency of interacting theory.

\[ D = 10 \]

\[ C = -\frac{1}{2} \quad \text{in NS sector} \bigg\} \quad \text{& Right sector} \]

\[ = 0 \quad \text{in R sector} \bigg\} \quad \text{& Left sector} \]

\[ \bar{C} = -\frac{1}{2} \quad \text{in NS sector} \bigg\} \quad \text{Left sector} \]

\[ = 0 \quad \text{in R sector} \bigg\} \quad \text{Right sector} \]

Note: \( |\ell k 3> \) is projected out in NS-NS sector.

\[ \Sigma_{\mu \nu} b^\mu_{-\frac{1}{2}} b^\nu_{-\frac{1}{2}} |\ell k 3>_{\text{NS-NS}} \quad \text{is GSO even} \]

Demanding that there are no negative norm states gives:

\[ C \geq -\frac{1}{2}, \quad \bar{C} \geq -\frac{1}{2} \]

Note: lowest \( (n_\text{max})^2 \) state has 0 mass

Massless states: \( \ell = 0 \) no tachyon

Bosons: NS NS & RR.

\[ m^2 = \frac{g}{\alpha'}, (\Sigma m_\alpha + \Sigma \bar{m}_\alpha - \frac{1}{2}) = 0 \]

\[ = \frac{g}{\alpha'}, (\Sigma \eta_\lambda + \Sigma \bar{\eta}_\lambda - \frac{1}{2}) = 0. \]

\[ \Rightarrow \Sigma m_\alpha = 0, \Sigma \eta_\lambda = 0, \quad \Sigma \bar{m}_\alpha = \frac{1}{2} \quad \text{for one} \ell, \]

\[ \Sigma \bar{\eta}_\lambda = \frac{1}{2} \quad \text{for one} \bar{\ell}. \]
\( \delta_{\mu \nu} \cdot b_{a\frac{1}{2}} \cdot b_{b\frac{1}{2}} \cdot f(R)_{3} \).

- Trace less

- Sym. part of \( f \rightarrow \) gravitational

- Anti-sym. part of \( f \rightarrow \) anti-symmetric tensor field

- Trace part of \( f \rightarrow \) scalar dilaton.

\[ m^2 = \frac{\Lambda}{\alpha}, \ (\sum \alpha_i^2 + \sum \beta_i^2) = 0 \]
\[ = \frac{\Lambda}{\alpha}, \ (\sum \eta_i^2 + \sum \delta_i^2) = 0 \]

\( \Rightarrow \sum \alpha_i = 0, \sum \beta_i = 0, \sum \eta_i = 0, \sum \delta_i = 0 \)

\( \Rightarrow \sum f_{k3} \otimes 1_{\mathbb{R}} \otimes 1_{\mathbb{R}} \) for IIA

\( \Rightarrow \sum f_{k3} \otimes 1_{\mathbb{R}} \otimes 1_{\mathbb{R}} \) for IIB.

IIA: vector, rank 3 anti-symmetric tensor.

IIB: scalar, rank 2 anti-symmetric tensor.

\( \Lambda \) is rank 4 anti-symmetric tensor with self-dual field strength.
Note: \( G_0 = 0, \overline{G}_0 = 0 \) constraint.

\[ \Rightarrow \text{Constraint on } \Phi \text{ linear in } k^\mu. \]

\[ \Rightarrow \text{Polarization tensor } \tilde{\Sigma}_{\alpha \beta} \text{ or } \tilde{\Sigma}_{\alpha \bar{\beta}} \text{ are Fourier transform of field strengths rather than gauge potentials.} \]

IIA: 2 form & 4 form field strength.

IIB: 1 form, 3 form & self-dual 5-form field strength.

\[ \Rightarrow \text{Massless fermions} \]

NS-R & R-NS sector:

\[ \text{NS-R} \]

\[ \frac{1}{\alpha'} (\Sigma m_x + \Sigma X_i) = 0. \]

\[ \frac{1}{\alpha'} (\Sigma X_i + \Sigma s_i - \frac{1}{2}) = 0. \]

\[ \Rightarrow 5^\mu, H_{\mu \nu}^3 \otimes \tilde{b}_{-\frac{1}{2}}, 10_L \otimes 1^a \]

A particle whose polarization tensor has one spinor & one vector index.

Rarita-Schwinger field.

Spin \( \frac{3}{2} \) particle in 4-dim.

Massless spin \( \frac{3}{2} \) particle, gravitino.
\( \frac{\eta}{\lambda}, (\Sigma \eta_x + \Sigma \eta_y - \frac{1}{2}) = 0 \)

\( \frac{\eta}{\lambda}, (\Sigma \eta_x + \Sigma \eta_y) = 0 \).

\[ S_{\mu} \left[ \left( k^a \right) \otimes \frac{1}{2} \right] \otimes \left( \alpha \right) \otimes b^{-\frac{1}{2}} 10 \right]_R \text{ for } \text{IIA} \]

\[ S_{\mu} \left[ \left( k^a \right) \otimes \frac{1}{2} \right] \otimes \left( \alpha \right) \otimes b^{-\frac{1}{2}} 10 \right]_R \text{ for } \text{IIB} \]

- another spin \( \frac{3}{2} \) particle.

Note: In IIB the two spin \( \frac{3}{2} \) particles have the same chirality.

In IIA they have opposite chirality.

One can also show that at each mass level there are equal and no.

of physical bosonic and fermionic states.

\( \Rightarrow \) Space-time Supersymmetry.

Gravitinoes \( \Rightarrow \) Super symmetric partner

of gravitons.

\( \Rightarrow \) Gravitinoes \( \Rightarrow \) \( \mathcal{N} = 2 \) Supersymmetry.

Massless spectrum identical to that of

type IIA and type IIB supergravity theories respectively.
Heterotic String Theory

- Combines superstring theory in the right-moving sector with bosonic string theory in the left-moving sector.

**Digression:** A bosonic string theory in 10 non-compact dimensions.

\[
S = -\frac{1}{4\pi \alpha'} \int d^2 \overline{\zeta} \eta^{\alpha \beta} \eta_{\mu \nu} \partial \overline{\zeta} X^\mu \partial \overline{\zeta} X^\nu \quad (k, \nu = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
\]

\[+ S_{CFT} \]

\[
\frac{2}{\pi \alpha'} \int d^2 \zeta \sum_{I=1}^{N} \lambda^I \overline{\chi}_I \partial \chi_I
\]

\[\lambda^I = \begin{pmatrix} \chi^I_+ \\ \chi^I_- \end{pmatrix} \rightarrow \text{Majorana fermions.} \]

\[N \Rightarrow \text{A conformal field theory} \]

- Energy momentum tensor \( T^{CFT} \)
- Central Charge. For \( (L_0, L_\infty) \)

\[C = \frac{N}{2}. \]
For a consistent bosonic string theory we need

\[ N = 32. \]

What kind of boundary conditions can we put on \( \lambda^\pm \)?

- Depends on what kind of identification we make.

(Constraints due to a requirement of modular invariance)

**Example:**

\[ (\lambda^+_+, \lambda^-_-) \rightarrow (-\lambda^+_+, \lambda^-_-) \]
\[ \rightarrow (\lambda^+_+, -\lambda^-_-) \rightarrow (-\lambda^+_+, -\lambda^-_-) \]

1. **Projection:** \((-1)^\lambda A \) & \((-1)^\lambda R \)

2. **4-sectors:** \((++, (-), (-), (--)) \) on \((\lambda^+_+, \lambda^-_-)\)

We can divide \( \lambda \) into groups of 16 & call them \((\phi^M, \eta^M)\) \(1 \leq M \leq 16\).

Identification: \((\mathbb{Z}_2)^4\)

- \( \phi^M \rightarrow -\phi^M \)
- \( \eta^M \rightarrow -\eta^M \)
- \( \eta^M \rightarrow -\eta^M \)
Separate GSO-like projection for $X \otimes \eta$.

Allow separately NS & R-like bosons $X^+, X^-, \eta^+, \eta^-$.

$2^4 = 16$ sectors.

Other choices possible.

Heterotic string theory.

Two possible modular invariant theory:

1. $\mathfrak{E} \left( X^\mu, \psi^\mu_\pm, \lambda^{I\pm} \right) = (X^\mu, -\psi^\mu_\pm, \lambda^{I+})$

   $= (X^\mu, \psi^\mu_-, -\lambda^{I-})$.

   $\text{SO}(32)$ heterotic. $\text{SO}(32)$ acts on $X^\mu_{\pm}$.

2. $(X^\mu, \psi^\mu_\pm, X^M, \eta^M) = (X^\mu, -\psi^\mu_\pm, X^M, \eta^M)$

   $= (X^\mu, \psi^\mu_-, -X^M, \eta^M)$.

   $E_8 \times E_8$ heterotic.

$\text{SO}(16) \times \text{SO}(16)$ acts on $X^\mu_\pm, \eta^M$. 

Degrees of Freedom

$X^\mu \quad \mu = 0, \ldots, 9$

$\psi^\mu_\pm \quad \mu = 0, \ldots, 9$

$\lambda^I_{\pm} \quad I = 0, 1, \ldots, 32$
Quantization straightforward.

Constraints: \( C_m \), \( \Gamma_m \), \( \Gamma_m \) + \( J^\perp \)

Tap Note: No \( \overline{\Gamma}_m \).

Examine the spectrum of \( SO(32) \) heterotic.

Fock vacuum:

\[ \mathbb{R}^{K^M_3} \times (1u\gamma) \times (1\overline{a}) \]

\[ \text{Spinor rep. of } SO(32) \]

\[ \text{in Ramond sector} \]

\[ \text{Spinor rep. of } SO(9,1) \]

\[ \text{in Ramond sector} \]

Mass shell constraint on

\[ a_{-m}, \ldots, a_{-mK}, \overline{\alpha}_{-n}, \ldots, \overline{\alpha}_{-nL}, b_{-\alpha}, \ldots, b_{-\alpha}, \overline{\eta}_{-\gamma}, \ldots, \overline{\eta}_{-\gamma}, \overline{\gamma}_{-\delta}, \ldots, \overline{\gamma}_{-\delta} \]

\[ 1 + (K^M_3) \times [ \cdots ] \]

\[ m^2 = \frac{g}{\lambda} (\sum m_n + \sum b_n + \overline{c}) \]

\[ = \frac{g}{\lambda} (\sum \overline{c} + \sum \overline{c} + \overline{c}) \]

\[ c = -\frac{1}{2} \text{ in NS}, \ 0 \text{ in R} \]

\[ \overline{c} = -1 \text{ in NS}, \ 21 \text{ in R} \]
GSO projection rule on the right hand sector → usual superstring.

GSO projection rule on the left hand sector (N oscillators)

1. NS sector: Fock vacuum is GSO even.

2. In \( R \) sector, one of the two spinor representations, one is GSO even, the other is GSO odd.

Take 1w) to be GSO even.

Absence of tachyons:

\[ m^2 = \frac{4}{\alpha'} (\Sigma m_x + \Sigma \tilde{A}_x + C) \]

In the \( R \) sector of right moving oscillators, \( C = 0 \).

\[ \Rightarrow m^2 \geq 0. \]

NS sector: \( C = -\frac{1}{2} \).

\[ \Rightarrow m^2 \geq -\frac{2}{\alpha'}; \] There can be a tachyon.

But the NS sector ground state on the right is odd under \((-1)^F\) \( \Rightarrow \) Projected out \( \Rightarrow \) no tachyon.
Massless states:

NS or R sector on the right.

NS sector on the left.

Since $m^2 \geq \frac{g}{2}$, $\omega$ and in the R-sector on the left $\omega \theta = 1$.

\[ b^{-\frac{1}{2}} \sim \chi \quad \frac{1}{2} \{ k^{+3} \} \]

\[ b^{-\frac{1}{2}} \chi^{-\frac{1}{2}} \phi^{-\frac{1}{2}} \frac{1}{2} \{ k^{+3} \} \]

\[(R,N) \quad (N,S - NS) : \quad \text{Bosonic} \]

\[(R,N) \quad (R,R) : \quad \text{Fermionic} \]

Massless states from $(N,S - NS)$ & $(N,S,R)$:

$(N,S - NS)$:

\[ \sum m_\lambda + \sum s_\lambda = \frac{1}{2} \]

\[ \sum \eta_\lambda + \sum s_\lambda = 1 \]

$m_\lambda$, $\eta_\lambda$ integers, $s_\lambda = \frac{1}{2}$ integer.

\[ b^{-\frac{1}{2}} \chi^{-\frac{1}{2}} \{ k^{+3} \} \rightarrow \text{graviton - dilaton - antisym tensor} \]

\[ b^{-\frac{1}{2}} \chi^{-\frac{1}{2}} \{ k^{+3} \} \rightarrow \text{SO(32) gauge bosons} \]
\[ R_0 (\text{NS R}) : \]

\[ \sum m_x + \sum n_x = 0. \]
\[ \sum n_x + \sum \bar{n}_x = 1 \]

\[ m_x, n_x, \bar{n}_x : \text{integer, } \delta x : \frac{1}{2} \text{ integer.} \]

\[ \psi^I_y (k\eta_5) \otimes \lambda^a_R \rightarrow \text{Gravitino.} \]

\[ \psi^{\frac{1}{2}}_y \psi^{-\frac{1}{2}}_y (k\eta_3) \otimes \lambda^a_R \rightarrow \text{Gaugino.} \]

In this theory, there is exact matching of the degeneracy of bosonic and fermionic states at all mass levels.

\[ \Rightarrow \text{space-time supersymmetry.} \]
First quantized bosonic string (closed)

Action: \[ S = -\frac{1}{2} \int d^2 \xi \sqrt{-\det \eta_{\alpha\beta}} \]

\((\xi^0, \xi^1) = (\tau, \sigma)\) - world-sheet coordinates

\(X^\mu\) - space-time coordinates

\[ \eta_{\alpha\beta} = \eta_{\mu\nu} = \text{diag}(-1, 1) \]

\[ \frac{1}{2} \int d^2 \xi \sqrt{-\det \eta_{\alpha\beta}} \text{ area of the world-sheet} \]

\[ T = \frac{1}{2\pi \alpha'} \text{ string tension} \]

Alternative action:

\[ \tilde{S} = -\frac{1}{2} \int d^2 \xi \sqrt{-\det \chi_{\alpha\beta}} \chi^\alpha_{\chi^\beta} \eta_{\mu\nu} \]

\(\chi^\alpha_{\chi^\beta}\) - A metric on the world-sheet

\(\eta_{\mu\nu}\) - New degree of freedom

\(\eta_{\mu\nu}\) - eqs. of motion:

\[ \chi^\alpha_{\chi^\beta} = F(\xi^0, \xi^1) \eta_{\mu\nu} \partial_\xi^\chi X^\mu \partial_\xi^\beta X^\nu \]

Substitute arbitrary \(\xi^\chi\) in the action \(\tilde{S}\)

\(\Rightarrow\) old action \(S\)

Classically, \(S\) and \(\tilde{S}\) are equivalent.
We use $S$ to quantize.

Cancel mass and

- At general coordinates $x^a$
- With coordinates $h_{ab}$, $h_{ab}$

\[ S = - \frac{1}{2} \int d^5 \eta \eta^{ab} \partial_a x^i \partial_b x^j \eta_{ij} \]

- See scalar field theory

But there are constraints

- Not all the $x^a$ and momenta $p_a$ are automatically satisfied
- \[ \Theta^a \equiv \eta_{a b} \partial_b x_i \partial^i - \frac{1}{2} \eta_{ij} \eta^{ab} \partial_a x^i \partial^j - \frac{1}{2} \eta_{ij} \eta^{ab} \partial_a x^i \partial_b x^j \eta_{ij} \]

= 0

Consequence + residual gauge covariance

\[ x^a (u, v) = x^a (g(u), g(v)) \]

arbitrarily functions

A generalization of $\eta^{ab}$ and local $b_i$

Independent constraints: \[ T_{a b} \sim T_{a b} \sim T_{a b} \sim \text{identically} \]

\[ T_{a b} = C \eta_{a b} \partial_a x^i \partial_b x^j - \epsilon \]

\[ T_{a b} = 0 \eta_{a b} \partial_a x^i \partial_b x^j - \epsilon \]
Strategy:

1. Quantize the theory ignoring the constraints.

2. Impose

\[ \langle \text{phys} | T_{\mu \nu} | \text{phys} \rangle = 0 = \langle \text{phys} | T_{\mu \nu} | \text{phys} \rangle \]

Step 1:

\[ X^\mu (\tau, \sigma + i\pi) = X^\mu (\tau, \sigma) \]

\[ \therefore X^\mu (\tau, \sigma) = \sum_{n \neq 0} \phi_n^\mu (\tau) e^{i n \sigma} + x^\mu (\tau). \]

a) Substitute in the action.

b) Define

\[ \Pi_{\mu n} = \frac{S_S}{S(\partial_\tau \phi_n^\nu)} = \frac{1}{\alpha^1} \eta_{\mu \nu} \partial_\tau \phi_{-n}^\nu \]

\[ p_\tau = \frac{S_S}{S(\partial_\tau x^\nu)} = \frac{1}{\alpha^1} \eta_{\mu \nu} \partial_\tau x^\nu \]

\[ \{ \text{conjugate momenta} \}

\[ H = \sum_{n \neq 0} \Pi_{\mu n} \partial_\tau \phi_n^\nu + p_\tau \partial_\tau x^\nu - \tilde{S} \]

c) \[ \Re \text{a } X^\mu \text{ is real} \]

d) \[ \Pi_{\mu n}^* = \Pi_{\mu (-n)}, (\phi_n^\mu)^* = \phi_{-n}^\mu \\
(x^\mu)^* = x^\mu, (p_\tau)^* = p_\tau \]
Quantization:

\[ [\phi_m^n, \Pi_m^n] = i \, \delta_{mn} \, \delta_{\nu \nu}, \quad [\chi^\nu_n, \psi^\mu_n] = i \, \delta^\nu_\mu \]

Define:

\[ \chi_m^n = \frac{1}{\sqrt{2}} \left( \eta^{\mu \nu} \Pi_m^n \sqrt{\alpha'} + \frac{i n}{\sqrt{\alpha'}} \phi_m^n \right) \]

\[ (\tilde{\chi}_m^n) = \frac{1}{\sqrt{2}} \left( \eta^{\mu \nu} \Pi_m^n \sqrt{\alpha'} - \frac{i n}{\sqrt{\alpha'}} \phi_m^n \right) \]

\[ (\chi^\mu_m)^+ = \chi_m^- \quad (\tilde{\chi}^\mu_m)^+ = \tilde{\chi}_m^- \]

**H:**

\[ [\chi_m^n, \tilde{\chi}_m^n] = \eta^{\mu \nu} \delta_{mn}, \quad \eta^{\mu \nu} \]

\[ [\tilde{\chi}_m^n, \tilde{\chi}_m^n] = \eta^{\mu \nu} \delta_{mn}, \quad \eta^{\mu \nu} \]

\[ [\chi_m^n, \tilde{\chi}_m^n] = 0 \]

\[ H = \sum_{n > 0} (\chi_m^n \tilde{\chi}^\mu_n + \tilde{\chi}_m^n \tilde{\chi}^\mu_n) \eta^{\mu \nu} + \frac{\alpha'}{2} \eta^{\mu \nu} \mu \nu \]

+ \text{K}

+ \text{normal ordering}

**Hilbert Space**

Define:

\[ \langle \{ k_n \} \rangle \subset (\mathbb{R}^3)^* \]

\[ \mathcal{B} \{ k_n \} = k_n \{ k_n \}, \quad \alpha_m^k \{ k_n \} = 0 = \tilde{\chi}^h \{ k_n \} \]

**General basis of states**

\[ \chi_{-m_1}^{\alpha_{m_1}} \cdots \chi_{-m_k}^{\alpha_{m_k}} \tilde{\chi}^{\nu_1}_{-n_1} \cdots \tilde{\chi}^{\nu_e}_{-n_e} \{ k_n \} \]
Constraints

\[ T_{\mu \nu} = (\text{const}) \sum L_m e^{-i\sigma} \]

\[ T_{\mu \nu} = (\text{const}) \sum \tilde{L}_m e^{-i\sigma} \]

\[ L_m = \frac{1}{2} \sum_{n \neq 0} \alpha^{r} \alpha^{s} \eta_{\mu \nu} + \frac{1}{2} \sqrt{2} \alpha^{i} \alpha^{m} \eta_{\mu \nu} \text{ for } m \neq 0 \]

\[ L_0 = \sum_{n > 0} \alpha^{r} \alpha^{s} \eta_{\mu \nu} + \frac{1}{4} \eta_{\mu \nu} \eta_{\rho \sigma} \eta_{\rho \sigma} + C \]

Normal ordering constant

\[ \chi = 1 \]

\( \text{[demanding consistency]} \)

Also \( D = 26 \)

\[ \tilde{L}_m = \alpha^{r} \rightarrow \tilde{\alpha}^{r} \]

\[ \tilde{L}_0 = \alpha^{r} \rightarrow \tilde{\alpha}^{r} \]

\[ L_m = L_m^+ \]

Physical state condition:

\[ L_m | \text{phys} > = \tilde{L}_m | \text{phys} > = 0 \]

∃ guarantees \[ < \text{phys} | T_{\mu \nu} | \text{phys} > = 0 \]

\[ < \text{phys} | T_{\mu \nu} | \text{phys} > \]
Acting on $X^{\mu}$, $X^{A\nu}$, $X^{\nu}$, $X^{\nu e}$ (R(K(r)))

$L_{0} = 0$ gives:
\[-\eta^{\mu \nu} k_{\mu} k_{\nu} = \frac{4}{\alpha'} (\Sigma m_{i} - 1)\]

$\tilde{L}_{0} = 0$ gives:
\[-\tilde{\eta}^{\mu \nu} k_{\mu} k_{\nu} = \frac{4}{2\alpha'} (\Sigma \tilde{n}_{i} - 1)\]

\[\Rightarrow \quad (M^{2})^{2} (\text{Mass})^{2}\]

\[\Rightarrow \quad \text{Mass shell constraint:}\]
\[M^{2} = \frac{4}{\alpha'} (\Sigma m_{i} - 1) - \frac{4}{2\alpha'} (\Sigma \tilde{n}_{i} - 1)\]

1. $\Sigma m_{i} = \Sigma \tilde{n}_{i} = 0 \Rightarrow$ tachyons
2. $\Sigma m_{i} = \Sigma \tilde{n}_{i} = 1 \Rightarrow$ gravititon, dilaton, anti-symmetric tensor
3. $\Sigma m_{i} = \Sigma \tilde{n}_{i} > 1 \Rightarrow$ Massive states
We shall make a slight change in notation:

New $L_0 = \text{old } L_0 + 1$
New $\tilde{L}_0 = \text{old } \tilde{L}_0 + 1$

$\mathcal{L}_0 | \{k_n = 0 \}\rangle = 0$
$\tilde{\mathcal{L}}_0 | \{k_n = 0 \}\rangle = 0$

$\Rightarrow$ Consistency:

$$(L_m - \delta_{m,0}) | \text{phys} \rangle = 0 = (\tilde{L}_m - \delta_{m,0}) | \text{phys} \rangle$$

for $m \geq 0$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}$$
$$[\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}$$

2-dim

Conformal algebra with central charge $c = 26$.

We can construct other string theories by replacing any of the bosonic fields $\phi^\mu (\mu = 28 - n, \ldots, 265)$ by a conformal field theory of central charge $n$.

$L_m = L_m^{\text{CFT}} + L_m^{\tilde{\text{CFT}}}$
$\tilde{L}_m = \tilde{L}_m^{\text{CFT}} + \tilde{L}_m^{\tilde{\text{CFT}}}$

- generates the full conformal algebras with central charge $c = 26$. 
Interpretation: $E_{m}^{(\tau=0)}$ generates

$u \to u + E_{m}^{(\tau=0)}u$, $\bar{u} \to u$

$E_{m}^{(\tau=0)}$ generate

$u \to u + E_{m}^{(\tau=0)}u$, $u \to u$

From now on we shall refer by $L_{m}, \bar{L}_{m}$ to the corresponding operators evaluated at $\tau=0$.

Using (17a) we get:

$L_{m}(\tau) = L_{m}(0)E^{\tau i\int_{m}^{+0}}$, $\bar{L}_{m}(\tau) = L_{m}(0)E^{-\tau i\int_{m}^{-0}}$

if we redefine $L_{m}(\tau), \bar{L}_{m}(\tau)$ by absorbing the factor of $E^{\pm\int_{m}}$, then they are time independent and $L_{m}, \bar{L}_{m}$ will refer to these time independent operators.

$T_{uw} = \sum \sum L_{m} E^{\tau i\int_{m}^{+0}} - \sum \sum L_{m} E^{-\tau i\int_{m}^{-0}}$

$T_{uw} = \sum \sum \bar{L}_{m} E^{\tau i\int_{m}^{+0}} - \sum \sum \bar{L}_{m} E^{-\tau i\int_{m}^{-0}}$

Each element $z_{e}$:

$z \to z$

$\omega = (z^{\frac{e}{i\pi}})$, $\bar{\omega} = (\bar{z}^{\frac{e}{i\pi}})$

$u \to -i\omega$, $\bar{u} \to -i\bar{\omega}$

$T_{uu} = \sum L_{m} E^{-\tau i\int_{m}}$, $T_{\bar{u}u} = \sum \bar{L}_{m} E^{\tau i\int_{m}}$

$T_{u\bar{u}} = T_{\bar{u}u} = 0$
\[ Z = e^\omega, \quad \tau_{zz} = \left( \frac{d\omega}{dx} \right)^2 \quad \tau_{ww} = e^{-2\omega} \tau_{ww} \]
\[ \omega = \sum \frac{2}{m} \tau_{w} \]
\[ \tau_{zz} = \left( \frac{d\omega}{dx} \right)^2 \quad \tau_{zz} = \sum \frac{2}{m} \tau_{w} \]
Bosonic string theory

A 2-dimensional CFT with central charge $c = 26$.

\[ [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} \]

\[ [L_m, \bar{L}_n] = (m-n) \bar{L}_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} \]

Complex coordinate on the cylinder:

\[ \omega = (\tau + i\sigma), \quad \bar{\omega} = (\bar{\tau} + i\bar{\sigma}) \]

\[ T_{\omega \bar{\omega}} = \sum L_m e^{-m\omega}, \quad T_{\bar{\omega} \bar{\omega}} = \sum \bar{L}_m e^{-m\bar{\omega}} \]

\[ T_{\omega \bar{\omega}} = 0 = T_{\bar{\omega} \omega} \]

\[ \tau : \text{Euclidean time.} \]

Physical state:

\[ L_n |\text{phys}\rangle = \bar{L}_n |\text{phys}\rangle = 0 \quad \text{for } n > 0 \]

\[ (L_{0-1}) |\text{phys}\rangle = 0 = (\bar{L}_{0-1}) |\text{phys}\rangle \]

A basis of physical states

What we want to compute is the $S$-matrix:

\[ S(\beta_{n_1}, \beta_{n_2}, \ldots, \beta_{n_k}) \]

Note: We are not distinguishing between incoming and outgoing particles.

\[ k_0 > 0 : \text{outgoing} \]

\[ k_0 < 0 : \text{incoming} \]
Suppose \( \{Q, \gamma \} \) denotes the collection of all elementary fields in terms of which the CFT is defined.

E.g. if we are dealing with the 26-dimensional bosonic string, then \( \{Q, \gamma, x^0, \ldots, x^{25} \} \).

Now every state in CFT (physical or not) can be given a wave-function representation:

\[ \{ |\gamma_k \rangle \} \text{ Complete set of states of CFT} \]

\[ |\gamma_k \rangle \rightarrow \Phi_{|\gamma_k \rangle} \left( \{ Q(x^0) \} \right) \]

A loop in \( q \)-space.

Configuration space \( \rightarrow \) Loop space

Note: it does not enter here.

Analogous with quantum mechanics. A complete set of basis states:

\[ \rightarrow \Phi_{n}(x) \]

Complete set of functions of the coordinate.

Thus, \( \{ |\gamma_k \rangle \} \subset \{ |\gamma_k \rangle \} \), we have

\[ |\gamma_k \rangle \rightarrow \Phi_{|\gamma_k \rangle} \left( \{ Q(x^0) \} \right) \]
Prescription for N-point S-matrix in $S(15_i, \ldots, 1A_n)$

$= S[\infty \ell_i(\sigma)] \cdot [\infty \ell_n(\sigma)] \cdot \Psi_{\xi_1}(\{\ell_1(\sigma)\}) \cdot \Psi_{\xi_2}(\{\ell_2(\sigma)\}) \cdots \Psi_{\xi_n}(\{\ell_n(\sigma)\}) \cdot K(\{\ell_1(\sigma)^2, \ell_2(\sigma)^2, \ldots, \ell_n(\sigma)^2\})$

$= \int \frac{d^4 \phi}{V(\text{gauge})} e^{-S(\phi)} \bigg| \phi = \bigcup_{i=1}^{N} D_{\xi_i}$

A choice of world-sheet conditions is shown.

Promising once and for all in a given coordinate system. (Choice of coordinate system that we use unit disk)

If we can do it if there are no furniture.

\[ ds^2 = 1 d\tilde{z}^2 \]

2 complex coordinate on the plane.

Residual gauge invariance.

Define an $SL(2, \mathbb{C})$ transformation

\[ z \rightarrow \frac{a z + b}{c z + d} \quad a, b, c, d \text{ complex and } ad - bc = 1 \]

\[ 1 \rightarrow \text{tr} \left[ T \right] \text{ on } \mathbb{C}^2 + \infty \]
\[ |d\tilde{z}|^2 = \frac{|dz|}{|z^2+d|^2} \]

Weyl tras: \[ ds^2 \rightarrow |z^2+d|^{-2} ds^2 \]

\[ \Rightarrow \text{New metric is } |dz|^2 \]

Thus \[ z \rightarrow z + \frac{a z + b}{c z + d} \] together with a Weyl tras. is a residual gauge tras. which has not been fixed by the gauge condition \[ \delta_{ab} = \delta_{ab} \]

- \# of parameters: 3
  - \( (a, b, c, d) \rightarrow (\lambda a, \lambda b, \lambda c, \lambda d) \)
  - Same tras.

Normalize: \[ ad - bc = 1 \]

Now return to the case with punctures.

Suppose we have \( N \) punctures.

We can always use a \( 2 \)-d j.t. to have them centered around some predetermined points \( z_i, \ldots, z_N \).

But after this the \( N \)-allowed group of different morphisms is smaller, since we are only allowed different morphisms which do not move the points \( z_i, \ldots, z_N \).

\[ \Rightarrow \text{Cannot fix } \delta_{ab} = \delta_{ab} \]
Instead we fix $g_{\text{ap}} = S_{\text{ap}}$ but now integrate over the locations of the disk centers $z_i$.

(This should really be treated as hidden degrees of freedom of the metric).

Note: We only need to integrate over $(N-3)$ of these coordinates, since we have enough gauge degrees of freedom to fix $3 = 3 - (N-3)$ of the centers and the metric to $S_{\text{ap}}$.

\[
\mathcal{K} = \left\{ \{q_i(\sigma)\}_{i=1}^{3}, \ldots, \{q_n(\sigma)\}_{i=1}^{3} \right\}
\]

\[
\mathcal{K} = \sum_{i=1}^{N-3} d^2 z_i F(z_1, \ldots, z_N) [\alpha q] e^{-S(q)} \bigg|_{C = U \cup D_\xi} \bigg|_{q = q_i(\sigma) \text{ on } D_\xi}
\]

Integration measure

(Jacobian of the from)

\[
\frac{[\alpha q]}{\text{Vol } \text{Gauge}} \rightarrow \frac{1}{(N-3)!} d^2 z_i
\]

\[
F(z_1, \ldots, z_N) = \left| z_1 - z_2 \right|^2 \left| z_2 - z_3 \right|^2 \left| z_3 - z_4 \right|^2
\]

Consistency check: $\mathcal{K}$ should be independent of the choice of $z_1, z_2, z_3$. The final
\{ \psi_{\alpha}^{(n)} \} : Basis of physical states in CFT

\psi_{\alpha}^{(n)}(\{\varphi(\sigma)\}). The functional describing the wave function of the state \langle \psi_{\alpha}^{(n)} \rangle

\Phi : set of basic fields describing the CFT

x^0, \ldots, x^{25} for bosonic string theory in flat space.

S-matrix:

\begin{align*}
S(\langle \psi_{\alpha}^{(n)} \rangle, \ldots, \langle \psi_{\alpha}^{(n)} \rangle) & = S[\varphi_1(\sigma)] S[\varphi_2(\sigma)] \cdots S[\varphi_{25}(\sigma)] \\
\psi_{\alpha_1}^{(n_1)}(\{\varphi_i(\sigma)\}) & \to \psi_{\alpha_n}^{(n)}(\{\varphi_i(\sigma)\}) \\
K(\{\varphi_i(\sigma)\}) & \to \{\varphi_i(\sigma)\}
\end{align*}

\begin{align*}
\int \frac{[\varphi]}{\text{vol (gauge)}} e^{-s(\varphi, g)} & \in \mathcal{C}_n - \cup_{x} \partial D_{\infty} \\
\int d^2 z_1 \cdots d^2 z_n |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2 & \in \mathcal{C}_n - \cup_{x} \partial D_{\infty} \\
\int [\varphi] e^{-s_{\varphi}} & \in \mathcal{C}_n - \cup_{x} \partial D_{\infty} \\
\varphi = \varphi_i(\sigma) & \text{ on } \partial D_{\infty}
\end{align*}
Next we need to find a convenient description of $\Psi_{\Phi}(\Phi(\sigma)^3)$.

Suppose $\{G_{\Phi}(2,\tau)\}$ denote a complete set of local operators constructed out of $G$ and its derivatives at $(2,\tau)$.

Define the following object:

$$\Phi_{\Phi}(\{\Phi(\sigma)^3\}) = \{ [x'q] \in \mathcal{H} : G_{\Phi}(0) \text{ unit disk } D \text{ around } 0 \}$$

$\Phi = \Phi(\sigma) \text{ mod } 2\pi$.

$\Rightarrow$ A functional $\Phi_{\Phi}(\{\Phi(\sigma)^3\})$.

Interpret this as a wave function of some state $|\Phi_{\Phi}\rangle$.

Fundamental Postulate of CFT: There is 1-1 correspondence between the set of local operators & set of states.

$|\Psi_k\rangle \Leftrightarrow V_{\Psi_k}(2,\tau)$ such that

$$\Psi_{\Psi_k}(\{\Phi(\sigma)^3\}) = \{ [x'q] \in \mathcal{H} : V_{\Psi_k}(0) \text{ unit disk } D \text{ around } 0 \}$$

$\Phi = \Phi(\sigma) \text{ mod } 2\pi$.

Special case:

$$\Psi_{\Phi_n}(\{\Phi(\sigma)^3\}) = \{ [x'q] \in \mathcal{H} : V_{\Phi_n}(0) \text{ unit disk } D \text{ around } 0 \}$$
Now substitute into the expression for the S-matrix:

\[
S(1 \phi_1, \ldots, 1 \phi_n) = \int d^2 \bar{z}_1 \ldots d^2 \bar{z}_n \left| z_1 - z_2 \right|^2 \left| z_2 - z_3 \right|^2 \left| z_3 - z_4 \right|^2 \\
\left[ \phi_1 (\bar{z}_1) \right]_{D_1} \ldots \left[ \phi_n (\bar{z}_n) \right]_{D_n} \left[ \phi \right]_{D_1} e^{-S(\phi)} \left| \psi_1 \right|_{D_1} \left| \psi_n \right|_{D_n} \\
\left[ \phi \right]_{D_1} e^{-S(\phi)} \left| \psi_1 \right|_{D_1} \left| \psi_n \right|_{D_n} \\
\left[ \phi \right]_{D_n} e^{-S(\phi)} \left| \psi_n \right|_{D_n} \\
\left[ \phi \right]_{D_1} e^{-S(\phi)} \left| \psi_1 \right|_{D_1} \left| \psi_n \right|_{D_n} \\
\left[ \phi \right]_{D_1} e^{-S(\phi)} \left| \psi_1 \right|_{D_1} \left| \psi_n \right|_{D_n} \\
\left[ \phi \right]_{D_1} e^{-S(\phi)} \left| \psi_1 \right|_{D_1} \left| \psi_n \right|_{D_n}
\]

Note: \( \phi_n(\bar{z}) \): Boundary value for \( \phi \) inside \( D_1 \) as well as that of \( \phi \) in \( D_1 \). Unrestricted integrals over \( \phi \) over the full complex plane.

\[
S(1 \phi_1, \ldots, 1 \phi_n)
\]

\[
= \int d^2 \bar{z}_1 \ldots d^2 \bar{z}_n \left| z_1 - z_2 \right|^2 \left| z_2 - z_3 \right|^2 \left| z_3 - z_4 \right|^2 \\
\left[ \phi_1 (\bar{z}_1) \right]_{D_1} \ldots \left[ \phi_n (\bar{z}_n) \right]_{D_n} \left[ \phi \right]_{\bar{D}_1} e^{-S(\phi)} \left| \psi_1 \right|_{\bar{D}_1} \left| \psi_n \right|_{D_n} \\
= \int d^2 \bar{z}_1 \ldots d^2 \bar{z}_n \left| (z_1 - z_2) (z_2 - z_3) (z_3 - z_4) \right|^2 \\
\left< \psi_1 \right|_{\bar{D}_1} \left| \psi_n \right|_{D_n} \\
\left< \psi_1 \right|_{\bar{D}_1} \left| \psi_n \right|_{D_n} \\
\left< \psi_1 \right|_{\bar{D}_1} \left| \psi_n \right|_{D_n}
\]

Correlation function of vertex operators in CFT.
we consider quantizing the theory, this would mean that the no function vanishes. After forcing $E_1, E_2$, there is a residue gauge invariance. e.g., $N = 2, E_2 \rightarrow E_2 + \frac{2}{E_2 - 2}$

$\text{Integrand} \rightarrow E_1 \cdot$ Complex.

$\text{Conformal law:} \frac{d\theta}{d\phi} = ab$.
N=2: Two point function:

Recall the rules for computing S-matrix:

a) Take N-point Green's f.

b) Multiply it by \((p^2 - m^2)\) for each external leg.

c) Set \(p_i^2 = m^2\).

For \(N=2\):

a) Two pt. Green's f.: \(f = \frac{1}{p^2 - m^2}\)

b) Multiply by \((p^2 - m^2)(p^2 - m^2)\).

\[\Rightarrow (p^2 - m^2)\]

c) Set \(p^2 = m^2 \Rightarrow 0\).

Thus, one-shall 2-point S-matrix is expected to vanish as is the case here.
How do we find the state corresponding to a given vertex operator, and vice versa?

Consider the correlation for of a set of local operators on the plane:

$$\langle T \prod \hat{G}_{\chi}(z_i, \bar{z}_i) \rangle = \int \mathcal{D}q \, e^{-S(q)} \prod \hat{G}_{\chi}(z_i, \bar{z}_i)$$

We want to give this an interpretation in terms of a vev.

In a normal field theory, this would be

$$\langle 0 \mid T \left( \prod \hat{G}_{\chi}(z_i, \bar{z}_i) \right) \mid 10 \rangle$$

Time order:

$$\int_{-\infty}^{\infty} \text{Im}(z) \quad \text{or} \quad \text{Re}(z)$$

In CFT we have other options:

$$w = e^{i \phi}$$

This defines a cylinder

Take (Im \(w\)) = time = \(e^{i\phi}\) on \(\mathbb{C} \times [0, 2\pi]\)

Time ordering ↔ ordering in \(\mathbb{C} \times [0, 2\pi]\)

(radii ordering)

$$\langle T \prod \hat{G}_{\chi}(z_i, \bar{z}_i) \rangle = \langle 0 \mid R(\hat{G}_{\chi}(z_i, \bar{z}_i)) \mid 10 \rangle$$

operators' acting on
How do we find a vertex operator $V$ corresponding to a given state $|\psi\rangle$?

To achieve this, we need to study some general properties of vertex operators:

Define coordinate $Z = e^{w}$. Let $w \to \infty$:

$w \to -\infty \Rightarrow Z \to 0$.

$w \to \infty \Rightarrow Z \to \infty$.

$E_{-} L_{n}$ generates:

$\tilde{w} \to \tilde{w} + E_{n} e^{\tilde{w}} \Rightarrow Z = e^{w} \Rightarrow Z \to e^{\infty}$.

$\overline{E}_{-} \overline{L}_{n}$ generates:

$\tilde{\omega} \to \tilde{\omega} + \overline{E}_{n} e^{\tilde{\omega}} \Rightarrow Z \to \infty$.

Consider a vertex operator $V(\omega, \overline{\omega})$.

It is a primary vertex operator if under a conformal transformation $w \to f(w), \tilde{w} \to f(\tilde{w})$:

$V(\omega, \overline{\omega}) \Rightarrow \tilde{V}(\tilde{\omega}, \overline{\tilde{\omega}}) = (f'(\omega))^{h} (f'(\tilde{w}))^{\overline{h}} V(f(\omega), f(\tilde{w}))$.

$V(f(\omega), f(\tilde{w})) = e^{\tilde{\omega} + E_{n} e^{\tilde{\omega}}}$.

$V(\omega, \overline{\omega}) - V(\omega, \overline{\omega}) = (1 + E_{n} e^{\tilde{\omega}})^{h} (1 + \overline{E}_{n} e^{\tilde{\omega}})^{\overline{h}}$.

$\{ L_{n}, V(\omega, \overline{\omega}) \} = n e^{nw} V(\omega, \overline{\omega}) + n e^{nw} \tilde{V}(\omega, \overline{\omega})$.
Define a new coordinate system: 

\[ z = e^{\omega t} \]

\[ \mathbf{V}_2(\omega, \mathbf{r}) = \left( \frac{d\omega}{dt} \right)^{\frac{1}{2}} \left( \frac{d\mathbf{r}}{dt} \right)^{\frac{1}{2}} \mathbf{V}(\omega, \mathbf{r}) \]

\[ = e^{-\omega t} e^{\omega t} \mathbf{V}(\omega, \mathbf{r}) \]

\[ = e^{-\omega t} e^{\omega t} \mathbf{V}(\omega, \mathbf{r}) \]

\[ = e^{-\omega t} e^{\omega t} [n \mathbf{e} e^{\imath \omega t} \mathbf{e}^{\omega t} \mathbf{V}_z + e^{\imath \omega t} \mathbf{e}^{\omega t} \mathbf{V}_z] \]

\[ = \{ (n+1) \hbar \mathbf{Z} \mathbf{V}_{\text{int}} + \mathbf{Z}^{n+1} \mathbf{e} \mathbf{V}_S \} \]

Similarly,

\[ \left[ \mathbf{L}_n, \mathbf{V}_z \right] = \left[ (n+1) \hbar \mathbf{Z} \mathbf{V}(\omega, \mathbf{r}) + \mathbf{Z}^{n+1} \mathbf{e} \mathbf{V}_z \right] \]

Now consider the state \( \mathbf{V} \rangle = \mathbf{V}_2(0) \mathbf{10} \).

\[ L_n \mathbf{V} \rangle = [L_n, \mathbf{V}(0)] \mathbf{10} \] for \( n > 0 \)

\[ = 0 \text{ for } n > 0 \]

\[ = \hbar \mathbf{V}(0) \mathbf{10} + \hbar \mathbf{V} \] for \( n = 0 \)

\[ L_n \mathbf{V} \rangle = 0 \text{ for } n > 0 \]

\[ L_n \mathbf{V} \rangle = \hbar \mathbf{V} \]

Similarly:

\[ L_0 \mathbf{V} \rangle = 0 \text{ for } n > 0 \]

\[ L_0 \mathbf{V} \rangle = \hbar \mathbf{V} \]

\[ \mathbf{V} \text{ primary states with weight } (\hbar \mathbf{1}) \]
Now suppose we have a relation:

\[ S[\phi] e^{-S[\phi]} T \tilde{G}_i(t_i, \bar{z}_i) = \langle \hat{\Psi}_i (t_i, \bar{z}_i) | \hat{v}(0) \rangle \]

On the one hand:

\[ \langle \prod_{i} \tilde{G}_i(t_i, \bar{z}_i) | v(0) \rangle = \langle 0 | R(\prod \tilde{G}_i(t_i, \bar{z}_i)) | \hat{v}(0) \rangle \]

\[ = \langle 0 | R(\prod \hat{G}_i(t_i, \bar{z}_i)) | \hat{v}(0) \rangle \]

On the other hand:

\[ S[\phi] e^{-S[\phi]} T \tilde{G}_i(t_i, \bar{z}_i) | v(0) \rangle \]

\[ = S[\phi_0(\sigma)] \left( \prod_{i} \tilde{G}_i(t_i, \bar{z}_i) \right) | v(0) \rangle \]

\[ = \langle 0 | R(\prod \hat{G}_i(t_i, \bar{z}_i)) | \hat{v}(0) \rangle \]

Compare both sides:

\[ \Rightarrow | v \rangle = \hat{v}(0) | 0 \rangle \]
Physical states:

Primary states of weight \( (1,1) \)

Primary, vector operators of weight \( (1,1) \)

The S-matrix:

\[
S(V_1, \ldots, V_n) = \int d^2 z_1 \ldots d^2 z_n \left| z_1 - z_2 \right|^2 \left| z_2 - z_3 \right|^2 \left| z_1 - z_2 \right|^2 \left| z_1 - z_3 \right|^2 \sum_i \langle \prod_{k=1}^{n} V_k \left( \bar{z}_k, \bar{z}_k \right) \rangle
\]

We shall now find the vertex operators for some physical states explicitly in 26-dimensional bosonic string theory in flat space-time.

Mode expansion of $X^k$.

$X^k(x, \tau) = \sum_{n \neq 0} \Phi_n(\tau) e^{i n \tau} + \tilde{X}^k(\tau)$

1. Express $\Phi_n(\tau)$ in terms $X^k_n(\tau), \tilde{X}^k_n(\tau)$.

2. Use eqs. of motion to express $X^k_n(\tau), \tilde{X}^k_n(\tau)$ in terms of $X^k_n(0) = x^k_n$, $\tilde{x}^k_n(0), x^k_0, p^k_0$.

3. Make Euclidean continuation $\tau \to i\tau$.

$\tau = \tau - i\pi$.

$X^k(0, \tau) = x^k + \frac{m^2}{2} \frac{\lambda}{i} \frac{1}{n} (x^k_n e^{-in\tau} + \tilde{x}^k_n e^{in\tau})$

$+ i \sqrt{\frac{\alpha'}{2}} \sum_{\pm} \frac{1}{n} (x^k_n e^{-in\tau} + \tilde{x}^k_n e^{in\tau})$

$= \sum x^k + \frac{m^2}{2} \frac{\lambda}{i} \frac{1}{n} (x^k_n e^{-in\tau} + \tilde{x}^k_n e^{in\tau})$

$\equiv \Lambda^k_n X^k_n$

$= \sum_{k,n} \Lambda^k_n X^k_n$

$\equiv \sum_{k,n} \Lambda^k_n X^k_n$

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$\equiv \sum_{k,n} \Lambda^k_n X^k_n$
\[ e^{i k \cdot x} |z = 10\rangle = e^{i k \cdot x_n} |10\rangle = |k\rangle \]

Note: \[ \hat{X}_< (z, \bar{z}) |10\rangle = 0 \quad \hat{P}_n |10\rangle = 0 \]
\[ \hat{X}_< (z, \bar{z}) = 0 \quad \text{at} \quad z = 0 \]

\[ e^{i k \cdot x} \hat{X}_<(z, \bar{z}) \] is the vertex operator associated with a tachyon state of momentum \( k \). \( e^{i k \cdot x} \) is not allowed since it violates the bound:

\[ \exists \chi^\mu = \frac{\alpha'}{2} p^\mu z^{-1} \quad 2 \sqrt{\chi^\mu} \geq x_n^\mu \]

Consider:

\[ S_{\mu \nu} = e^{\hat{X}_\mu} \bar{e}^{\hat{X}_\nu} e^{i k \cdot \hat{x}} \]

\[ S_{\mu \nu} = \left( e^{\hat{X}_\mu} \bar{e}^{\hat{X}_\nu} : e^{i k \cdot \hat{x}} : + e^{\hat{X}_\mu} e^{i k \cdot \hat{x}} \bar{e}^{\hat{X}_\nu} \right) \]

\[ + e^{\hat{X}_\mu} e^{i k \cdot \hat{x}} e^{\frac{\alpha'}{2} k \cdot \hat{p} \cdot z^{-1} (\alpha' \hat{p}^\nu z^{-1})} \]

\[ + e^{i k \cdot \hat{x}} e^{\hat{X}_\nu} e^{\frac{\alpha'}{2} k \cdot \hat{p} \cdot z^{-1} (\alpha' \hat{p}^\nu z^{-1})} e^{i k \cdot \hat{x}} \]

\[ + \ldots \]

\( S_{\mu \nu} : \hat{x}^\mu \hat{x}^\nu \in e^{i k \cdot \hat{x}} : (0^2 = 0, \vec{z} = 0) 10 \) \\
\( = S_{\mu \nu} \hat{x}_<(^0) \hat{x}_<(^0) \in e^{i k \cdot \hat{x}} : (0) 10 \) \\
\( \approx S_{\mu \nu} \left( -\frac{\alpha'}{2} \right) \hat{x}_<^- \hat{x}_<^- \| k^3 \) \\

Thus the vertex operators for the massless states are given by:

\( S_{\mu \nu} : \hat{x}^\mu \hat{x}^\nu \in e^{i k \cdot \hat{x}} \)

Physical state condition:

\( k \cdot S_{\mu \nu} = 0 : \quad k \cdot S_{\mu \nu}, \quad k^2 = 0. \)

We shall now try to compute amplitudes using these vertex operators:

N: tachyon amplitude:

\[ A(k_1, \ldots, k_n) \]

\[ = \sum_{n} \prod_{i=1}^{n} d^2z_i \left| z_i - z_i\right|^2 \left| z_i - z_{i+1}\right|^2 \left| z_i - z_3\right|^2 \left| z_i - z_4\right|^2 \]

\[ \left\langle \prod_{i} e^{i k_i \cdot x_i} (z, \bar{z}) \right\rangle \]

6

Compute this using path integral.
Strategy:

1. Evaluate:
\[
\langle \epsilon \rangle \left\langle \int \frac{d^2 z}{z} \times \epsilon(z, \bar{z}) \right\rangle
\]
\[
= \int \left\{ \epsilon \epsilon, \int d^2 z \cdot \epsilon \right\} \epsilon \epsilon
\]
\[
+ i \int \int \mu(z, \bar{z}) \times \epsilon(z, \bar{z}) \epsilon \epsilon
\]

2. Substitute:
\[
J_{\mu}(z, \bar{z}) = \frac{1}{\theta x} k_{2\mu} \delta(z - z_2) \delta(\bar{z} - \bar{z}_2)
\]
So that:
\[
i \int J_{\mu}(z, \bar{z}) \times \epsilon(z, \bar{z}) d^2 z \epsilon \epsilon
\]
\[
= i \sum_{k_{2\mu}} k_{2\mu} \times \epsilon(z, \bar{z})
\]

3. The evaluation of this integral is standard:
\[
\langle \epsilon \rangle \left\langle \int \frac{d^2 z}{z} \times \epsilon(z, \bar{z}) \right\rangle
\]
\[
= \int \left\{ \epsilon \epsilon, \int d^2 z \cdot \epsilon \right\} \epsilon \epsilon
\]
\[
- \int \int d^2 z' \cdot d^2 z \Delta_{\mu\nu}(z, z') J(z) J(z')
\]
\[
\Delta_{\mu\nu}(z, z') = -\alpha' \ln |z - z'| \quad \text{Propagator of 2 dim. free scalar field}
\]

4. 
\[
\langle \prod \epsilon k_{2\mu} \times \epsilon(z, \bar{z}) \rangle = e^{i \alpha' \sum k_{2\mu} k_{2\nu} \ln |z - z'|}
\]
Normal ordering: Removes the $k \cdot k'$ terms.

\[ \langle T(k) \rangle = e^{i k \cdot x} \langle \sum_{k \ll \bar{k}'} k \cdot k' \rangle \ln |z_k - z_{\bar{k}'}| \]

\[ = e^{i \alpha' \sum_{k \ll \bar{k}'} k \cdot k' \ln |z_k - z_{\bar{k}'}|} \]

\[ = \prod_{k \ll \bar{k}'} \frac{e^{i \alpha' k \cdot k'}}{|z_k - z_{\bar{k}'}|} \]

\[ \Rightarrow \text{N-tachyon amplitude:} \]

\[ A(k_1, \ldots, k_N) \]

\[ = \int \prod_{x=0}^{N-1} d^2 z_{x+1} |z_{x+1} - z_x|^2 |z_x - z_{x+2}|^2 |z_{x+2} - z_x|^2 \]

\[ \prod_{k \ll \bar{k}'} \frac{e^{i \alpha' k \cdot k'}}{|z_k - z_{\bar{k}'}|} \]

\[ \ll k \ll N \]

\[ 10 \leq \bar{k}' \leq N \]

Check that the answer is independent of $z_1, z_2, z_3$.

$SL(2, \mathbb{R})$ trans.: $y_x \hat{\otimes} z_x = \frac{a z_x + b}{c z_x + d}$

\[ d^2 \hat{y}_x = d^2 y_x = \frac{1}{(c z_x + d)^2} \]

\[ \otimes y_x \otimes z_x = (z_x - z_y) \frac{1}{(c z_x + d)(c z_y + d)} \]
\[ A \left( k_1, \ldots, k_N \right) \]

\[ = \int \prod_{x=1}^{N} d^2 y_k \prod_{x=1}^{N} \left| Cz_x + d \right|^\theta \left| y_1 - y_2 \right|^2 \left| y_2 - y_3 \right|^2 \left| y_1 - y_3 \right|^2 \]

\[ \left| Cz_1 + d \right| \left| Cz_2 + d \right| \left| Cz_3 + d \right|^\theta \]

\[ \prod_{x \neq i} \left| y_{x} - y_i \right| \quad x' \neq k' \quad k' \]

\[ \sum_{k' \neq k} \omega' k' \cdot k' \]

\[ \prod_{x} \left| Cz_x + d \right| \]

\[ \prod_{x} \left| Cz_x + d \right| \cdot (-k'_x) \quad \alpha' k'_x \cdot k_x = \theta \]

\[ \Rightarrow \text{Man Shell Constraint.} \]

\[ \Rightarrow A \left( k_1, \ldots, k_N \right) \]

\[ = \int \prod_{x=1}^{N} d^2 y_k \left| y_1 - y_2 \right|^2 \left| y_2 - y_3 \right|^2 \left| y_1 - y_3 \right|^2 \left| y_1 - y_2 \right|^2 \]

\[ \prod_{x \neq i} \left| y_{x} - y_i \right| \quad x' \neq k' \quad k' \]

\[ \text{Note: } y_i \text{'s for } i = 1, \ldots, N \text{ are dummy variables.} \]

\[ \Rightarrow \text{can be changed to } z_i \text{'s.} \]

\[ \text{But } (y_1, y_2, y_3) \neq (z_1, z_2, z_3) \]

\[ \text{Adjusting } a, b, c \text{ at we can change } y_i, y_2, y_3 \text{ to be arbitrary points.} \]

\[ \Rightarrow \text{The answer is independent of the choices of } y_1, y_2, y_3 \]
Permutation symmetry of $t_1, \ldots, t_n$.

$\psi = \frac{1}{r y + z}$, \hspace{1cm} \psi' - 2 \psi = 0$

Make $f, g, r, s$ depend on $\epsilon_0, \epsilon_1, \ldots, \epsilon_5$.

$\psi_4 = \frac{b^2 + z}{r y + s} = 0$, \hspace{1cm} \psi_5 = 1$, \hspace{1cm} \psi_6 = \infty$.

$\psi_1 = \frac{b^2 + z}{s}$, \hspace{1cm} \psi_2 = \frac{b + z}{r + s}$, \hspace{1cm} \psi_3 = \frac{1}{r}$

now depend on $\epsilon_0, \epsilon_1, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8$.

Trade on $d^2 y_1, \ldots, d^2 y_8 \rightarrow d^2 \psi_0, d^2 \psi_2, d^2 \psi_3, \ldots, d^2 \psi_8$.

Ex. Check that the integrals in terms of $\psi_i$ have the same for as in terms of $y_i$, except for the replacement:

\[ (y_1 - y_2)^2, (y_2 - y_3)^2, (y_3 - y_1)^2 \rightarrow \psi_i (y_4 - \psi_5)^2 (\psi_5 - \psi_6)^2 (\psi_6 - \psi_1)^2 \]
\[ y_1 = 0, \quad y_2 = 1, \quad y_3 = \infty \]

**Example:** \( N = \nu \)

\[ A(k_1 \ldots k_n) \]

\[ = \int d^2 z_y \left( \begin{array}{c} (1-z_y)^2 \left( 1-\frac{y_3}{y_3} \right)^2 \\ 1 \end{array} \right) \alpha' k_1 \cdot k_2 \alpha' k_2 \cdot k_3 \alpha' k_3 \cdot k_4 \alpha' k_4 \cdot k_5 \alpha' k_5 \cdot k_6 \]

As \( y_3 \to \infty \):

\[ y + \alpha' k_3 \cdot (k_1 + k_2 + k_4) = y - \alpha' k_3^2 - \alpha' k_3 \cdot k_4 \]

\[ = \int d^2 z_y \left( \begin{array}{c} (1-z_y)^2 \left( 1-\frac{y_3}{y_3} \right)^2 \\ 1 \end{array} \right) \alpha' k_1 \cdot k_2 \alpha' k_2 \cdot k_3 \alpha' k_3 \cdot k_4 \alpha' k_4 \cdot k_5 \alpha' k_5 \cdot k_6 \]

\[ = \int d^2 z_y \left( \begin{array}{c} (1-z_y)^2 \left( 1-\frac{y_3}{y_3} \right)^2 \\ 1 \end{array} \right) \alpha' k_1 \cdot k_2 \alpha' k_2 \cdot k_3 \alpha' k_3 \cdot k_4 \]

\[ = \int d^2 z_y \frac{1}{z_y} \left( 1-\frac{y_3}{y_3} \right)^2 \alpha' k_1 \cdot k_2 \alpha' k_2 \cdot k_3 \alpha' k_3 \cdot k_4 \]

\[ = \int d^2 z_y \frac{1}{z_y} \left( 1-\frac{y_3}{y_3} \right)^2 \alpha' k_1 \cdot k_2 \alpha' k_2 \cdot k_3 \alpha' k_3 \cdot k_4 \]

**Note:** \( \alpha' k_1 \cdot k_2 = \frac{\alpha'^2}{2} (k_1 + k_2)^2 - k_1^2 - k_2^2 \) could be less than 2 in some range of momenta.

\( \Rightarrow \) The integral diverges from \( z = 0 \)

Similarly, there can be divergences from 1 and \( \infty \) for other ranges of momenta.

We need to define the integral in a range of momenta where it is finite and then do analytic continuation.
Formamplitude involving in doing this we shall set poles at

\[ \alpha', \mathbf{k}, \mathbf{k}_4 = -2 \]

\[ \frac{\alpha'}{2} \left( (k_1 + k_4)^2 - k_1^2 - k_4^2 \right) = \frac{\alpha'}{2} (k_1 + k_4)^2 - \cdots \]

\[ \mathcal{Y} \quad (k_1 + k_4)^2 = \frac{4}{\alpha'} \]

\[ \text{Tachyon pole} \]

\[ \text{Pole at} \quad (k_1 + k_4)^2 = -M_T^2 = \frac{4}{\alpha'} \]

The integral also has poles at

\[ \alpha', \mathbf{k}, \mathbf{k}_4 = -4, -6, -8, \cdots \]

\[ \Rightarrow (k_1 + k_4)^2 = 0, \quad \mathcal{Y} = \frac{4}{\alpha'}, \quad -\frac{8}{\alpha'} \]

\[ \Rightarrow \left( \text{massless intermediate} \right)^2 = 0, \quad \frac{4}{\alpha'} \]

- Second massive states
- First massive states
- States
- States
- \ldots

Similarly, divergence from \( z = 1 \) produces the \( t \)-channel poles (in \( (k_2 + k_4)^2 \)), and divergence from \( \infty \) produces the \( u \)-channel poles (in \( (k_1 + k_3)^2 \)).
The same complete product anti (E)
falls on all three channels

Unlike in (E) (here, where there
are separate Feynman diagrams produced)
falls on the I,F and 1 channels

More generally, in an N-point multiloop
the singularities are \( E - E_l \); formally we
write \( n \rightarrow (E + E_l)^n \) as \( E + E_l \) has a line
of \( \mathbb{R} \).

For \( g \leq 3 \), the pole comes from
all finite orders (an \( E_l \) and \( E \); where
comes close together. Here together. The loop energy is \( E + E_l \), typical.

The origin of these singularities can be
treated to be singularities in the operator
S-matrix expansion of vertex operators in
an \( \mathbb{F} \).

\[
V_{\text{reg}} (\omega_i) \ V_{\text{reg}} (\omega_j) = \sum_{\mathbb{F}} C_{\omega_i \omega_j} V_{\text{reg}} (\omega_i + \omega_j)
\]

\( \rightarrow \) A solution which is sufficient;
 hold inside any correlation function

The dependence \( \mathbb{F} \) on \( E + E_l \) can be
found from the correlation function, i.e., \( \mathbb{F} \).
Today we shall try to reinterpret these results in terms of singularities in operator product expansion.

But first we need to study carefully the origin of the singularities.

\[ \int_0^R d^2 \zeta \left| \zeta \right|^{-\eta + \epsilon} = f^{(s)}(\eta, \epsilon) + f^{(r)}(R, \eta). \]

\[ \text{singular regular} \]

\[ Z = \lambda y \]

\[ \lambda y \int_0^{R/\lambda} d^2 y \left| y \right|^{-\eta + 2\epsilon} \]

\[ = \lambda^{2-\eta + 2\epsilon} \left( \int_0^R d^2 z \right) \left( \int_0^{R/\lambda} d^2 y \right) \]

\[ = \lambda^{2-\eta + 2\epsilon} \left( f^{(s)}(\eta, \epsilon) + f^{(r)}(R, \eta) \right) + \lambda^{2-\eta + 2\epsilon} \phi^{(r)}(\eta, R, \lambda) \]

\[ \Rightarrow \left( \lambda^{2-\eta + 2\epsilon} - 1 \right) f^{(s)}(\eta, \epsilon) + \left( \lambda^{2-\eta + 2\epsilon} - 1 \right) f^{(r)}(R, \eta) + \lambda^{2-\eta + 2\epsilon} \phi^{(r)}(\eta, R, \lambda) = 0 \]

Singular part must vanish by itself:

\[ \Rightarrow f^{(s)}(\eta, \epsilon) = 0 \text{ when } \eta = 2. \]

\[ \Rightarrow f^{(s)}(\theta, \epsilon) = \sin \int_0^R d^2 \zeta \left| \zeta \right|^{-2 + 2\epsilon} \]

\[ = 4\pi \int_0^R d\theta \int_0^1 r dr \left| r \right|^{-2 + 2\epsilon} = \frac{4\pi \eta}{2\epsilon} \eta^{-1 + 2\epsilon} \]
Use this to check the expression:

\[
\int d^2 z \: |z|^2 a^{-2} (1 - z)^{-2 b - 2} = \frac{2\pi \Gamma(a) \Gamma(b) \Gamma(1 - a - b)}{\Gamma(1 - a) \Gamma(1 - b) \Gamma(b + a)}
\]

Take \(a = -n + \epsilon\)

\[
\text{l.h.s.} = \int d^2 z \: |z|^{-2 n + 2 \epsilon - 2} (1 - z)^{2 b - 2}
\]

\[
\sum_{m=0}^{\infty} \sum_{m=1}^{\infty} \frac{2^m}{m!} \frac{m^m}{2^m} (1 - z)^{-2 m - 2} \sum_{b=0}^{\infty} \frac{(2^* b! \frac{2^{b+1}}{2^{b+1}} (1 - z)^{-b - 2}}{z^{*} 0}
\]

Only \(m = b - n\) term will contribute to sum.

\[
\text{l.h.s.} = \left( \int d^2 z \: |z|^{2 \epsilon - 2} \right) \sum_{n} \frac{1}{n!} \left\{ \frac{b! (b - 2) \cdots (b - n - 2)}{2\pi \epsilon} \right\}^2
\]

\[
\text{s.h.s.} = 2\pi \frac{\Gamma(-n + \epsilon) \Gamma(b) \Gamma(1 + n - b - \epsilon)}{\Gamma(1 + n - \epsilon) \Gamma(1 - b) \Gamma(-n + \epsilon + b)}
\]

\[
= \frac{2\pi}{\epsilon} \left\{ \frac{1}{n!} (b - 1) \cdots (b - n) \right\}^2 + \text{finite}
\]

(\text{Check})
In general we need to study singularities of

\[ \langle \prod_{x} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) \rangle \quad \text{as} \quad z_{x} \to z; \]

Take

\[ \text{Take unit disk around } z_{i}; \]

\[ \Rightarrow D_{z_{i}}; \]

\[ \langle \prod_{x} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) \rangle \]

\[ = \sum_{\Phi_{0}} e^{-S(\Phi)} \prod_{x} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) \]

\[ = \sum_{\Phi_{0}} \Phi_{0} \left( \sigma \right) \sum_{\Phi} e^{-S(\Phi)} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) V_{\mathcal{R}_{y}} (z_{y}, \overline{z}_{y}) \]

\[ \left\{ \begin{array}{l}
\sum_{\Phi} e^{-S(\Phi)} \prod_{x} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) \\
(\Phi_{0} \left( \sigma \right))
\end{array} \right. \]

\[ \sum_{\Phi} e^{-S(\Phi)} \prod_{x} V_{\mathcal{R}_{x}} (z_{x}, \overline{z}_{x}) \]

\[ = \sum_{Y_{\mathcal{R}}} C_{Y_{\mathcal{R}}} \left( z_{i}, \overline{z}_{i}, z_{j}, \overline{z}_{j} \right) Y_{R} \Phi_{0} \left( \sigma \right) \]

\[ = \sum_{Y_{\mathcal{R}}} C_{Y_{R}} \left( z_{i}, \overline{z}_{i}, z_{j}, \overline{z}_{j} \right) \text{Wave state associated with state } | \Phi_{0} \left( \sigma \right) \rangle \]
Final result:

\[ \langle \prod_{k \neq i,j} \mathcal{V}_{R_k}(z_k, \bar{z}_k) \rangle \]

= \sum_{\gamma_R} C_{\gamma_R} \left( z_i, \bar{z}_i, z_j, \bar{z}_j, \bar{z}_i \right)

\[ \int_{-D_z} e^{-s(\varphi)} \sum_{\gamma_R} \prod_{k \neq i,j} \mathcal{V}_{R_k}(z_k, \bar{z}_k) \Big|_{\mathcal{C} - D_z} \]

= \sum_{\gamma_R} C_{\gamma_R} \left( z_i, \bar{z}_i, z_j, \bar{z}_j, \bar{z}_i \right)

\[ \int_{-D_z} e^{-s(\varphi)} \sum_{\gamma_R} \prod_{k \neq i,j} \mathcal{V}_{R_k}(z_k, \bar{z}_k) \]

Net result:

\[ \mathcal{V}_{R_i}(z_i, \bar{z}_i) \mathcal{V}_{R_j}(z_j, \bar{z}_j) \]

= \sum_{\gamma_R} C_{\gamma_R} \left( z_i, \bar{z}_i, z_j, \bar{z}_j, \bar{z}_i \right) \mathcal{V}_{R_k}(z_k, \bar{z}_k) \text{ inside} \left( z_i - z_j, \bar{z}_i - \bar{z}_j \right) \]
New coordinate $z \rightarrow f(z)$.

Inversion of inner new coordinate

Choose $f(z) = \lambda z$.

$V_{Y_x}(z, \bar{z}) \rightarrow \lambda \overline{h_{Y_x}} \lambda^{-1} \overline{h_{Y_x}} V_{Y_x}(\lambda z, \lambda \bar{z})$

$= \sum_{k} C_{Y_x Y_x Y_k} (z - \omega, \bar{z} - \bar{\omega}) \lambda^{- \lambda h_{Y_k}} \overline{h_{Y_k}} \overline{h_{Y_k}} V_{Y_k}(\lambda \omega, \lambda \bar{\omega})$

$\Rightarrow V_{Y_x}(\lambda z, \lambda \bar{z}) \rightarrow V_{Y_k}(\lambda \omega, \lambda \bar{\omega})$

$= \sum_{k} C_{Y_x Y_x Y_k} (z - \omega, \bar{z} - \bar{\omega}) \lambda^{- \lambda h_{Y_k}} \overline{h_{Y_k}} \overline{h_{Y_k}} V_{Y_k}(\lambda \omega, \lambda \bar{\omega})$

On the other hand

$V_{Y_x}(\lambda z, \lambda \bar{z}) \rightarrow V_{Y_k}(\lambda \omega, \lambda \bar{\omega})$

$= \sum_{k} C_{Y_x Y_x Y_k} (\lambda(z - \omega), \lambda(\bar{z} - \bar{\omega})) V_{Y_k}(\lambda \omega, \lambda \bar{\omega})$

$\Rightarrow C_{Y_x Y_k Y_k}(\lambda(z - \omega), \lambda(\bar{z} - \bar{\omega}))$

$= \lambda^{- \lambda h_{Y_k}} \overline{h_{Y_k}} \overline{h_{Y_k}} C_{Y_x Y_k Y_k}(\lambda(z - \omega), \lambda(\bar{z} - \bar{\omega}))$

$= C^{(n)}_{Y_x Y_k Y_k}(z - \omega, \bar{z} - \bar{\omega}) = C^{(n)}_{Y_x Y_k Y_k}(\omega - z, \bar{\omega} - \bar{z})$
\[ \sum_{k} \mathcal{O}(c) \left( \bar{V}_{r_{1}} V_{r_{2}} \right) \left( \bar{V}_{r_{3}} \right) \frac{(\bar{r}_{k} - \bar{r}_{r_{1}} - \bar{r}_{r_{2}})(\bar{r}_{r_{3}} - \bar{r}_{r_{1}} - \bar{r}_{r_{2}})}{(\bar{z} - \bar{\omega})} \]\n
Involve (momentum)^2.

Then in the N-point amplitude:

\[ \int \prod_{n=1}^{N} d^{2}z_{n} \left| z_{1} - z_{2} \right|^{2} \left| z_{2} - z_{3} \right|^{2} \left| z_{1} - z_{3} \right|^{2} \]

We can use OPE in any order.

E.g., \( V_{r_{1}} V_{r_{2}} \) first.

\[ \Rightarrow \text{poles singularity in } (z_{3} - z_{2}) \]

\[ \Rightarrow \text{poles in } (k_{r_{1}} + k_{r_{2}})^{2} \]

Then take the OPE of the result with \( V_{r_{3}} \) to get singularity in \( (z_{3} - z_{2}) \).

\[ \Rightarrow \text{poles in } (k_{r_{1}} + k_{r_{2}} + k_{r_{3}})^{2} \text{ etc.} \]
But in the same amplitude, taking the OPE in a different order, we can recover the singularities in \((z_j - z_j')^2\) for other \((j', j)\) and hence poles in \((k_{j'} - k_j)^2\).

\[\text{Note: Only } h_{j'k} - h_{j'k} - h_{j'k} = 0, \quad \overline{h}_{j'k} = \overline{h}_{j'k} = \overline{h}_{j'k} = 0\]

states contribute to the pole.

\[h_{j'k} = h_{j'k} = 1, \quad \overline{h}_{j'k} = \overline{h}_{j'k} = 0\]

\[\Rightarrow h_{j'k} = 1, \quad \overline{h}_{j'k} = 1\]

\[\Rightarrow |\rho_{j'}\rangle \text{ satisfies physical state condition for } L_0\]

A more careful analysis can be given to show that only physical state contribute to the sum over poles.
The net result is that any $N$-point amplitude can be expressed as a sum over intermediate states.

\[
\sum_i \sum_j \sum_k \frac{V_{ijk}}{(k^2 - m_i^2)^3} \frac{V_{ijkl}}{(k^2 - m_j^2)^3} \frac{V_{ijklm}}{(k^2 - m_k^2)^3}
\]

The same amplitude may be enumerated differently:

\[
\sum_i \sum_j \sum_k \frac{V_{ijk}}{(k^2 - m_i^2)^3} \frac{V_{ijkl}}{(k^2 - m_j^2)^3} \frac{V_{ijklm}}{(k^2 - m_k^2)^3}
\]

- Known as channel duality.

Coupling constant in string theory.

To determine the amplitude accurately, we can account for factors that reflect an ambiguity in defining the correct measure over the metric.
$A_N$ is accompanied by $g^{N-2}$

3-point $\propto g$

4-point $\propto g^2$ etc.

consistent with the rules relating $A_N$ to $A_m$ via sum over intermediate states

$g$ is known as the string coupling constant.

We shall soon see that it can be also added into the definition of a field.

$S$-matrix involving massless fields

$S_{\mu \nu} \propto k^\mu \propto k^\nu$

vector of $\propto S_{\mu \nu} \propto k^\mu \propto k^\nu$ can be used to calculate $S$-matrix.

Need correl. $\propto \langle \propto k^\mu \propto k^\nu \rangle$ involving

\[
\langle \propto k^\mu \propto k^\nu \rangle \propto 1^k_i x (t_1 \propto x \propto k^\mu \propto k^\nu \rangle x (t_2) \propto x \propto k^\mu \propto k^\nu \rangle
\]
\[
\frac{\delta}{\delta J^M (\vec{z}, \bar{\vec{z}})} \frac{\delta}{\delta J^N (\vec{z}, \bar{\vec{z}})} \left< \sum \int \mathcal{J} (\vec{z}, \bar{\vec{z}}) \times \mathcal{K} (\vec{z}, \bar{\vec{z}}) \, d^2 \vec{z} \, d^2 \bar{\vec{z}} \right>
\] = \sum K_a \delta (\vec{z} - \bar{\vec{z}})

\begin{align*}
\int \mathcal{J} (\vec{z}, \bar{\vec{z}}) \times \delta (\vec{z} - \bar{\vec{z}}) \, d^2 \vec{z} \, d^2 \bar{\vec{z}} &= 0
\end{align*}

Can be computed

5 matrix elements involving Massless field

For massless states we can second order take external momenta to be small

Expansion in powers of \( \times' K^2 \)

Notion of low energy effective action

Find an "effective action" such that the tree level Feynman rules reproduce the string theory amplitudes

The low energy effective action will involve expansion in \( \times K^2 \)

\( \times' \) space-time derivatives and\( K^2 \)

(not to be confused with \( \partial, \partial' \) derivatives)
Low energy effective action for bosonic string theory:

\[ S = K \int d^{26}x \left[ R - \frac{1}{12} H_{\mu\nu\rho} \right] \]

\[ \sim \frac{1}{g^2} \]

\[ e^{-2\phi} \]

\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \]
Recall the procedure for deriving the effective action:

1. Calculate scattering amplitudes
2. Try to write down a field theory action which reproduces the scattering amplitude.

Today we shall discuss another method for computing the S-matrix.

Consider 

\[ S = \frac{T}{2} \int d^2 \xi \, \bar{\psi} \gamma^\mu \partial_{\mu} \psi \eta_{\mu\nu} \]

Now consider its propagation in curved background.

\[ S = \frac{T}{2} \int d^2 \xi \, \bar{\psi} \gamma^\mu \partial_{\mu} \psi \sqrt{-g} \, g^{\alpha\beta} \, G_{\mu\nu}(X) \]

→ an interacting field theory.

Set \( T = \frac{1}{2\pi} \) so \( \alpha = 1 \) \( \Rightarrow \) \( S = \frac{2 \pi}{T} \int \) \( G_{\mu\nu}(X) - G_{\mu\nu}(0) + \bar{\psi} \gamma^\mu \partial_{\mu} \psi \)

All operators have dimension 2
\[ \text{dim } \psi = 0 \Rightarrow G_{\mu\nu}(X) \text{ is dimensionless.} \]

The theory is renormalizable.

What other dimension 2 operator can we add to the action without destroying
\[
S = \frac{1}{4H^2} \int d^8z \, \frac{\partial}{\partial x} \frac{\partial}{\partial \bar{x}} \left[ \sqrt{-g} \, \bar{y}^{a^3} \, C_{\mu
u}(x) + \bar{\epsilon}^{a^3} B_{\mu
u}(x) \right]
+ \sqrt{-g} \, R_{\mu
u} \bar{\Phi}(x)
\]

8 Scaler curvature associated with $\bar{\Phi}$

Note: 1. The action is manifestly general coordinate invariant.

2. The first two terms are also weak invariant (under $Y_i \to e^{\lambda_i} Y_i$) but the last term is not.

3. When we fix the gauge $Y_{\mu} = \eta_{\mu}$, the last term drops out of the action.

But its effect appears in the definition of the energy momentum tensor $T_{\mu\nu}$:

\[
T_{\mu\nu} \propto \frac{\delta S}{\delta Y^{a^3}} \rightarrow \text{contains a term}
\]

\[
\propto \tilde{\epsilon}^a \tilde{\epsilon}^b \tilde{\Phi}(x)
= \tilde{\epsilon}^a \tilde{\epsilon}^b \tilde{\Phi} \tilde{\epsilon}^a \tilde{\epsilon}^b X^X
+ \tilde{\epsilon}^a \tilde{\epsilon}^b \tilde{\epsilon}^a \tilde{\epsilon}^b X^X
\]

Contribution from this term to $\eta^{\mu} T_{\mu
u}$ does not vanish.

\rightarrow \text{reflects lack of weak invariance.}
The reason is that quantum correction destroys two dimensional field theory any way.

So we sought as non-vanishing $\beta$-

Note: $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ & $\Phi(x)$ are all the coupling constants in the theory.

If $G_{\mu\nu}(x) = G_{\mu\nu}(0) + \epsilon \Phi \cdot G_{\mu\nu}(0) \Phi^T$, then the action contains

$$ L \propto \int d^4x \left( \partial_\mu x^\nu \partial_\nu x^\mu + \frac{1}{2} G_{\mu\nu}(0) \partial_\mu x^\nu \partial_\nu x^\mu + \cdots \right) $$

de coupling constants.

The $\beta$-function specifying how these coupling constants change with the renormalization scale $\mu$ will also be

(renormalization)

R.G. eqn.:

$$ \mu \frac{d G_{\mu\nu}(x)}{d\ln \mu} = \beta \frac{G_{\mu\nu}(x)}{\mu} $$

$$ \mu \frac{dB_{\mu\nu}(x)}{d\ln \mu} = \beta \frac{B_{\mu\nu}(x)}{\mu} $$

$$ \mu \frac{d\Phi(x)}{d\ln \mu} = \beta \frac{\Phi(x)}{\mu} $$
Put another way,

\[ \gamma^{\alpha} T_{\alpha \beta} x^0 (\beta^6 \alpha^x x^0 \eta^{\alpha} x^0 \sqrt{x} \gamma^x \beta^6 + \beta^6 \nu \alpha^x x^0 \eta^{\alpha} x^0 \epsilon^{\alpha} \beta^6 + \beta^6 \frac{\sqrt{\gamma}}{\gamma} R^{(3)}_Y) \]

\( \beta^6, \beta^8 \) and \( \beta^9 \) can be computed in

\[
\begin{align*}
\exists & \beta^6 \nu^0 = 0 \} & \text{Conformal invariance} & \text{In gauge fixed version} \\
\beta^6 \nu^0 = 0 & \} & \text{Central charge 26} & \text{In gauge fixed version} \\
\beta^9 = 0 & \}
\end{align*}
\]

\( \beta^6, \beta^8 \) and \( \beta^9 \) can be computed using

\[
\text{(Note: The perturbation theory is in 2-d conforming constant} \ \alpha' \ \text{factor of} \ \text{Each} \ \alpha' \ \text{will come with} \ \text{a) }}
\]
Thus in the $x' = 1$ limit, higher and higher order terms in the perturbation theory will contain more and more derivatives (e.g. $x^{10}$).

Result at one loop (for $x' = 1$)

\[ \beta_{\mu \nu} = 2 \, \gamma_{\mu \nu} - 1 \, H_{\mu \nu} - \frac{1}{2} \, \nabla_{\mu} \nabla_{\nu} \Phi = 0 \]

\[ \beta^B_{\mu \nu} = - \frac{1}{2} \, \nabla^B_{\mu} \nabla^B_{\nu} + (\nabla^B \Phi) \, H_{\mu \nu} = 0 \]

\[ \beta_{\Phi} = - \frac{1}{2} \, \nabla_{\tau} \Phi + (\nabla_{\tau} \Phi) \, (\nabla_{\tau} \Phi) - \frac{1}{24} \, H_{\mu \nu \rho} = 0 \]

These equations can be derived from the effective action

\[ S = (\frac{1}{12} \, H_{\mu \nu \rho} + \frac{1}{4} \, \partial_{\mu} \Phi \, \partial_{\nu} \Phi) \]

\[ \rightarrow \text{identical to the effective action described earlier.} \]

Conclusions:
- Equations of motion of string theory
- Conformal invariance of 2-d field theories
- Solution of the equations of motion of string theory
- Conformally invariant 2-d field theory with central charge $= 26$.

This provides an very efficient way skimming for solutions of the equations of motion without even knowing them.
Heterotic string theory:

\[ S = \frac{1}{4\pi \alpha'} \int d^2 \bar{z} \left( \eta^{\alpha\beta} \eta_{\alpha\beta} \partial \bar{X}^\alpha \partial \bar{X}^\beta \right) \]

\[ + \frac{2}{4\pi \alpha'} \left( \bar{\psi}^\alpha \gamma^z \psi^z \eta_{\alpha\mu} + \frac{i}{4\pi \alpha'} \bar{\lambda}^z \gamma^z \rho^\alpha \right) \]

\[ \mu, \nu = 0, \ldots, 32, \quad I, J = 1, \ldots, 32 \]

\[ \psi^\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \quad \lambda^z = \begin{pmatrix} z^- \\ z^+ \end{pmatrix}, \quad \rho^\alpha = \begin{pmatrix} \rho^- \\ \rho^+ \end{pmatrix}, \quad \rho' = \begin{pmatrix} 0 \\ \rho^+ \end{pmatrix} \]

Constraints:

Supergravity:

\[ \epsilon = \begin{pmatrix} \epsilon_- \\ 0 \end{pmatrix} \in \text{real.} \]

\[ \delta_\epsilon \chi^\mu \equiv \epsilon \chi^\mu - \epsilon^+ \rho^\alpha \psi^\mu = \epsilon \chi^\mu \]

\[ \delta_\epsilon \left( \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \right) = -i \epsilon \sigma^\alpha \partial_x \psi^\alpha \epsilon = \left( (\sigma_- - \sigma_+) \chi^\mu \right) \epsilon_- \]

\[ \Rightarrow \delta_\epsilon \psi_+ = (\sigma_- - \sigma_+) \chi^\mu \epsilon_- \]

Constraints:

\[ \text{Conserved current energy momentum tensor associated with susy trs.} \]
\[ (L_n \hat{\theta} c \in \mathbb{C}, \nu \in \mathbb{Z}) | \nu \in \mathbb{Z} \cap \mathbb{Z}_0 \]

\[ (L_n \hat{\theta} c \in \mathbb{C}, \nu \in \mathbb{Z}) | \nu \in \mathbb{Z} \cap \mathbb{Z}_0 \]

\[ A_n | \nu \in \mathbb{Z} \cap \mathbb{Z}_0 \]

We have two sectors: NS and on right.

Two sectors on left for $\mathfrak{so}(32)_n$

Four sectors on left for $(\mathfrak{so} \times \mathfrak{so})_n$

**Mode Expansion**

\[ \phi^\mu = \sum \hat{b}_k^\mu (x) \in \mathbb{C} \text{ for } \mathfrak{so}(32)_n \]

For $\mathfrak{so}(32)_n$

\[ \lambda \in \mathbb{Z}, \sum \check{b}_k^\mu (x) \in \mathbb{C} \text{ for } \mathfrak{so}(32)_n \]

\[ \mathbf{L}_n = \frac{1}{2} \sum \limits_{n \neq 0} \mathbf{x}^\mu \mathbf{x}_n \mathbf{2}_{\nu} + \frac{1}{2} \sum \check{b}_k^\mu \mathbf{x}_n \mathbf{b}_k^\nu \]

\[ L_0 = \sum \mathbf{x}^\mu \mathbf{x}_n \mathbf{2}_{\nu} + \frac{1}{2} \check{b}_k^\mu \mathbf{b}_k^\nu \]
\[ G_{ab} = \sum_{n=0}^{\infty} \frac{\alpha^r}{x + n} b_n a_n + \sqrt{x^+} \frac{b_n b^+}{2} \]

\[ C_n = \frac{1}{2} \text{ in } N_S, \quad 0 \text{ in } R. \]

\[ \gamma = -1 \text{ in } N_S, \quad 10 \text{ in } R. \]

\[ \text{ASC projection rules: On the right it is the same as superstring.} \]

\[ \text{NS vacuum } |\psi_k\rangle \text{ has - parity.} \]

\[ \text{R vacuum: } |\psi_k, a\rangle \text{ has + parity.} \]

\[ |\psi_k, a\rangle \text{ has - parity.} \]

\[ \text{ASC projection rule on the left:} \]

\[ \text{NS vacuum: } |\psi_k\rangle \text{ has + parity.} \]

\[ \text{R vacuum: Two spinors of } SO(32): \]

\[ |x\rangle, \quad |\bar{x}\rangle \]

\[ \text{Chiral, } \quad \text{Anti-chiral,} \]

\[ + \text{ parity, } \quad - \text{ parity.} \]
General State:

\[ a_{m_1} \ldots a_{m_k} \overline{\chi}_{n_1} \ldots \overline{\chi}_{n_l} b_{\tilde{n}_1} \ldots b_{\tilde{n}_m} \]

\[ a_i \ldots a_i \overline{\chi}^\dagger_{\overline{n}_1} \ldots \overline{\chi}^\dagger_{\overline{n}_3} \rightarrow \]

Spinor indices for R sector.

\[ L_0 \text{ constraint:} \]

\[ \sum_{i=1}^{N} \chi_i + \sum_{i=1}^{N} \tilde{\chi}_i = 0 \]

\[ \sum_{i=1}^{N} \chi_i + \sum_{i=1}^{N} \tilde{\chi}_i = 0 \]

\[ m^2 = \frac{1}{2} \left( N + C \right) = \frac{1}{2} \left( N + \tilde{C} \right) \]

\[ \tilde{C} = -1 \text{ in NS, } 1 \text{ in } R \]

\[ \Rightarrow \text{Tachyon may only come from NS - NS.} \]

\[ \Rightarrow \text{Need } N - \frac{1}{2} < 0. \text{ Since } N = 0, \frac{1}{2}, \ldots, \text{ we need } N = 0. \]
Fork vacuum on NS right sector.
This is also odd $\Rightarrow$ No tachyon.

Massless states:

$$N + C = 0 = \tilde{N} + \tilde{C}$$

Note: If we have R sector on the left, then

$$\tilde{C} > \tilde{C} \Rightarrow \text{No massless sector}.$$ 

Choose NS sector on the left.

$$\tilde{C} = -1$$

$$m^2 = (N - I)^2 \Rightarrow \text{Need } N = \frac{1}{2}.$$

On the right we can have NS or R sector.

On right:

**NS sector:** $\tilde{C} = -\frac{1}{2} \Rightarrow \text{Need } N = \frac{1}{2}.$

Possible states:

$$S_{\mu \nu} b_{-\frac{1}{2}} \chi_{-1} l_{k3} \rightarrow g_{\mu \nu}, B_{\mu \nu}$$

$$S_{IJ} b_{-\frac{1}{2}} \chi_{-\frac{1}{2}} \tilde{d}_{-\frac{1}{2}} \tilde{l}_{k3} \rightarrow \frac{32 \times 3!}{2} \times \text{combinations}$$

$\Rightarrow 50(32)$ gauge bosons.
Ramond sector on right: \( C = 0 \). Need \( N = 0 \).

\( \chi, 1 \{ k^3, a \} \rightarrow \) gravitino

\( \psi^{-\frac{1}{2}} \psi^{-\frac{1}{2}} 1 \{ k^3, a \} \rightarrow \) Fermions in the adjoint of \( SO(32) \).

The massless spectrum is that of \( N = 1 \) supergravity in a 10-dimensional space with \( SO(32) \) gauge group.

After imposing physical state constraint, the full spectrum becomes supersymmetric.

Equal number of fermions and bosons at each level.

Note: At the massless level there are no spinors of \( SO(32) \) but there are massive states carrying spinor representation, \( e.g. \) 

\[
\begin{align*}
\psi^{-\frac{1}{2}} \phi^{-\frac{1}{2}} \{ \phi^{-\frac{1}{2}} \} & : N = \frac{3}{2}, \quad N = 0; \\
C = \frac{1}{2}, \quad Z = 2 \\
\Rightarrow \quad m^2 = \frac{y}{2}, (\bar{N} + 0); \quad \frac{y}{2}, (\bar{N} + \frac{1}{2}); \quad \frac{y}{2}, \frac{y}{2}; \\
& \quad \frac{y}{2}; (\bar{N} + \frac{1}{2}); \quad \frac{y}{2}, \frac{y}{2}; \\
\end{align*}
\]
$E_8 \times E_8$ heterotic string theory:

On the right there is no change.

On the left we now need to divide the oscillators into two parts:

$\sum_{s \in \mathbb{Z}}$ and $\sum_{s \in \mathbb{Z}+1/2} \sum_{t=0}^{16}$.

On the left itself we have four sectors:

- **NS-NS**: $s, t \in \mathbb{Z} + \frac{1}{2}$
- **NS-R**: $s \in \mathbb{Z} + \frac{1}{2}, t \in \mathbb{Z}$
- **R-NS**: $s \in \mathbb{Z}, t \in \mathbb{Z} + \frac{1}{2}$
- **NSR-R**: $s, t \in \mathbb{Z}$

Twisting of each fermion gives a contribution of $\frac{1}{16}$.

On the right:

- $C = -\frac{1}{2}$ in NS-sector
- $= 0$ in R-sector
### General State

\[ \alpha_{n_1} \rightarrow \alpha_{-n_1}, \quad \alpha_{n_2} \rightarrow \alpha_{-n_2}, \quad \alpha_{k} \rightarrow \alpha_{-k}, \quad \beta_{n_1} \rightarrow \beta_{-n_1}, \quad \beta_{n_2} \rightarrow \beta_{-n_2}, \quad \beta_{k} \rightarrow \beta_{-k} \]

\[ d_{n_1} \rightarrow d_{-n_1}, \quad f_{n_1} \rightarrow f_{-n_1}, \quad g_{n_1} \rightarrow g_{-n_1}, \quad \{k^3\} \rightarrow \{k^3\} \]

### GSO Projection on the Right

Identical (RR, \(1 + k^3, 1 + k^3\))

### GSO Projection on the Left

\[ (-1)^{f_1} \text{ family} \]

### Table

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Absence of tachyons follows from

\[ m^2 = \frac{1}{2} \left( \sum m_i + \sum \gamma_i + \sum \epsilon \right) = \frac{1}{2} (N + \epsilon) \]

\[ = \frac{1}{2} \left( \sum n_i + \sum 3 \epsilon_i + 2 \epsilon \right) = \left( \frac{N + \epsilon}{2} \right) \]

Absence of tachyons: Follows from analysis on the right sector.

\[ N + \epsilon \geq 0 \text{ for } GSO \text{ even states.} \]

Massless states: Need \( N + \epsilon = 0 \implies N + \epsilon \)

Bosonic states (NS on the right):

\[ \text{NS - NS - NS: } b_{-\frac{1}{2}}, \xi_{-\frac{1}{2}}, \phi \]

\[ \text{with } \epsilon = -\frac{1}{2}, \phi = -1 \]

\[ \text{so } N + \epsilon = 1 \implies \phi_{\mu \nu}, B_{\mu \nu}, \phi \]

\[ b_{-\frac{1}{2}}, \xi_{-\frac{1}{2}}, \chi_{-\frac{1}{2}} \text{ (k3)} \]

\[ \rightarrow \text{SO}(16)_{1} \text{ gauge bosons.} \]

\[ b_{-\frac{1}{2}}, \phi_{-\frac{1}{2}}, \xi_{-\frac{1}{2}}, \phi \text{ (k3)} \]

\[ \rightarrow \text{SO}(16)_{2} \text{ gauge bosons.} \]

Ex: What about \[ b_{-\frac{1}{2}}, \xi_{-\frac{1}{2}}, \phi_{-\frac{1}{2}} \text{ (k3)}? \]
RNS-NS \( \alpha = -\frac{1}{2}, \gamma = 0 \)

No need \( \hat{N} = \frac{1}{2}, \hat{N} = 0 \).

\[ b = \frac{1}{2}, x, \{ k \} \]

A vector carrying spinor representations of \( \text{SO}(16) \).

Take \( \psi \), how can this be?

The only consistent way we know of coupling a vector is using non-abelian gauge theory.

Answer: The spinor of \( \text{SO}(16) \) + adjoint of \( \text{SO}(16) \rightarrow \) adjoint of \( E_8 \); \( \text{SO}(16) \).

The gauge group is actually \( E_8 \) and not \( \text{SO}(16) \).

(Analog of self-dual point in circle compactification of bosonic string theory.)
\( \text{NS - R - NS} \quad C = -\frac{1}{2} \quad \tilde{C} = 0. \)

* Need \( N = \frac{1}{2}, \quad \tilde{N} = 0. \)

\[ b^{1/2} \quad | \beta, k \rangle \]

- Spinor of \( \text{SO}(16) \).

- Combined with adjoint to give adjoint of \( (E_8)_2 \).

\( \text{R - R - NS} \quad \quad C = -\frac{1}{2} \quad \tilde{C} = 1 \quad \rightarrow \quad \text{no massless states}. \)

- R sector on the right \( \rightarrow \) generates fermionic supersymmetry of all these bosons.

Result: Again a supersymmetric spectrum.

- Massless sector: \( N = 1 \) supergravity in \( D = 10 \) with \( E_8 \times E_8 \) gauge group.
D-branes in superstring theory:

- Defined in identical manner as in the case of bosonic string theory.

\[ x^n \uparrow \]

- Dirichlet b.c. on \( x_1 \)
- Neumann b.c. on \( x_n \)

- World-sheet

What about b.c. on \( y^m \)?

- Should relate left moving \( y^m \) to right moving \( y^m \)

\( \text{i.e.} \quad y^m_+ \text{ to } y^m_- \text{ at } \tau = 0, \pi \)

- Choice: \( y^m_+ = \pm y^m_- \text{ at } \tau = 0, \pi \)

- \( \text{Diagonalizability} \) chooses \( y^m_+ \text{ and } y^m_- \) to be

- \( \text{World-sheet} \) susy relates b.c. on \( x^k \) to that on \( y^m \)
Consistent choice

\[ \begin{align*}
\Psi_{\|+} &= \Psi_{\|-} \quad \Psi_{\perp+} = -\Psi_{\perp-} \\
\Psi_{\|+} &= -\Psi_{\|-} \quad \Psi_{\perp+} = \Psi_{\perp-}
\end{align*} \]

At each end \( \sigma = 0, \pi \).

Note: we can make a field redefinition
\[ \Psi_{\|+} \rightarrow -\Psi_{\|+} \quad \text{to choose at } \sigma = 0: \]
\[ \begin{align*}
\Psi_{\|+} &= \Psi_{\|-} \quad \Psi_{\perp+} = -\Psi_{\perp-} \\
\Psi_{\|+} &= -\Psi_{\|-} \quad \Psi_{\perp+} = \Psi_{\perp-}
\end{align*} \]

Then at \( \sigma = \pi \) and we still have two cases:
\[ \begin{align*}
\Psi_{\|+} &= \Psi_{\|-} \quad \Psi_{\perp+} = -\Psi_{\perp-} & \Rightarrow \text{ R sector} \\
\Psi_{\|+} &= -\Psi_{\|-} \quad \Psi_{\perp+} = \Psi_{\perp-} & \Rightarrow \text{ NS sector}
\end{align*} \]

Quantize using usual procedure.

\( \Psi_{\|} \) will have zero modes in R sector.

\( \Rightarrow \) Spinor representation of \( SO(\hat{p}, 1) \)

\( \Rightarrow \) Boser space-time fermions. (101-dimensional)

Also projection: as usual (projects out NS vacua).
Note: Construction of D-branes require symmetry between left- and right-moving modes on the world sheet. Otherwise we cannot impose boundary conditions.

- No D-branes in heterotic string theory.

Further consistency condition: Collision of open strings may produce closed strings.

- We need to make sure that all of these are physical closed string states.

E.g., in type IIA theory, collision of two open strings should not produce \( (a \, b) \) \( a, b \) type RR sector state.

In type IIB theory, this process should not produce \( (a \, a) \) type RR sector state.
o gives restriction on what kind of D-p-branes we can have.

**IIA**

Dp-branes with even p:

\[ 0, 2, 4, 6, 8 \]

**IIB**

Dp-branes with odd p:

\[ -1, 1, 3, 5, 7, 9 \]

D-instanton

These D-branes carry charge.

Consider a pair of parallel D-branes:

One loop effective action:

\[ \circ \rightarrow \text{closed string}. \]
Then if we compute the contribution

$$\sum \int \frac{dk^{*} \ln (k^2 + m^2)}{\sigma}$$

of open string states

Can be interpreted as the potential \( V(a) \) induced between the two D-branes due to closed string exchange.

Includes graviton exchange.

\( T_\phi = \frac{1}{(\alpha')^{k+1/2}} \frac{1}{(2\pi)^k} \) gives tension \( \phi \) (energy/area).

Also includes exchange of \((b+1)\) form RR field

\[ c_1 \rightarrow b+1 \]

In \( \mathbb{Z}^A \): \( b = 0 \) even \( \Rightarrow c_{(b+1)} = \text{odd} \) matches the RR fields.
\[ C^{(2)} \rightarrow A_\mu \]

\[ C^{(3)} \rightarrow C_{\nu \epsilon \epsilon} \]

\[ C_{\mu \nu \epsilon \epsilon} \rightarrow \text{dual to } C^{(3)}_{\nu \epsilon \epsilon} \]

\[ dC^{(5)} = \ast dC^{(3)} \]

\[ C^{(7)} \rightarrow \text{dual to } C^{(1)}_\mu \]

\[ dC^{(9)} = \ast (dC^{(1)}) \]

\[ C_{\nu \epsilon \epsilon \epsilon} = N m - \text{ dynamical} \]

Similarly in IIB:

\( \beta = \text{odd}, \quad (\beta + 1) = \text{even} \)

no matches, no known RR fields.

\[ C^{(0)}_\mu \rightarrow A_\mu \rightarrow \text{RR scalars} \]

\[ C^{(2)} \rightarrow B^\dagger_{\nu \mu} \rightarrow 0 \]

\[ C_{\mu \nu \epsilon} \rightarrow D_{\mu \nu \epsilon} \]

\[ dC^{(3)} = \ast (dC^{(2)}) \]

\[ dC^{(8)} = \ast (dC^{(4)}) \]

\( C^{(10)} \) is non-dynamical.
How does a D-p-brane carry RR (p+1) form charge?

Particle: carries 1 form charge (A_{\mu}).

Contrived gauge:

$$A_\mu \propto \frac{\epsilon}{r^{D-3}} \text{ far away.}$$

D-p-brane along $i_1, \ldots, \bar{p}$

$$C^{(p)}_{\mu_1 \cdots \mu_p} \propto \frac{\epsilon}{r^{D-p+3}} \text{ far away from the brane.}$$

How does a & D-p-brane feel the effect of & form field?

Particle: world-line action $+ \int A_\mu dx^\mu$ world line

D-p-brane: world volume action $+ \int C^{(p)}_{\mu_1 \cdots \mu_p} \wedge dx^{\mu_1} \cdots dx^{\mu_p}$ world volume
N - coincident D-branes

→ supersymmetric U(n) Yang-Mills theory at low energy.

→ generalization of the result from bosonic string theory.
A D-p brane carries (p+1) form charge $\mathcal{C}_{p+1}$.

No act as sources for these fields.

For a particle moving along a curve $x^k$, the action is:

$$\text{Action} = \int A_\mu \frac{dx^\mu}{ds} ds + \text{Free particle action} + \text{g.m. field action}$$

A D-p brane spans a (p+1) dimensional world-volume labelled by $\Sigma_{p+1} = x^k(S_0, S_1)$.

We have been sources up to D7-branes.

In DB,

D7-brane carries 8 form charge.

Field strength 9 form.

Dualize:

1 form field strength $\rightarrow$ 0 form field

RR scalar $\rightarrow$ 0.
What about D8 & D9-branes in IIA/IB

g from $C^{(9)}$ 10-form $C^{(10)}$

If we quantize free field theory of $C^{(9)}$ and $C^{(10)}$ and see what physical states they produce, we find that there are no physical states associated with these fields.

(Just like a gauge field in 4+1 dimensions has no physical states)

This is why we have not discovered them so far in analyzing the spectrum of IIA/IB.

But their existence can be guaranteed seen in scattering amplitudes.

(They appear as internal lines).
Focus on the D3-brane of IIB

C. space (giving) \( \chi^{a} = \lambda \).

C. filling: \( N \int_{\partial M} C^{(10)} \sim \chi_{I} \sim \chi_{I0} \times_{0} d^{10}x \).

\# of D3-brane

as will act as source of \( C^{(10)} \) field.

Unfortunately, this is the only term in the action involving \( C^{(10)} \).

No kinetic term.

\( \partial C^{(10)} \) vanishes identically.

Then \( C^{(10)} \) eq. of motion gives

\( N = 0 \).

Besides, D3-brane also carries energy density \( \rho \).

Cosmological constant.

Putting D3-branes in flat space-time is in consistent.

(Note: both involve back reaction of closed strips and is higher order effect.)
Type I String Theory

Recent gaps The type II world-sheet theory has a symmetry under left-right exchange:

\[ \sigma \to -\sigma, \quad \tau \to -\tau, \quad \Lambda^+ \to \Lambda^- \]

or world-sheet parity \( \Omega \).

Are the GSO projection rules invariant under such a symmetry?

\[ \text{NS-NS} \leftrightarrow \text{NS-NS}, \quad \text{NS-R} \leftrightarrow \text{R-NS}, \quad \text{RR} \leftrightarrow \text{RR} \]

No problem.

\[ \begin{align*}
\text{(I)} \quad & \text{NS} & \leftrightarrow & \text{NS} \\
\text{(II)} \quad & \text{R} & \leftrightarrow & \text{R}
\end{align*} \]

In \( \Pi B \) and not in \( \Pi A \). In \( \Pi A \) we have \( b_{1,2} \parallel \{ a, \{ k \} \} \).
Thus the $SO$ symmetry is broken by the $GSO$ projection rules of IIA but preserved by the $GSO$ projection rules of IIB.

We can now construct a new theory:

(Unoriented string theory)

Mod out IIB by $SO$

i.e., keep only those states of IIB which are invariant under $SO$

Massless Bosons:

NSNS sector: C, B, $\Phi$

RR sector: $X$, $B'$, $\varphi$, $\phi$,

Massless fermions: Come from linear combination of states in (NS-RR) and (RR-NS) sectors.
Is that a consistent theory?

No.

Calculate scattering amplitudes and effective action

- A term \( \propto \int \frac{d^{10}x}{k_i - k_{10}} \)

- A negative cosmological constant:

\[ -\Lambda \]

\[ \text{S fiat} \]

1. \( \mathcal{O}^{(10)} \) eq. of motion is not satisfied.

2. Flat background is not a consistent solution.

Possible remedy: Can we cancel these terms in the effective action by putting in some \( D9 \)-branes?

Ans. Yes, with 32 \( D9 \)-branes.
In (9+1) dimension D9-branes are always coincident. 

\[ \Rightarrow \mathbb{U}(32) \text{ gauge group, fermions in adjoint} \]

But we now need to determine which of these gauge bosons survive.

2 projection

\[ \Rightarrow \mathfrak{so}(32) \text{ gauge group with fermions in the adjoint representation of the gauge group.} \]

Type I string theory.

5 consistent string theories in (9+1) dim. 

\[ \text{IIA, IIB, } (E_8 \times E_8)_H, \ (\mathfrak{so}(32))_H, \ \text{F}_{10} \]

\[ \overset{\text{Self-duality}}{\text{Fundamental string} \leftrightarrow \text{D-string}} \]

Non-perturbatively equivalent:

\[ \mathfrak{so}(32) \rightarrow \mathfrak{g}_3 \]