(1)

What is string theory and why should we study it?

Current understanding of elementary constituents of matter

→ Standard Model (Glashow - Weinberg - Salam)

→ Based on Quantum Field Theory.

Generalization of Quantum Electrodynamics (QED)

In QED we have

electron & photon

↓ Dirac field ψ

↓ Gauge field A_μ

Action: $\int d^4x \, (\bar{\psi}_e \not{D} \psi_e + \frac{1}{4} F_{\mu\nu} F^{\mu\nu})$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Generalization:

ψ

↓

Quarks & Leptons

u c t

d s b

e μ τ

ν_e ν_μ ν_τ

$(1+\gamma^5)\psi_{\nu_e} = 0$

A_μ

A_μ (Photon)

G_μ^a (Gluon) ↓ Strong interaction

$W_\mu^\pm$, Z_μ ↓ We[ak] inte[raction] (violat[ing])

Higgs Scalar ϕ

Problem Inclusion of gravity.

→ An important force at long distances (Astrophysical / Terrestrial scale).

Non-relativistic limit + weak gravitational field ⇒ Newtonian gravity.

Otherwise General Theory of Relativity.

Every known object gravitates.
⇒ Their constituent elementary particles must also gravitate.

~~Then how can we~~ But the interactions among elementary particles described by the standard model does not include gravity.

Q. Why then is standard model so successful in explaining observed experimental results?

Compare gravitational and electromagnetic force between two protons:

$$\frac{G_N m_p^2/r^2}{e_p^2/r^2} = \frac{G_N m_p^2}{e_p^2} \simeq 10^{-38}$$

Thus Grav. force << Electromagnetic & other forces.

But however small it may be, gravitational ~~force~~ interaction is certainly present in nature and must be included in a complete theory.

Also, gravitational interaction does not always remain small.

Consider two protons ~~and~~ moving opposite to each other with large energy and colliding.

$$\xrightarrow[E]{p} \quad \xleftarrow[E]{p}$$

Effective mass of each proton: (E/c^2)

$$= (E/m_p c^2) \cdot m_p$$

$\Rightarrow$ Gravitational interaction between two protons get magnified by a factor of

$$(E/m_p c^2)^2$$

Electromagnetic interaction between two protons also grows with E but only logarithmically

$$\sim \ln(E/m_p c^2)$$

If $E/m_p c^2 \sim 10^{19}$, then the gravitational and electromagnetic interactions become comparable.

Result of such an experiment cannot be predicted by the standard model.

Note: At present we cannot carry out this experiment due to lack of powerful enough accelerators.

But a complete theory must be able to predict the results of all experiments, including those which cannot be performed due to practical limitations.

Conclusion: A complete theory must include the effect of gravitational interaction between elementary particles.

What do we know about gravity?

classical gravity is described by general theory of relativity

Dynamical variables:

Metric $g_{\mu\nu}(\vec{x}, t)$ describing geometry of space-time

$\vec{x}$ $\quad$ t

$\mu, \nu : \underset{t}{0}, \underbrace{1, 2, 3}_{\vec{x}}$

$g_{\mu\nu}$ satisfies Einstein's equations

$$\underset{\text{Ricci tensor}}{R_{\mu\nu}} - \frac{1}{2} R g_{\mu\nu} = -8\pi G_N \underset{\text{Energy momentum tensor}}{T_{\mu\nu}}$$

~~→~~ General Relativity is a classical field theory with $g_{\mu\nu}(\vec{x},t)$ as the classical fields.

In contrast standard model is a quantum field theory with quark + lepton fields, Gauge fields etc. as dynamical fields.

At present there is no direct experimental evidence that gravity satisfies the rules of quantum mechanics.

Could it be that the complete theory is

Standard Model

+

classical General Relativity ?

Not possible since elementary particles like electrons, which are known to ~~satisfy~~ obey the laws of quantum mechanics, act as the source of gravity.

Note: ① An electron produces gravitational field.

② If we could measure all components of the metric and their time derivatives exactly, ~~at~~ ~~at~~ ~~to~~ then we could reconstruct the position and momentum of the electron to arbitrary accuracy.

⇒ will violate the rules of quantum mechanics for the electron.

⇒ Gravity must also satisfy the rules of quantum mechanics, i.e. there should be an uncertainty principle involving the measurement of the metric and its time derivative.

An aside: The same argument may be used to show that the electromagnetic field must be quantized.

Electric field of an electron depends on its coordinate.

Magnetic field of an electron depends on its velocity.

If we could measure the electric and the magnetic fields at every point to arbitrary accuracy, then we could use this information to find the coordinate and momentum of an electron to arbitrary accuracy.

⇒ Violation of uncertainty principle.

⇒ The electric and magnetic fields must satisfy an uncertainty principle among themselves.

$$[E_i(\vec{x},t), B_j(\vec{y},t)] \neq 0$$

Conclusion: A complete theory must contain:

1. Standard Model.
2. Quantum General Relativity.

Why can't we simply quantize the general relativity and append it to the standard model?

Follow the same path as QED.

Dynamical variables: $g_{\mu\nu}(\vec{x},t)$.

Alternatives:

1. operator approach: $g_{\mu\nu}(\vec{x},t)$ are operators satisfying appropriate commutation relations.
2. Path integral approach:

$$\int [\mathcal{D} g_{\mu\nu}] \, e^{-\underbrace{S[g]}_{\text{Action}}} \prod_i \mathcal{O}_i$$

In either approach, quantization of $g_{\mu\nu}$

→ A particle mediating gravitational interaction (Analog of photon).

⇒ Graviton

Photon comes from a vector field $A_\mu(\vec{x},t)$.

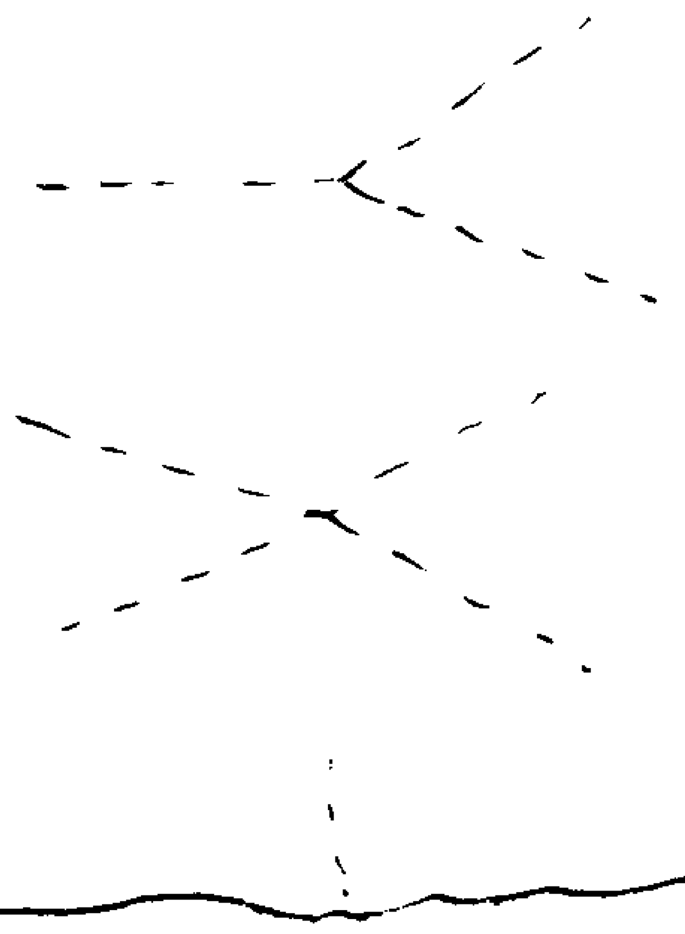
$\rightarrow$ has spin 1.

Graviton comes from a tensor field

$g_{\mu\nu}(\vec{x},t)$

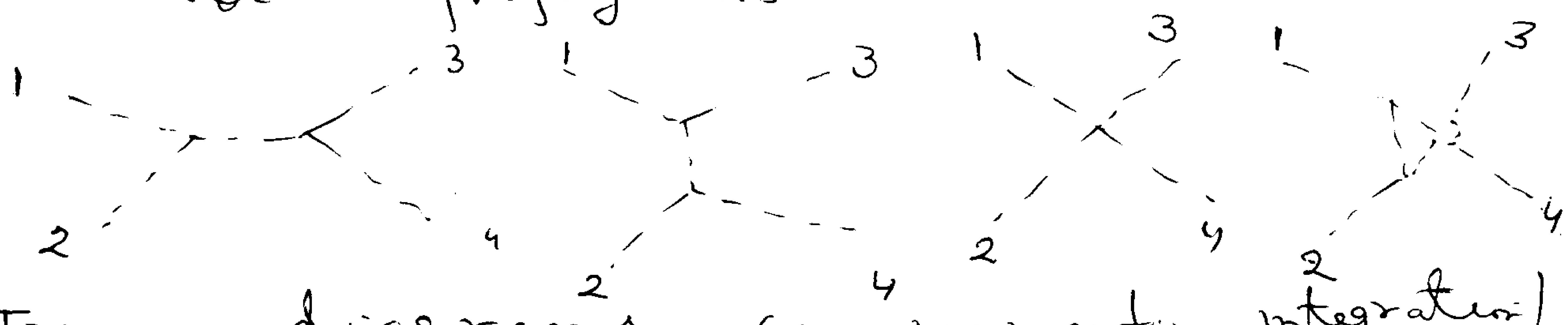
$\rightarrow$ has spin 2.

Graviton propagator

Graviton interaction vertex 

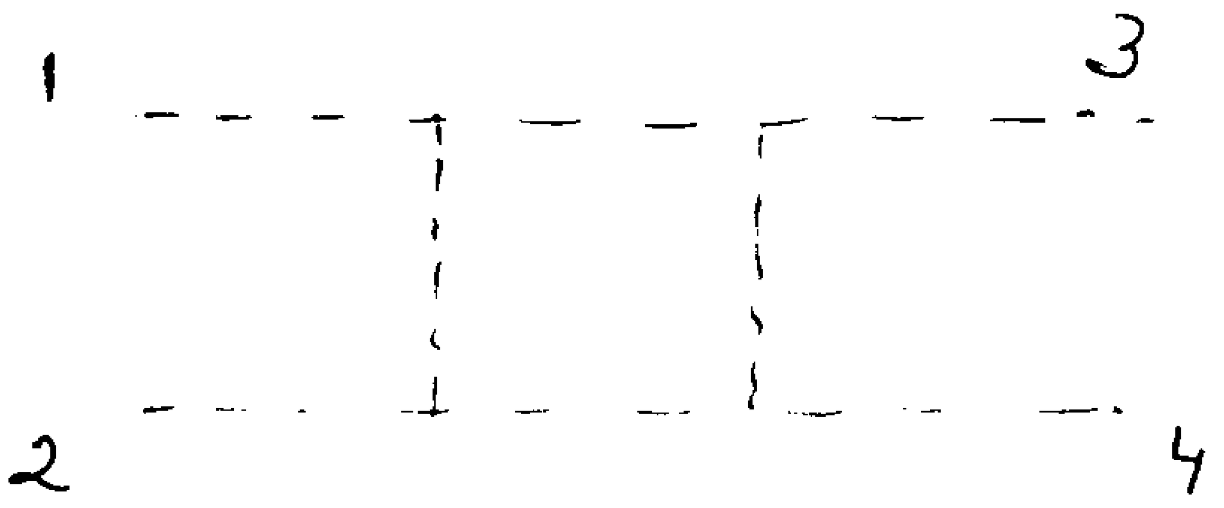
The form of the propagator as well as the vertices can be derived from knowing the eqs. of motion / Lagrangian of classical general relativity.

Graviton scattering amplitude: can be ~~studied~~ computed by putting together the vertices and propagators in the usual fashion



$\rightarrow$ Tree diagrams. (no momentum integration)

Loop diagrams:

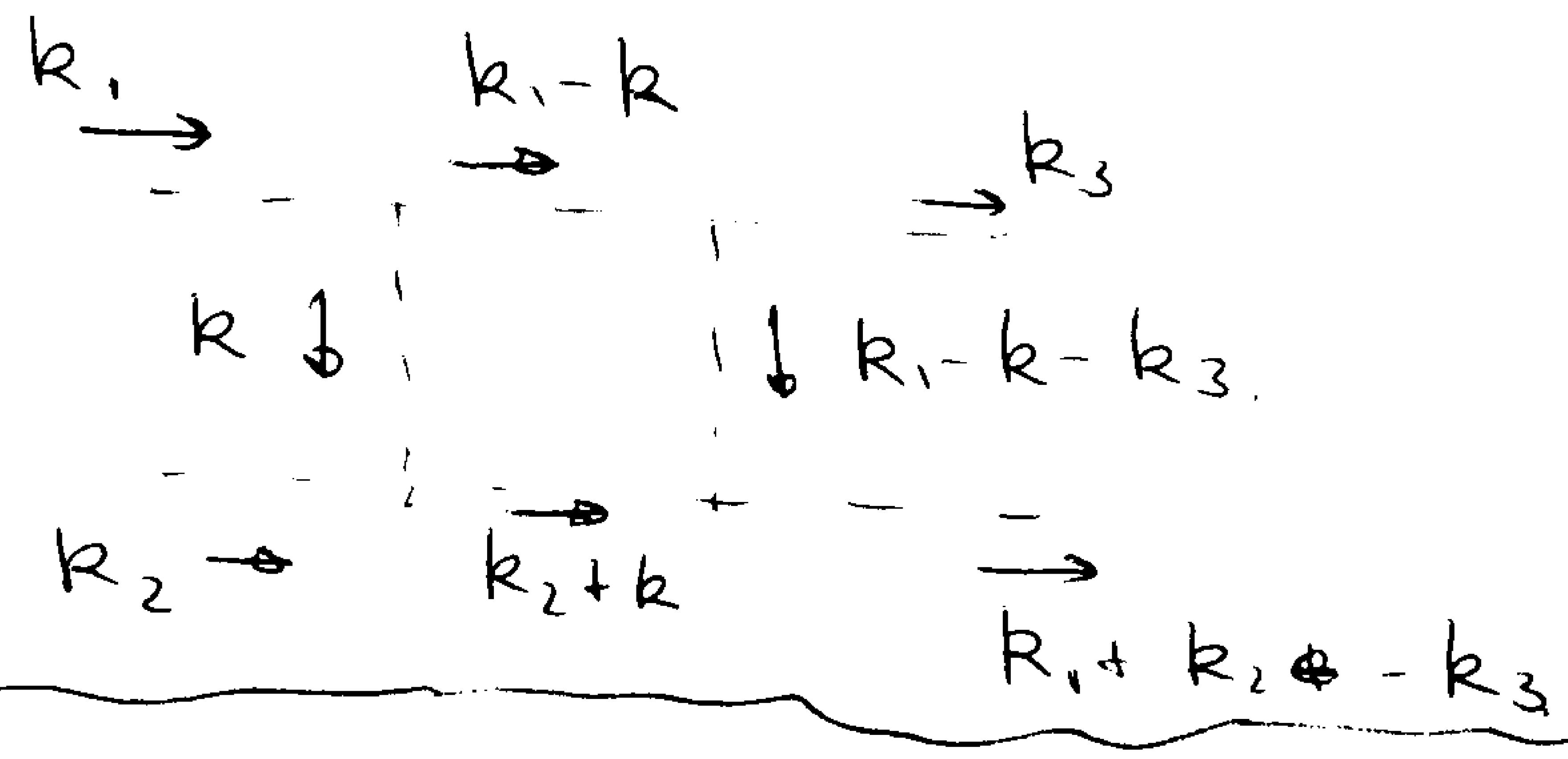


→ Infinite answer.

(Ultraviolet divergence).

Origin:

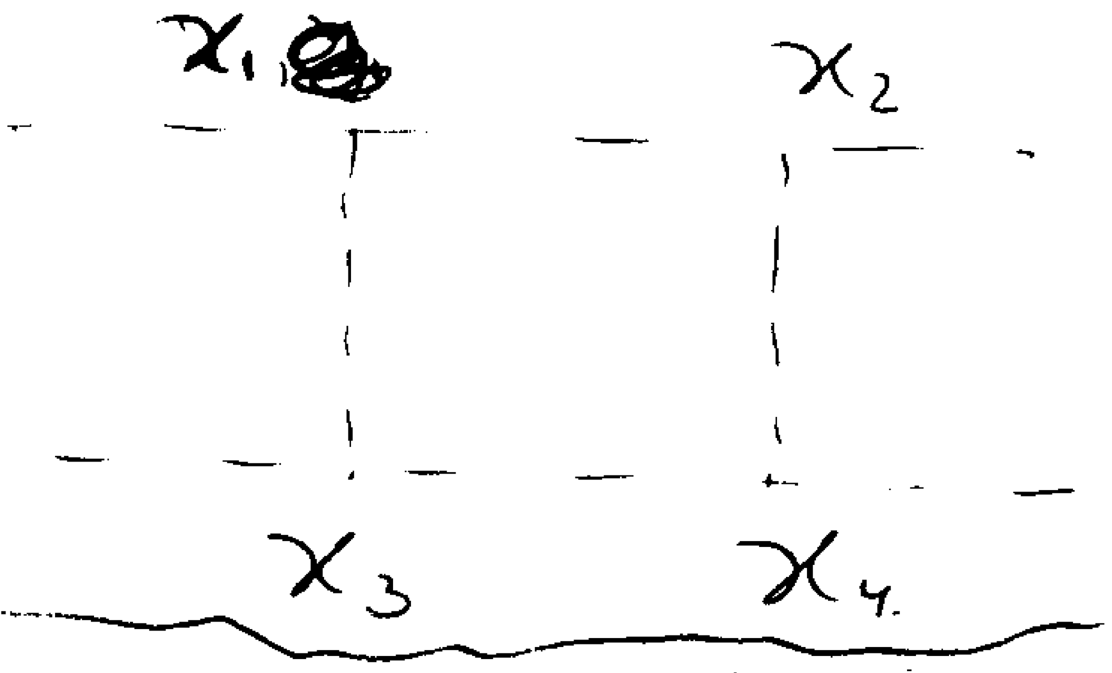
a) In momentum space:



$\int d^4k$ × Propagators × Vertex factors.

→ Divergent for large k.

b) In the position space:



$\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4$ × Propagators × Vertex factors

→ Divergent from the region where all x_i's come close.

Note : such divergences also exist in QED and the standard model.

But they can be removed by the procedure of renormalization.

(Final answer is not divergent although in the intermediate stages there is divergence).

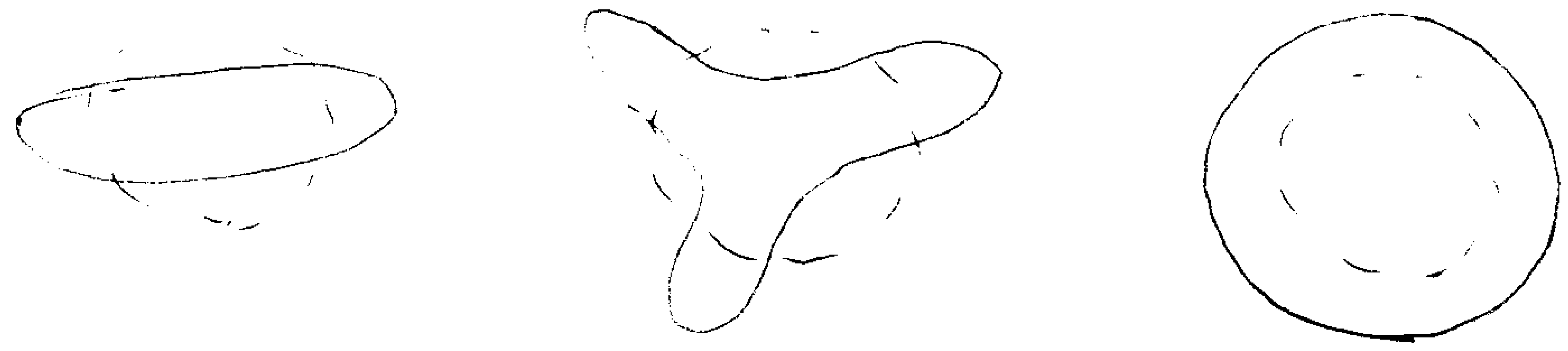
But the renormalization procedure does not work for gravity.

$\Rightarrow$ ~~Quantum~~ Naive quantization of general relativity does not give a consistent theory.

String theory is an attempt to resolve this problem.

Basic idea in string theory:

Different elementary particles are all vibrational modes of a single type of string.



→ Closed strings.

→ Open strings.

Shouldn't we have seen this in the experiment involving elementary particles?

Not if the typical size of a string is smaller than the resolution that an accelerator can provide.

l_s : string length.

To see the stringy nature of elementary particles we need "light" of wavelength $\lambda < l_s$.

$\Rightarrow$ Energy $\frac{\hbar}{\lambda} c > \frac{\hbar}{l_s} c$.

For small l_s we need very high energy particle.

Standard scenario : $l_s \sim 10^{-33}$ cm.

$\Rightarrow \frac{\hbar c}{l_s} \sim 10^{19}$ GeV. $\gg 10^{3}$ GeV.
(Present accelerator energy).

$\Rightarrow$ At present day experiments strings will appear to be point like (particle-like) objects.

$\Rightarrow$

A given string ~~has energy~~

$\Rightarrow$ collection of ∞ no. of harmonic oscillators (normal modes of oscillation).

$\Rightarrow$ ∞ no. of energy levels.

Each such energy level

$\Rightarrow$ a type of elementary particle.

Thus it would appear that a string theory contains infinite number of elementary particles.

$\rightarrow$ Seem to contradict what we know about nature.

Resolution : Most of these ~~elementary~~ states particles have mass $\sim \frac{\hbar c}{l_s} \sim 10^{19}$ GeV

$\rightarrow$ Cannot be produced in present day accelerator.

Only those states which have mass $\ll \frac{\hbar c}{l_s}$ will be observable at current accelerators.

One particular quantum state of the string represents a massless spin 2 particle.

Recall that graviton is a massless spin 2 particle.

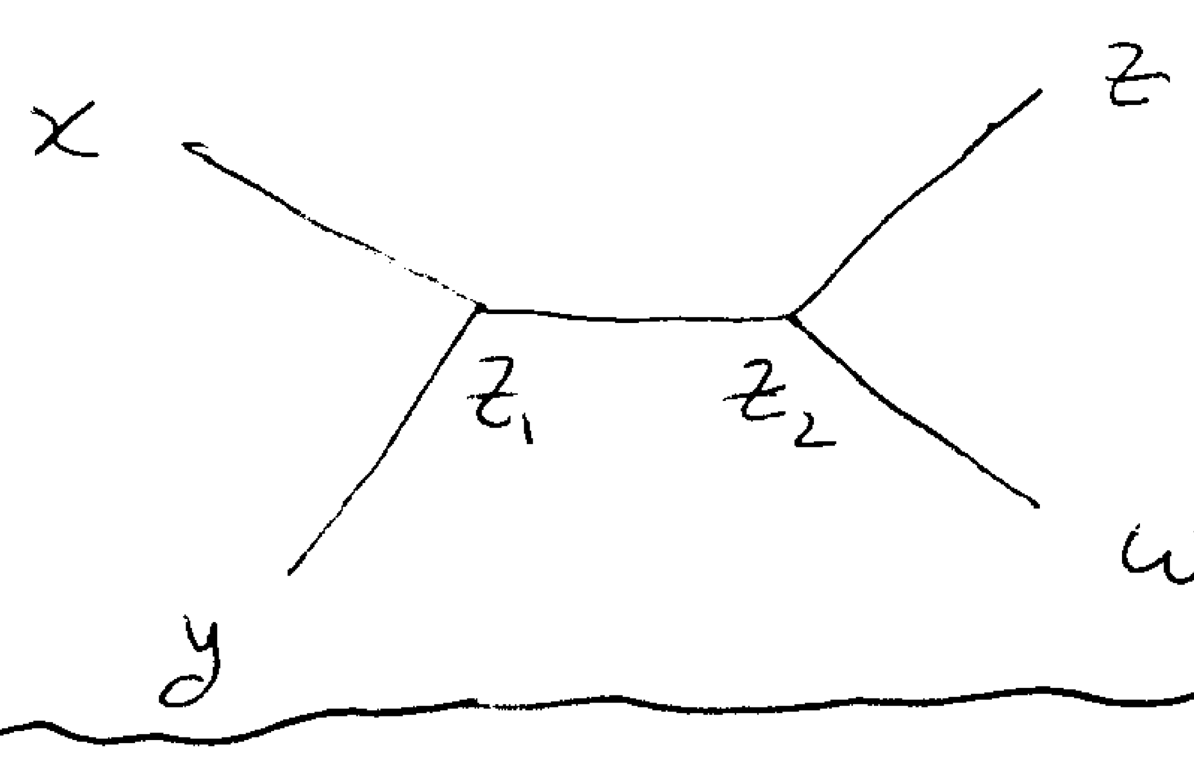
$\Rightarrow$ string theory automatically contains a graviton.

But ~~this~~ before we can say that string theory contains gravity, we ~~must~~ should see if this ~~grav~~ massless spin 2 particle interacts in the same way as a graviton.

String interaction

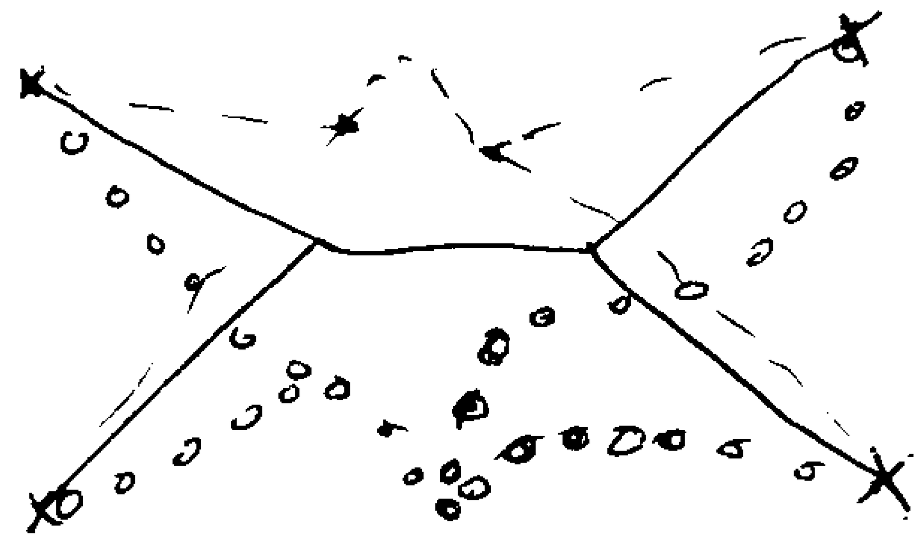
Interaction between point particles.

$\rightarrow$ determined by Feynman diagrams.

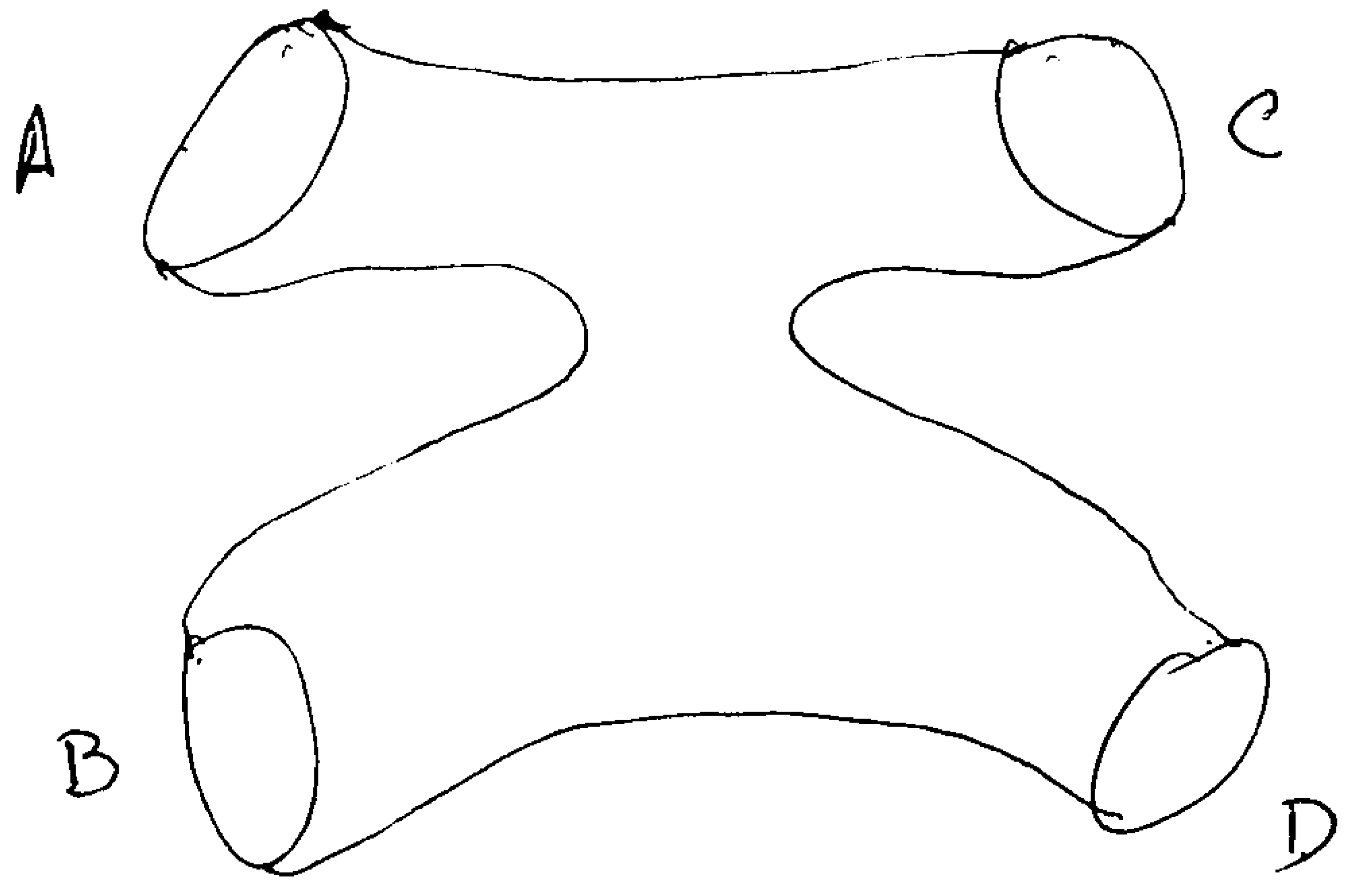


Propagator : $\Delta(x, z_1)$ $\rightarrow$ obtained by summing over all paths between x & z_1 etc.

Full Feynman diagram obtained by summing over all paths with this topology.



We follow the same philosophy for describing string interactions.



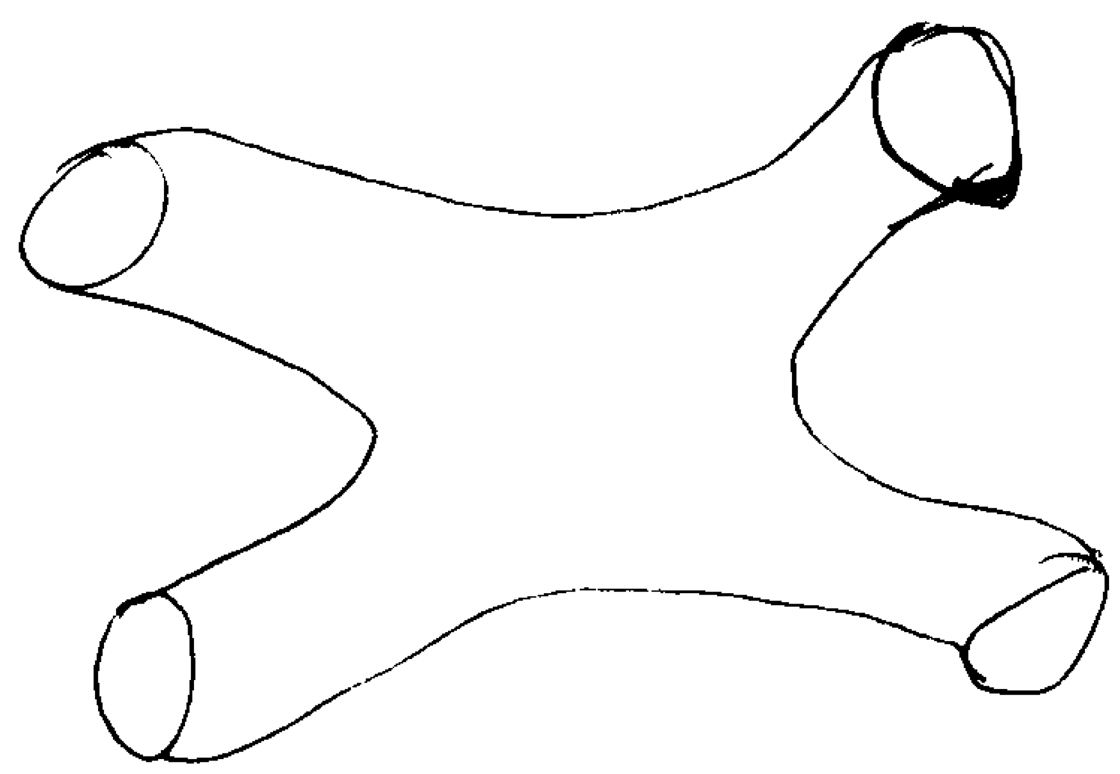
→ Sum over all surfaces describing the process A + B → C + D.

⇒ Scattering amplitude for strings A & B → strings C & D.

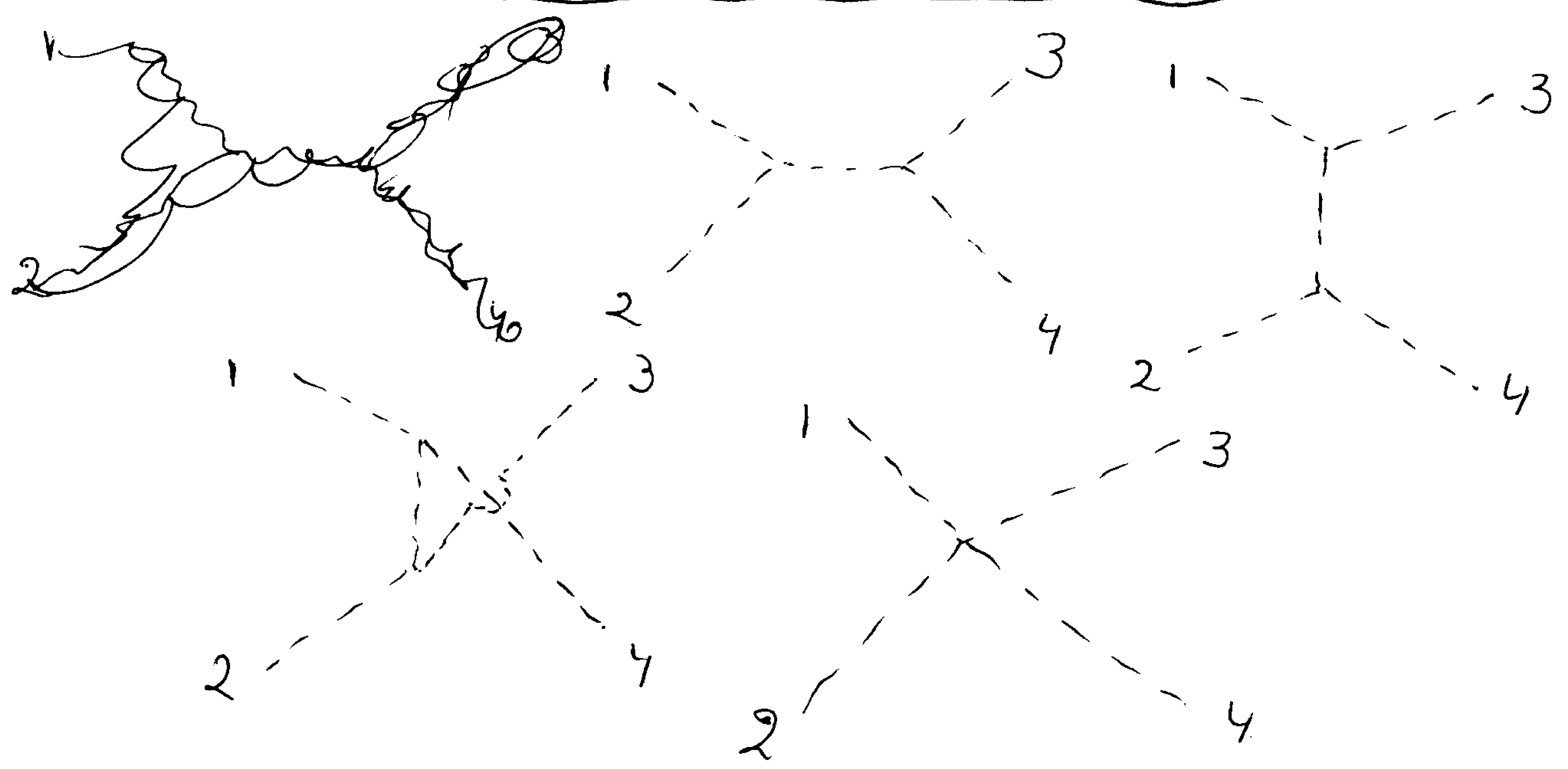
~~But~~ ~~of~~ ~~@~~ If we want scattering of strings in specific energy eigenstates (e.g. the graviton state) we need to take the convolution of the ~~external~~ position space amplitude with the corresponding wave fn.

This way we can calculate the graviton scattering amplitude.

Result from the diagram:



When the energies of the external states are small compared to $\hbar c/l_s$, the ~~grav~~ scattering amplitude calculated from string theory agrees exactly with that calculated from tree level graphs in general relativity.

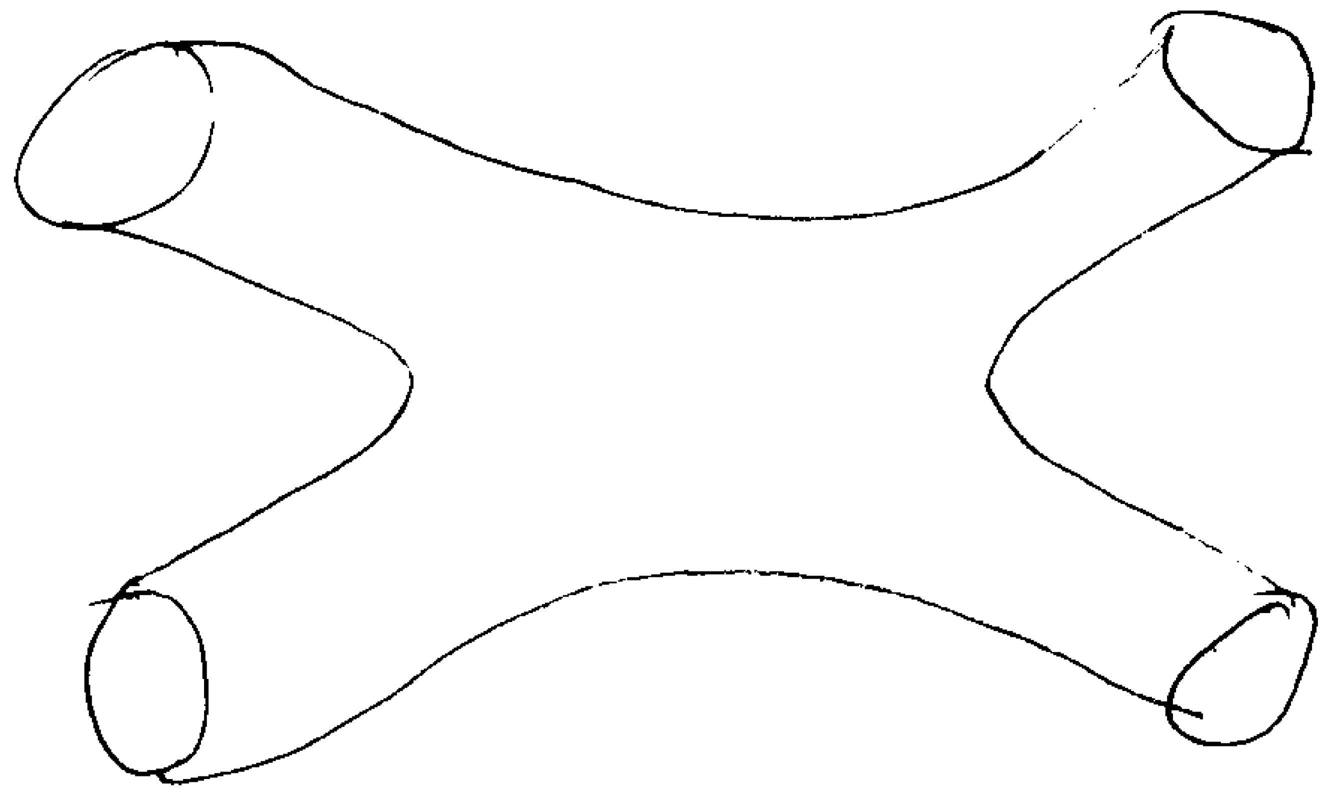


$\Rightarrow$ the spin 2 particle in string theory has interactions exactly like that of the graviton.

This is also true for higher point functions

Note: In quantum general relativity the form of the vertex is fixed by the action giving the Einstein's equation.

In string theory there is no vertex as the surface is smooth.



$\Rightarrow$ We have no freedom in specifying how the strings should interact.

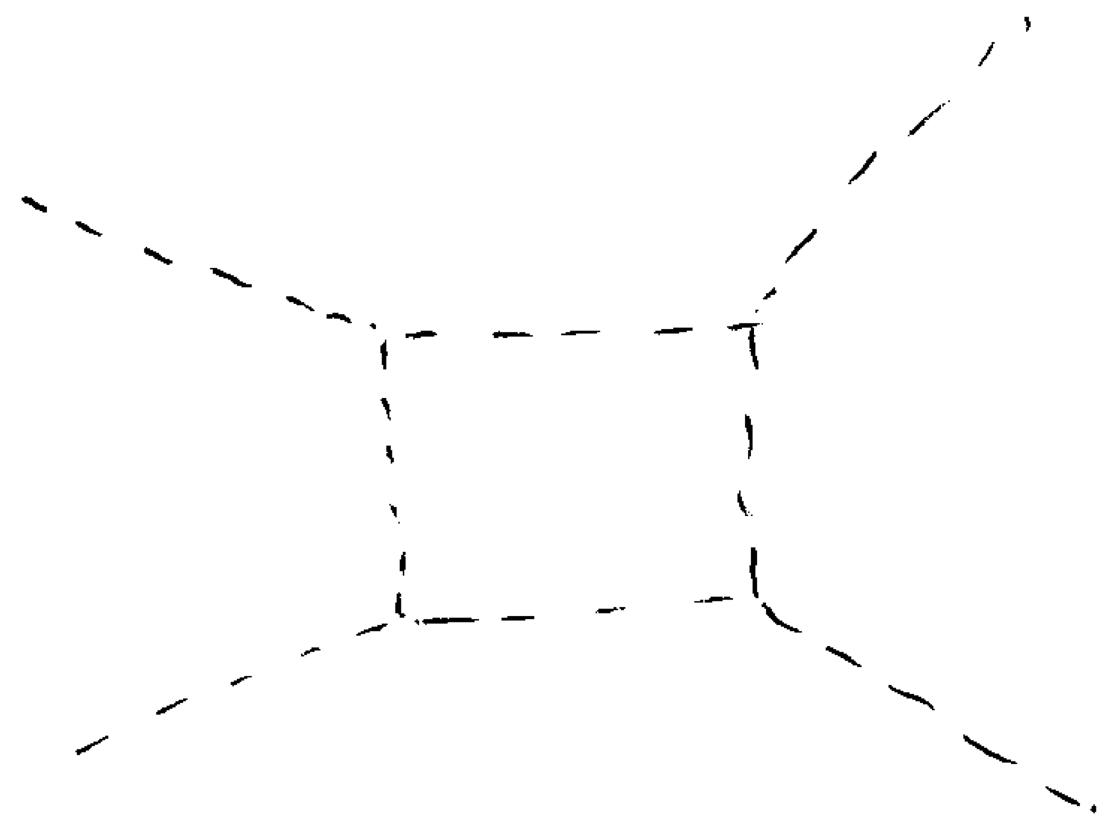
Yet we get out an interaction that ~~exactly~~ ~~describes~~ contains ~~general~~ quantum general relativity.

Thus in string theory

① Quantum mechanics and principles of special relativity are inputs.

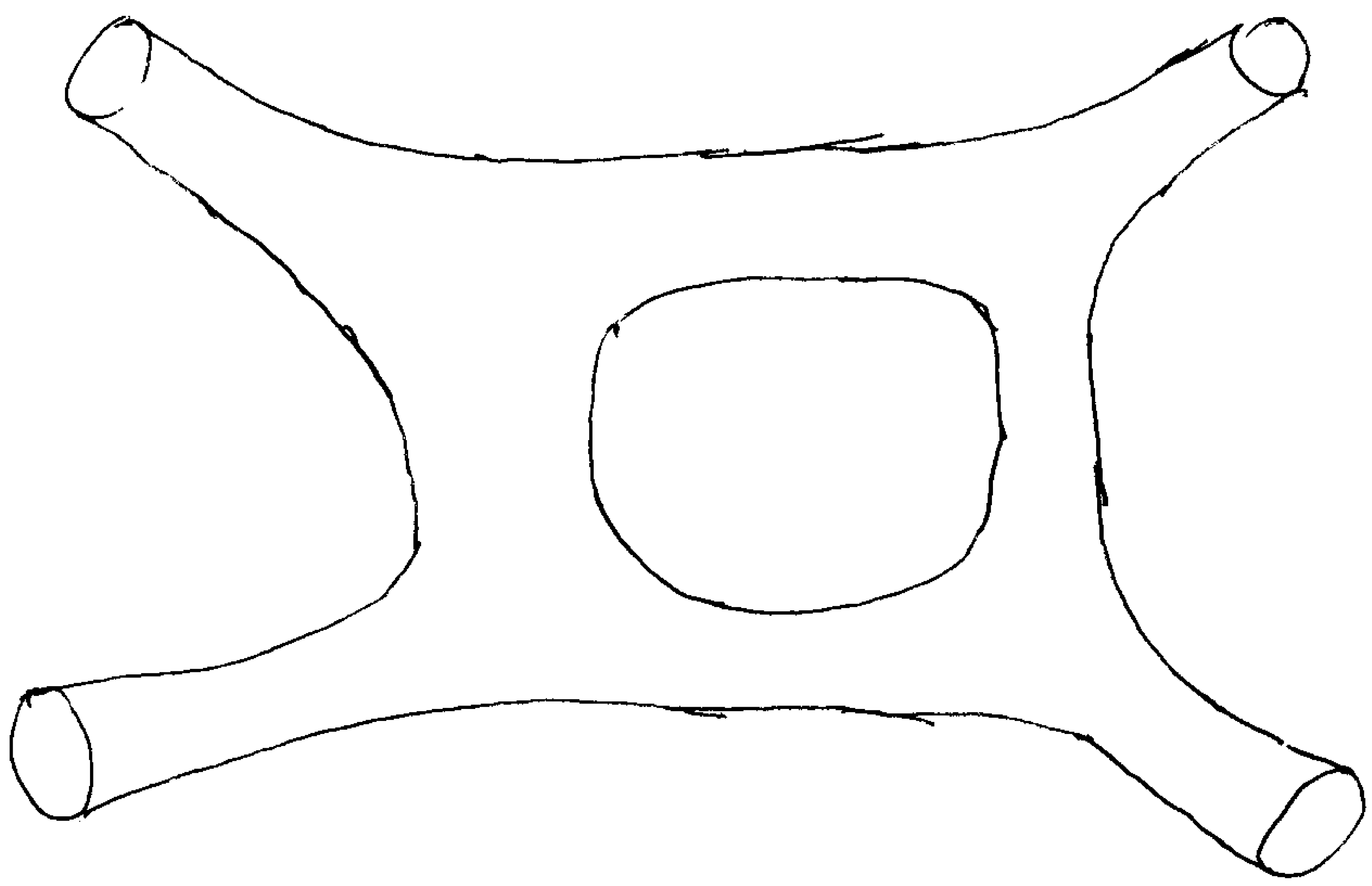
② General Relativity is an output.

What about the problem of ultraviolet divergences?



The divergence comes when all four vertices come close to each other.

What will be the corresponding diagram in string theory?



This diagram has no vertices.

Thus when we sum over all surfaces (rather than paths) we does not encounter configurations analogous to collapsed vertices.

⇒ String theory amplitudes have no ultraviolet (short distance) divergence!

Result: string theory provides a finite quantum theory of gravity!

Note: This is even better than renormalizable quantum field theories since here there is no divergence in the first place.

Problems.

1. This theory is consistent only when the number of space-time dimensions is 26 (instead of 4)!

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2. Besides the graviton and other massless states, this theory contains a particle with negative $mass^2$.
$\Rightarrow$ A "tachyon"

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3. ~~Th~~ This theory contains gravity but not other known interactions in nature (strong, weak, electromagnetic).

Furthermore this does not have fermions.

~~Thus~~ ~~the~~

Conclusion: This does not look like the theory we observe in nature.

The problem of not having gauge interactions can be resolved by including open strings in the theory.

Just like one of the states ~~modes~~ of a closed string represent a massless spin 2 particle and resembles a graviton, some of the modes of open string ~~resemble~~ ~~xx~~ correspond to massless spin 1 particles and resembles non-abelian gauge bosons in their interaction.

But in order to resolve the other problems we need to turn to superstring theory.

(The theory described here is called the bosonic string theory)

Superstring theory:

Strings, besides ~~instead of~~ vibrating in usual space-time, also has some internal degree of freedom.

These internal degrees of freedom are fermionic.

Thus a superstring could be regarded as a collection of infinite no of bosonic and fermionic harmonic oscillators.

→ Can be quantized in a manner similar that that for bosonic strings.

The good features (e.g emergence of gravity and UV finiteness) are preserved, but some (not all) bad features go away.

① The theory is consistent in 10 space-time dimensions instead of 4.

② The tachyonic mode is absent.

③ There are five fully consistent string theories in D = 10

Type IIA, Type IIB, Type I, $E_8 \times E_8$ heterotic, SO(32) heterotic

↓

open + closed string

④ The last 3 theories have non-abelian gauge interaction besides gravity.

⑤ Except for type IIA, the other theories also contain ~~chiral~~ Weyl fermions. (chiral fermions)

Recall that Nature contains chiral fermions like the neutrinoes.

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(5) All of these theories have a special kind of symmetry, known as supersymmetry, which relates bosonic states to fermionic states and vice versa.

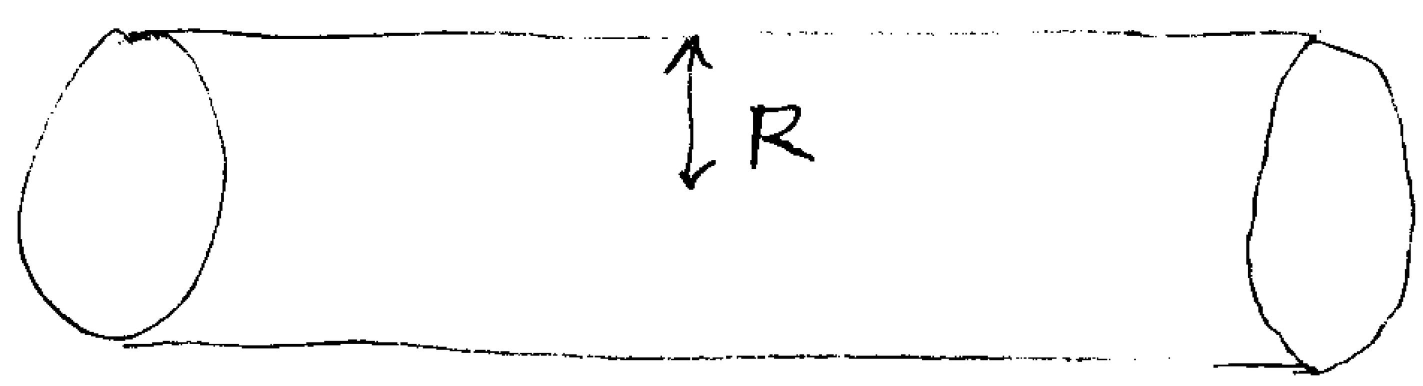
This is closer to nature compared to the bosonic string theory described earlier, but we are still far from a realistic model.

How do we reduce the number of dimensions from 10 to 4 ?

Kaluza - Klein mechanism / Compactification

As a toy example we shall start from a (2+1) dimensional theory and show how one can get a (1+1) dimensional theory from there.

Consider the two dimensional space to describe the surface of an infinite cylinder rather than an infinite plane.

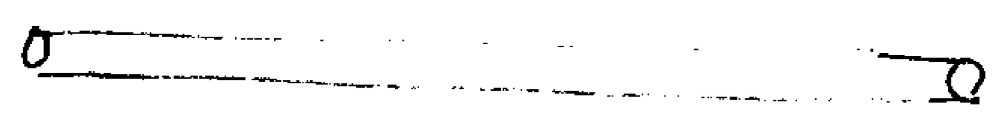


If $R >>$ the ~~radius~~ range of the most powerful telescope, then the scientists living in this space will not know that they are living on a cylinder and not on a plane.

If R is not so large then scientists with powerful telescopes will see it.

As R becomes smaller and smaller more and more people can feel the effect.

But now suppose R is so small that it is smaller than the resolution of the most powerful microscope.



$\Rightarrow$ The space will look one dimensional.

Thus the same space in one limit may look 2-dimensional, and in another limit may look 1-dimensional.

This is the same principle that can be used to make a 9+1 dimensional theory look (3+1)-dimensional.

Take the ten dimensional space of the form:

$$M^4 \times K$$

A 4 dimensional Minkowski space — A 6-dimensional compact space.

Size (K) $\ll$ Resolution of most powerful microscopes (accelerator) $\sim 10^{-17}$ cm.

~~When K is more than~~ When K is more than one dimensional then there are many different possible choices for K.

e.g. for 2-dimensional manifold we can have sphere, torus, double torus etc.

Similarly there are many different six dimensional manifolds.

Not all of them are consistent. (e.g. the metric does not satisfy Einstein's equation)

A certain subset of six dimensional manifolds satisfying all consistency requirements are known as Calabi-Yau manifolds.

Taking K to be a Calabi-Yau manifolds gives ~~a~~ four dimensional theories with all the qualitative features of the standard model.

For example, for a suitable choice of K, we can get four dimensional theories containing

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1. Gravity.
2. $SU(3) \times SU(2) \times U(1)$ gauge interactions.

~~(3) Weyl fermions~~

3. 3 generation of quarks and leptons ~~with~~ including neutrinos which are Weyl fermions.

etc.

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* However so far nobody has found a suitable manifold K which gives exactly the standard model with all the masses and couplings correctly.

More importantly, there is as yet no theoretical principle ~~which~~ for deciding which K is chosen by nature.

Developments during the last 5-Years.

1. All the five apparently different string theories have been found to be different limits of the same underlying theory

→ M-theory.

2. ~~String theory has been used to~~

~~We~~ Our aim will be to quantize a string consistent with the principles of quantum mechanics and special relativity.

We shall do this in analogy with the quantization of relativistic point particle.

Set $\hbar = 1$, $c = 1$

Starting point: classical dynamics of relativistic point particle.

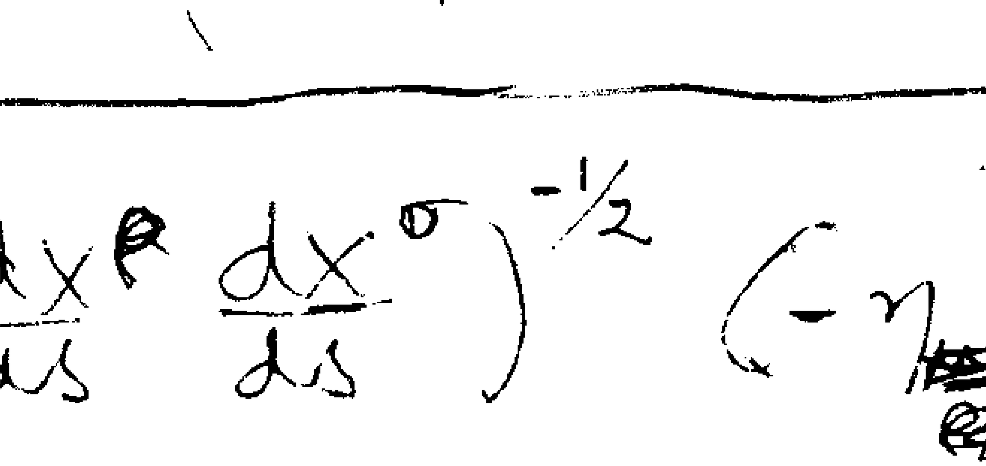
Eq. of motion: $\dfrac{d^2 x^\mu}{d\tau^2} = 0$.

τ: proper time

$$d\tau^2 = -\eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Action:



$$S = c\int_a^b d\tau = c\int_a^b \left(-\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{1/2} ds.$$

$$\delta S = \frac{c}{2}\int_a^b \left(-\eta_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right)^{-1/2}\left(-\eta_{\mu\nu}\frac{d(\delta x^\mu)}{ds}\frac{dx^\nu}{ds} - \eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{d(\delta x^\nu)}{ds}\right)$$

$$= c\int_a^b ds \frac{d}{ds}\left\{\left(-\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{-1/2}\eta_{\mu\nu}\frac{dX^\nu}{ds}\right\}\delta X^\mu(s)$$

$\Rightarrow$ Eqs. of motion:

$$\frac{d}{ds}\left\{\left(-\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{-1/2}\frac{dX^\nu}{ds}\right\} = 0.$$

$$\frac{d\tau}{ds} = \left(-\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{1/2}$$

$$\Rightarrow \frac{d}{ds} = \left(-\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{1/2}\frac{d}{d\tau}$$

$$\Rightarrow \left(-\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{1/2}\frac{d^2X^\nu}{d\tau^2} = 0$$

Either $\eta_{\rho\sigma}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds} = 0$ ~~(light-like parti~~ $\rightarrow$ Massless particle

or $\frac{d^2X^\nu}{d\tau^2} = 0$ $\rightarrow$ Massive particle.

Thus the action that we started with correctly reproduces the equations of motion.

Our next task is to quantize it

This is made difficult by the square root in the action.

$\Rightarrow$ We first need to simplify it.

Strategy: ~~Re~~ Replace the action $\int d\tau$ by another **action** which is classically equivalent to the action $\int d\tau$, i.e. which gives the same eqs. of motion.

Then quantize it.

Consider the action

$$S = \frac{1}{2} c \left[\int ds \left[\left(- \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \eta_{\mu\nu} \right) \lambda(s) + \lambda(s)^{-1} \right] \right]$$

$$\frac{\delta S}{\delta \lambda(s)} = 0.$$

$$\Rightarrow \quad \lambda(s)^{-2} = \left(- \eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)$$

$$\Rightarrow \quad S = c \int ds \left(- \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \eta_{\mu\nu} \right)^{1/2}$$

$\Rightarrow$ Same action.

Alternatively look at X^μ eqs. of motion.

$$\frac{\delta S}{\delta X^\mu(s)} = 0. \qquad \frac{d}{ds} \left(\lambda(s) \, \eta_{\mu\nu} \frac{dX^\nu}{ds} \right) = 0$$

$$\Rightarrow \quad \frac{d}{ds} \left(\left(- \eta_{\rho\sigma} \frac{dX^\rho}{ds} \frac{dX^\sigma}{ds} \right)^{1/2} \eta_{\mu\nu} \frac{dX^\nu}{ds} \right) = 0.$$

$\Rightarrow$ Same eqs. of motion.

A reinterpretation of the action:

Define:

$$g_{ss} = \lambda(s)^{-2}$$

$\rightarrow$ A 1×1 matrix.

$$\det g = \lambda(s)^{-2} \qquad g^{ss} \equiv (g_{ss})^{-1} = \lambda(s)^{2}$$

$$S = \frac{1}{2}c\int ds\, \sqrt{\det g_{ss}}\left[-g^{ss}\,\eta_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} + 1\right]$$

$\Rightarrow$ A one dimensional general covariant action.

g_{ss}: Metric in the space labelled by s.

Reparametrization invariance:

$$s \to f(s), \quad g_{ss} \to g_{ss}\,(f'(s))^{-2}$$

Choosing $f'(s) = (g_{ss})^{-1/2}$ we can fix the gauge

$$g_{ss} = 1.$$

$$\Rightarrow \boxed{S = \frac{1}{2}c\int ds\left[-\eta_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} + 1\right].}$$

g_{ss} eqs. of motion:

$$g_{ss} = -\eta_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} \Rightarrow \boxed{-\eta_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} = 1 \\ \Rightarrow s = \tau}$$

S : Free field action.

$\Rightarrow$ Quantization is straight forward.

$$P_\mu = \frac{\partial S}{\partial(\partial_s X^\mu)} = -c\, \eta_{\mu\nu} \frac{dX^\nu}{ds}$$

$$\Rightarrow \mathcal{H} = P_\mu \frac{dX^\mu}{ds} - L$$

$$= -\frac{1}{2c} P_\mu P_\nu \eta^{\mu\nu} - \frac{1}{2} c$$

Basis : Momentum eigenstates

$$P_\mu |k\rangle = k_\mu |k\rangle$$

k_μ : ~~Momen~~ 4-momentum eigenvalue

P_μ : 4-momentum operator.

Constraint: $\frac{1}{c^2} \eta^{\mu\nu} P_\mu P_\nu = -1$

$$\Rightarrow \eta^{\mu\nu} k_\mu k_\nu = -c^2$$

~~Compare with the~~ This agrees with the expected property of a relativistic point particle if

$$c = m.$$

Light cone gauge

→ A different gauge fixing which destroys manifest covariance, but the constraints are easier to solve at the operator level.

$$S = \frac{1}{2} m \left[\int ds \left[\left(-\frac{dX^\mu}{ds} \frac{dX^\nu}{ds} \eta_{\mu\nu} \right) \lambda(s) + (\lambda(s))^{-1} \right] \right]$$

$$= \frac{dX^+}{ds}\frac{dX^-}{ds} - \frac{dX^i}{ds}\frac{dX^i}{ds}$$

$i = 1, \ldots D-2$

Choose s such that

$$s = X^+ \equiv X^0 + X^3$$

$$X^- \equiv X^0 - X^3$$

$$\Rightarrow \quad S = \frac{1}{2} m \left[\int ds \left[\left(\frac{dX^-}{ds} - \frac{dX^i}{ds}\frac{dX^i}{ds} \right) \lambda(s) + (\lambda(s))^{-1} \right] \right]$$

X^+ eqn. of motion:

$$\frac{d}{ds}\left(\lambda(s) \frac{dX^-}{ds} \right) = 0 .$$

Construct the Hamiltonian:

$$P_- \equiv \frac{\delta S}{\partial(\partial_s X^-)} = \frac{1}{2} m \lambda$$

$$P_\lambda = \frac{\delta S}{\partial(\partial_s \lambda)} = 0$$

$$P_i = \frac{\delta S}{\partial(\partial_s X^i)} = -\lambda m \frac{dX^i}{ds}$$

$$\mathcal{H} = P_- \partial_s X^- + P_\lambda \partial_s \lambda + P_i \partial_s X^i - S$$

$$= -\frac{1}{2m\lambda} P_i P_i - \frac{m}{2\lambda} = -\frac{P_i P_i}{4 P_-} - \frac{m^2}{4 P_-}$$

Commutation relations: can be worked out using the constrained Hamiltonian formalism.

In this case they are the ~~same as the original~~ usual ones.

$$[X^-(t), P_-(t)] = i, \quad [X^i(t), P^j(t)] = i\delta_{ij}$$

Other commutators vanish.

~~Interpretation of $\mathcal{H}$: Generator of s evolution.~~

~~But $s = X^+$.~~

~~$\Rightarrow$ $\mathcal{H}$ should generate X^+ evolution.~~

~~$\Rightarrow$ $\mathcal{H} = P_+$.~~

Eqs of motion:

$$\partial_s(P^i) = 0, \quad \partial_s(P_-) = 0.$$

X^+ eqn

$$\frac{d}{ds}\left(\frac{P_-}{m}\frac{dX^-}{ds}\right) = 0.$$

$$\Rightarrow \frac{dX^-}{ds} = \text{constant}$$

$$\frac{dX^-}{ds} = [\mathcal{H}, X_-] = \frac{P_i P_i}{4P_-^2} + \frac{m^2}{4P_-^2}$$

Since P_i & P_- are constants, this eqn. is automatically satisfied.

Hilbert space state:

$$e^{i(k_- x^- + k^i x^i)} \equiv |k_-, \vec{k}\rangle$$

$$P_- |k_-, \vec{k}\rangle = k_- |k_-, \vec{k}\rangle$$
$$P^i |k_-, \vec{k}\rangle = k^i |k_-, \vec{k}\rangle$$

Note: k_- & k^i are unconstrained variables.

~~Interpretation~~

$$\mathcal{H} |k_-, \vec{k}\rangle = -\frac{\vec{k}^2}{2k_-} - \frac{m^2}{2k_-}$$

Interpretation of $\mathcal{H}$:

$$\mathcal{H} |\,\rangle = i \frac{\partial}{\partial s} |\,\rangle = i \frac{\partial}{\partial x^+} |\,\rangle$$

On the other hand

$$P_+ |\,\rangle = -i \frac{\partial}{\partial x^+} |\,\rangle \quad \text{by definition}$$

$$\Rightarrow \quad \mathcal{H} = -P_+$$

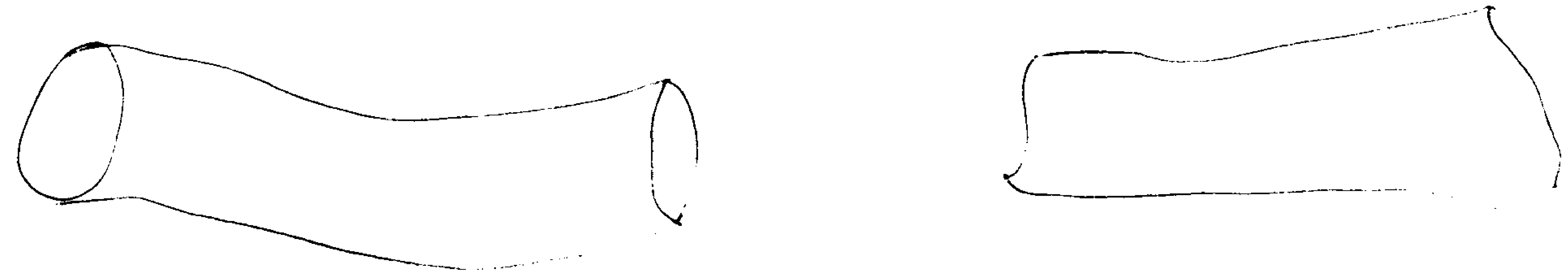
$\mathcal{H}$ eigenvalue $\equiv k_+$.

$$\Rightarrow \quad k_+ = +\frac{\vec{k}^2}{4k_-} + \frac{m^2}{4k_-}$$

$$\rightarrow \quad 4k_+ k_- - \vec{k}^2 = m^2$$

$k_\pm = \frac{1}{2}(k^0 \pm k^{D-1})$ $\quad \Rightarrow \quad (k^0)^2 - (k^{D-1})^2 - \vec{k}^2 = m^2$ → usual relation.

Quantization of a relativistic string:



Action $S = -T \int dA$

T: Tension, dA: area element

Parametrize the surface by two parameters (σ, τ)

$$\xi^0 = \tau \quad , \quad \xi^1 = \sigma$$

$$S = -T \int d^2\xi \sqrt{-\det(h_{\alpha\beta})}$$
→ Reparametrization invariant
$\xi^\alpha \to f^\alpha(\vec{\xi})$

$$h_{\alpha\beta} = (\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}).$$
$X^\mu(\sigma,\tau)$: 2-d field theory

First we need to remove $\sqrt{\ }$.

Alternative action:

$$\tilde{S} = -\frac{T}{2} \int d^2\xi \sqrt{-\det\gamma}\ \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\gamma_{\alpha\beta}$: 2×2 symmetric matrix valued field.

$\gamma^{\alpha\beta}$: Matrix inverse of $\gamma_{\alpha\beta}$.

Eq of motion for γ:

$$\delta \tilde{S} = -\frac{T}{2} \int d^2\xi \, \left[\delta\sqrt{-\det\gamma}\, \gamma^{\alpha\beta} + \sqrt{-\det\gamma}\, \delta\gamma^{\alpha\beta} \right] \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\delta\sqrt{-\det\gamma} = \frac{1}{2}(-\det\gamma)^{+1/2} \left(+\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}\right)$$

$$\delta\gamma^{\alpha\beta} = -\gamma^{\alpha\alpha'} \delta\gamma_{\alpha'\beta'} \gamma^{\beta'\beta}$$

$$\Rightarrow \delta\tilde{S} = -\frac{T}{2} \int d^2\xi \, \delta\gamma_{\alpha\beta} \left[\frac{1}{2}(-\det\gamma)^{-1/2} \gamma^{\alpha\beta} \gamma^{\alpha'\beta'} - (-\det\gamma)^{1/2} \gamma^{\alpha'\alpha} \gamma^{\beta'\beta} \right] \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu \eta_{\mu\nu}$$

$$= +\frac{T}{2} \int d^2\xi \, (-\det\gamma)^{1/2} \gamma^{\alpha'\alpha} \gamma^{\beta'\beta} \delta\gamma_{\alpha\beta} \left[\eta_{\mu\nu} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu - \frac{1}{2} \gamma_{\alpha'\beta'} \gamma^{\alpha''\beta''} \partial_{\alpha''} X^\mu \partial_{\beta''} X^\nu \eta_{\mu\nu} \right]$$

$$T_{\alpha'\beta'} = \eta_{\mu\nu} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu - \frac{1}{2} \eta_{\mu\nu} \gamma^{\alpha''\beta''} \partial_{\alpha''} X^\mu \partial_{\beta''} X^\nu \gamma_{\alpha'\beta'}$$

$$\Rightarrow \gamma_{\alpha'\beta'} = F(\vec{\xi})\, \eta_{\mu\nu} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu = F(\vec{\xi})\, h_{\alpha'\beta'}$$

solves $T_{\alpha'\beta'} = 0$ for any $F(\vec{\xi})$

Substitute into the expression for $\tilde{S}$:

$$(-\det \gamma) = -F^2(\vec{\xi}) \det h$$

$$\tilde{S} = -\frac{T}{2} \int d^2\xi \, (-\det h)^{1/2} F(\vec{\xi}) \underbrace{\gamma^{\alpha\beta} \gamma_{\alpha\beta}}_{2} f^{-1}(\vec{\xi})$$

$$= -T \int d^2\xi \, (-\det h)^{1/2}$$

Thus $\tilde{S}$ and S describes the same action after we eliminate $\gamma_{\alpha\beta}$ in $\tilde{S}$ by its equations of motion.

Symmetries of $\tilde{S}$.

① Reparametrization invariance:

$$X^{\mu\prime}(\vec{\xi}') = X^{\mu}(\vec{\xi}) \qquad \xi'^{\alpha} = f^{\alpha}(\vec{\xi})$$

$$\gamma'_{\alpha\beta}(\vec{\xi}') = \gamma_{\alpha'\beta'} \frac{\partial \xi^{\alpha'}}{\partial \xi'^{\alpha}} \frac{\partial \xi^{\beta'}}{\partial \xi'^{\beta}}$$

$$\tilde{S}(X^{\mu}, \gamma_{\alpha\beta}) = \tilde{S}(X'^{\mu}, \gamma'_{\alpha\beta})$$

$\Rightarrow$ Two parameter family of gauge symmetry.

② Weyl invariance:

$$X'^{\mu}(\vec{\xi}) = X^{\mu}(\vec{\xi})$$

$$\gamma'_{\alpha\beta}(\vec{\xi}) = e^{\phi(\vec{\xi})} \gamma_{\alpha\beta}(\vec{\xi})$$

$\phi(\vec{\xi})$ arbitrary function

$$\tilde{S}(X^{\mu}, \gamma_{\alpha\beta}) = \tilde{S}(X'^{\mu}, \gamma'_{\alpha\beta})$$

$\Rightarrow$ A one parameter family of local symmetries.

---

(3) Lorentz transformation:

$$X'^{\mu}(\vec{\xi}) = \Lambda^{\mu}{}_{\nu} X^{\nu}(\vec{\xi})$$

$$\gamma'_{\alpha\beta}(\vec{\xi}) = \gamma_{\alpha\beta}(\vec{\xi})$$

Λ : Lorentz trs. matrix.

$$\Lambda \eta \Lambda^{T} = \eta.$$

$$S(X'^{\mu}, \gamma_{\alpha\beta}) = S(X^{\mu}, \gamma_{\alpha\beta})$$

$\to$ global symmetry transformation.

3 gauge symmetries

$\Rightarrow$ We can fix three of the D+3 fields X^{μ}, $\gamma_{\alpha\beta}$.

We have different possible choices.

1 Covariant gauge fixing:

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$\Rightarrow \tilde{S} = -\frac{T}{2}\int d^2\xi \; \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\Rightarrow$ A free field theory.

Constraints: $\gamma_{\alpha\beta}$ equations of motion:

$$T_{\alpha'\beta'} = 0.$$

$$\parallel$$

$$\eta_{\mu\nu}\partial_{\alpha'}X^\mu\partial_{\beta'}X^\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu \eta_{\alpha'\beta'}$$

Note: $\eta^{\alpha'\beta'}T_{\alpha'\beta'} = 0$ identically.

$\Rightarrow$ One of the three constraints is redundant.

Define: $u \equiv \xi^0 + \xi^1 \equiv \tau + \sigma$, $v = \xi^0 - \xi^1 = \tau - \sigma$

Independent constraints:

$$T_{uu} \equiv \frac{1}{\alpha'}\eta_{\mu\nu}\partial_u X^\mu \partial_u X^\nu = 0. \qquad \underline{\text{check}}$$

$$T_{vv} \equiv \frac{1}{\alpha'}\eta_{\mu\nu}\partial_v X^\mu \partial_v X^\nu = 0.$$

Strategy:

① Quantize the free scalar field theory of the X^μ's

② Impose the physical state condition

$$\langle phys | T_{uu} | phys' \rangle = 0$$
$$\langle phys | T_{vv} | phys' \rangle = 0$$

Step 1 for closed string theory

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$$

$$X^\mu(\sigma, \tau) = \sum_{n\neq 0} \phi^\mu_n(\tau)\, e^{in\sigma} + x^\mu$$

Substitute into the action:

$$\tilde{S} = + \frac{T}{2} 2\pi \Big[\sum_{n \neq 0} \left(\partial_\tau \phi^\mu_{-n}\, \partial_\tau \phi^\nu_{+n} - n^2 \phi^\mu_{-n} \phi^\nu_n \right) \eta_{\mu\nu} + \partial_\tau x^\mu \partial_\tau x^\nu \eta_{\mu\nu} \Big]$$

$$T = \frac{1}{2\pi\alpha'} \qquad \phi^\mu_{-n} = (\phi^\mu_n)^*$$

$$\Rightarrow \tilde{S} = \frac{1}{2\alpha'} \Big[2\sum_{n>0} \left[(\partial_\tau \phi^\mu_n)^* (\partial_\tau \phi^\nu_n) - n^2 (\phi^\mu_n)^* \phi^\nu_n \right] \eta_{\mu\nu} + \eta_{\mu\nu}\, \partial_\tau x^\mu \partial_\tau x^\nu \Big]$$

⇒ A collection of harmonic oscillators.

Conjugate variable to ϕ^ν_n:

$$\Pi_{\nu n} = \frac{\delta \tilde{S}}{\delta(\partial_\tau \phi^\nu_n)} = \frac{1}{\alpha'}\left(\partial_\tau \phi^\mu_{-n}\right) \eta_{\mu\nu}, \qquad p_\nu = \frac{\delta S}{\delta(\partial_\tau x^\nu)} = \frac{1}{\alpha'}\eta_{\mu\nu}\partial_\tau x^\nu$$

$$\left[\phi^\mu_n, \Pi_{\nu m}\right] = i\,\delta^\mu_\nu\, \delta_{m,n} \qquad \left[x^\mu, p_\nu\right] = i\,\delta^\mu_\nu$$

$$H = \sum_{n\neq 0} \Pi_{\nu n}\,\partial_\tau \phi^\nu_n - \tilde{S} + \Pi_\nu\, \partial_\tau x^\nu$$

$$= (\alpha') \sum_{n>0} \Pi_{\mu(-n)}\, \Pi_{\nu n}\, \eta^{\mu\nu} + \frac{1}{\alpha'} \sum_{n>0} \left(\phi^\mu_{-n}\right)^* \phi^\nu_n\, \eta_{\mu\nu}$$

$$+ \frac{\alpha'}{2}\eta^{\mu\nu} p_\mu p_\nu$$

$$\Pi^*_{\mu n} = \Pi_{\mu(-n)}, \qquad \left(\phi^\mu_n\right)^* = \phi^\mu_{-n}$$

$$\Rightarrow H = \alpha' \sum_{n>0} \Pi_{\mu(-n)}\, \Pi_{\nu n}\, \eta^{\mu\nu} + \frac{1}{\alpha'}\sum_{n>0} \phi^\mu_{-n}\, \phi^\nu_n\, \eta_{\mu\nu}$$

$$+ \frac{\alpha'}{2}\eta^{\mu\nu} p_\mu p_\nu$$

$$\alpha^\mu_{-n} = \cancel{\phi^\mu_n + i\eta^{\mu\nu}\Pi_{\nu n}}\ \frac{\sqrt{\alpha'}}{\sqrt{2}}\left(\eta^{\mu\nu}\,\Pi_{\nu n} + \frac{i n}{\sqrt{\alpha'}}\phi^\mu_{-n}\right)$$

$$\left(\alpha^\mu_{-n}\right)^\dagger = \frac{1}{\sqrt{2}}\left(\eta^{\mu\nu}\sqrt{\alpha'}\,\Pi_{\nu(-n)} - \frac{i n}{\sqrt{\alpha'}}\phi^\mu_n\right) = \alpha^\mu_{+n}$$

$$\tilde{\alpha}^\mu_{+n} = \frac{1}{\sqrt{2}}\left(\eta^{\mu\nu}\sqrt{\alpha'}\,\Pi_{\nu n} - \frac{i n}{\sqrt{\alpha'}}\phi^\mu_{-n}\right), \qquad \left(\tilde{\alpha}^\mu_{+n}\right)^* = \tilde{\alpha}^\mu_{-n}$$

$$[\alpha^\mu_{-n}, \alpha^\nu_{-m}] = \tfrac{1}{2}[\eta^{\mu\rho}\Pi_{\rho n} + in\phi^\mu_{-n}, \eta^{\nu\sigma}\Pi_{\sigma m} + im\phi^\nu_{-m}]$$

$$= \tfrac{1}{2}[im\,\eta^{\mu\nu}(-i\delta_{m+n,0}) + in\,\eta^{\mu\nu}(i\delta_{m+n,0})]$$

$$= m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\tilde\alpha^\mu_{+n}, \tilde\alpha^\nu_{+m}] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\alpha^\mu_n, \tilde\alpha^\mu_n] = 0$$

$$\mathcal{H} = \sum_{n>0}(\alpha^\mu_{-n}\alpha^\nu_n + \tilde\alpha^\mu_{-n}\tilde\alpha^\nu_n)\eta_{\mu\nu} + K$$

$$+ \frac{\alpha'}{2}\eta^{\mu\nu}p_\mu p_\nu$$

K: normal ordering const.

$$p^\mu|\{k\}\rangle = k^\mu|\{k\}\rangle \qquad \alpha^{\mu_1}_{-n_1}\cdots\alpha^{\mu_k}_{-n_k}\tilde\alpha^{\nu_1}_{-m_1}\cdots\tilde\alpha^{\nu_l}_{-m_l}|\{k\}\rangle$$

Constraint eqs:

$$T_{uu} = 0$$

$\alpha^\mu_{-n}, \tilde\alpha^\mu_{-n}$: creation op.

$\alpha^\mu_n, \tilde\alpha^\mu_n$: Annihilation op.

$$\Rightarrow \frac{1}{4}\eta_{\mu\nu}(\partial_\tau - \partial_\sigma)X^\mu(\partial_\tau - \partial_\sigma)X^\nu = 0$$

$$\Rightarrow (\partial_\tau - \partial_\sigma)X^\mu$$

$$= \sum(\partial_\tau\phi^\mu_n - in\phi^\mu_n)e^{in\sigma} + \partial_\tau x^\mu$$

$$= \sum_n(\alpha'\Pi_{\nu(n)}\eta^{\mu\nu} - in\phi^\mu_n)e^{in\sigma} + \alpha'\eta^{\mu\nu}p_\nu$$

$$= \sqrt{2\alpha'}\sum_n\alpha^\mu_n e^{in\sigma} + \alpha'\eta^{\mu\nu}p_\nu$$

T_{uu}

$$= \alpha' \sum_n L_m e^{2im\sigma}$$

$$L_m = \frac{\alpha'}{2} \sum_{n \neq 0} \alpha^{\mu}_{m-n} \alpha^{\nu}_{n} \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{\mu}_{m} p_\mu$$

$L^{+}_{m} = L_{-m}$ for $m \neq 0$.

$$L_0 = \sum_{n>0} \alpha^{\mu}_{-n} \alpha^{\nu}_{n} \eta_{\mu\nu} + \frac{(\alpha')^2}{4} \eta^{\mu\nu} p_\mu p_\nu$$

$+ c$ ⟶ normal ordering constant.

Similarly

$$T_{v\theta} = \alpha' \sum \tilde{L}_n e^{-2in\sigma}$$

$$\tilde{L}_n = \frac{1}{2} \sum_n \tilde{\alpha}^{\mu}_{m-n} \tilde{\alpha}^{\nu}_{n} \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \tilde{\alpha}^{\mu}_{m} p_\mu$$

$\tilde{L}^{+}_{n} = \tilde{L}_{-n}$ for $m \neq 0$

$$\tilde{L}_0 = \sum_{n>0} \tilde{\alpha}^{\mu}_{-n} \tilde{\alpha}^{\nu}_{n} \eta_{\mu\nu} + \frac{(\alpha')^2}{4} \eta^{\mu\nu} p_\mu p_\nu$$

$+ \tilde{c}$

⟶ normal ordering constant

Procedure for Implementing the constraint:

$$L_n \,|\text{phys}\rangle = 0 \quad \text{for} \quad n \geq 0$$
$$\tilde{L}_n \,|\text{phys}\rangle = 0 \quad \text{for} \quad n \geq 0$$

$$\Rightarrow \quad \langle \text{phys}|\, L_n^{+} = 0 \quad \Rightarrow \quad \langle \text{phys}|\, L_{-n} = 0 \quad \text{for } n \geq 0$$

$$\langle \text{phys}'|\, \tilde{L}_n^{+} = 0 \quad \Rightarrow \quad \langle \text{phys}'|\, \tilde{L}_{-n} = 0$$

Then
$$\left.\begin{array}{l} \langle \text{phys}'|\, L_n \,|\text{phys}\rangle = 0 \\ \langle \text{phys}'|\, \tilde{L}_n \,|\text{phys}\rangle = 0 \end{array}\right\} \text{ for all } n.$$

Summary

Hilbert space states are given by

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k}\, \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell}\, \underbrace{|\{k\}\rangle}_{\text{Fock vacuum}}$$

Physical states satisfy

$$L_n |\text{phys}\rangle = 0, \quad \tilde{L}_n |\text{phys}\rangle = 0 \quad \text{for} \quad n > 0$$

Example

$|\{k\}\rangle$

$$L_m |\{k\}\rangle = \frac{1}{2} \sum_{n \neq 0} \alpha^{\mu}_{m-n}\, \alpha^{\nu}_{n}\, \eta_{\mu\nu}\, |\{k\}\rangle$$
$$+ \frac{1}{2} \sqrt{2\alpha'}\, \alpha^{\mu}_{m}\, k_\mu\, |\{k\}\rangle \quad \text{for } m > 0$$

Either $m-n$ or $n > 0$ in first term.

$$\Rightarrow \quad L_m |\{k\}\rangle = 0 \quad \text{for} \quad m > 0.$$

$$L_0 |\{k\}\rangle = \sum_{n>0} \alpha^\mu_{-n}\alpha^\mu_n \eta_{\mu\nu} |\{k\}\rangle + \frac{\alpha'}{4}\eta^{\mu\nu}k_\mu k_\nu |\{k\}\rangle + c\,|\{k\}\rangle$$

$\alpha^\mu_n |\{k\}\rangle = 0$ for $n>0$

$$\Rightarrow \quad L_0 |\{k\}\rangle = \left(\frac{\alpha'}{4}\eta^{\mu\nu}k_\mu k_\nu + c\right)|\{k\}\rangle$$

Physical state condition:

$$\frac{\alpha'}{4}\eta^{\mu\nu}\underbrace{k_\mu k_\nu}_{=(-k^0)^2+(k^1)^2} + c = 0$$

$$\Rightarrow \quad (k^0)^2 - (k^1)^2 = \frac{4}{\alpha'}c.$$

$$\tilde{L}_0 |\{k\}\rangle = 0 \quad \Rightarrow \quad (k^0)^2 - (k^1)^2 = \frac{4}{\alpha'}\tilde{c}.$$

$$\Rightarrow \quad c = \tilde{c}$$

$\Rightarrow$ describes a single particle state with

$$m_0^2 = \frac{4}{\alpha'}c$$

Thus the state $|\{k\}\rangle$ of the string represents a particle with $\text{mass}^2 = \frac{4}{\alpha'}c$

We shall now examine the physical state condition on other states.

Check:

$$[L_m, \alpha^\mu_{-n}] = n\,\alpha^\mu_{m-n} \quad \text{for } m \neq n.$$

$$= \frac{1}{2}\sqrt{2\alpha'}\, n\, p^\mu \quad \text{for } m = n$$

$$[L_m, p_\mu] = 0.$$

$$[\tilde{L}_m, \tilde{\alpha}^\mu_{-n}] = n\,\tilde{\alpha}^\mu_{m-n} \quad \text{for } m \neq n$$

$$= \frac{1}{2}\sqrt{2\alpha'}\, n\, p^\mu \quad \text{for } m = n.$$

$$L_0\, \alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k}\, \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k} |\{k\}\rangle$$

$$= \left(\sum_{i=1}^{k} m_i + c + \frac{\alpha'}{4}\eta^{\mu\nu} k_\mu k_\nu \right) \alpha^\mu_{-1} \cdots \tilde{\alpha}_{-n_k} |\{k\}\rangle$$

$$\Rightarrow \sum_{i=1}^{k} m_i + c + \frac{\alpha'}{4}\eta^{\mu\nu} k_\mu k_\nu = 0.$$

$$\Rightarrow -\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'}\left(c + \sum_{i=1}^{k} m_i\right)$$

$$\parallel$$

$$(k^0)^2 - \vec{k}^2$$

$$\parallel$$

$$(\text{mass})^2$$

$\tilde{L}_0 = 0$ constraint

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'}\left(c + \sum_{i=1}^{l} n_i\right)$$

$$\Rightarrow \sum_{i=1}^{k} m_i = \sum_{i=1}^{l} n_i \rightarrow \text{level}.$$

Thm

$$L_m \alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k} |\{k\}\rangle = 0$$

$$\text{if} \quad m > \sum_{i=1}^{k} m_i$$

Proof :

$$L_m \alpha^{\mu_1}_{-m_1} = m_1 \alpha^{\mu_1}_{m-m_1} + \alpha^{\mu_1}_{-m_1} L_m$$

$\Rightarrow$ we get

$$m_1 \alpha^{\mu_1}_{m-m_1} \alpha^{\mu_1}_{-m_2} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k} |\{k\}\rangle$$

$$+ \alpha^{\mu_1}_{-m_1} L_m \alpha^{\mu_1}_{-m_2} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k} |\{k\}\rangle.$$

In the first term:

$$m > \sum_{i=1}^{k} m_i$$

$$\Rightarrow m - m_1 > \sum_{i=2}^{k} m_i$$

$$\Rightarrow m - m_1 > m_i \quad \forall\ i = 2, \cdots k.$$

$\alpha^{\mu_1}_{m-m_1}$ can be taken to the ~~left~~ right and made to annihilate $|\{k\}\rangle$.

Second term can be analyzed as before.

Continuing this way we can bring L_m to the extreme ~~left~~ right & annihilate $|\{k\}\rangle$.

Similarly $\tilde{L}_m$ annihilates the state for $n > \sum_{i=1}^{k} n_i$.

Net result: For a level n state we only need to check the condition for $L_0, L_1, \cdots L_n$ and $\tilde{L}_0, \tilde{L}_1, \cdots \tilde{L}_n$.

We shall focus on the ~~level~~ level 1 states.

$$\zeta_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}\, |\{k\}\rangle$$

$L_0, \tilde{L}_0$ condition:

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'}(c+1)$$

$$\underset{m_1^2}{\parallel}$$

---

3 possibilities:

① $c+1 < 0 \Rightarrow$ tachyonic state

② $c+1 = 0 \Rightarrow$ Massless state

③ $c+1 > 0 \Rightarrow$ Massive state

~~And if we can choose a frame where $k^i = 0$~~

$$L_1\, \zeta^{\mu\nu} \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |\{k\}\rangle = \frac{1}{2}\sqrt{2\alpha'}\, n\, k_\mu\, \zeta^{\mu\nu}\, \tilde{\alpha}^{\nu}_{-1} |\{k\}\rangle$$

$$\Rightarrow \quad k_\mu \zeta^{\mu\nu} = 0.$$

$\tilde{L}_1$ constraint: $\quad k_\nu \zeta^{\mu\nu} = 0.$

Possibility 1.

$$C+1<0 \Rightarrow (k^0)^2-(\vec{k}^0)^2 = \frac{4}{\alpha'}(C+1) < 0.$$

$\Rightarrow$ We can choose a frame in which

$$k^0=0, \quad (k^1)^2 = \frac{4}{\alpha'}(C+1), \quad k^2=k^3=\cdots=0$$

Condition on $\zeta^{\mu\nu}$:

$$\zeta^{1\nu}=0 \qquad \zeta^{\mu 1}=0 \qquad \forall\ \mu,\nu.$$

Choose ζ such that $\zeta^{02}=1$.

All other components of ζ vanish.

State: $\alpha^0_{-1}\,\tilde{\alpha}^2_{-1}\,|\{k\}\rangle \to$ physical.

Norm: $\langle\{k\}|\,\tilde{\alpha}^2_1\,\alpha^0_1\,\alpha^0_{-1}\,\tilde{\alpha}^2_{-1}\,|\{k\}\rangle$

$$=\underbrace{(-1)\cdot(1)}_{(-1)}\langle\{k\}|\{k\}\rangle$$

$\Rightarrow$ Negative norm state $\Rightarrow$ inconsistent.

Conclusion: $C+1 \geq 0$.

One can show that $C+1>0 \Rightarrow$ inconsistent interacting theory. $|\{k\}\rangle$

$\Rightarrow C+1=0. \Rightarrow$ Level zero state has $(\text{mass})^2 = \frac{4}{\alpha'}C = -\frac{4}{\alpha'} \to$ tachyonic.

Hilbert space of a closed string:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}\ ; \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[x^\mu, p^\nu] = i\,\eta^{\mu\nu}$$

Generic state in the Hilbert space:

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k}\ \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k}\ |\{k\}\rangle\ ; \quad m_i, n_i > 0.$$

$$\alpha^\mu_m\,|\{k\}\rangle = 0, \qquad \tilde{\alpha}^\mu_m\,|\{k\}\rangle = 0 \qquad \text{for } m > 0$$

$$p_\mu\,|\{k\}\rangle = k_\mu\,|\{k\}\rangle$$

$$L_m \equiv \frac{1}{2}\sum_{n\neq 0} \alpha^\mu_{m-n}\,\alpha^\nu_n\,\eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\,\alpha^\mu_m\,p_\mu \quad \text{for } m \neq 0$$

$$L_0 \equiv \sum_{n>0} \alpha^\mu_{-n}\,\alpha^\nu_n\,\eta_{\mu\nu} + \frac{\alpha'}{4}\,\eta^{\mu\nu}\,p_\mu\,p_\nu + C$$

$$\tilde{L}_m, \ \tilde{L}_0 \qquad \alpha^\mu_m \to \tilde{\alpha}^\mu_m, \quad C \to \tilde{C}.$$

Physical state constraint:

$$L_m\,|\text{phys}\rangle = 0, \qquad \tilde{L}_m\,|\text{phys}\rangle = 0 \qquad \text{for } m \geq 0$$

Acting on $|\{k\}\rangle$, physical state constraint gives:

$$(k^0)^2 - (\vec{k})^2 = \frac{4}{\alpha'}\,C = \frac{4}{\alpha'}\,\tilde{C} \qquad \Rightarrow \qquad C = \tilde{C}$$

$\Rightarrow$ A particle of $(\text{mass})^2 = \frac{4}{\alpha'}\,C$

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_k}_{-n_k} |\{k\}\rangle$$

L_0 & $\tilde{L}_0$ constraints.

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'}\left(C + \sum m_i\right) = \frac{4}{\alpha'}\left(C + \sum n_i\right)$$

$$\underset{(k^0)^2 - \vec{k}^2}{\underbrace{-\eta^{\mu\nu} k_\mu k_\nu}} \Rightarrow \sum_{i=1}^{k} m_i = \sum_{i=1}^{l} n_i = N$$

Level

Thm. L_m, $\tilde{L}_m$ acting on a level N state automatically gives zero for $m > N$

$\Rightarrow$ We only need to check the constraints for $m \leq N$

Example: Level 1 state:

$$S_{\mu\nu}\, \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |\{k\}\rangle.$$

L_0, $\tilde{L}_0$ constraint:

$$(k^0)^2 - \vec{k}^2 = \frac{4}{\alpha'}(C+1)$$

$$\underset{(\text{Mass})^2}{\underbrace{(k^0)^2 - \vec{k}^2}}$$

Three cases

① $C + 1 < 0$

② $C + 1 = 0$

③ $C + 1 > 0$

Case 1 Choose a frame in which

$$k^0 = 0, \quad k^1 = \sqrt{-\frac{4}{\alpha'}(c+1)}, \quad k^2 = \cdots k^{D-1} = 0.$$

Constraint: $\zeta_{\mu 1} = 0 = \zeta_{01\beta\nu}$

Take two states:

(a) $\zeta_{02} = 1$, All other $\zeta_{\mu\nu}$'s $= 0$.

~~Norm~~ $\alpha^0_{-1} \tilde{\alpha}^2_{-1} |\{k\}\rangle.$

Norm: $- \langle \{k\} | \{k\} \rangle.$

(b) $\zeta_{32} = 1$, All other $\zeta_{\mu\nu}$'s $= 0$.

$\alpha^3_{-1} \tilde{\alpha}^2_{-1} |\{k\}\rangle$

Norm $\langle \{k\} | \{k\} \rangle.$

$\Rightarrow$ Some physical states will have negative norm.

$\Rightarrow$ Unacceptable.

Case (2) $\rightarrow$ Massless spin 2 particle.
Case (3) $\Rightarrow$ Massive spin 2 particle

$\Downarrow$

In this case one cannot have a consistent interacting string theory.

$\Rightarrow$ $c + 1 = 0$ $\Rightarrow$ $c = -1$.

Mass^2 of $|\{k\}\rangle$ $= \frac{4}{\alpha'} c = -\frac{4}{\alpha'}$ $\Rightarrow$ Tachyonic.

Net result :

① Level 1 state is massless

② Described by a rank 2 polarization tensor $\zeta_{\mu\nu}$ satisfying

$$k^{\mu}\zeta_{\mu\nu} = 0$$

$\Rightarrow$ 4 constraints.

~~Suppose~~ Consider a state

$|\psi\rangle = \sum_n L_{-n}|\chi_n\rangle$ ← any state $n > 0$

$\langle\psi|\text{phys}\rangle = \sum_n \langle\chi_n| L_n |\text{phys}\rangle = 0.$

Similarly if $|\tilde{\psi}\rangle = \sum_n \tilde{L}_{-n}|\tilde{\chi}_n\rangle$ then

$$\langle\tilde{\psi}|\text{phys}\rangle = 0.$$

States of the form $|\psi\rangle, |\tilde{\psi}\rangle$ are known as spurious states.

A state which is both spurious and physical is called null.

Suppose $|s_{phys}\rangle$ is a physical state

$|s'_{phys}\rangle \equiv |s_{phys}\rangle + |\chi\rangle$ is also a physical state if $|\chi\rangle$ is null.

$\langle s'_{phys}|\psi_{phys}\rangle = \langle s_{phys}|\psi_{phys}\rangle$ $\forall$ physical state $|\psi_{phys}\rangle$

We say that $|s_{phys}\rangle$ and $|s'_{phys}\rangle$ describe equivalent physical states, as they have the same inner product with all physical states.

$$|s_{phys}\rangle \simeq |s'_{phys}\rangle$$

Example of spurious state at level 1:

$$\frac{2}{\sqrt{2\alpha'}}\left[L_{-1}\, \xi_\nu \tilde{\alpha}^\nu_{-1} |\{k\}\rangle + \tilde{L}_{-1} \tilde{\xi}_\mu \alpha^\mu_{-1} |\{k\}\rangle\right]$$

$$= [k_\mu \xi_\nu \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} |\{k\}\rangle + k_\nu \tilde{\xi}_\mu \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} |\{k\}\rangle]$$

$$= (k_\mu \xi_\nu + k_\nu \tilde{\xi}_\mu)\, \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} |\{k\}\rangle$$

This state is physical if

$$k^2 = 0.$$

$$k^\mu (k_\mu \xi_\nu + k_\nu \tilde{\xi}_\mu) = 0 \quad \text{i.e. } k^\mu \tilde{\xi}_\mu = 0$$

$$k^\nu (k_\mu \xi_\nu + k_\nu \xi_\mu) = 0 \quad \text{i.e. } k^\nu \xi_\nu = 0$$

$$\Rightarrow \quad \zeta_{\mu\nu} \cong \xi_\mu k_\nu + \tilde{\xi}_\nu k_\mu + \zeta_{\mu\nu}$$

$$\text{if} \quad k^\mu \xi_\mu = 0 = k^\nu \xi_\nu$$

Write $\zeta_{\mu\nu} = \frac{1}{2}(\zeta^S_{\mu\nu} + \zeta^A_{\mu\nu})$

Symmetric; Antisymmetric

$$\zeta^S_{\mu\nu} = \zeta_{\mu\nu} + \zeta_{\nu\mu}, \quad \zeta^A_{\mu\nu} = \zeta_{\mu\nu} - \zeta_{\nu\mu}$$

$k^\mu \zeta^S_{\mu\nu} = 0 \quad , \quad k^\mu \zeta^A_{\mu\nu} = 0$

$\zeta^S_{\mu\nu} \equiv \zeta^S_{\mu\nu} + k_\mu \xi^S_\nu + k_\nu \xi^S_\mu \quad ; \quad k^\mu \xi^S_\mu = 0.$

$\zeta^A_{\mu\nu} \equiv \zeta^A_{\mu\nu} + k_\mu \xi^A_\nu - k_\nu \xi^A_\mu \quad ; \quad k^\mu \xi^A_\mu = 0$

We shall first focus on the states obtained from $\zeta^S_{\mu\nu}$.

Compare with the polarization tensor of the graviton obtained by quantizing general relativity:

$$S_{GR} = K \int d^D x \sqrt{-\det g}\; R$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \to$ Ricci scalar

~~metric~~

$g_{\mu\nu}$: metric

Flat Minkowski space : $g_{\mu\nu} = \eta_{\mu\nu}$

Fluctuations around the Minkowski space.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Express S_{GR} as a power series in $h_{\mu\nu}$

Note : $g^{\mu\nu} = \eta^{\mu\nu} + O(h) + O(h^2) + \cdots$

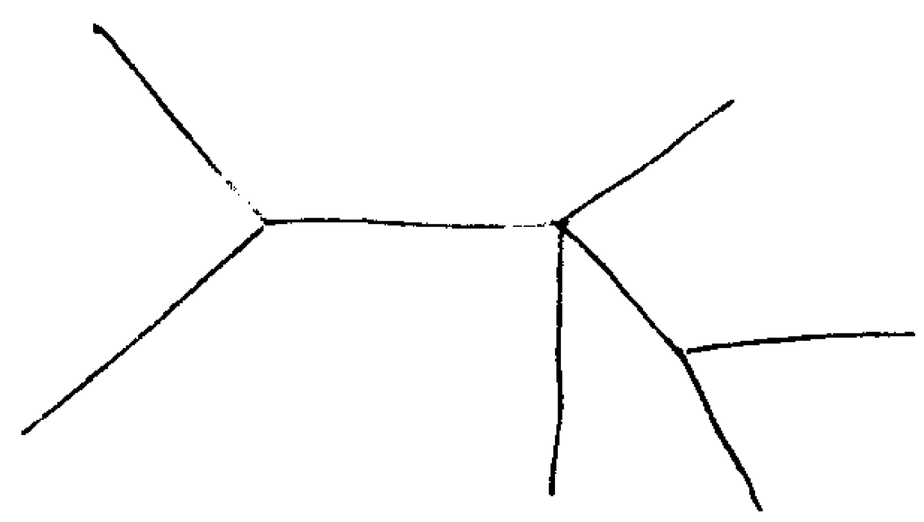
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \to$ Infinite order

---

$\Rightarrow$ S_{GR} is an infinite order polynomial in $h_{\mu\nu}$.

Propagator → Only quadratic term is important.

N-point function at tree-level:



Contains at most vertex of order h^N.

⇒ For a ~~given~~ tree level N-point amplitude we can truncate the expansion to order h^N.

We can find the spectrum of physical states by examining the quadratic term.

⇒ Massless particles with polarization vector $\zeta^{G}_{\mu\nu} = \zeta^{G}_{\nu\mu}$ (since $h_{\mu\nu} = h_{\nu\mu}$)

But general coordinate invariance ⇒ Constraints on $\zeta^{G}_{\mu\nu}$.

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

Take $x'^\mu = x^\mu + \epsilon f^\mu$

$$\Rightarrow \delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_\mu f_\nu + \nabla_\nu f_\mu$$

$$f_\mu \equiv g_{\mu\nu} f^\nu$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\delta h_{\mu\nu} = \partial_\mu f_\nu + \partial_\nu f_\mu + O(h).$$

$\rightarrow$ gauge transformation.

Effect on polarization tensor:

$$\delta \zeta^G_{\mu\nu} = k_\mu \xi^G_\nu + k_\nu \xi^G_\mu$$

$$\Rightarrow \zeta^G_{\mu\nu} \cong \zeta^G_{\mu\nu} + k_\mu \xi^G_\nu + k_\nu \xi^G_\mu$$

Gauge fixing: Lorentz covariant gauge:

$$\eta^{\mu\mu'} \partial_{\mu'} h_{\mu\nu} = 0$$

$$\Rightarrow k^\mu \zeta^G_{\mu\nu} = 0$$

Linearized Einstein's eq. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$ in $\partial^\mu h_{\mu\nu} = 0$ gauge:

$$\partial_\nu \partial_\sigma h^\mu{}_\mu - \eta_{\nu\sigma} \Box h^\mu{}_\mu + \Box h_{\nu\sigma} = 0.$$

$\times\ \eta^{\nu\sigma}$ $\quad (2-D)\Box h^\mu{}_\mu = 0 \Rightarrow \Box h^\mu{}_\mu = 0$ for $D \neq 2$

$$\partial_\nu \partial_\sigma h^\mu{}_\mu + \Box h_{\nu\sigma} = 0.$$

$$h_{\mu\nu} = \int d^D k \; \zeta^G_{\mu\nu}(k)\, e^{ik\cdot x}$$

$$k_\nu k_\sigma \zeta^\mu{}_\mu + k^2 \zeta_{\nu\sigma} = 0$$

$$k^2 \zeta^\mu{}_\mu = 0.$$

$\Downarrow$

Either $k^2 = 0$ or $\zeta^\mu{}_\mu = 0$.

If $\zeta^\mu{}_\mu \neq 0$ then $k^2 = 0$.

$\Rightarrow \quad k_\sigma k_\nu \zeta^\mu{}_\mu = 0 \quad \Rightarrow \zeta^\mu{}_\mu = 0$ (unless $k=0$).

For $\zeta^\mu{}_\mu = 0$ we get

$$k^2 \zeta_{\nu\sigma} = 0$$

$\Rightarrow \quad k^2 = 0 \quad$ if $\zeta_{\nu\sigma} \neq 0$.

Conclusion: States of a graviton are characterized by momentum k & polarization tensor $\zeta^G_{\mu\nu}$ satisfying:

$$k^2 = 0$$

$$\zeta^{G\mu}{}_\mu = 0, \quad k^\mu \zeta^G_{\mu\nu} = 0$$

$$\zeta^G_{\mu\nu} \cong \zeta^G_{\mu\nu} + (k_\mu \xi^G_\nu + \xi^G_\mu k_\nu)$$

$\leftrightarrow$ correspondence with level 1 states described by $\zeta^S_{\mu\nu}$ except that $\zeta^{S\mu}{}_\mu \neq 0$.

$\Rightarrow$ An extra state.

String theory spectrum has a graviton and a scalar.

$S^A_{\mu\nu}$ → describes states of a rank 2 anti-symmetric tensor field $B_{\mu\nu}$.

Gauge invariance:

$$\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu.$$

Thus the bosonic string theory contains 3 sets of massless fields.

Graviton: $h_{\mu\nu}$.

Scalar (dilaton): ϕ

Anti-symmetric tensor: $B_{\mu\nu}$.

Spectrum of closed string theory

Level 0 Scalar with $m^2 = -\frac{4}{\alpha'}$

Level 1 Massless $g_{\mu\nu}, B_{\mu\nu}, \phi$ — scalar

metric anti-symmetric tensor (dilaton)

Level N :

$$m^2 = \frac{4}{\alpha'}(N-1) \rightarrow \text{Massive tower of particles}$$

~~Ex~~ ~~Fair~~ Analysis for level 2

General state

$$\zeta^{(1)}_{\mu\nu\rho\sigma}\, \alpha^{\mu}_{-1}\, \alpha^{\nu}_{-1}\, \tilde{\alpha}^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1} + \zeta^{(2)}_{\mu\nu\rho}\, \alpha^{\mu}_{-2}\, \tilde{\alpha}^{\nu}_{-1}\, \tilde{\alpha}^{\rho}_{-1}$$

$$+ \zeta^{(3)}_{\mu\nu\rho}\, \alpha^{\mu}_{-1}\, \alpha^{\nu}_{-1}\, \tilde{\alpha}^{\rho}_{-2} + \zeta^{(4)}_{\mu\nu}\, \alpha^{\mu}_{-2}\, \tilde{\alpha}^{\nu}_{-2} \,) \, |k\rangle$$

$L_0, \tilde{L}_0$ constraint :

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'}(2-1) = \frac{4}{\alpha'} \rightarrow (\text{mass})^2$$

$L_1, \tilde{L}_1, L_2, \tilde{L}_2$ ~~co~~ constraints

$\Rightarrow$ Restriction on $\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \zeta^{(4)}$

Q. Are there negative norm states satisfying the physical state condition?

Answer: For $D > 26$, there are negative norm states satisfying the physical state conditions.

For $D \leq 26$ there are no negative norm states satisfying these conditions.

$\Rightarrow$ ~~But~~ Free string theory is consistent for $D \leq 26$.

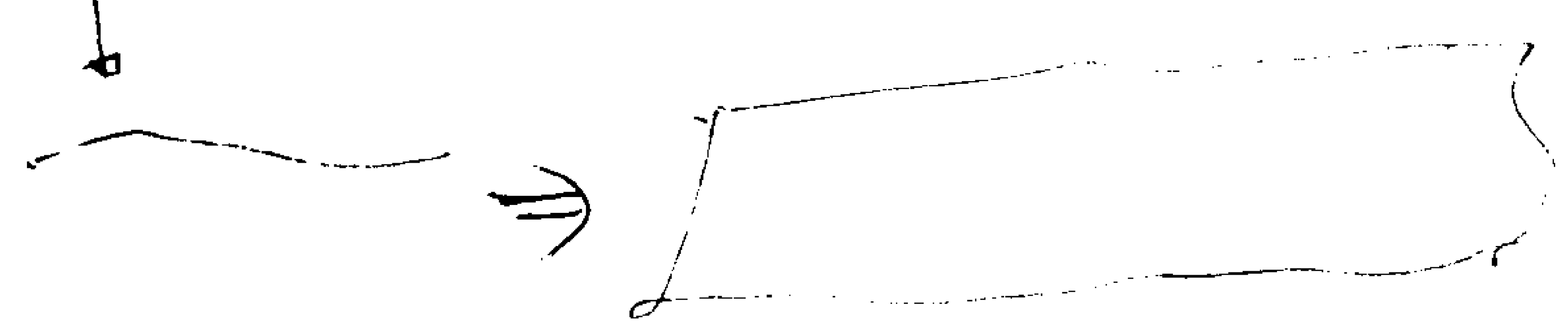
But interacting string theory is consistent for only $D = 26$.

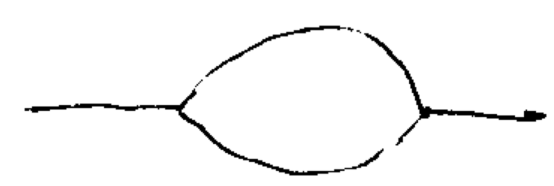
$\Rightarrow$ Critical dimension.

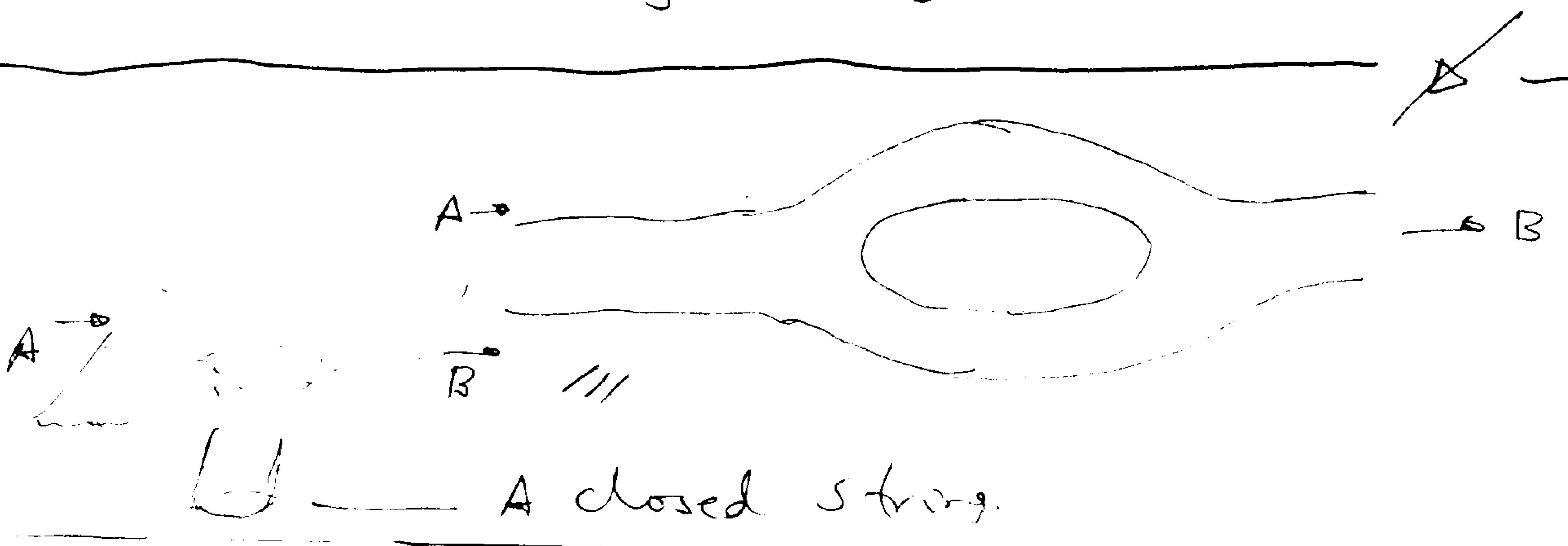
Open string theory

open string theories automatically contain closed strings.

Particle Open string world-sheet:



Consider a self energy diagram: 

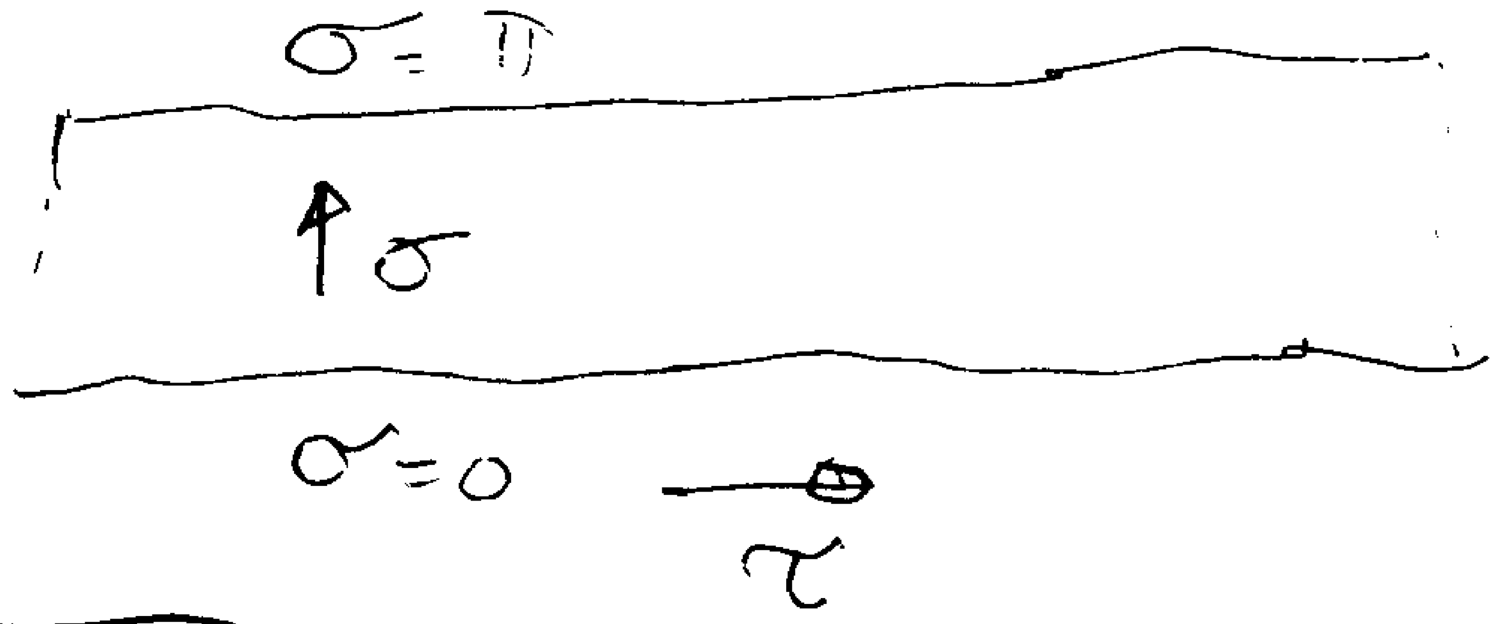


Thus these theories must also be defined for $D=26$.

If open string consistency condition gives different conditions on D then there would be a contradiction.

Fortunately this does not happen.

~~Non-interacting~~ ~~Free~~ Open string : spectrum



$\xi^0 = \tau, \quad \xi^1 = \sigma$

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \; \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\eta^{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$$

Constraint : $T_{\alpha\beta} = 0$

$$T_{\alpha\beta} \equiv K\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\, \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu \eta^{\alpha'\beta'} \eta_{\mu\nu}\right)$$

K = constant

$\eta^{\alpha\beta} T_{\alpha\beta} = 0 \Rightarrow$ Two independent constraint.

$u = \tau - \sigma, \quad v = \tau + \sigma$

$$T_{uu} \equiv \eta_{\mu\nu}\, \partial_u X^\mu\, \partial_u X^\nu = 0$$
$$T_{vv} \equiv \eta_{\mu\nu}\, \partial_v X^\mu\, \partial_v X^\nu = 0$$

In quantum theory

$$\langle phys | T_{uu} | phys' \rangle = 0 = \langle phys | T_{vv} | phys' \rangle$$

What ~~are~~ are the boundary conditions on X^μ at $\sigma = 0, \pi$?

$$\delta S = -\frac{1}{2\pi\alpha'} \int d^2\xi\, \eta^{\alpha\beta}\, \partial_\alpha(\delta X^\mu)\, \partial_\beta X^\nu\, \eta_{\mu\nu}$$
$$= \frac{1}{2\pi\alpha'} \int d^2\xi\, \eta^{\alpha\beta}\, \delta X^\mu\, \partial_\alpha \partial_\beta X^\nu\, \eta_{\mu\nu}$$
$$- \frac{1}{2\pi\alpha'} \int d\tau\, \delta X^\mu\, \partial_\sigma X^\nu \Big|_{\sigma=0}^{\sigma=\pi}$$

In order that $\delta S = 0$ reproduces the correct local eqs. of motion we need

$$\partial_\sigma X^\nu = 0 \quad \text{at } \sigma = 0, \pi \to \text{Neumann b.c.}$$
$$\text{or} \quad X^\nu = x_0^\nu \quad \text{at } \sigma = 0, \pi \to \text{Dirichlet b.c.}$$

Also possible ① Dirichlet at $\sigma = 0$, Neumann at $\sigma = \pi$

② vice versa.

③ $X^\nu = x_0^\nu$ at $\sigma = 0$, $X^\nu = x_1^\nu$ at $\sigma = \pi$

If we want to maintain D dimensional ~~~~ Poincare invariance, then all the X^μ's must satisfy Neumann b.c. at both ends.

$X^\mu = x_0^\mu$ ~~break~~ at $\sigma = 0$ or π breaks translational symmetry. $X^\mu \to X^\mu + a^\mu$.

If we want to maintain D-dimensional ~~~~ Poincare invariance we must put Neumann b.c. on all coordinates.

We first consider this case.

$$\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0, \pi$$

$$X^\mu = \sum_{n>0} \phi_n^\mu \cos n\sigma + x^\mu$$

$$S = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left(\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} - \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$= \frac{1}{8\alpha'} \sum_{n>0} \left(\partial_\tau \phi_n^\mu \, \partial_\tau \phi_n^\nu \, \eta_{\mu\nu} - n^2 \phi_n^\mu \phi_n^\nu \eta_{\mu\nu} \right)$$

$$+ \frac{1}{4\alpha'} \partial_\tau x^\mu \partial_\tau x^\nu \eta_{\mu\nu}$$

$$\Pi_{\mu n} = \frac{\delta S}{\delta(\partial_\tau \phi_n^\mu)} = \frac{1}{4\alpha'} \partial_\tau \phi_n^\nu \eta_{\mu\nu}$$

$$p_\mu = \frac{\delta S}{\delta(\partial_\tau x^\mu)} = \frac{1}{2\alpha'} \partial_\tau x^\nu \eta_{\mu\nu}$$

In quantum theory:

$$[\phi^\mu_n, \Pi_{\nu m}] = i\,\delta_{mn}\,\delta^\mu_\nu$$

$$[x^\mu, p_\nu] = i\,\delta^\mu_\nu$$

Define: $\alpha^\mu_n = \eta^{\mu\nu}\sqrt{2\alpha'}\,\Pi_{\nu n} - \frac{i n}{2\sqrt{2\alpha'}}\,\phi^\mu_n$

$$\alpha^\mu_{-n} = \eta^{\mu\nu}\sqrt{2\alpha'}\,\Pi_{\nu n} + \frac{i n}{2\sqrt{2\alpha'}}\,\phi^\mu_n$$ for $n > 0$.

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\,\delta_{m+n,0}$$

$\frac{1}{\sqrt{m}}\alpha^\mu_m \to$ annihilation operator
$\frac{1}{\sqrt{m}}\alpha^\mu_{-m} \to$ creation operator
$\Big\}\ m > 0.$

Define $|\{k\}\rangle$ such that

$$p_\mu |\{k\}\rangle = 0, \qquad \alpha^\mu_m |\{k\}\rangle = 0 \text{ for } m > 0$$

Basis of states:

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} |\{k\}\rangle$$

Constraints:

$$0 = T_{uu} = K\,\partial_u X^\mu \partial_u X^\nu \eta_{\mu\nu} \qquad u = \tau - \sigma$$

$$0 = T_{vv} = K\,\partial_v X^\mu \partial_v X^\nu \eta_{\mu\nu}. \qquad v = \tau + \sigma$$

Substitute expansion of X^μ

$$T_{uu} \propto \sum_m L_m e^{2im\sigma}$$

$$T_{vv} \propto \sum_m L_m e^{-im\sigma}$$

$$L_m = \frac{1}{2} \sum_n \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu} + \sqrt{2\alpha'}\, p_\mu \alpha^\mu_m$$

for $m \neq 0$.

$$L_0 = \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} + \alpha' p_\mu p_\nu \eta^{\mu\nu} + C_0$$

normal ordering constant

Physical state condition:

$$L_m |phys\rangle = 0 \quad \text{for } m > 0.$$

Spurious state:

$$\sum_n L_{-n} |\chi_n\rangle$$

A state which is spurious + physical is called null.

$$|S_{phys}\rangle \cong |S_{phys}\rangle + |null\rangle$$

since $\langle null | phys\rangle = 0 \quad \forall \; |phys\rangle$.

L_0 constraint on

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} |\{k\}\rangle$$

$$\alpha' \eta^{\mu\nu} k_\mu k_\nu + \sum_{i=1}^{k} m_i + C_0 = 0$$

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{1}{\alpha'}\Big(\underbrace{\sum_{i=1}^{k} m_i}_{\text{Level } N} + C\Big) = \frac{1}{\alpha'}(N+C)$$

On a level N state we need to check $L_0, L_1, \ldots L_N$ constraints.

Level 0 : $|\{k\}\rangle$

$$-\eta^{\mu\nu} k_\mu k_\nu = \underbrace{\frac{1}{\alpha'} C}_{(\text{mass})^2}$$

Level 1 : $\zeta_\mu \alpha^\mu_{-1} |\{k\}\rangle$

L_0 constraint : $-\eta^{\mu\nu} k_\mu k_\nu = \underbrace{\frac{1}{\alpha'}(C+1)}_{(\text{mass})^2}$

L_1 constraint : $k^\mu \zeta_\mu = 0$

Case 1 : $(C+1) < 0$ tachyonic

2. $(C+1) = 0$ massless

3. $(C+1) > 0$ massive

If $(c+1) < 0$ we can choose

$$k^0 = 0, \quad k^1 = \sqrt{-\frac{1}{\alpha'}(c+1)}, \quad k^2 = k^3 = \cdots = 0.$$

L_1 Constraint : $\zeta^1 = 0$.

$\alpha^0_{-1} |\{k\}\rangle$ & $\alpha^2_{-1} |\{k\}\rangle$ are both physical.

$\alpha^0_{-1} |\{k\}\rangle$ has norm $-\langle\{k\}|\{k\}\rangle$

$\alpha^2_{-1} |\{k\}\rangle$ has norm $\langle\{k\}|\{k\}\rangle$

Both cannot be +ve norm $\to$ contradiction

$(c+1) \geq 0$: No -ve norm state.

But interacting theory is consistent only for $c+1 = 0$.

$\Rightarrow$ Massless particle.

Note : $|\{k\}\rangle \to (\text{mass})^2 = \frac{1}{\alpha'} c = -\frac{1}{\alpha'}$

$\Rightarrow$ Tachyon.

Spurious state at level 1 :

$$\frac{a}{\sqrt{2\alpha'}} L_{-1} |\{k\}\rangle = a k_\mu \alpha^\mu_{-1} |\{k\}\rangle$$

Physical since $k^\mu k_\mu = 0$

Thus $\zeta^\mu \equiv \zeta^\mu + a k^\mu$

Physical state condition : $k_\mu \zeta^\mu = 0$ $\Big\} \Rightarrow$ A massless vector particle $\Big\}$ Gauge field

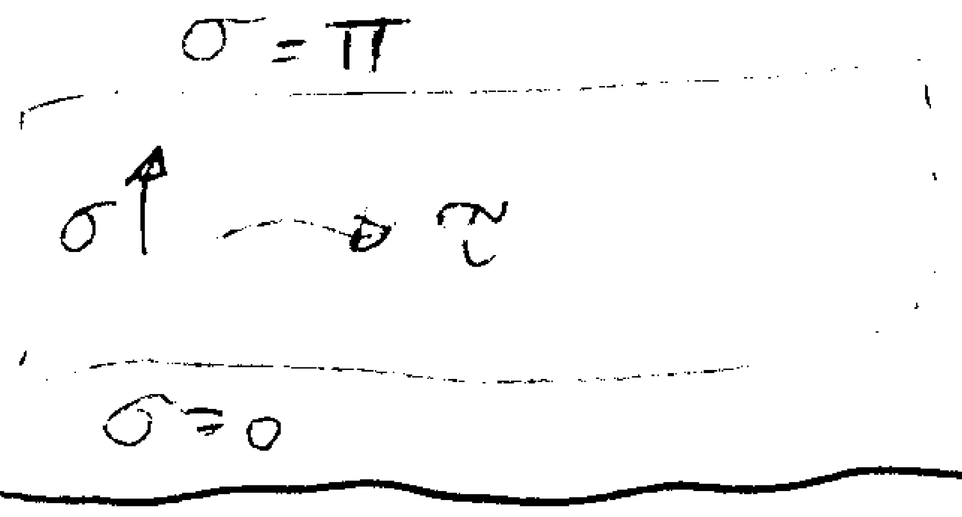
Absence of -ve norm physical states at level 2

$$\Rightarrow \quad D \leq 26$$

Consistent interacting theory:

$$D = 26.$$

Open strings:



$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad \begin{cases} \xi^0 = \tau \\ \xi^1 = \sigma \end{cases}$$

Possible boundary conditions on X^μ:

Dirichlet : $X^\mu(\sigma_0) = x_0^\mu$ at $\sigma = 0, \pi$

Neumann : $\partial_\sigma X^\mu(\sigma) = 0$ at $\sigma = 0, \pi$

~~Ma~~ Neumann b.c. on all X^μ's $\to$ only ~~such~~ Poincaré invariant b.c.

We shall now study ~~the~~ Dirichlet b.c.

Choose b.c:

Neumann for $\mu = 0, 1, \cdots p$

Dirichlet for $\mu = p+1, p+2, \cdots 25$.

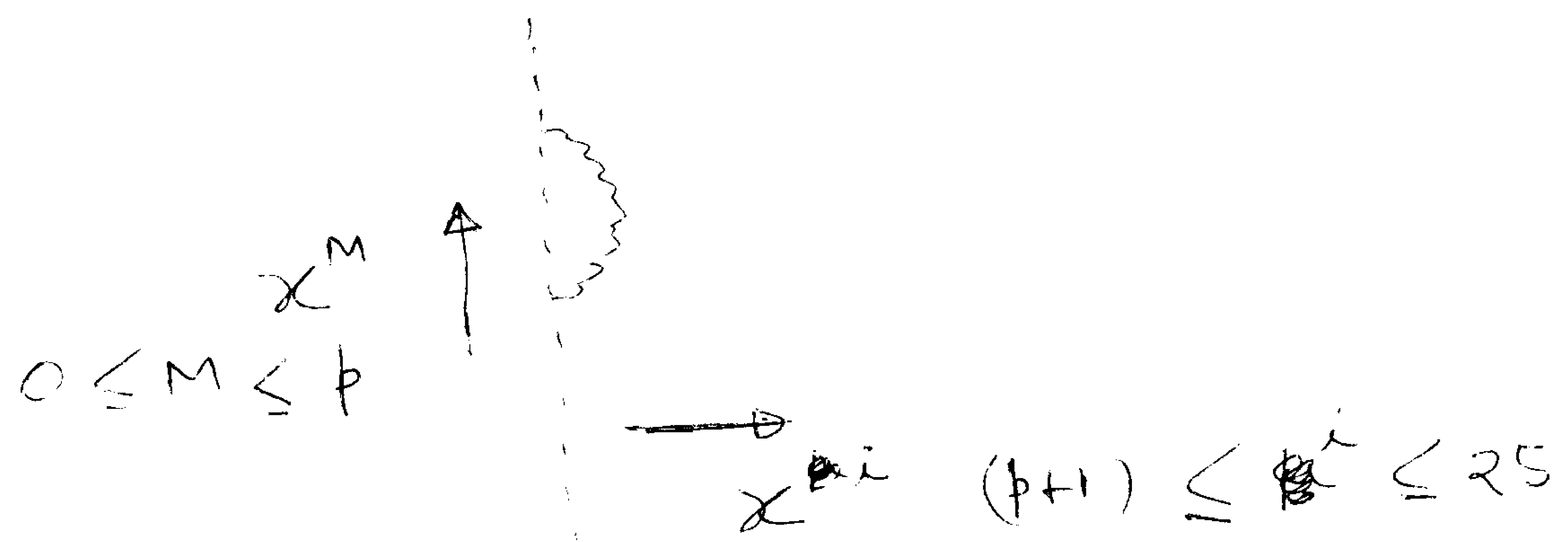
$$\partial_\sigma X^M = 0 \quad \text{at} \quad \sigma = 0, \pi \quad \text{for} \quad M = 0, \cdots p$$

$$X^i = x_0^i \quad \text{at} \quad \sigma = 0, \pi \quad \text{for} \quad i = p+1, \cdots 25$$

Interpretation

$x^\mu = x_0^\mu$ ~~describes~~ for $\mu = p+1, \dots 25$ describes a $(p+1)$ dimensional ~~hyper~~ subspace of the 26 dimensional space time.

Eg. if $p = 24$, the eqn. $x^{25} = x_0^{25}$ is a $(24+1)$ dimensional subspace.



Ends of open strings are constrained to lie on this $(p+1)$-dimensional hypersurface.

→ Breaks translational invariance.

Thus there is something special about this $(p+1)$ dimensional hypersurface.

→ allows open strings to end there.

Interpretation: There is a "soliton" sitting along this hypersurface.

→ Dirichlet p-brane (p-space, 1 time direction)

$p = 0$: Point-like object

$p = 1$: String like object

$p = 2$: Membrane like object etc.

Interpretation of the open strings with ends on the D-brane

$\Rightarrow$ Oscillation modes of the D-brane.

~~Expected~~ A soliton has various modes of vibration:

① Translational modes.

② Internal modes

Typically each mode $\leftrightarrow$ a $(p+1)$ dimensional field.

ϕ_n : A particular mode. string-1

Value of ϕ may differ from point to point on the brane.

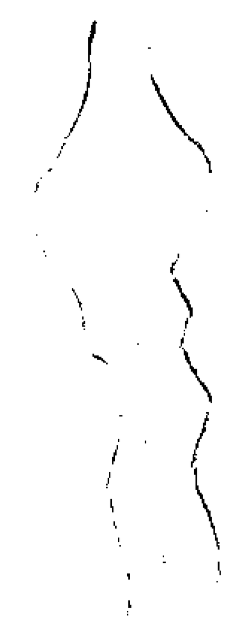
$\Rightarrow$ $\phi_n(x^M)$: A $(p+1)$ dimensional field.

E.g. If $\phi_0 \leftrightarrow$ translational mode

Then x^M dependent ϕ_0

$\leftrightarrow$

If $\phi_1 \leftrightarrow$ size of the soliton in the $\perp$ direction, then

$\phi_1(x^M) \to$  etc.

Thus if the open strings satisfying Dirichlet b.c. describe modes of the p-brane soliton, then each mode must correspond to a $(p+1)$ dimensional fields.

Try to check if this is correct.

Mode expansion of X^μ:

X^M. Neumann b.c. $\to$ identical mode expansion.

$$X^{i} = x_0^{i} + \sum_n \phi_n^{(i)} \sin n\sigma$$

x_0^i $\to$ fixed const.

Note: No zero mode $x^\mu(\tau)$

$$S = \frac{1}{4\pi\alpha'}\,\frac{\pi}{2}\int d\tau\left(\partial_\tau \phi_n^{(i)}\,\partial_\tau \phi_n^{(i)} - n^2\,\phi_n^{(i)}\,\phi_n^{(i)}\right) + X^M\text{-contribution}$$

Note $\eta_{ii} = 1$ as all Dirichlet direction are ~~Euclidean~~ space-like.

Conjugate momentum

$$\Pi_n^{(i)} = \frac{\delta S}{\delta(\partial_\tau \phi_n^{(i)})} = \frac{1}{4\alpha'}\,\partial_\tau \phi_n^{(i)}$$

$$\alpha_n^i = \sqrt{2\alpha'}\,\Pi_n^{(i)} - \frac{in}{2\sqrt{2\alpha'}}\,\phi_n^{(i)}$$

$$\alpha_{-n}^i = \sqrt{2\alpha'}\,\Pi_n^{(i)} + \frac{in}{2\sqrt{2\alpha'}}\,\phi_n^{(i)}$$

$$[\phi_n^{(i)}, \Pi_m^{(j)}] = i\,\delta_{ij}\,\delta_{mn}$$

$$\Rightarrow [\alpha_m^i, \alpha_n^j] = m\,\delta_{m+n,0}\,\delta_{ij}$$

Note: No x^i & p^j

$$[\alpha_m^M, \alpha_n^N] = m\,\delta_{m+n,0}\,\eta^{MN}$$

$$[x^M, p_N] = i\,\delta^M_N$$

$0 \leq M, N \leq p$

$p+1 \leq i, j \leq 25$

$|\{k\}\rangle$

$\hookrightarrow (k^0, k^1, \ldots k^p)$

$$\alpha_m^M |\{k\}\rangle = 0, \quad \alpha_m^i |\{k\}\rangle = 0 \quad \text{for } m > 0$$

$$p_M |\{k\}\rangle = k_M |\{k\}\rangle$$

General basis state

$$\alpha^{M_1}_{-m_1} \cdots \alpha^{M_k}_{-m_k}\, \alpha^{i_1}_{-n_1} \cdots \alpha^{i_\ell}_{-n_\ell}\, |\{k\}\rangle$$

Note: States are labelled by $(\cancel{}\, p+1)$ dimensional momenta

$\leftrightarrow$ quanta of $(\cancel{}\, p+1)$ dimensional field.

We shall now determine what kind of physical states / fields we get.

Constraints:

$$\langle phys | T_{uu} | phys' \rangle = 0 = \langle phys | T_{vv} | phys' \rangle$$

$$T_{uu} = (\text{Const})\, \partial_u X^\mu \partial_u X^\nu \eta_{\mu\nu}$$

$$T_{vv} = (\text{Const.})\, \partial_v X^\mu \partial_v X^\nu \eta_{\mu\nu}$$

$u = \tau \cancel{+} \sigma, \quad v = \tau + \sigma$

Expressed in terms of α^μ_n

$$T_{uu} = (\text{const.}) \sum_{m=-\infty}^{\infty} L_m e^{imo}$$

uu & vv have been exchanged!

$$T_{vv} = (\text{const}) \sum_{m=-\infty}^{\infty} L_m e^{-imo}$$

T_{uu} should be fn. of $u = \tau + \sigma$

$T_{uu} = \tilde{L}_m e^{imo}$

$$L_m = \frac{1}{2}\sum_n \left(\alpha^M_{m-n}\,\alpha^N_n\,\eta_{MN} + \alpha^i_{m-n}\,\alpha^i_n\right) + \sqrt{2\alpha'}\,p_M\,\alpha^M_m$$

for $m \neq 0$.

$$L_0 = \sum_{n>0}\left(\alpha^M_{-m}\,\alpha^N_m\,\eta_{MN} + \alpha^i_{-m}\,\alpha^i_m\right)$$

$$+\,\alpha'\,p_M\,p_N\,\eta^{MN} + \underbrace{C_0}_{\text{normal ordering constant}}$$

Physical state condition:

$$L_m\,|phys\rangle = 0 \quad \text{for } m>0.$$

Spurious state $\sum_n L_{-m}\,|\chi_m\rangle$.

A state is null if it is spurious as well as physical.

$$|s_{phys}\rangle \cong |s_{phys}\rangle + |null\rangle$$

since $\langle null\,|\,phys\rangle = 0 \quad \forall$ phys.

$L_0 = 0$ constraint on $\alpha^{M_1}_{-m_1}\cdots\alpha^{M_k}_{-m_k}\,\alpha^{i_1}_{-n_1}\cdots\alpha^{i_l}_{-n_l}\,|k s\rangle$

$$\Rightarrow\ -\eta^{MN}\,p_M\,p_N = \frac{1}{\alpha'}\Big(C_0 + \underbrace{\sum m_i + \sum n_i}_{N \text{ (level)}}\Big)$$

$\Rightarrow$ a $(p+1)$ dimensional particle of $(mass)^2 = \frac{1}{\alpha'}(C_0 + N)$

Level 0: Scalar with $(\text{mass})^2 = \frac{c_0}{\alpha'}$

Level 1: $(\text{Mass})^2 = \frac{c_0+1}{\alpha'}$

~~Analysis of~~ Absence of negative norm physical states + Consistent interacting theory

$$\Rightarrow \quad c_0 + 1 = 0.$$

$\Rightarrow$ Level 0 $\Rightarrow$ Tachyonic state

Level 1 $\Rightarrow$ ~~Re~~ Massless state.

$$\zeta_M \alpha^M_{-1} |\{k\}\rangle + \xi_i \alpha^i_{-1} |\{k\}\rangle.$$

Physical state constraint:

$$\zeta_M k^M = 0.$$

Equivalence relation: $\zeta_M \cong \zeta_M + a k_M$.

$\zeta_M \to$ Massless ~~vector~~ gauge field living on the D-brane.

~~&~~ ξ_i: A set ~~a~~ of $(25-p)$ massless scalars ϕ^i

$\Leftrightarrow$ ~~the~~ Translation mode of the D-brane in $(25-p)$ transverse direction. $\langle\phi^i\rangle = c^i \Rightarrow$ D-brane located at $\vec{x}_0 + \vec{c}$.

Interpretation: Consider p-brane solutions in an ordinary D-dimensional field theory.

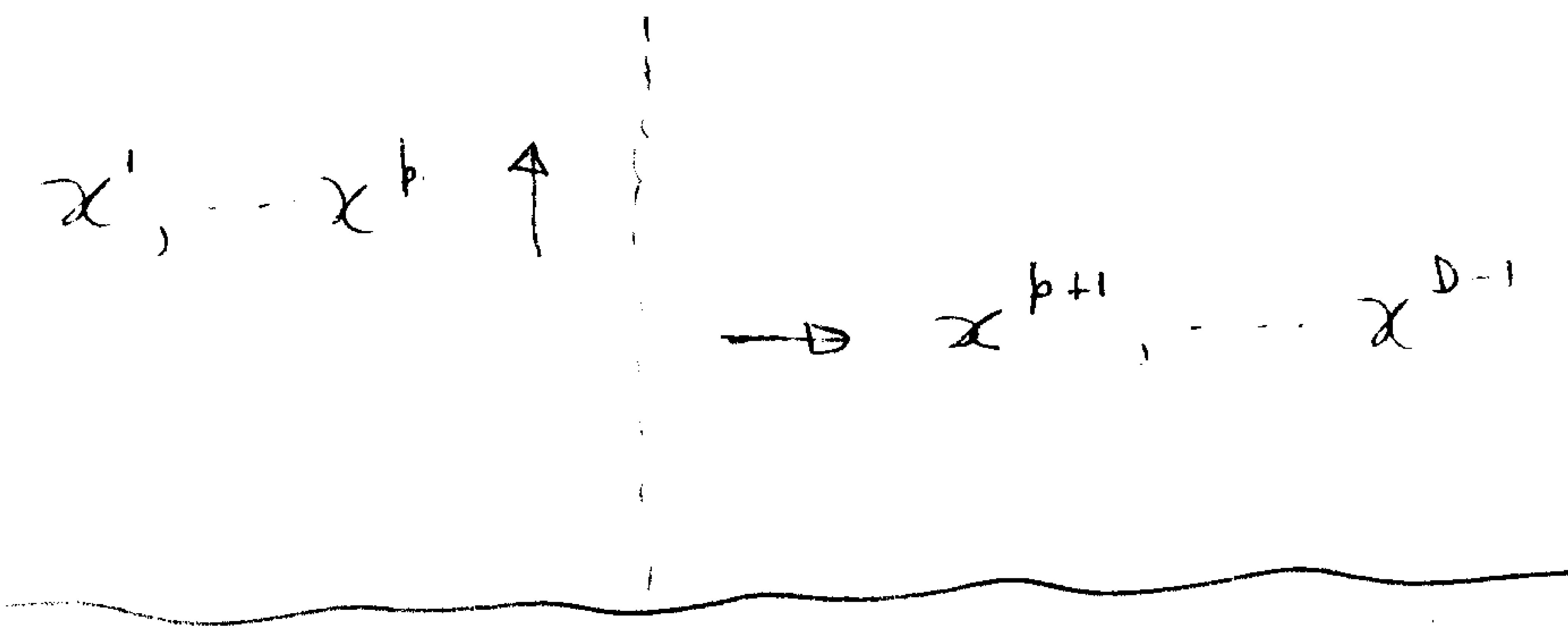
$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + V'(\phi) = 0$$

classical solution: $\phi = \phi_{cl}(x^{p+1}, \ldots x^{D-1})$

$\boxed{\text{Independent of } x^0, \ldots x^p.}$ $\phi_{cl} \to \phi_{vac}$ far away

Suppose energy density is concentrated around $x^{p+1} \simeq 0,\ x^{p+2} \simeq 0, \ldots\ x^{D-1} \simeq 0$.

$\Rightarrow$ A p-brane soliton.



Fluctuations around the classical soln:

$$\phi(x^0, \ldots x^D) = \phi_{cl}(x^{p+1}, \ldots x^{D-1}) + \chi(x^0, \ldots x^p)$$

Eqs. of motion

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\chi + V''(\phi_{cl})\,\chi + G(\chi^2) = 0$$

$$\parallel$$

$$\eta^{MN}\partial_M\partial_N\chi + \partial_i\partial_i\chi + V''(\phi_{cl})\,\chi + G(\chi^2) = 0$$

~~Define~~ $(x^{p+1}, \ldots x^{D-1}) \to (y^{p+1}, \ldots y^{D-1})$

Define $F_n(\vec{y})$

$$\partial_i\partial_i F_n(\vec{y}) + V''(\phi_{cl})\,F_n(\vec{y}) = -\lambda_n F_n(\vec{y})$$

$\Rightarrow$ Complete set of states.

Expand X as:

$$X(\vec{x},\vec{y}) = \sum_n F_n(\vec{y})\,\Psi_n(\vec{x})$$

$$\Rightarrow \eta^{MN}\partial_M\partial_N\Psi_n - \lambda_n\Psi_n + G(\Psi_n^2) = 0$$

$\Rightarrow$ A $(p+1)$ dimensional field of mass λ_n

Note: ~~The~~ Only those ~~bulk~~ modes corresponding to normalizable F_n ($\int |F_n(\vec{y})|^2 d^{D-p-1}y$ = finite) correspond to $(p+1)$ dimensional fields localized on the brane.

$\Rightarrow$ Excitation modes of the brane.

For non-normalizable modes λ_n is continuous & actually describe D dimensional bulk fields. (F_n are scattering states labelled by $D-p-1$ dimensional transverse momentum)

The open strings living on the D-brane describe the normalizable modes of the soliton.

Note that unlike for ordinary solitons where we first specify the solution and then compute the spectrum of modes living on the brane, D-branes are specified by the spectrum of modes living on the brane.

For ordinary solitons the $O(\psi_n^2)$ and higher terms describe interaction of the excitation modes of the soliton with each other & also the bulk fields.

For D-branes the interaction of the excitation modes of the brane among each other and the bulk fields is described by the fundamental rules for string interactions

(Sum over all 2-d surfaces bounded by external string states).

Translation mode:

$$\phi_{cl}(\vec{y}+\vec{a}) - \phi_{cl}(\vec{y}) = a^i \partial_i \phi_{cl}(y)$$

A soln. of linearized eqs. of motion.

$\Rightarrow$ Translation modes

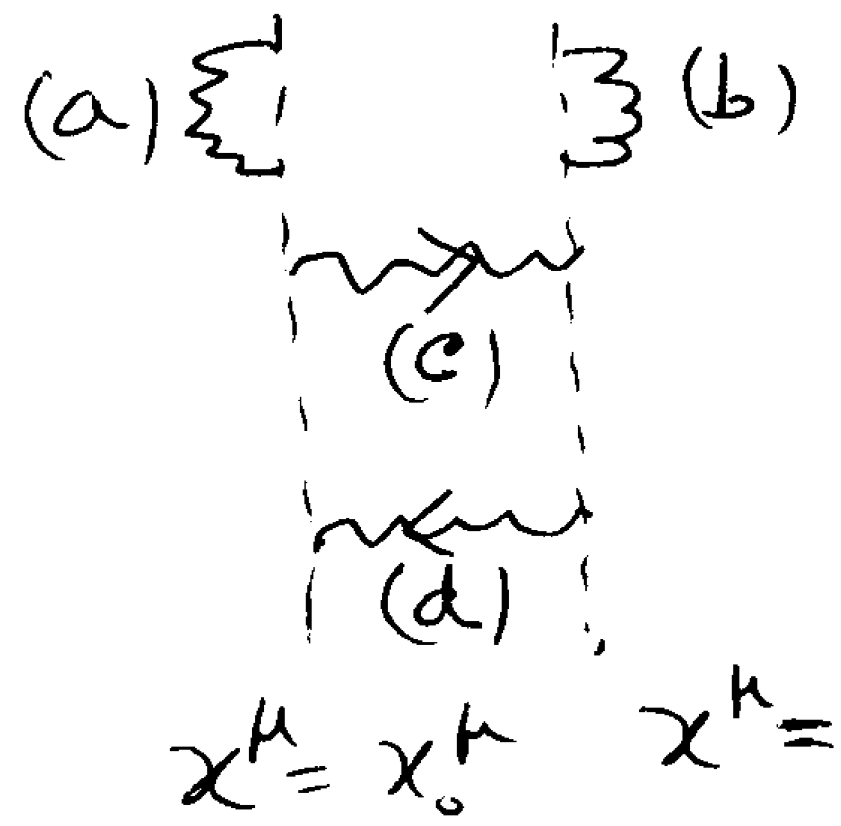
Eigenvalue $= 0$

For conventional solitons in field theory, if we have a pair of solitons, then the spectrum of normalizable modes around the solution representing the pair cannot be easily determined by ~~known~~ any simple rules.

→ We need to solve the eigenvalue equation around the new classical solution representing the pair of branes.

But for D-branes the analysis is simple.

Consider a pair of $\parallel$ D-branes at $x^\mu = x_0^\mu$ & $x^\mu = x_1^\mu$ respectively.



b.c. $\partial_\sigma X^M = 0 \qquad 0 \le M \le p$

(a) $X^i = x_0^i$ at $\sigma = 0, \pi$

(b) $X^i = x_1^i$ at $\sigma = 0, \pi$

(c) $X^i = x_0^i$ at $\sigma = 0$
$\qquad = x_1^i$ at $\sigma = \pi$

(d) $X^i = x_1^i$ at $\sigma = 0$
$\qquad = x_0^i$ at $\sigma = \pi$

Spectrum of open strings in sectors (a) and (b)

→ same as that in the case of a single D-brane.

(c) : Mode expansion of X^μ → same.

~~Mod~~ Mode expansion of X^i

$$X^i = (x_1^i - x_0^i)\frac{\sigma}{\pi} + x_0^i + \sum_n \phi_n^{(i)} \sin n\sigma$$

$$S = \frac{1}{4\pi\alpha'}\int d\tau\left[\frac{(\vec{x}_1 - \vec{x}_0)^2}{\pi} + \frac{\pi}{2}\left(\partial_\tau \phi_n^{(i)}\partial_\tau \phi_n^{(i)} - n^2 \phi_n^{(i)}\phi_n^{(i)}\right)\right]$$

Extra constant term.

$$\Pi_n^{(i)} = \frac{\delta S}{\delta(\partial_\tau \phi_n^{(i)})} = \frac{1}{4\alpha'}\partial_\tau \phi_n^{(i)}$$

→ Same commutation relation as in the case of sector (a)

→ same Fock space structure.

Difference comes in the expressions for L_m.

$$T_{uu} = \text{Const}\sum_m L_m e^{im\sigma}$$

$$T_{vv} = \text{Const}\sum_m L_m e^{-im\sigma}$$

$$L_m = \frac{1}{2}\sum_{n}\left(\alpha^M_{m-n}\,\alpha^N_n\,\eta_{MN} + \alpha^i_{m-n}\,\alpha^i_n\right) + \sqrt{2\alpha'}\,p_M\alpha^M_m.$$

for $m \neq 0$

$$L_0 = \sum_{n>0}\left(\alpha^M_{-m}\,\alpha^N_m\,\eta_{MN} + \alpha^i_{-m}\alpha^i_m\right)$$
$$+ \alpha' p_M p_N \eta^{MN} + C_0 + \frac{\left|\vec{x}_1 - \vec{x}_0\right|^2}{4\pi^2\alpha'}$$

C_0 takes the same value -1

$\to$ can be seen by analyzing the system at $\vec{x}_1 = \vec{x}_0$.

$\Rightarrow$ Same open string spectrum except that each state has an additional contribution to $(\text{mass})^2$

$$m^2_{(c)} = m^2_{(a)} + \frac{\left|\vec{x}_1 - \vec{x}_0\right|^2}{4\pi^2\alpha'^2}$$

Similarly $m^2_{(d)} = m^2_{(a)} + \dfrac{\left|\vec{x}_1 - \vec{x}_0\right|^2}{4\pi^2\alpha'^2}$

$$m^2_{(b)} = m^2_{(a)}$$

Now consider the case $\vec{x}_1 = \vec{x}_0$

$\Rightarrow$ 4-fold degeneracy.

4 Gauge fields $\to$ $U(2)$ gauge fields.

4 sets of $(25-p)$ scalars $\to$ $(25-p)$ ~~adj~~ scalar fields in the adjoint of $U(2)$.

Each state is 4 fold degenerate

$\Rightarrow$ adjoint of $U(2)$

~~generated~~

$\rightarrow$ 2×2 hermitian matrix, Λ.

$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ sector (a)

$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ sector (b)

$\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow$ sector (c)

$\Lambda = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow$ sector (d).

Λ: Chan-Paton factor.

Look at an adjoint scalar associated with a particular scalar (say ϕ^{25}).

$\langle \phi^{25} \rangle = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ moving first brane at $\vec{x}_0$ to $\vec{x}_0 + (0,0,\cdots 0,a)$

$\langle \phi^{25} \rangle = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \rightarrow$ moving second brane at $\vec{x}_0$ to $\vec{x}_0 + (0,0,\cdots 0,b)$

$\langle \phi^{25} \rangle = \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \rightarrow$ branes at $\vec{x}_0 + (0,0,\cdots 0,a)$ & at $\vec{x}_0 + (0,0,\cdots 0,b)$.

If $a = b$, the two branes are at the same location $\Rightarrow$ still 4 massless gauge fields

$\leftrightarrow$ $U(2)$ is unbroken since all Λ commute with $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

If $a \neq b$, ~~...~~ ~~...~~ is ~~...~~ only sectors (a) & (b) give massless gauge fields.

$\leftrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ commutes with only two

generators : $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow U(2) \xrightarrow[\text{broken to}]{} U(1) \times U(1)$.

Write $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} a+b & 0 \\ 0 & a+b \end{pmatrix}}_{\text{commutes with all } \Lambda} + \frac{1}{2}\underbrace{\begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix}}_{\text{does not commute}}$

$\Rightarrow$ Expect the ~~...~~ mass of the off-diagonal gauge bosons to be proportional to $|b-a|$

Can be seen from the fact that the mass of the vectors from sectors (c) & (d) are proportional to

$$\left| \vec{x}_0 + (0,0,\cdots 0, a) - \vec{x}_0 - (0,0,\cdots 0, b) \right|$$

$$= |a-b|.$$

Similarly N coincident D-branes

$\Rightarrow U(N)$ gauge theory will all states in the adjoint representation of $U(N)$

* Similarly N coincident D-branes
⇒ U(N) gauge theory
All states are in the adjoint representation of U(N) — N^2-fold degenerate

We can now consider closed string theories with N space-filling D-25 branes.
⇒ open strings with Neumann b.c. on all coordinates and carrying N×N Chan-Paton factors.

Massless open string spectrum
⇒ U(N) gauge fields in (25+1) dimension.

Note: There are no transverse directions
⇒ No scalar fields.

⇒ U(N) gauge theory + $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ fields.

It turns out that once interactions are taken into account this theory becomes inconsistent.

Reason: Dp-branes carry finite ~~mass~~ energy/unit p-volume.

⇒ D-25-branes ⇒ finite energy density in (25+1) dim. space.

⇒ A cosmological constant term in the Einstein-Hilbert action.

⇒ (25+1)-Dim Minkowski space is no longer a solution to the eqs. of motion. (~~...~~ Other soln.s may exist).

Unoriented strings

Consider the transformation:

$\Omega : \sigma \to -\sigma , \ \tau \to \tau$ for closed strings

A symmetry of the action.

$$X'^{\mu}(\sigma, \tau) = X^{\mu}(-\sigma, \tau)$$

$$X^{\mu} = \sum \phi^{\mu}_{n} e^{in\sigma}$$

$$\phi^{\mu}_{n} \to \phi^{\mu}_{-n}.$$

$$\Rightarrow \quad \alpha^{\mu}_{n} \leftrightarrow \tilde{\alpha}^{\mu}_{n}.$$

We can define a new string theory by restricting only to Ω invariant states.

Quotient of the original theory by Ω.

$\Rightarrow$ Unoriented string theory.

Ω: changes orientation:

$|\{k\}\rangle$ is Ω invariant.

$\Rightarrow$ Tachyon survives.

$S_{\mu\nu}\, \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |\{k\}\rangle$ survives for symmetric $S_{\mu\nu}$.

$\Rightarrow$ $g_{\mu\nu}$ & ϕ are present

$B_{\mu\nu}$ is absent.

It turns out that the process of Ω projection generates a negative cosmological constant.

Thus once interactions are taken into account, the ~~unoriented string~~ Minkowski space is no longer, a solution of the equations of motion.

Can be cancelled by putting in space-filling 25-branes.

Need 2^{13} 25-branes.

$\Rightarrow$ $2^{13} \times 2^{13}$ matrix Λ for Chan-Paton factors.

Ω projection on the open string states:

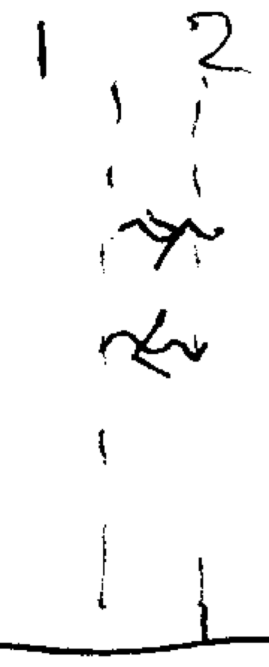
$$\sigma \to \pi - \sigma$$

$$X^\mu = \sum \phi_n^\mu \sin n\sigma$$

$$\Rightarrow \phi_n^\mu \to (-1)^n \phi_n^\mu$$

$$\Rightarrow \alpha_n^\mu \to (-1)^n \alpha_n^\mu$$

Action on Λ: $\Lambda \to \Lambda^T$



$$12 \to 21$$

Consider action on tachyon:

$$\Omega\left(|\{k\}\rangle \otimes \Lambda\right) = |\{k\}\rangle \otimes \Lambda^T$$

$\Rightarrow$ $\Lambda = \Lambda^T$ for Ω invariant states.

Level I state:

$$\Omega \left(\zeta_\mu \alpha^\mu_{-1} |k\rangle \otimes \Lambda \right) = -\zeta_\mu \alpha^\mu_{-1} |k\rangle \otimes \Lambda^T$$

$\Lambda = -\Lambda^T$ for Ω invariant state.

Also $\Lambda = \Lambda^+$

$\Rightarrow \Lambda = -\Lambda^*$ $\Rightarrow$ Λ is purely imaginary.

$\Lambda \leftrightarrow$ generators of $SO(N)$ group.

$\Rightarrow$ Gauge symmetry is $SO(N)$.

Even levels: Symmetric $N \times N$ representation of $SO(N)$

Odd levels: **Anti-symmetric** $N \times N$ representation of $SO(N)$

Thus we get a theory with gauge group $SO(2^{13})$.

Meaning of the $\langle \text{phys} | T_{uu} | \text{phys}' \rangle = 0$ constraint
T_{uv}

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, \sqrt{-\det \gamma} \, \gamma^{\alpha\beta} \, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\gamma_{\alpha\beta}$: Two dimensional metric.

Gauge symmetries.

① Reparametrization invariance:

$$X'^\mu(\vec{\xi}') = X^\mu(\vec{\xi}) \qquad \xi'^\alpha = f^\alpha(\vec{\xi})$$

$$\gamma'_{\alpha\beta}(\vec{\xi}') = \gamma_{ab}(\vec{\xi}) \frac{\partial \xi^a}{\partial \xi'^\alpha} \frac{\partial \xi^b}{\partial \xi'^\beta}$$

② Weyl Invariance:

$$\gamma'_{\alpha\beta}(\vec{\xi}) = e^{\phi(\vec{\xi})} \gamma_{\alpha\beta}(\vec{\xi})$$

$$X'^\mu(\vec{\xi}) = X^\mu(\vec{\xi})$$

Gauge fixing : $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$

$$\Rightarrow \quad S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$

$\gamma_{\alpha\beta}$ eqs. of motion:

$$0 = T_{\alpha\beta} \equiv \text{const.} \left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} \eta_{\alpha\beta} \eta^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu \eta_{\mu\nu} \right)$$

$\eta^{\alpha\beta} T_{\alpha\beta} \equiv 0 \Rightarrow$ Two inequivalent constraints.

$$0 = T_{uu} = (\text{const.}) \, \partial_u X^\mu \partial_u X^\nu \eta_{\mu\nu} \qquad u = \tau - \sigma$$

$$0 = T_{vv} = (\text{const.}) \, \partial_v X^\mu \partial_v X^\nu \eta_{\mu\nu} \qquad v = \tau + \sigma$$

S has ~~a~~ two residual gauge invariances:

① Rep. $u \to f(u)$, $v \to v$, — reparametrization

$-du\,dv = -du'\,dv'\,f'(u)^{-1}$

$\Rightarrow \gamma'_{u'v'} = f'(u)^{-1}$

Together with a Weyl trs:

$\gamma_{\alpha\beta} \to f'(u)\,\gamma_{\alpha\beta}$, $X^\mu \to X^\mu$

Net result:

$$X'^\mu(f(u), v) = X^\mu(u, v)$$

$$\gamma'_{\alpha\beta}(f(u), v) = \eta_{\alpha\beta}$$

② Reparametrization:

$v \to g(v)$, $u \to u$, Together with a Weyl trs.

$$\gamma_{\alpha\beta} \to g'(v)\,\gamma_{\alpha\beta}$$

Net result:

$$X'^\mu(u, g(v)) = X^\mu(u, v)$$

$$\gamma'_{\alpha\beta}(u, g(v)) = \gamma_{\alpha\beta}(u, v) \to \eta_{\alpha\beta}$$

Since these gauge trs. are not fixed, we must demand that physical states are invariant under these gauge trs

Generator for first trs. (conformal trs.)

$u \to u + \epsilon\,\phi(u)$, $v \to v$

$\int d\sigma\,\phi(u)\,T_{uu}(u, v)$

If $\delta\mathcal{L} = \partial_\alpha K^\alpha$ then $J^\alpha = \frac{\delta\mathcal{L}}{\delta(\partial_\alpha\phi_i)}\delta\phi_i$

$\delta\int J^0 d\sigma\,{-}\,K^\alpha$

Generator of second symmetry.

$$v \to v + \epsilon \psi(v)$$

$$\int d\sigma\, \psi(v)\, T_{vv}(u,v)$$

Closed string:

$$\phi(u = \tau - \sigma) = \phi(u = \tau - \sigma - 2\pi)$$

$$\Rightarrow \quad \phi(u) = \phi(u - 2\pi)$$

$$\Rightarrow \quad \phi(u) = \sum_n \phi_n e^{+in u}$$

$\Rightarrow$ ~~Now~~ Similarly

$$\psi(v) = \sum_n \psi_n e^{+in v}$$

$\Rightarrow$ ~~Also the~~ Generators of conformal trs.

$$\sum_n \phi_n \int_0^{2\pi} d\sigma\, e^{+in(\tau-\sigma)}\, T_{uu} \propto \sum_n \phi_n L_n e^{in\tau}$$

$$\& \quad \sum_n \psi_n \int_0^{2\pi} d\sigma\, e^{in(\tau+\sigma)}\, T_{vv} \propto \sum_n \psi_n \tilde{L}_n e^{in\tau}$$

ϕ_n, ψ_n arbitrary coefficients.

$\Rightarrow$ Gauge invariance requires:

$$\langle phys | L_n | phys' \rangle = 0 = \langle phys | \tilde{L}_n | phys' \rangle.$$

$L_n, \tilde{L}_n$ - Generators of conformal trs. $u \to u + \epsilon\phi_n e^{inu}$, $v \to v + \epsilon\psi_n e^{inv}$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}$$

$$[\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}.$$

$c = D \Rightarrow$ central charge.

Open strings

$$u \to u + \epsilon \phi(u)$$
$$v \to v + \epsilon \psi(v)$$

$$\Rightarrow \quad \tau = \tfrac{1}{2}(u+v) \to \tau + \epsilon(\phi(u) + \psi(v))$$
$$\sigma = \tfrac{1}{2}(v-u) \to \sigma + \epsilon(\psi(v) - \phi(u))$$

$\sigma = 0$ should map to $\sigma = 0$

$\sigma = \pi$ should map to $\sigma = \pi$

$\Rightarrow$ At $\sigma = 0$, $u = \tau$, $v = \tau$

At $\sigma = \pi$, $u = \tau - \pi$, $v = \tau + \pi$

$$\Rightarrow \quad \psi(\tau) - \phi(\tau) = 0.$$
$$\psi(\tau + \pi) - \phi(\tau - \pi) = 0.$$

$$\Rightarrow \quad \psi(\tau) = \phi(\tau).$$

$$\psi(\tau + 2\pi) = \psi(\tau)$$

$$\Rightarrow \quad \psi(v) = \sum \psi_n e^{inv}$$
$$\phi(u) = \sum \psi_n e^{inu}$$

Noether current:

$$\sum_n \int_0^\pi d\sigma \, \psi_n \left(e^{inu} T_{uu} + e^{inv} T_{vv}\right) \propto$$

$$T_{uu} = \sum L_m e^{im\sigma}$$
$$T_{vv} = \sum L_m e^{-im\sigma}$$

$$\sum_n \psi_n e^{in\tau} L_m \int_0^\pi d\sigma \sum_m \left(e^{i(m-n)\sigma} + e^{i(n-m)\sigma} \right) \ldots$$

$$\propto \sum_n \psi_n L_n e^{in\tau}$$

Thus L_n are the generators of the conformal transformation:

$$u \to u + \epsilon \psi_n e^{inu}$$
$$v \to v + \epsilon \psi_n e^{inv}$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\, \delta_{m+n,0}$$

Note: $L_m = \sum_{i=0}^{D-1} L_m^{(i)}$, $\tilde{L}_m = \sum_{i=0}^{D-1} \tilde{L}_m^{(i)}$

Thus the theory has extra symmetries generated by $L_m^{(i)}$ & $\tilde{L}_m^{(i)}$ for each i, although it is not part of the original gauge invariance.

$\Rightarrow \langle | L_m^{(i)} | \rangle \neq 0$, $\langle | \tilde{L}_m^{(i)} | \rangle \neq 0$ for individual i

But we can still use these additional symmetries to our advantage.

We shall see that in the light cone gauge we do not have any constraints, but we still have conformal symmetry which we can use.

$\tau = X^+$, $\partial_\sigma \gamma_{\sigma\sigma} = 0$, $\det \gamma = -1$

Light cone gauge

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{-\det\gamma}\;\gamma^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}$$

Three gauge transformation parameters:

→ 2 from 2-d g.c.t.

1 from Weyl trs.

⇒ Need three gauge fixing conditions.

Define: $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$

Take the 3 gauge fixing conditions to be:

$$X^+ = \tau \equiv \xi^0$$

$$\partial_\sigma \gamma_{\sigma\sigma} = 0$$

$$\det\gamma = -1$$

$$\gamma_{\tau\tau}\gamma_{\sigma\sigma} - (\gamma_{\tau\sigma})^2 = -1$$

$$\Rightarrow \quad \gamma_{\tau\tau} = \{(\gamma_{\tau\sigma})^2 - 1\}/\gamma_{\sigma\sigma}$$

$$\gamma^{\alpha\beta} = -\begin{pmatrix} \gamma_{\sigma\sigma} & -\gamma_{\sigma\tau} \\ -\gamma_{\sigma\tau} & \gamma_{\tau\tau} \end{pmatrix}$$

$$\Rightarrow \quad S = +\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma$$

$$\left[\gamma_{\sigma\sigma}\,\partial_\tau X^\mu\,\partial_\tau X^\nu\,\eta_{\mu\nu} - 2\gamma_{\sigma\tau}\,\partial_\tau X^\mu\,\partial_\sigma X^\nu\,\eta_{\mu\nu} + \gamma_{\tau\tau}\,\partial_\sigma X^\mu\,\partial_\sigma X^\nu\,\eta_{\mu\nu}\right]$$

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$$-\partial_\alpha X^- \partial_\beta X^+$$

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = -\partial_\alpha X^+ \partial_\beta X^- + \sum_{i=2}^{D-1} \partial_\alpha X^i \partial_\beta X^i$$

i, j repeated $\Rightarrow$ sum over $2, \ldots D-1$

$$\partial_\tau X^+ = 1, \quad \partial_\sigma X^+ = 0$$

$$\Rightarrow S = \frac{1}{4\pi\alpha'} \int d\tau\, d\sigma \left[\gamma_{\sigma\sigma}(-2\partial_\tau X^- + \partial_\tau X^i \partial_\tau X^i) \right.$$

$$- 2\gamma_{\sigma\tau}(-\partial_\sigma X^- + \partial_\tau X^i \partial_\sigma X^i)$$

$$\left. + \gamma_{\sigma\sigma}^{-1}((\gamma_{\tau\sigma})^2 - 1)\, \partial_\sigma X^i \partial_\sigma X^i \right]$$

$$X^-(\tau,\sigma) = x^-(\tau) + \underbrace{\sum_{n\neq 0} \phi_n^-(\tau) e^{in\sigma}}_{y^-(\tau,\sigma)}$$

ℓ = 2π for closed, π for open

$$S = \frac{1}{4\pi\alpha'} \int d\tau \left[-2\gamma_{\sigma\sigma} \partial_\tau x^- \ell \right.$$

$$+ \gamma_{\sigma\sigma} \int d\sigma\, \partial_\tau X^i \partial_\tau X^i - 2 \int d\sigma\, \gamma_{\sigma\tau}(-\partial_\sigma y^- + \partial_\sigma X^i \partial_\tau X^i)$$

$$\left. + \gamma_{\sigma\sigma}^{-1} \int d\sigma\, ((\gamma_{\tau\sigma})^2 - 1)\, \partial_\sigma X^i \partial_\sigma X^i \right]$$

Note: y^- & hence $\phi_n^-(\tau)$ has no time derivative in S.

$\Rightarrow \phi_n^-(\tau)$ eqs of motion should give constraints.

$$\int d\sigma\, e^{in\sigma} (in)\, \gamma_{\sigma\tau} = 0 \quad \forall\, n \neq 0$$

$\Rightarrow$ ~~$\gamma_{\sigma\tau}$ = constant~~ i.e. $\partial_\sigma \gamma_{\sigma\tau} = 0$.

We shall now restrict ourselves to the open string sector with Neumann b.c. but similar analysis can be carried out for other b.c.'s as well.

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\delta S = -\frac{1}{2\pi\alpha'}\int d^2\xi \sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta(\delta X^\nu)\eta_{\mu\nu}$$

$$= +\frac{1}{2\pi\alpha'}\int d^2\xi\, \partial_\beta\left(\sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu\right)\eta_{\mu\nu}\delta X^\nu$$

$$- \frac{1}{2\pi\alpha'}\int d\tau \sqrt{-\det\gamma}\,\gamma^{\alpha\sigma}\partial_\alpha X^\mu \delta X^\nu \eta_{\mu\nu}\Big|_{\sigma=0}^{\pi}$$

$\Rightarrow$ b.c.

$$\gamma^{\alpha\sigma}\partial_\alpha X^\mu = 0 \quad \text{at } \sigma = 0, \pi$$

$$\Rightarrow \gamma^{\sigma\tau}\partial_\tau X^\mu + \gamma^{\sigma\sigma}\partial_\sigma X^\mu = 0$$

$$\Rightarrow \gamma_{\sigma\tau}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0, \pi$$

Take $\mu = +$.

$$\Rightarrow \gamma_{\sigma\tau} = \gamma_{\tau\tau}\partial_\sigma X^+ = 0.$$

$$\Rightarrow \gamma_{\sigma\tau} = 0 \quad \text{everywhere since } \partial_\sigma\gamma_{\sigma\tau} = 0$$

$$S = \frac{1}{4\pi\alpha'}\int d\tau \Big[-2\gamma_{\sigma\sigma}\partial_\tau x^- \cdot \pi$$
$$+ \gamma_{\sigma\sigma}\int d\sigma\, \partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\int d\sigma\, \partial_\sigma X^i \partial_\sigma X^i\Big]$$

Eqs. of motion:

$\gamma_{\sigma\sigma}$ eqn.: $\quad -2\pi\,\partial_\tau x^- + \int d\sigma\, \partial_\tau X^i \partial_\tau X^i$

$$+ \gamma_{\sigma\sigma}^{-2} \int d\sigma\, \partial_\sigma X^i \partial_\sigma X^i = 0.$$

x^- eqn.: $\quad \partial_\tau \gamma_{\sigma\sigma} = 0.$

X^i eqn.: $\quad \gamma_{\sigma\sigma}\, \partial_\tau^2 X^i - \gamma_{\sigma\sigma}^{-1}\, \partial_\sigma^2 X^i = 0$

Q. Do these equations contain all the original eqs. of motion?

We have gauge fixed X^+, $\gamma_{\sigma\sigma}$ & $\det\gamma$.

Check the eqs. of motion of X^+ & $\gamma_{\alpha\beta}$.

X^+ eqn:

$$\partial_\alpha \left(\sqrt{-\det\gamma}\, \gamma^{\alpha\beta}\, \partial_\beta X^- \right) = 0.$$

In the gauge:

$$\partial_\tau \left(\gamma^{\tau\tau} \partial_\tau X^- \right) + \partial_\sigma \left(\gamma^{\sigma\sigma} \partial_\sigma X^- \right) = 0$$

$$\Rightarrow \quad -\gamma_{\sigma\sigma}\, \partial_\tau^2 X^- + \gamma_{\sigma\sigma}^{-1}\, \partial_\sigma^2 X^- = 0$$

$$\Rightarrow \quad -\gamma_{\sigma\sigma}\, \partial_\tau^2 x^- = 0.$$

$$\& \quad -\gamma_{\sigma\sigma}\, \partial_\tau^2 Y^- + \gamma_{\sigma\sigma}^{-1}\, \partial_\sigma^2 Y^- = 0.$$

First eq. a consequence of the eqs. of motion.

2nd eq. constraint on Y^-. But Y^- has disappeared from the dynamics.

$\Rightarrow$ No extra constraint on the physical degrees of freedom

$\gamma_{\alpha\beta}$ eqs:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\alpha'\beta'}\partial_{\alpha'}X^\mu\partial_{\beta'}X^\nu \eta_{\mu\nu}$$

$$T_{\sigma\sigma} = \partial_\sigma X^i \partial_\sigma X^i - \frac{1}{2}\gamma_{\sigma\sigma}\{-\gamma_{\sigma\sigma}\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + (\gamma_{\sigma\sigma})^{-1}\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\}$$

$$= -\frac{1}{2}\gamma_{\sigma\sigma}\{+2\gamma_{\sigma\sigma}\partial_\tau X^- - \gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i \pm (\gamma_{\sigma\sigma})^{-1}\partial_\sigma X^i \partial_\sigma X^i\}$$

$\partial_\tau x^-$ eq. is a consequence of the eqs of motion.

$$\partial_\tau Y^- = \frac{1}{2}\{\partial_\tau X^i \partial_\tau X^i + (\gamma_{\sigma\sigma})^{-2}\partial_\sigma X^i \partial_\sigma X^i\}$$

Determines Y^- but no constraints on the physical degrees of freedom.

$$T_{\tau\tau} = \partial_\tau X^i \partial_\tau X^i - 2\partial_\tau X^- + \frac{1}{2}\gamma_{\sigma\sigma}^{-1}(-\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i + 2\gamma_{\sigma\sigma}\partial_\tau X^- + \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\}$$

$$= \frac{1}{2}(-2\partial_\tau X^- + \partial_\tau X^i \partial_\tau X^i + \gamma_{\sigma\sigma}^{-2}\partial_\sigma X^i \partial_\sigma X^i)$$

Vanishing of $T_{\tau\tau}$ gives the same constraint.

Finally:

$\partial_\tau X^i = \frac{\pi}{p_-} P_i$

$\frac{1}{2\pi} p_- \frac{\pi^2}{p_-^2}$

$$T_{\tau\sigma} = \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}$$

$$= -\partial_\sigma X^- + \partial_\tau X^i \partial_\sigma X^i$$

$$\Rightarrow \underset{\partial_\sigma Y^-}{\underset{\parallel}{}} \qquad \partial_\sigma X^- = \partial_\tau X^i \partial_\sigma X^i$$

$\Rightarrow$ Determines ϕ_n^{-} in terms of X^i $\to$ satisfies all other eq.

These two equations are compatible as a consequence of the X^i equation.

Now back to the gauge fixed action:

$$S = \frac{1}{4\pi\alpha'} \int d\tau \left[-2\pi\gamma_{\sigma\sigma} \partial_\tau x^- + \gamma_{\sigma\sigma} \int d\sigma\, \partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1} \int d\sigma\, \partial_\sigma X^i \partial_\sigma X^i \right]$$

$$p_\gamma = \frac{\delta S}{\delta(\partial_\tau \gamma_{\sigma\sigma})} = 0 \,; \quad p_- = \frac{\delta S}{\delta(\partial_\tau x^-)} = -\frac{\gamma_{\sigma\sigma}}{2\alpha'}, \quad P_i(\sigma) = \frac{\delta S}{\delta(\partial_\tau X^i)} = \gamma_{\sigma\sigma} \frac{\partial_\tau X^i}{2\pi\alpha'}$$

$$\mathcal{H} = p_\gamma \partial_\tau \gamma_{\sigma\sigma} + p_- \partial_\tau x^- + \int d\sigma\, P_i(\sigma)\, \partial_\tau X^i(\sigma) - L$$

$$= -\frac{\pi}{2p_-} \int d\sigma\, P_i(\sigma) P_i(\sigma) - \frac{1}{8\pi\alpha'^2 p_-} \int d\sigma\, \partial_\sigma X^i \partial_\sigma X^i$$

b.c. $\partial_\sigma X^i = 0$ at $\sigma = 0, \pi$

$\partial_\sigma X^- = 0$ at $\sigma = 0, \pi$ $\to$ automatically satisfied

Mode expansion

$$X^i = x^i(\tau) + \sum_{n=1}^{\infty} \phi_n^i(\tau) \cos n\sigma$$

$$S = \frac{1}{4\pi\alpha'} \int d\tau \Big[-2\pi \gamma_{\sigma\sigma} \partial_\tau x^- + \pi \gamma_{\sigma\sigma} \partial_\tau x^i \partial_\tau x^i$$
$$+ \frac{\pi}{2} \gamma_{\sigma\sigma} \sum_{n=1}^{\infty} \partial_\tau \phi_n^i \partial_\tau \phi_n^i - \frac{\pi}{2} (\gamma_{\sigma\sigma})^{-1} \sum_{n=1}^{\infty} n^2 \phi_n^i \phi_n^i \Big]$$

$$p_\gamma = \frac{\delta S}{\delta(\partial_\tau \gamma_{\sigma\sigma})} = 0, \qquad p_- = \frac{\delta S}{\delta(\partial_\tau x^-)} = -\frac{1}{2\alpha'} \gamma_{\sigma\sigma},$$

$$\Pi_n^i = \frac{\delta S}{\delta(\partial_\tau \phi_n^i)} = \frac{1}{4\alpha'} \gamma_{\sigma\sigma} \partial_\tau \phi_n^i, \qquad p_i = \frac{\delta S}{\delta(\partial_\tau x^i)} = \frac{1}{2\alpha'} \gamma_{\sigma\sigma} \partial_\tau x^i$$

$$\Rightarrow \mathcal{H} = p_\gamma \partial_\tau \gamma_{\sigma\sigma} + p_- \partial_\tau x^- + \Pi_n^i \partial_\tau \phi_n^i - L + p_i \partial_\tau x^i$$
$$= \frac{2\alpha'}{\gamma_{\sigma\sigma}} \sum_{n=1}^{\infty} \Pi_n^i \Pi_n^i + \frac{1}{8\alpha' \gamma_{\sigma\sigma}} \sum_{n=1}^{\infty} n^2 \phi_n^i \phi_n^i + \frac{\alpha'}{\gamma_{\sigma\sigma}} p_i p_i$$
$$= -\frac{1}{p_-} \sum_{n=1}^{\infty} \Pi_n^i \Pi_n^i - \frac{1}{16\alpha'^2 p_-} \sum_n n^2 \phi_n^i \phi_n^i - \frac{1}{2p_-} p_i p_i$$

Interpretation of $\mathcal{H}$:

Generator of τ translation.

But $\tau = x^+$

$\Rightarrow$ Generator of x^+ translation

$\Rightarrow -p_+$

p_+ : generator of passive x^+ ~~transformation~~ translation

$\mathcal{H}$: generator of active τ ~~transformation~~ translation

$$\Rightarrow p_+ = +\frac{p_i p_i}{2p_-} + \frac{1}{16\alpha'^2 p_-} \sum_n n^2 \phi_n^i \phi_n^i + \frac{1}{p_-} \sum_{n=1}^{\infty} \Pi_n^i \Pi_n^i$$

Now quantize:

$$[x^-, p_-] = i, \quad [x^i, p_j] = i\delta^i_j$$

$$[\phi^i_n, \Pi^j_m] = i\,\delta_{mn}\,\delta^{ij}$$

Define:

$$\left.\begin{aligned} \alpha^j_n &= \sqrt{2\alpha'}\,\Pi^j_n - \frac{in}{2\sqrt{2\alpha'}}\phi^j_n \\ \alpha^j_{-n} &= \sqrt{2\alpha'}\,\Pi^j_n + \frac{in}{2\sqrt{2\alpha'}}\phi^j_n \end{aligned}\right\} \text{ for } n>0.$$

Then

$$p_+ = \frac{p_i p_i}{2p_-} + \frac{1}{2\alpha' p_-}\Big[\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + c_0\Big]$$

normal ordering constant

$$[\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\,\delta_{m+n,0}$$

Define $|k_-, \vec{k}\rangle$ s.t. $\vec{k}$: $D-2$ dim. vector $(k^2, \dots k^{D-1})$

$$p_-|k_-, k\rangle = k_-|k_-, k\rangle$$

$$p_i|k_-, k\rangle = k_i|k_-, k\rangle$$

$$\alpha^i_n|k_-, k\rangle = 0 \quad \text{for } n>0$$

Generic state:

$$\alpha^{i_1}_{-m_1}\alpha^{i_2}_{-m_2}\cdots\alpha^{i_k}_{-m_k}|k_-, \vec{k}\rangle$$

$$p_+\,\alpha^{i_1}_{-m_1}\cdots\alpha^{i_k}_{-m_k}|k_-, \vec{k}\rangle$$

$$= \underbrace{\Big\{\frac{k_i k_i}{2k_-} + \frac{1}{2\alpha' k_-}\Big(\sum_{l=1}^{k} m_l + c_0\Big)\Big\}}_{k_+}\alpha^{i_1}_{-m_1}\cdots\alpha^{i_k}_{-m_k}|k_-, \vec{k}\rangle$$

$$\Rightarrow \quad \underbrace{2k_+k_- - k_i k_i}_{\Downarrow} = \frac{1}{\alpha'}\left(\sum_{l=1}^{k} m_l + C_0\right)$$

$\sum_{l=1}^{k} m_l \equiv N$ (level)

$$(k^0)^2 - \sum_{i=1}^{D-1} k_i k_i = \frac{1}{\alpha'}(N + C_0)$$

$\Downarrow$

m^2.

$\Rightarrow$ A state with

$$m^2 = \frac{1}{\alpha'}(N + C_0)$$

Note: Now all states are physical.

But manifest symmetry

$$SO(D-2) \subset SO(D-1,1)$$

Determination of C_0:

Note: At level 1 there are $(D-2)$ states

$$\alpha^i_{-1} |k_-, \vec{k}\rangle$$

$$m^2 = \frac{1}{\alpha'}(1 + C_0)$$

Must form a representation of $SO(D-1,1)$

A massive $SO(D-1,1)$ vector has $(D-1)$ helicity states.

A massless $SO(D-1,1)$ vector has $(D-2)$ helicity states.

$\Rightarrow$ State must be massless. $\Rightarrow C_0 = -1$

$\Rightarrow$ Level 0 state is tachyonic.

In order to establish Lorentz invariance of the quantum theory we should be able to explicitly construct the Lorentz generators and show that they satisfy the usual commutation relations.

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-\det \gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\delta X^\mu = \Lambda^\mu{}_\nu X^\nu.$$

$$\delta S = 0.$$

Noether current:

$$J = \frac{1}{2\pi\alpha'} \int d\sigma \sqrt{-\det\gamma}\, \gamma^{\alpha\tau} \partial_\alpha X^\mu \eta_{\mu\nu} \Lambda^\nu{}_\rho X^\rho$$

$$= \frac{1}{2\pi\alpha'} \Lambda_{\mu\rho} J^{\mu\rho}$$

$$J^{\mu\rho} = \frac{1}{4\pi\alpha'} \int d\sigma \sqrt{-\det\gamma} \left[\gamma^{\alpha\tau} (\partial_\alpha X^\mu) X^\rho - \gamma^{\alpha\tau} (\partial_\alpha X^\rho) X^\mu \right]$$

Now we consider the Noether current in the gauge fixed theory.

Note: In the gauge fixed theory the Lorentz trs. need to be accompanied by a compensating gauge trs.

But gauge invariance

$\Rightarrow$ The compensating gauge trs. leaves the physical states invariant.

$\Rightarrow$ We can continue to use the same generators if the gauge symmetry is valid.

$$\det \gamma = -1, \quad \gamma^{\sigma\tau} = 0.$$

$$\gamma^{\tau\tau} = -\gamma_{\sigma\sigma} = 2\alpha' p_-$$

$$\gamma^{\sigma\sigma} = -\gamma_{\tau\tau} = \frac{(\gamma_{\tau\sigma})^2 - 1}{\gamma_{\sigma\sigma}} = -(2\alpha' p_-)^{-1}$$

$$J^{\mu\rho} = \frac{1}{4\pi\alpha'} \int d\sigma \, [(2\alpha' p_-) \, \partial_\tau X^\mu \, X^\rho - (2\alpha' p_-) \, \partial_\tau X^\rho \, X^\mu]$$

$$= \frac{p_-}{2\pi} \int d\sigma \, [X^\rho \partial_\tau X^\mu - X^\mu \partial_\tau X^\rho]$$

$$J^{+-} = \frac{p_-}{2\pi} \int d\sigma \, [X^+ \partial_\tau X^- - X^- \partial_\tau X^+]$$

$$J^{i+} = \frac{p_-}{2\pi} \int d\sigma \, [X^i \partial_\tau X^+ - X^+ \partial_\tau X^i]$$

$$J^{i-} = \frac{p_-}{2\pi} \int d\sigma \, [X^i \partial_\tau X^- - X^- \partial_\tau X^i]$$

$$J^{ij} = \frac{p_-}{2\pi} \int d\sigma \, [X^i \partial_\tau X^j - X^j \partial_\tau X^i]$$

Note: $\delta X^+ = \Lambda^+{}_\nu X^\nu = -\Lambda^{+-} X^+ + \Lambda^{+i} X^i$

Thus we need compensating gauge trs for Λ^{+-} and Λ^{+i}

Generator:

$$\Lambda_{\mu\nu} J^{\mu\nu} = 2\left[\Lambda^{+-} J^{-+} - \Lambda^{-i} J^{+i} - \Lambda^{+i} J^{-i} + \Lambda^{ij} J^{ij}\right]$$

$$\eta^{\mu\nu} = \begin{array}{c|c} & \begin{matrix} + & - & 2 & 3 \end{matrix} \\ \hline \begin{matrix} + \\ - \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

Thus we expect J^{+i}, J^{ij} to be safe since these do not require compensating gauge transformation.

Problematic cases J^{-+}, J^{-i}.

We need to check that these $J^{\mu\nu}$'s satisfy the usual Lorentz algebra.

It turns out that all the $J^{\mu\nu}$ satisfy the Lorentz algebra except the $[J^{i-}, J^{j-}]$ commutator.

Expected answer $= 0$.

$$J^{i-} = \frac{p_-}{2\pi}\int d\sigma\,[X^i \partial_\tau X^- - X^- \partial_\tau X^i]$$

$\partial_\tau X^i, X^i$: Expressible in terms of x^i, p^i, α^i_n.

$$X^- = x^-(\tau) + \sum_n \phi^-_n(\tau) \cos n\sigma$$

$\gamma_{\sigma\sigma}$ eq. of motion:

$$\partial_\tau X^- = \frac{1}{2}\{\partial_\tau X^i \partial_\tau X^i + (\gamma_{\sigma\sigma})^{-2} \partial_\sigma X^i \partial_\sigma X^i\}$$

$$\gamma_{\sigma\sigma} = -2\alpha' p_-$$

Thus $\partial_\tau X^-$ can be expressed in terms of α^i_n, p^i, p_- etc.

$\gamma_{\tau\sigma}$ eq. of motion:

$$\partial_\sigma X^- = \partial_\tau X^i \partial_\sigma X^i$$

$$\Downarrow$$

$$-\sum_{n\neq 0} \phi^-_n(\tau)\, n \sin(n\sigma)$$

$\Rightarrow \phi^-_n(\tau)$ can be determined in terms of α^i_n, p^i etc.

$\Rightarrow X^-$ can be determined in terms of x^-, α^i_n, p^i etc.

Net result: J^{i-} can be expressed in terms of $x^i, p^i, \alpha^i_n, x^-, p_-$

Compute $[J^{i-}, J^{j-}]$

Net result:

$$[J^{i-}, J^{j-}] = \text{const} \sum_{m=1}^{\infty} \Delta_m (\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m)$$

$$\Delta_m = m\,\frac{26-D}{12} + \frac{1}{m}\left(\frac{D-26}{12} + 2(1+C_0)\right)$$

$$\Rightarrow \quad C_0 = -1, \quad D = 26$$

Light-cone gauge for closed strings

$$X^+ = \tau$$
$$\partial_\sigma \gamma_{\sigma\sigma} = 0$$
$$\det \gamma = -1$$

$$X^- = \sum_n \phi_n^-(\tau)\, e^{in\sigma} + x^-(\tau)$$

ϕ_n^- eqs. of motion become algebraic.

Gives $\partial_\sigma \gamma_{\tau\sigma} = 0$.

For open string we saw that $\gamma_{\tau\sigma} = 0$ at $\sigma = 0$ & π due to b.c.

$$\Rightarrow \quad \gamma_{\tau\sigma} = 0$$

This is no longer possible for closed string.

Instead here we proceed in a slightly different manner.

Residual gauge inv.

$$\sigma' = \sigma + f(\tau)$$

Not possible for open strings since $\sigma = 0, \pi$ need to be fixed.

$$\gamma_{\tau\tau}\, d\tau^2 + \gamma_{\sigma\sigma}\, d\sigma^2 + 2\gamma_{\sigma\tau}\, d\sigma\, d\tau$$

$$\to \gamma_{\tau\tau}\, d\tau^2 + \gamma_{\sigma\sigma}\left(d\sigma^2 + 2f'(\tau)\, d\tau\, d\sigma + (f'(\tau))^2 d\tau^2\right)$$

$$+ 2\gamma_{\sigma\tau}\left(d\sigma + f'(\tau)\, d\tau\right) d\tau$$

$$= \left(\gamma_{\tau\tau} + 2\gamma_{\sigma\tau} f'(\tau) + \gamma_{\sigma\sigma}(f'(\tau))^2\right) d\tau^2$$

$$+ 2\left(\gamma_{\sigma\tau} + f'(\tau)\gamma_{\sigma\sigma}\right) d\tau\, d\sigma$$

$$+ \gamma_{\sigma\sigma}\, d\sigma^2$$

$$\Rightarrow \quad \gamma_{\tau\tau} \to \gamma_{\tau\tau} + 2\gamma_{\sigma\tau} f'(\tau) + \gamma_{\sigma\sigma}(f'(\tau))^2$$

$$\gamma_{\sigma\tau} \to \gamma_{\sigma\tau} + f'(\tau)\gamma_{\sigma\sigma}$$

$$\gamma_{\sigma\sigma} \to \gamma_{\sigma\sigma}.$$

Note $\gamma_{\sigma\sigma}$ & $\det \gamma$ does not change.

$\Rightarrow$ Gauge conditions are preserved.

Since $\gamma_{\sigma\tau}$ & $\gamma_{\sigma\sigma}$ are independent of σ, we can make new $\gamma_{\sigma\tau}$ vanish by choosing $f'(\tau) = -\gamma_{\sigma\tau}/\gamma_{\sigma\sigma}$.

Rest of the analysis proceeds as in the open string case.

Result: Final dynamical variables:

$x^-,\ p_-,\ x^i,\ p^i,\ \alpha^i_n,\ \tilde{\alpha}^i_n$

$$-p_+ = H = -\frac{p^i p^i}{2p_-} - \frac{2}{\alpha' p_-}\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + c + \tilde{c}\right)$$

Note: There is still a residual gauge freedom $\sigma \to \sigma +$ constant.

$\Rightarrow$ We get an additional constraint:

$$\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + c = \sum_{n=1}^{\infty}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \tilde{c}$$

Symmetry $\alpha^i_n \leftrightarrow \tilde{\alpha}^i_n \Rightarrow c = \tilde{c}$

Analysis of Lorentz invariance

$\Rightarrow \quad c = \tilde{c} = -1$

$\Rightarrow$

$$H = -\frac{p^i p^i}{2p_-} - \frac{2}{\alpha' p_-}\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - 2\right)$$

$$\left(\sum_n \alpha^i_{-n}\alpha^i_n - \sum_n \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\right)|phys\rangle = 0$$

Acting on $|s\rangle = \alpha^{i_1}_{-m_1}\cdots\alpha^{i_k}_{-m_k}\,\tilde{\alpha}^{j_1}_{-n_1}\cdots\tilde{\alpha}^{j_l}_{-n_l}\,|k_-,k\rangle$

Constraint $\Rightarrow \sum m_i = \sum n_j \equiv N$

$$H\,|s\rangle = \left\{-\frac{k^i k^i}{2k_-} - \frac{2}{\alpha' k_-}(2N-2)\right\}|s\rangle$$

$$k_+ = \frac{k^i k^i}{2k_-} + \frac{4}{\alpha' k_-}(N-1)$$

$$\underbrace{2k_+ k_- - k^i k^i}_{m^2} = \frac{4}{\alpha'}(N-1)$$

This gives the spectrum of closed strings.

→ Same spectrum as in the covariant formulation.

Compactification: Reducing the number of dimensions.

---

(1) Take a D dimensional theory.

(2) Take d of the space-like dimensions to be compact & small.

For sufficiently small size, these directions will be invisible to the ~~present~~ ~~day~~ people and the experimentalists.

$\Rightarrow$ The theory will appear $(D-d)$ dimensional.

Field theory example

(1) Start with a scalar field in D dimensions.

$$(\Box - m^2)\,\phi = 0$$

$$\sum_{\mu,\nu=0}^{D-1} \eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0$$

$$\eta^{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, \cdots 1)$$

Take the x^{D-1} direction to describe a circle of radius R

---

x^M : $0 \le M \le D-2$ Remaining coordinates.

$$\phi(\{x^M\}, x^{D-1}) = \sum_n \phi_n(\{x^M\})\, e^{i n x^{D-1}/R}$$

$$\Box\phi = \sum_{\mu=0}^{D-2} \eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \partial_{D-1}^2\phi$$

$$= \sum_n \left(\sum_{M=0}^{D-2} \eta^{MN}\partial_M\partial_N\phi_n - \frac{n^2}{R^2}\phi_n \right) e^{i n x^{D-1}/R}$$

Klein Gordan eqn.

$$0 = (\Box - m^2)\,\phi = \sum_n \left(\sum_{M=0}^{D-2} \eta^{MN} \partial_M \partial_N \phi_n - \frac{n^2}{R^2}\phi_n - m^2 \phi_n \right) e^{inx^{D-1}/R}$$

$$\Rightarrow \sum_{M,N=0}^{D-2} \eta^{MN} \partial_M \partial_N \phi_n - \left(\frac{n^2}{R^2} + m^2\right) \phi_n = 0$$

$\Rightarrow$ ϕ_n describes a $(D-1)$ dimensional scalar field of $(\text{mass})^2 = \left(m^2 + \frac{n^2}{R^2}\right)$

Infinite no of such scalar fields since $-\infty < n < \infty$.

$$\phi_{-n} = \phi_n^*$$

Note: For small R, $\left(m^2 + \frac{n^2}{R^2}\right)$ is large except for $n = 0$.

$\Rightarrow$ As $R \to 0$ only the $n=0$ mode will be observable by experimentalists.

$\Rightarrow$ A single scalar field in $(D-1)$ dimension

This way by compactifying d dimensions we'll get a $(D-d)$ dimensional field theory at energy $\ll \frac{1}{R_i}$. $(i = D-1, D-2, \ldots D-d)$

Gauge field: A_μ

$$A_M = \sum_n A_M^{(n)}(\{x^M\})\, e^{inx^{D-1}/R}$$

$$A_{D-1} = \sum_n A_{D-1}^{(n)}(\{x^M\})\, e^{inx^{D-1}/R}$$

Substitute in Maxwell's eq:

$$\partial^\mu F_{\mu\nu} = 0 .$$

$A_M^{(n)}$: A vector field of $(\text{mass})^2 = \frac{n^2}{R^2}$

in $(D-1)$ dimension for $0 \leq M \leq D-2$.

$A_{D-1}^{(n)}$: A scalar field of $(\text{mass})^2 = \frac{n^2}{R^2}$

in $(D-1)$ dimensions, $\rightarrow$ gets absorbed in $A_M^{(n)}$ for $n \neq 0$

Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h_{MN} = \sum_n h_{MN}^{(n)} e^{i n x^{D-1}/R} \qquad 0 \leq M, N \leq D-2$$

$\rightarrow$ spin 2 field of $(\text{mass})^2 = \frac{n^2}{R^2}$

$$h_{M(D-1)} = \sum_n h_{M(D-1)}^{(n)} e^{i n x^{D-1}/R}$$

$\rightarrow$ vector field of $(\text{mass})^2 = \frac{n^2}{R^2}$ $\rightarrow$ absorbed in h_{MN}

$$h_{(D-1)(D-1)} = \sum_n h_{(D-1)(D-1)}^{(n)} e^{i n x^{D-1}/R}$$

$\rightarrow$ scalar field of $(\text{mass})^2 = \frac{n^2}{R^2}$

As $R \rightarrow 0$, only $n=0$ modes survive.

$\Rightarrow$ a massless gauge field, a massless scalar, and a massless vector

We now want to implement this procedure in string theory.

String theory → ∞ no of states of different $(\text{mass})^2$.

We can apply this procedure to each of these states

⇒ For each single particle state in (26) dim. string theory ~~with (mass)²~~ we get ∞ no. of single particle states in the compactified theory, labelled by momenta in the compactified directions.

But besides these states string theory contains extra states associated with "winding modes".

To study these we need to analyze directly strings ~~theory~~ ~~directly~~ in the compact space-time.

X^{D-1} is compact say.

Recall mode expansion of X^μ in the uncompactified theory:

$$X^\mu = x^\mu(\tau) + \sum_n \phi_n^\mu e^{in\sigma}$$

(Focus on closed string theory)

New mode expansion:

$$X^M = x^M(\tau) + \sum_n \phi_n^M e^{in\sigma}$$

$$X^{D-1} = Rw\,\sigma + x^{D-1}(\tau) + \sum_n \phi_n^{D-1} e^{in\sigma}$$

w : An integer.

As $\sigma \to \sigma + 2\pi$, $X^{D-1} \to X^{D-1} + 2\pi R w$

$\Rightarrow$ The string winds w times along the compact direction

$\Rightarrow$ Winding number

① Substitute this expansion into the worldsheet action $-\frac{1}{4\pi\alpha'} \int \eta^{\alpha\beta} \eta_{\mu\nu} d^2\xi \, \partial_\alpha X^\mu \partial_\beta X^\nu$ $\quad \alpha, \beta = 0, 1$ $\quad \xi^0 = \tau, \; \xi^1 = \sigma$

② Construct canonically conjugate momenta

w does not affect the definition of conjugate momenta.

$\Rightarrow$ We can proceed exactly as in the non compact case

$$\Pi_{\mu n} = \frac{1}{\alpha'} \eta_{\mu\nu} \partial_\tau \phi^\nu_{-n}$$

$$p_\mu = \frac{1}{\alpha'} \eta_{\mu\nu} \partial_\tau x^\nu$$

$$\alpha^\mu_{-n} = \frac{1}{\sqrt{2}} \left(\eta^{\mu\nu} \sqrt{\alpha'}\, \Pi_{\nu n} + \frac{i n}{\sqrt{\alpha'}} \phi^\mu_{-n} \right)$$

$$\tilde{\alpha}^\mu_n = \frac{1}{\sqrt{2}} \left(\eta^{\mu\nu} \sqrt{\alpha'}\, \Pi_{\nu n} - \frac{i n}{\sqrt{\alpha'}} \phi^\mu_{-n} \right)$$

x^μ, p_ν hermitian; $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$, $(\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$

$$[x^\mu, p_\nu] = i \delta^\mu{}_\nu, \quad [\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{m+n,0}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m \eta^{\mu\nu} \delta_{m+n,0}$$

Define $|\{k\}, w\rangle$ such that

$$p_\mu |\{k\}, w\rangle = k_\mu |\{k\}, w\rangle, \quad \alpha^\mu_n |\{k\}, w\rangle = 0 = \tilde{\alpha}^\mu_n |\{k\}, w\rangle$$

for $n > 0$

Note $k^{D-1} = n/R$; $n \in \mathbb{Z}$

$|\{k\}, w\rangle \to |\{k^M\}, n, w\rangle$;

Hilbert space:

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_l}_{-n_l} |\{k^M\}, n, w\rangle$$

$\{k^M\}$: (D-1) dim momenta

w: winding no

n: $\frac{n}{R}$ = momentum along x^{D-1}

(String theory in D-dim with x^D compactified on a circle of radius R)

Constraints:

$$L_m |phys\rangle = 0 \quad m \geq 0 \; ; \quad \tilde{L}_m |phys\rangle = 0 \quad m \geq 0$$

$$L_m = \frac{1}{2} \sum_{n \neq 0, m} \alpha^{\mu}_{m-n} \alpha^{\nu}_{n} \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{M}_{m} p_M + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{D-1}_{m}\left(-\frac{Rw}{\alpha'} + \frac{n}{R}\right) \quad m > 0$$

$$L_0 = \sum_{n>0} \alpha^{\mu}_{-n} \alpha^{\nu}_{n} \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{MN} p_M p_N + \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)^2 + c$$

$$\tilde{L}_m = \frac{1}{2} \sum_{n \neq 0, m} \tilde{\alpha}^{\mu}_{m-n} \tilde{\alpha}^{\nu}_{n} \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \tilde{\alpha}^{M}_{m} p_M + \frac{1}{2}\sqrt{2\alpha'}\, \tilde{\alpha}^{D-1}_{m}\left(\frac{n}{R} + \frac{Rw}{\alpha'}\right) \quad m > 0$$

$$\tilde{L}_0 = \sum_{n>0} \tilde{\alpha}^{\mu}_{-n} \tilde{\alpha}^{\nu}_{n} \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{MN} p_M p_N + \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{Rw}{\alpha'}\right)^2 + \tilde{c}$$

Generic state:

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell} |\{k^M\}, n, \omega\rangle$$

We need to impose constraints.

$$\langle \text{phys} | T_{uu} | \text{phys}' \rangle = 0 = \langle \text{phys} | T_{vv} | \text{phys}' \rangle$$

$$u = \tau - \sigma, \quad v = \tau + \sigma$$

$$T_{uu} = (\text{const})\, \eta_{\mu\nu}\, \partial_u X^\mu \partial_u X^\nu$$

$$= (\text{const}) \sum_m L_m e^{im\sigma}$$

$$T_{vv} = (\text{const})\, \eta_{\mu\nu}\, \partial_v X^\mu \partial_v X^\nu$$

$$= (\text{const}) \sum_m \tilde{L}_m e^{-im\sigma}$$

$$L_m = \frac{1}{2} \sum_{n \neq 0} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^\mu_m p_\mu - \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{D-1}_m \frac{R\omega}{\alpha'} \qquad m \neq 0$$

$$L_0 = \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{MN} p_M p_N + \frac{\alpha'}{4}\left(p_{D-1} - \frac{R\omega}{\alpha'}\right)^2 + C$$

C ↳ normal ordering const.

$$\tilde{L}_m = \frac{1}{2} \sum_{n \neq 0} \tilde{\alpha}^\mu_{m-n} \tilde{\alpha}^\nu_n \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^\mu_m p_\mu + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{D-1}_m \frac{R\omega}{\alpha'} \qquad m \neq 0$$

$$\tilde{L}_0 = \sum_{n>0} \tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{MN} p_M p_N + \frac{\alpha'}{4}\left(p_{D-1} + \frac{R\omega}{\alpha'}\right) + \tilde{C}$$

$\tilde{C}$ ↳ normal ordering const.

$$C = \tilde{C} = -1$$

Constraint: $L_m |\text{phys}\rangle = 0 = \tilde{L}_m |\text{phys}\rangle$ for $m \geq 0$

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell} |\{k^M\}, n, w\rangle$$

L_0 constraint:

$$\underbrace{\sum_{i=1}^{k} m_i}_{N} + \frac{\alpha'}{4}\eta^{MN} k_M k_N + \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)^2 + C = 0$$

$$-\eta^{MN} k_M k_N = \frac{4}{\alpha'}(N + C) + \left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)^2$$

$\tilde{L}_0$ constraint:

$$\underbrace{-\eta^{MN} k_M k_N}_{(\text{Mass})^2} = \frac{4}{\alpha'}(\tilde{N} + \tilde{C}) + \left(\frac{n}{R} + \frac{Rw}{\alpha'}\right)^2$$

$$(\text{Mass})^2 = \frac{4}{\alpha'}(N + C) + \left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)^2$$

$$= \frac{4}{\alpha'}(\tilde{N} + \tilde{C}) + \left(\frac{n}{R} + \frac{Rw}{\alpha'}\right)^2$$

Note

① For $w = 0$

$$(\text{Mass})^2 = \frac{4}{\alpha'}(N + C) + \frac{n^2}{R^2} = \frac{4}{\alpha'}(\tilde{N} + \tilde{C}) + \frac{n^2}{R^2}$$

$$\Rightarrow C + N = \tilde{N} + \tilde{C}$$

$$(\text{Mass})^2 = \underbrace{\frac{4}{\alpha'}(N + C)}_{\text{26-dim (mass)}^2} + \underbrace{\frac{n^2}{R^2}}_{\text{contribution from momentum along 25th direction}}$$

Thus $w = 0$ states correspond to the expected Kaluza-Klein states which we ~~have~~ get by compactifying an ordinary field theory on a circle.

~~Absence~~ Consistency $\Rightarrow$ $D = 26$, $C = \tilde{C} = -1$

$w \neq 0$ states are new states coming from "stringy" considerations

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(2) Spectrum is symmetric under

$$R \to \frac{\alpha'}{R}, \quad w \to n, \quad n \to w.$$

$$\frac{n}{R} + \frac{Rw}{\alpha'} \to \frac{wR}{\alpha'} + \frac{n}{R}$$

Not only $L_0, \tilde{L}_0$, but also $L_m, \tilde{L}_m$ are symmetric

---

$$L_m = \frac{1}{2}\sum_{n\neq 0} \alpha^{\mu}_{m-n}\, \alpha^{\nu}_{n}\, \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{M}_{m}\, p_M$$
$$+ \frac{1}{2}\sqrt{2\alpha'}\, \alpha^{D-1}_{m} \left(\frac{n}{R} - \frac{Rw}{\alpha'}\right)$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n\neq 0} \tilde{\alpha}^{\mu}_{m-n}\, \tilde{\alpha}^{\nu}_{n}\, \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \tilde{\alpha}^{M}_{m}\, p_M$$
$$+ \frac{1}{2}\sqrt{2\alpha'}\, \tilde{\alpha}^{D-1}_{m} \left(\frac{n}{R} + \frac{Rw}{\alpha'}\right)$$

$L_m \to L_m$, $\tilde{L}_m \to \tilde{L}_m$ under $R \to \frac{\alpha'}{R}$, $n \to w$, $w \to n$. $\alpha^{D-1}_{m} \to -\alpha^{D-1}_{m}$

One can show that the interaction rules also obey this symmetry.

Conclusion: The 26 dimensional bosonic string theory with one direction compactified on a circle of radius R is equivalent to 26 dimensional bosonic string theory with one direction compactified on a circle of radius α'/R.

$R \leftarrow \alpha'/R$ duality.

Momentum along $x^{25} \leftrightarrow$ Winding along x^{25}

Spectrum of massless states

$$(\text{Mass})^2 = \frac{4}{\alpha'}(N-1) + \left(\frac{n}{R} - \frac{R\omega}{\alpha'}\right)^2$$
$$= \frac{4}{\alpha'}(\tilde{N}-1) + \left(\frac{n}{R} + \frac{R\omega}{\alpha'}\right)^2$$

First consider $n=0$, $\omega=0$

$(\text{Mass})^2 = 0 \Rightarrow$ level 1 state.

$$\zeta_{MN}\, \alpha^M_{-1}\, \tilde{\alpha}^N_{-1}\, |\{k^M\}, 0, 0\rangle$$

$\Rightarrow$ graviton, dilaton, anti-symmetric tensor

$(\zeta_{MN} + \zeta_{NM})$ $\quad$ $(\zeta_{MN} - \zeta_{NM})$ in $D=25$

$$\zeta_M\, \alpha^M_{-1}\, \tilde{\alpha}^{25}_{-1}\, |\{k^M\}, 0, 0\rangle$$
$$\tilde{\zeta}_M\, \alpha^{25}_{-1}\, \tilde{\alpha}^M_{-1}\, |\{k^M\}, 0, 0\rangle$$

$\Rightarrow$ Two ~~vect~~ massless vector fields.

Field theory interpretation:

$g_{25\,M}$ & $B_{25\,M}$

Finally:

$$\alpha^{25}_{-1}\, \tilde{\alpha}^{25}_{-1}\, |\{k^M\}, 0, 0\rangle \rightarrow \text{scalar}$$

Field theory interpretation:

$g_{25\,25}$

Thus this part of the spectrum coincides with ~~the~~ the expected spectrum of a 26-dim. field theory of $g_{\mu\nu}$, ϕ & $B_{\mu\nu}$ & compactifying one direction on S^1.

But there are other massless states for $n \neq 0$, $\omega \neq 0$.

$$(\text{Mass})^2 = \frac{4}{\alpha'}(N-1) + \left(\frac{n}{R} - \frac{R\omega}{\alpha'}\right)^2$$
$$= \frac{4}{\alpha'}(\tilde{N}-1) + \left(\frac{n}{R} + \frac{R\omega}{\alpha'}\right)^2$$

$(\text{Mass})^2 = 0$

$$\Rightarrow \left(\frac{n}{R} - \frac{R\omega}{\alpha'}\right)^2 = \frac{4}{\alpha'}(1-N)$$
$$\left(\frac{n}{R} + \frac{R\omega}{\alpha'}\right)^2 = \frac{4}{\alpha'}(1-\tilde{N})$$

$\Rightarrow$ $N \leq 1$ $\quad \tilde{N} \leq 1$

(1) $N = \tilde{N} = 1 \Rightarrow n = \omega = 0. \to$ studied already

(2) $N = 1$, $\tilde{N} = 0$.

$$\frac{n}{R} - \frac{R\omega}{\alpha'} = 0.$$
$$\frac{n}{R} + \frac{R\omega}{\alpha'} = \pm \frac{2}{\sqrt{\alpha'}}$$

A ~~possi~~ $\Rightarrow \dfrac{2n}{R} = \pm \dfrac{2}{\sqrt{\alpha'}}$

$R = |n|\sqrt{\alpha'}$ $\quad\Big|\quad$ $\omega = \dfrac{n\alpha'}{R^2} = \dfrac{1}{n} \Rightarrow \omega = n = \pm 1$

$\Rightarrow R = \sqrt{\alpha'}$

states

$\zeta_M \, \alpha_{-1}^{M} |\{k^M\}, n = \pm 1, \omega = n\rangle \to$ Massless vector

$\alpha_{-1}^{25} |\{k^M\}, n = \pm 1, \omega = n\rangle \to$ Massless scalar

satisfy L_1 constraint.

$\Rightarrow$ 2 massless vectors & 2 massless scalars.

(3) $\tilde{N} = 1, \quad N = 0.$

$$\frac{n}{R} + \frac{R\omega}{\alpha'} = 0.$$

$$\frac{n}{R} - \frac{R\omega}{\alpha'} = \pm \frac{2}{\sqrt{\alpha'}}$$

Same analysis $\Rightarrow$ $R = \sqrt{\alpha'}$, $\omega = -n = \pm 1$

States

$\tilde{\zeta}_M \, \tilde{\alpha}^M_{-1} \, |\{k^M\}, n = \pm 1, \omega = -n\rangle \to$ Massless vectors

$\tilde{\alpha}^{25}_{-1} \, |\{k^M\}, n = \pm 1, \omega = -n\rangle \to$ Massless scalars.

$\Rightarrow$ 2 massless vectors + 2 massless scalars.

(4) $\tilde{N} = 0, \quad N = 0$

$$\left(\frac{n}{R} + \frac{R\omega}{\alpha'}\right)^2 = \frac{4}{\alpha'}$$

$$\left(\frac{n}{R} - \frac{R\omega}{\alpha'}\right) = \frac{4}{\alpha'}$$

Subtract: $4\dfrac{n\omega}{\alpha'} = 0 \Rightarrow n = 0$ or $\omega = 0$.

If $n = 0$, $R^2\omega^2 = 4\alpha'$

For $R = \sqrt{\alpha'}$ $\omega = \pm 2$

If $\omega = 0$, $\dfrac{n^2}{R^2} = \dfrac{4}{\alpha'}$

For $R = \sqrt{\alpha'}$, $n = \pm 2$.

$\Rightarrow$ Four ~~massless scalar~~ states:

$\cancel{\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}} \, |\{k^M\}, n = \pm 2, \omega = 0\rangle \to$ scalars

$\cancel{\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}} \, |\{k^M\}, n = 0, \omega = \pm 2\rangle \to$ scalars

Complete spectrum of massless states at $R = \sqrt{2\alpha'}$:

1. 25 dim. metric
2. 25 dim. anti-symmetric tensor
3. 25 dim. scalar dilaton.

4. 9 scalars.

1 from $\alpha^{25}_{-1} \tilde{\alpha}^{25}_{-1} |\{k^M\}, n=0, w=0\rangle$
2 from $\alpha^{25}_{-1} |\{k^M\}, n=\pm 1, w=n\rangle$
2 from $\tilde{\alpha}^{25}_{-1} |\{k^M\}, n=\pm 1, w=-n\rangle$
2 from $|\{k^M\}, n=\pm 2, w=0\rangle$
2 from $|\{k^M\}, n=0, w=\pm 2\rangle$

5. 6 vectors:

1 from $\alpha^{M}_{-1} \tilde{\alpha}^{25}_{-1} |\{k^M\}, n=0, w=0\rangle$
1 from $\tilde{\alpha}^{M}_{-1} \alpha^{25}_{-1} |\{k^M\}, n=0, w=0\rangle$
2 from $\alpha^{M}_{-1} |\{k^M\}, n=\pm 1, w=n\rangle$
2 from $\tilde{\alpha}^{M}_{-1} |\{k^M\}, n=\pm 1, w=-1\rangle$

Study of interactions

$\Rightarrow$ The 6 massless vectors correspond to gauge bosons of $SU(2) \times SU(2)$ gauge theory.

9 scalars $\to$ (3,3) representation of $SU(2) \times SU(2)$

~~Stringy~~ compactifications

~~Ms~~ Compactification to lower dimensions:

Take d of the directions to be circles of ~~B~~ radii $R_1, \cdots R_d$.

$\Rightarrow$ Momentum & winding modes along each of the d directions.

$\Rightarrow$ ① $R_i \to 1/R_i$ duality for each i

② Larger gauge symmetry groups.

$\Rightarrow$ A theory in $(26-d)$ uncompactified dimensions.

More general compactification

Note: 26 dimensional critical string theory has constraints

$$L_m \,|\,phys\rangle = \tilde{L}_m \,|\,phys\rangle = 0 \qquad m \geq 0.$$

$$L_m = \sum_{\mu=0}^{25} L_m^{(\mu)} \quad \text{for } m \neq 0$$

$$L_0 = \sum_{\mu=0}^{25} L_0^{(\mu)} - 1$$

$\hookrightarrow$ normal ordering constant.

$$\tilde{L}_m = \sum_{\mu=0}^{25} \tilde{L}_m^{(\mu)} \qquad m \neq 0$$

$$\tilde{L}_0 = \sum_{\mu=0}^{25} \tilde{L}_0^{(\mu)} - 1$$

$$L_m^{(\mu)} = \frac{1}{2}\sum_{n\neq 0,m} \alpha^\mu_{m-n}\,\alpha^\mu_n\,\eta_{\mu\mu} + \frac{1}{2}\sqrt{2\alpha'}\,p_\mu\,\alpha^\mu_m \quad \text{for } m\neq 0.$$

no sum over μ.

$$L_0^\mu = \sum_{n>0} \alpha^\mu_{-m}\,\alpha^\mu_m\,\eta_{\mu\mu} + \frac{\alpha'}{4}\,p_\mu\,p_\mu\,\eta^{\mu\mu}$$

$$\tilde{L}_m^{(\mu)} = \frac{1}{2}\sum_{n\neq 0,m} \tilde{\alpha}^\mu_{m-n}\,\tilde{\alpha}^\mu_n\,\eta_{\mu\mu} + \frac{1}{2}\sqrt{2\alpha'}\,p_\mu\,\tilde{\alpha}^\mu_m \quad \text{for } m\neq 0.$$

$$\tilde{L}_0^{(\mu)} = \sum_{n>0} \tilde{\alpha}^\mu_{-m}\,\tilde{\alpha}^\mu_m\,\eta_{\mu\mu} + \frac{\alpha'}{4}\,p_\mu\,p_\mu\,\eta^{\mu\mu}.$$

Dynamical variables:

$$x^\mu, \quad p_\mu, \quad \alpha^\mu_n, \quad \tilde{\alpha}^\mu_n$$

$$\alpha^{\mu_1}_{-m_1}\cdots\alpha^{\mu_k}_{-m_k}\,\tilde{\alpha}^{\nu_1}_{-n_1}\cdots\tilde{\alpha}^{\nu_\ell}_{-n_\ell}\,|\{k^\alpha_\nu\}\rangle \rightarrow \text{A state}$$

$$p_\mu\,|\{k^\alpha_\nu\}\rangle = k_\mu\,|\{k^\alpha_\nu\}\rangle.$$

$$\alpha^\mu_n\,|\{k^\alpha_\nu\}\rangle = 0 = \tilde{\alpha}^\mu_n\,|\{k^\nu\}\rangle \quad \text{for } n>0.$$

$L_m^{(\mu)}, \tilde{L}_m^{(\mu)}$: normal ordered Contribution to $L_m, \tilde{L}_m$ due to the modes of X^μ.

$$\left[L_m^{(\mu)}, L_n^{(\mu)}\right] = (m-n)\, L_{m+n} + \frac{1}{12}\,\delta_{m+n,0}\,(m^3-m)$$

$$\left[\tilde{L}_m^{(\mu)}, \tilde{L}_n^{(\mu)}\right] = (m-n)\,\tilde{L}^{(\mu)}_{m+n} + \frac{1}{12}\,\delta_{m+n,0}\,(m^3-m)$$

Now consider a string theory based on the following gauge fixed action:

$$S = -\frac{1}{4\pi\alpha'}\left[\int d^2\xi \sum_{\mu,\nu=0}^{D-1} \eta_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta}\right]$$

$$+ S_{CFT}$$

$S_{CFT} \to$ describes a conformal field theory of central charge c.
(internal degrees of freedom of the string)

$\Rightarrow S_{CFT}$ contains symmetry algebra generated by $L_m^{CFT}, \tilde{L}_m^{CFT}$:

$$\left[L_m^{CFT}, L_n^{CFT}\right] = (m-n)\, L^{CFT}_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m+n,0}$$

$$\left[\tilde{L}_m^{CFT}, \tilde{L}_n^{CFT}\right] = (m-n)\,\tilde{L}^{CFT}_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m+n,0}$$

$$T^{CFT}_{++} \propto \sum L_m\, e^{2im\sigma^+}$$

$$T^{CFT}_{--} \propto \sum \tilde{L}_m\, e^{-2im\sigma^-}$$

Now quantize the X^{μ}'s as usual and define $L_m^{(\mu)}, \tilde{L}_m^{(\mu)}$.

$$L_m = \sum_{\mu=0}^{n} L_m^{(\mu)} + L_m^{CFT} \qquad m \neq 0.$$

$$L_0 = \sum_{\mu=0}^{n} L_0^{(\mu)} + L_0^{CFT} - 1$$

$$\tilde{L}_m = \sum_{\mu=0}^{n} \tilde{L}_m^{(\mu)} + \tilde{L}_m^{CFT} \qquad m \neq 0.$$

$$\tilde{L}_0 = \sum_{i=0}^{n} \tilde{L}_0^{(\mu)} + \tilde{L}_0^{CFT} - 1$$

Hilbert space states:

D dim momentum

$$\alpha_{-m_1}^{\mu_1} \cdots \alpha_{-m_k}^{\mu_k} \tilde{\alpha}_{-n_1}^{\nu_1} \cdots \tilde{\alpha}_{-n_\ell}^{\nu_\ell} |\{k^\mu\}\rangle \otimes |s\rangle_{CFT}$$

Physical state constraint:

$$L_m |phys\rangle = 0, \quad \tilde{L}_m |phys\rangle = 0 \quad \text{for } m \geq 0.$$

Q. Does it get rid of the $-ve$ normed states & give a consistent theory?

A. Yes if

$$D + c = 26.$$

Thus for a given n we need to choose a conformal field theory with

$$c = 26 - D$$

→ stringy "compactification".

Example: for $n = 4$, $c = 22$.

Ising model has $c = \frac{1}{2}$ ⇒ Can use 44 copies of Ising model.

Physical insight:

In the presence of 2-d metric $\gamma_{\alpha\beta}$ the action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, \eta_{\mu\nu} \, \partial_\alpha X^\mu \partial_\beta X^\nu \sqrt{\det\gamma} \, \gamma^{\alpha\beta}$$

$$+ \; S_{CFT}(\gamma, \underbrace{\cdots}_{\text{other fields}}) \longrightarrow \int d^2\xi \, \mathcal{L}(\gamma_{\alpha\beta}, \underbrace{\varphi^i}_{\text{other fields}})$$

$$\frac{\delta S}{\delta\gamma^{\alpha\beta}} = \text{const} \left(\sum_\mu T^{(\mu)}_{\alpha\beta} + T^{CFT}_{\alpha\beta} \right)$$

$$T^{CFT}_{\alpha\beta}(\vec{\xi}) \equiv \frac{\delta S}{\delta\gamma^{\alpha\beta}(\vec{\xi})}$$

$\gamma_{\alpha\beta}$ eqs. of motion

$$\Rightarrow \sum_\mu T^{(\mu)}_{\alpha\beta} + T^{CFT}_{\alpha\beta} = 0$$

$\rightarrow$ must be imposed as constraints after gauge fixing $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$

Conformal invariance

$$\Rightarrow \gamma^{\alpha\beta} T^{(\mu)}_{\alpha\beta} = 0$$

$$\gamma^{\alpha\beta} T^{(CFT)}_{\alpha\beta} = 0.$$

(Invariance under $\gamma_{\alpha\beta} \to \lambda\gamma_{\alpha\beta}$)

$u = \tau + \sigma$
$v = \tau - \sigma$

$\Rightarrow$ Two conditions:

$$\sum_\mu T^{(\mu)}_{uu} + T^{(CFT)}_{uu} = 0, \qquad \sum_\mu T^{(\mu)}_{vv} + T^{(CFT)}_{vv} = 0$$

$\Rightarrow$ the physical state conditions.

$$T^{CFT}_{uu} = \sum_m L^{CFT}_m e^{im\sigma}, \qquad T^{CFT}_{vv} = \sum_m \tilde{L}^{CFT}_m e^{-im\sigma}$$

Compactification on tori discussed earlier are special cases of the more general compactification.

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma\, d\tau \sum_{\mu,\nu=0}^{D-1} \eta_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta}$$

$$-\frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \sum_{i,j=D}^{25} \partial_\alpha Y^i \partial_\beta Y^i\, \eta^{\alpha\beta}$$

$\Rightarrow$ S_{CFT}

Internal coordinates

$$Y^i \equiv Y^i + 2\pi R^i$$

$$L_m^{(CFT)} = \sum_{i=D}^{25} L_m^{(i)}$$

Generalize:

$$S_{CFT} = -\frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \sum_{i,j=D}^{25} \big(G_{ij}\, \partial_\alpha Y^i \partial_\beta Y^j \eta^{\alpha\beta} + B_{ij}\, \partial_\alpha Y^i \partial_\beta Y^j \epsilon^{\alpha\beta} \big)$$

$\epsilon^{\tau\sigma} = 1, \quad \epsilon^{\sigma\tau} = -1$

$\epsilon^{\tau\tau} = 0, \quad \epsilon^{\sigma\sigma} = 0$

$= -2\left(G_{ij}\, \partial_u Y^i \partial_v Y^j + B_{ij}\, \partial_u Y^i \partial_v Y^j \right)$

G, B : constant matrices.

Check invariance under:

$$Y^{i\prime}(\sigma,\tau) = Y^i\left(f_1(\sigma,\tau),\, f_2(\sigma,\tau) \right)$$

$$f_1(\sigma,\tau) + f_2(\sigma,\tau) = f(\sigma+\tau)$$

$$f_1(\sigma,\tau) - f_2(\sigma,\tau) = g(\sigma-\tau)$$

$$S_{CFT} = \frac{1}{2\pi\alpha'} \int d\sigma\, d\tau \sum_{i,j} \left[G_{ij}\, \partial_u X^i \partial_v X^j + B_{ij}\, \partial_u X^i \partial_v X^j \right]$$

$u = \tau + \sigma, \quad v = \tau - \sigma$

Scale invariance:

$$x^0 \to \lambda x, \quad x \to \lambda x$$

$$(\sigma \to \lambda\sigma, \quad \tau \to \lambda\tau).$$

In general quantum effects may break this symmetry.

(UV divergence $\to$ regulator
$\to$ broken scale invariance)

But the theory at hand has action quadratic in the fields y^i

---

$\Rightarrow$ free field theory

$\Rightarrow$ ~~scale~~ no renormalization necessary.

$\to$ scale invariance is not broken by quantum corrections.

What about the central charge?

For $G_{ij} = \delta_{ij}$ & $B_{ij} = 0$, $c = 26 - D$.

In the presence of G & B one can explicitly construct the $L_m^{(CFT)}$, $\tilde{L}_m^{(CFT)}$ & calculate their commutators.

$\Rightarrow \quad c = 26 - D$ (independent of G_{ij} & B_{ij})

A single ising model

$|0\rangle$, ~~$\frac{1}{2}$~~ $|\epsilon\rangle$, $|\sigma\rangle$ $\to$ primary

$L^{(ising)}_n |0\rangle = 0$ for $n > 0$
$|\epsilon\rangle$
$|\sigma\rangle$

$L^{(ising)}_0 |0\rangle = 0$
$|\epsilon\rangle = \frac{1}{2}|\epsilon\rangle$
$|\sigma\rangle = \frac{1}{16}|\sigma\rangle$

$\tilde{L}^{(ising)}_0 |0\rangle = 0$
$|\epsilon\rangle = \frac{1}{2}|\epsilon\rangle$
$|\sigma\rangle = \frac{1}{16}|\sigma\rangle$

Other states $\tilde{L}^{(ising)}_{-n_1} \cdots \tilde{L}^{(ising)}_{-n_k}$ $\to$ secondary

$L^{(ising)}_{-m_1} L^{(ising)}_{-m_2} \cdots L^{(ising)}_{-m_k} \left\{ \begin{array}{l} |0\rangle \\ \wedge |\epsilon\rangle \\ |\sigma\rangle \end{array} \right.$

Secondary.
States which are secondary & primary
$\Rightarrow$ null $\Rightarrow$ removed.

$\Rightarrow$ Constitute the full Hilbert space of a single ising CFT.

A basis state in the Hilbert space of 44 ising model

$\Rightarrow$ direct product of states from each of the 44 sectors.

Light cone gauge Compactification

26 non-compact dimension:

Dynamical variables: $x^-, p_-, \alpha^i_m, \tilde{\alpha}^i_m, x^i, p_i$

$2 \le i \le 25$

~~Light cone Hamiltonian~~

$\cancel{-p_+ = \frac{1}{\alpha' p_-}\left(\sum_{i=2}^{25} L_0^{(i)} + \sum\right.}$

$$\alpha^{i_1}_{-m_1}\cdots\alpha^{i_k}_{-m_k}\,\tilde{\alpha}^{j_1}_{-n_1}\cdots\tilde{\alpha}^{j_l}_{-n_l}\,|k_-, \{k^i\}\rangle$$

$$p_-|k_-, \{k_i\}\rangle = k_-|k_-, \{k_i\}\rangle$$

$$p_i|k_-, \{k_j\}\rangle = k_i|k_-, \{k_j\}\rangle$$

$$\alpha^i_n|k_-, \{k_j\}\rangle = 0 = \tilde{\alpha}^i_n|k_-, \{k_j\}\rangle \quad \text{for } n>0.$$

Light-cone Hamiltonian:

$$-p_+ = -\frac{4}{\alpha' p_-}\left(\sum_{i=2}^{25} L_0^{(i)} + \sum_{i=2}^{25}\tilde{L}_0^{(i)} - 2\right)$$

$L_0^{(i)}, \tilde{L}_0^{(i)}$ have the same expression in terms of $\alpha^i_m, \tilde{\alpha}^i_m, p_i$.

Compactification: ~~Replace~~ Take n free bosons $X^0, X^1, \ldots X^{D-1}$ and a CFT of central charge c.

Light-cone gauge: Dynamical variables $x^-, p_-, \alpha^i_m, \tilde{\alpha}^i_m, x^i, p_i$ $(2 \le i \le D-1)$ & the variables of CFT.

Define light-cone Hamiltonian:

$$-p_+ = -\frac{4}{\alpha' p_-}\left(\sum_{i=2}^{D-1} L_0^{(i)} + \sum_{i=2}^{D-1} \tilde{L}_0^{(i)} + L_0^{(CFT)} + \tilde{L}_0^{(CFT)} - 2\right)$$

Does it work?

Need to construct Lorentz generators $J^{\mu\nu}$ for $0 \leq \mu, \nu \leq D-1$ & check that they satisfy the usual Lorentz algebra.

$\alpha_n^i, \tilde{\alpha}_n^i$ for $i \geq D$ enters $J^{\mu\nu}$ for $0 \leq \mu, \nu \leq D-1$ through

$\Rightarrow$ Can be done.

(Replace $\sum_{i=D}^{25} L_m^{(i)}$ & $\sum_{i=D}^{25} \tilde{L}_m^{(i)}$ inside the Lorentz generators by $L_m^{(CFT)}$ & $\tilde{L}_m^{(CFT)}$ respectively).

$\Rightarrow$ The algebra is satisfied if

$$D + C = 26.$$

~~Non-compact~~ $\Rightarrow$ Same result as in the covariant gauge.

One loop ~~vacuum~~ partition function ~~energy~~ . Why are string amplitudes finite?

Field theory example:

→ A free massless scalar field:

$$S = \frac{1}{2}\int d^D x \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) \quad \text{(Euclidean)}$$

$$e^{+Z} = \int \mathcal{D}\phi \, e^{-S}$$

$$= \int [\mathcal{D}\tilde{\phi}(k)] \, e^{-\frac{1}{2}\int \tilde{\phi}(k)(k^2+m^2)\tilde{\phi}(-k)}$$

$$= \int \prod_n d\tilde{\phi}(k_n) \, e^{-\frac{1}{2}\sum_n \tilde{\phi}(-k_n)(k_n^2+m^2)\tilde{\phi}(k_n)}$$

$$= C' \prod_n (k_n^2+m^2)^{-1/2} = C' e^{-\frac{1}{2}\sum_n \ln(k_n^2+m^2)}$$

$$\Rightarrow \quad Z = -\frac{1}{2}\frac{V}{(2\pi)^D}\int d^D k \, \ln(k^2+m^2) \rightarrow \text{UV divergent}$$

$$\frac{1}{k^2+m^2} = \int_0^\infty ds \, e^{-s(k^2+m^2)}$$

$$\frac{\partial}{\partial m^2}\ln(k^2+m^2) \qquad -\frac{\partial}{\partial m^2}\int_0^\infty \frac{ds}{s} e^{-s(k^2+m^2)}$$

$$\Rightarrow \ln(k^2+m^2) = -\int_\epsilon^\infty \frac{ds}{s} e^{-s(k^2+m^2)} + \text{constant}$$

ϵ → cutoff

constant: indep. of m → ignore

$$\bullet \int d^D k \, \ln(k^2+m^2) = -\int_\epsilon^\infty \frac{ds}{s}\int d^D k \, e^{-s(k^2+m^2)}$$

Note: Large k divergence is regulated.

Large k → small s

⇒ ϵ is the UV - cut off.

$Z =$ (129)

$$\frac{1}{2}\frac{V}{(2\pi)^D}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\left(\frac{s}{\pi}\right)^{-D/2} \xrightarrow[\epsilon\to\pi\epsilon\alpha']{s\to\pi s\alpha'} \frac{V(\alpha')^{-\frac{D}{2}}}{(2\pi)^D}\int_\epsilon^\infty \frac{ds}{2s}\, e^{-\pi s m^2\alpha'}$$

Note Large the value of D

$\Rightarrow$ ~~stronger~~ stronger is the divergence near $s=0$.

$\epsilon \sim \frac{1}{\Lambda^2}$ $\quad\Lambda$: UV ~~mass~~ cut off mass.

We shall compute one loop vacuum amplitude in string theory and will show that it gives a form identical to ~~the~~ field theory form with two differences:

① An infinite sum over many terms with different m^2.

$\Rightarrow$ Infinite no of ~~states~~ particle like states in string theory.

② There is a natural lower cut-off to the s-integral.

$\to$ does not have to be produced by hand.

In light cone gauge: (Naive answer)

$$m^2 = \frac{4}{\alpha'}\left(\sum m_i - 1\right) = \frac{4}{\alpha'}\left(\sum n_j - 1\right) \quad \left(\sum m_i = \sum n_j\right)$$

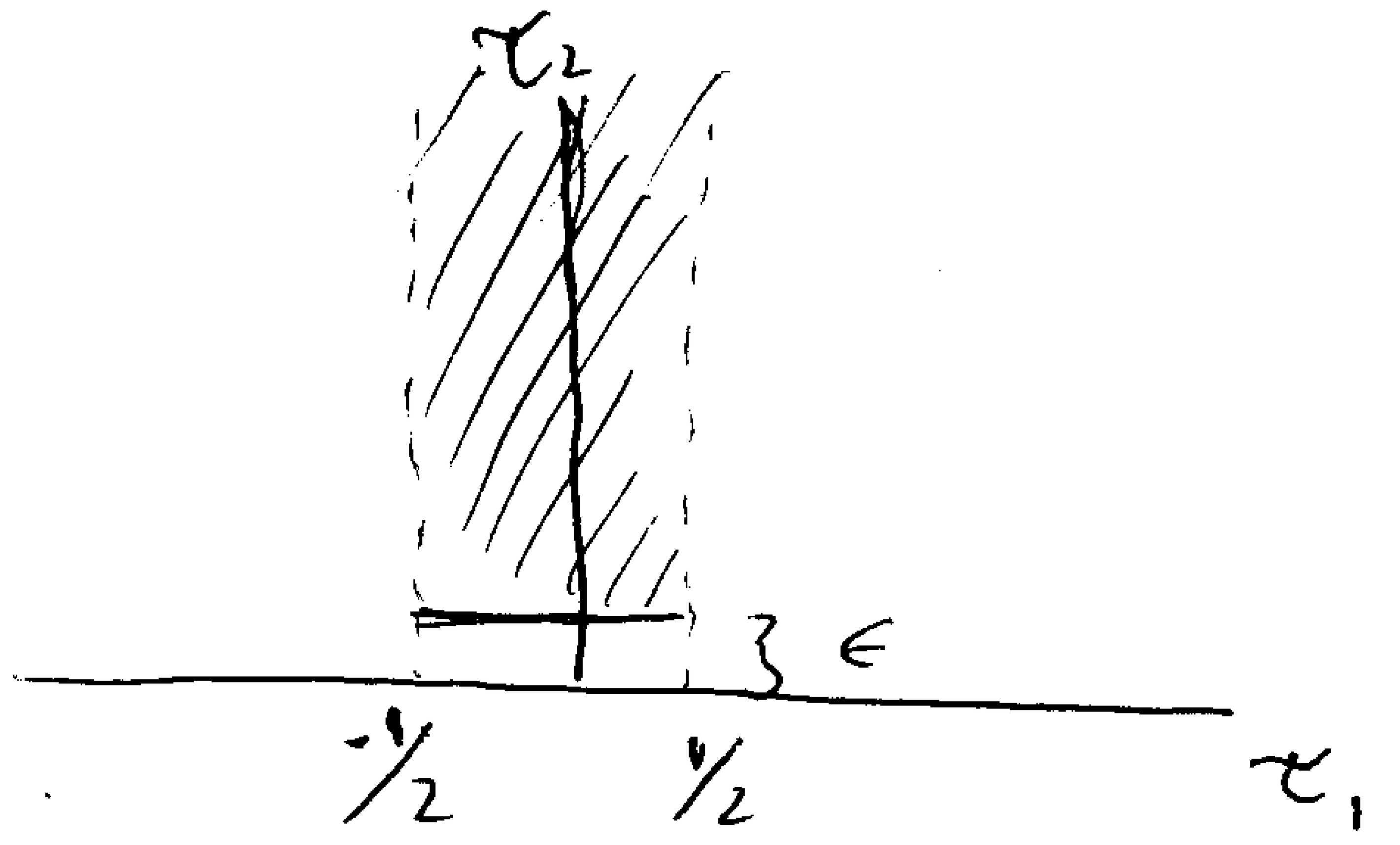
for $\alpha^{i_1}_{-m_1}\cdots\alpha^{i_k}_{-m_k}\,\tilde\alpha^{j_1}_{-n_1}\cdots\tilde\alpha^{j_l}_{-n_l}\,|k_-,\vec{k}\rangle$

$$\Rightarrow Z = \frac{V(\alpha')^{-13}}{(2\pi)^{26}}\int_\epsilon^\infty \frac{ds}{2s}\sum_{\substack{\text{all states}\\ \text{in light cone}}} e^{-2\pi s(\sum m_i - 1)}\, e^{-2\pi s(\sum n_j - 1)}\,\delta_{\sum m_i, \sum n_j}$$

$$\delta_{\Sigma m_i, \Sigma n_j} = \int_{-\frac{1}{2}}^{1/2} d\theta \; e^{2\pi i \theta (\Sigma m_i - \Sigma n_j)}$$

$\theta + i s = \tau \rightarrow$ complex parameter.

$$Z = \frac{1}{2} \frac{V (\alpha')^{-13}}{(2\pi)^{26}} \int d^2\tau \sum_{\{n_i\},\{m_j\}} e^{2\pi i \tau (\Sigma m_i - 1)} \, e^{-2\pi i \bar{\tau} (\Sigma n_j - 1)}$$



~~estimated for any time independent classical solution of the equations of~~

Details of the computation:

$$Z = \int \underbrace{[\mathcal{D}g\,\gamma]\,[dX^\mu]}_{\mathcal{D}(\text{gauge})}\, e^{-S}$$ → Remove gauge degrees of freedom.

$$S = \frac{1}{4\pi\alpha'}\int \sqrt{\det\gamma}\;\gamma^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\mu$$

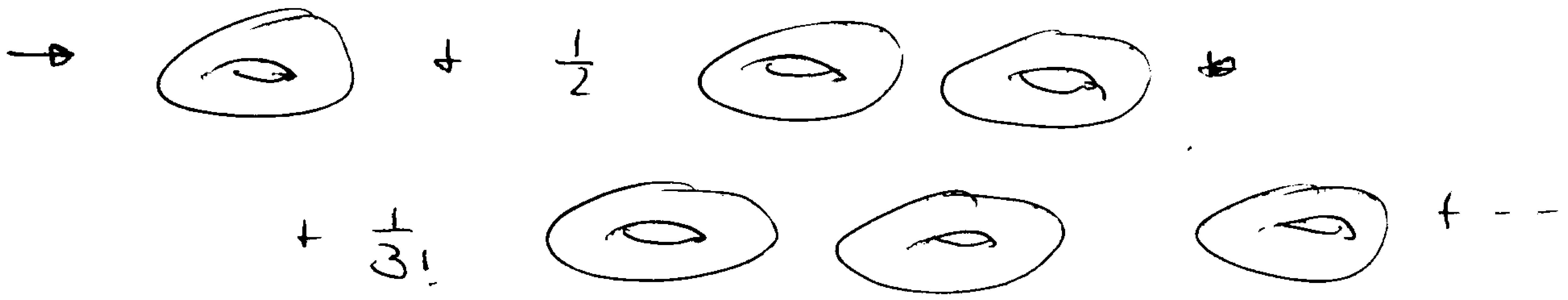
→ Euclidean continuation both in ~~real~~ space-time and world-sheet.

⇒ $\gamma_{\alpha\beta}$ has Euclidean signature.

$\eta_{\mu\nu} = \delta_{\mu\nu}$

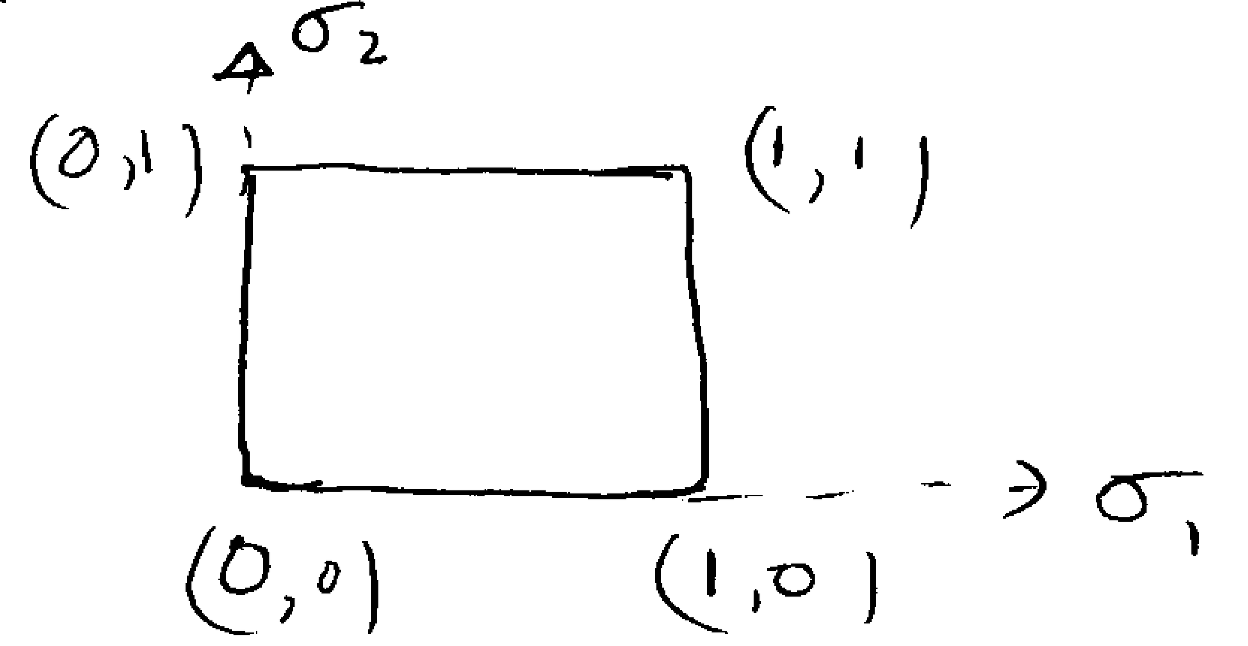
Note: This computation gives Z and not e^Z.

e^Z will require $\bigcirc + \frac{1}{2}\bigcirc\,\bigcirc + \frac{1}{3!}\bigcirc\,\bigcirc\,\bigcirc + \cdots$



Try to fix a gauge.

Coordinate system: $(\sigma_1, \sigma_2) \equiv (\sigma_1+1, \sigma_2) \equiv (\sigma_1, \sigma_2+1)$



(1) Allowed reparametrization:

$$\sigma_1 \to f_1(\sigma_1, \sigma_2)$$
$$\sigma_2 \to f_2(\sigma_1, \sigma_2)$$

$$\left(f_1(\sigma_1+1, \sigma_2),\ f_2(\sigma_1+1, \sigma_2)\right)$$
$$= \left(f_1(\sigma_1, \sigma_2),\ f_2(\sigma_1, \sigma_2)\right) + (m_1, n_1)$$

$$\left(f_1(\sigma_1, \sigma_2+1),\ f_2(\sigma_1, \sigma_2+1)\right)$$
$$= \left(f_1(\sigma_1, \sigma_2),\ f_2(\sigma_1, \sigma_2)\right) + (m_2, n_2)$$

f_1, f_2 are arbitrary functions subject to this restriction.

(2) Weyl transformation:

$$\gamma_{\alpha\beta}(\sigma_1,\sigma_2) \to e^{\phi(\sigma_1,\sigma_2)}\, \gamma_{\alpha\beta}(\sigma_1,\sigma_2)$$

$$\phi(\sigma_1+1, \sigma_2) = \phi(\sigma_1, \sigma_2) = \phi(\sigma_1, \sigma_2+1).$$

Gauge fixing:

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} \; ?$$

Not possible.

We can bring an arbitrary $\gamma_{\alpha\beta}$, via gauge trs., to the form:

$$ds^2 = |d\sigma_1 + \tau\, d\sigma_2|^2$$

τ: a complex parameter; $\text{Im}\,\tau > 0$.

$\tau = i, \Rightarrow ds^2 = (d\sigma_1)^2 + (d\sigma_2)^2 \Rightarrow \gamma_{\alpha\beta} = \delta_{\alpha\beta}$.

But different values of τ cannot be related by gauge trs.

$\rightarrow$ $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$ cannot be implemented

Residual gauge trs. which preserves the form of the metric:

(1) $\sigma_1 \to \sigma_1 + a$
$\sigma_2 \to \sigma_2 + b$

$ds^2 \to ds^2$.

(2) $\sigma_1 \to -\sigma_1$
$\sigma_2 \to -\sigma_2$

The final path integral must be divided by the volume of this gauge gp.

$\Rightarrow$ A factor of $\frac{1}{2}$.

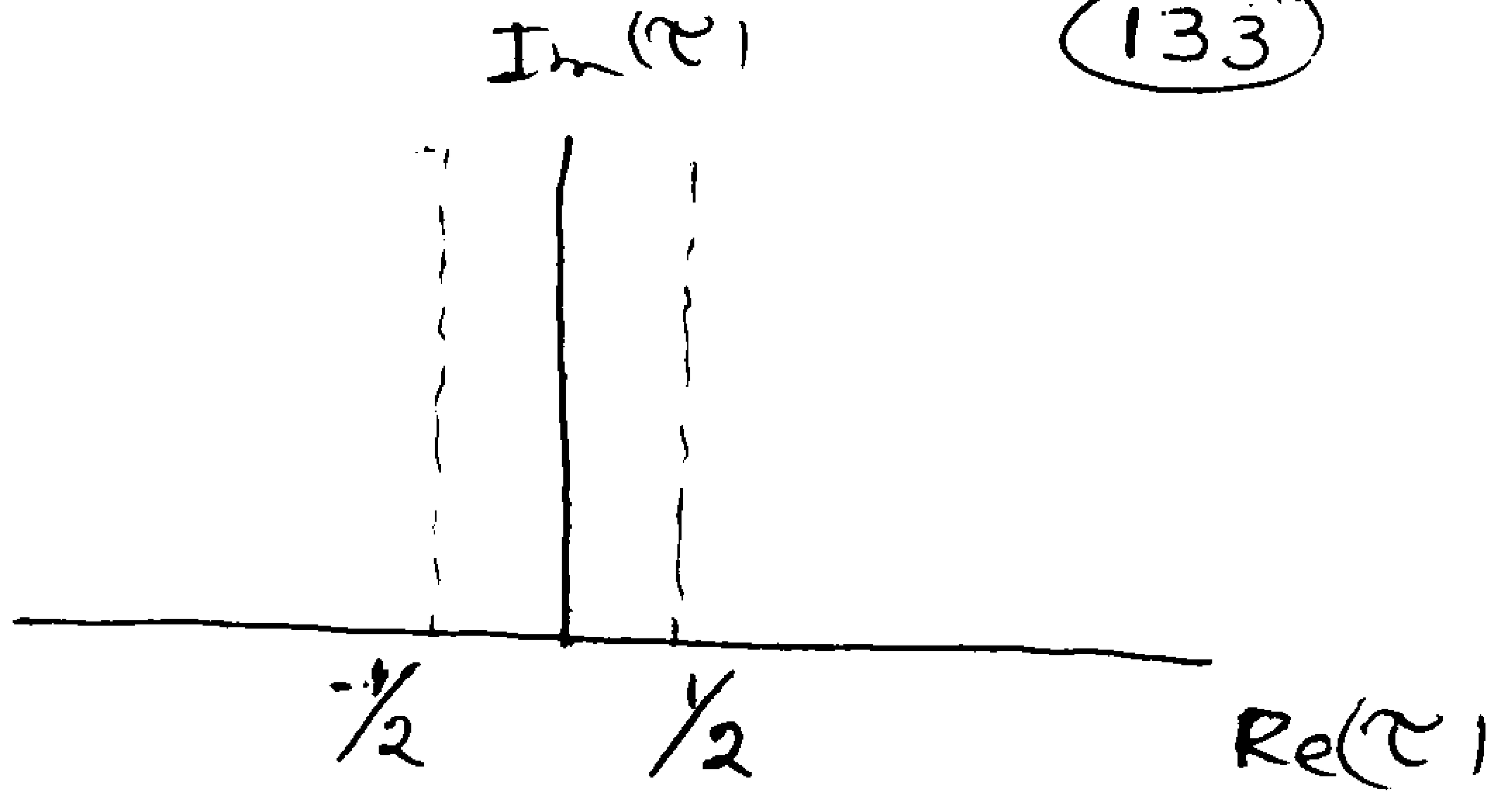
(2) $\sigma_1 \to \sigma_1 + \sigma_2$.
$\sigma_2 \to \sigma_2$

$$ds^2 \to |d\sigma_1 + (\tau+1)\, d\sigma_2|^2.$$

$\Rightarrow$ τ and $\tau+1$ describe the same metric.

Restrict τ to.

$$-\tfrac{1}{2} < \text{Re}\,\tau < \tfrac{1}{2}.$$



③ $\sigma_1 \to \sigma_2$

$\sigma_2 \to -\sigma_1$

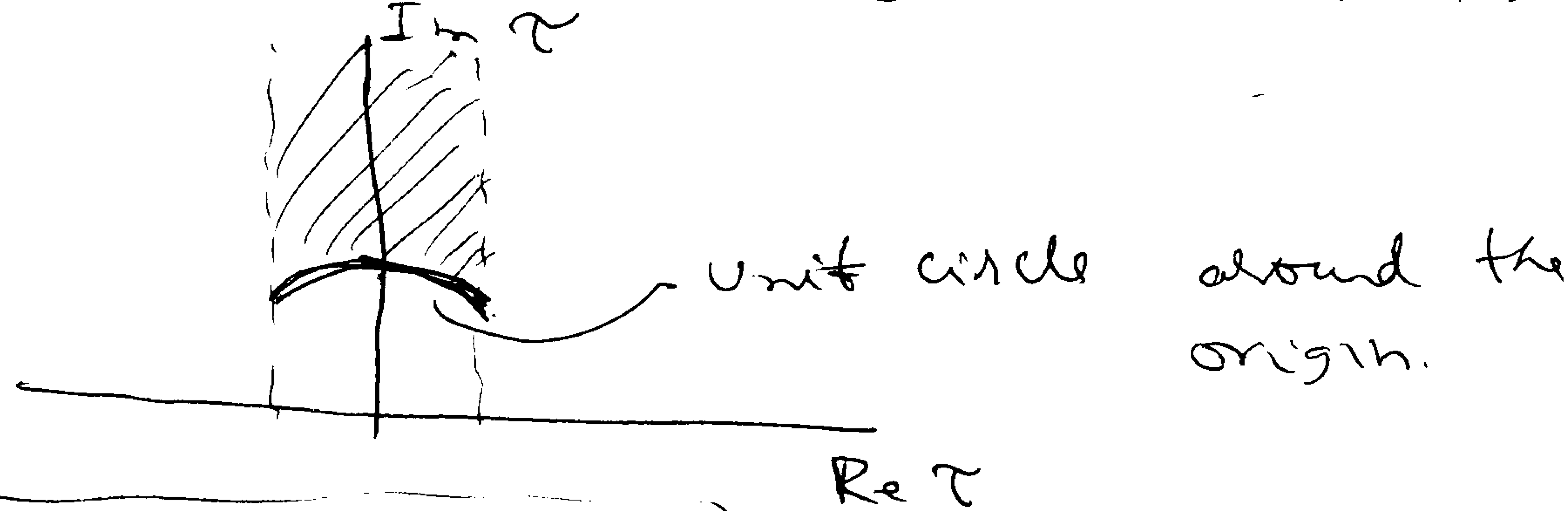
$$ds^2 \Rightarrow |d\sigma_2 - \tau\, d\sigma_1|^2 = |\tau|^2\, |d\sigma_1 - \tfrac{1}{\tau} d\sigma_2|^2$$

Weyl trs: $\gamma_{\alpha\beta} \to |\tau|^{-2}\, \gamma_{\alpha\beta}$

$\Rightarrow ds^2 \to |d\sigma_1 - \tfrac{1}{\tau}\, d\sigma_2|^2$

$\Rightarrow$ τ and $-\frac{1}{\tau}$ describe gauge equivalent metric.

Restrict τ to a "fundamental region"



$\Rightarrow$ Final amplitude:

$$\int_{\mathcal{F}} d\tau_1\, d\tau_2 \times \text{Integrand. } f(\tau, \bar\tau)$$

↳ Identical to the expected integral from field theory calculation.

① Jacobian of trs from $\mathcal{D}(\gamma_{\alpha\beta})$ to $d^2\tau\, \mathcal{D}(\text{gauge})$
② $\mathcal{D}X^\mu$ integral

Thus string theory automatically regularizes the ultraviolet divergences.

Note For this to work, $f(\tau,\bar\tau)$ must be covariant under

$$\tau \to \tau+1, \quad \tau \to -1/\tau$$

i.e.

$$f(\tau+1, \bar\tau+1) = f(\tau,\bar\tau)$$

$$d^2\tau \; f(\tau,\bar\tau) = d^2\left(-\frac{1}{\tau}\right) f\left(-\frac{1}{\tau}, -\frac{1}{\bar\tau}\right)$$

$$\frac{d^2\tau}{|\tau|^4}$$

$\Rightarrow$ Need $f\left(-\frac{1}{\tau}, -\frac{1}{\bar\tau}\right) = |\tau|^4 f(\tau,\bar\tau)$

→ can be explicitly checked.

(Consequence of g.c.i & Weyl inv.)

Now suppose we want to do this for an arbitrary string compactification based on D bosonic ~~coordinates~~ fields X^μ and an arbitrary CFT with $c = 26-D$

$\Rightarrow f(\tau,\bar\tau)$ has to be computed again from the spectrum of the theory in light-cone gauge

Q. ~~Does~~ Is it modular inv.?

If we know for sure that the CFT comes from integral of a local Lagrangian then modular inv. follows from g.c.i & Weyl inv.

Otherwise we need to check it explicitly.

$\Rightarrow$ Constraints on which irreps of the Virasoro algebra should be included in the spectrum.

Superstring Theory

Bosonic string theory in covariant gauge:

Action $S = -\frac{1}{4\pi\alpha'} \int \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu d^2\xi$.

$\eta^{\alpha\beta} = \text{diag}(-1, 1)$, $\eta_{\mu\nu} = \text{diag}.(-1, \underbrace{1, 1, \cdots 1}_{D-1})$

Constraint: $T_{\alpha\beta} = 0$

↓

Energy-Momentum tensor

Superstring theory is obtained by supersymmetrizing the action as well as the constraint.

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \, [\eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu}].$$

ψ^μ: Two component real fermion.

$$\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix} \rightarrow \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}$$

Exchange $\pm$ to be consistent with later convention.

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta} \rightarrow \gamma$ matrices.

Note: $i\rho^\alpha \partial_\alpha$ → real

⇒ Can take $\psi^\mu_\pm$ to be real.

⇒ Majorana ~~spinor~~ fermion

Supersymmetry:

$\delta_\epsilon S = 0$ under

$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu$ $\qquad \bar{\epsilon} = \epsilon^\dagger \rho^0 = \epsilon^T \rho^0$

$\delta_\epsilon \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon$

ϵ: A Majorana spinor (anti-commuting)

→ supersymmetry

(Mixes bosons with fermions]

Note:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] X^\mu = \delta_{\epsilon_1}\delta_{\epsilon_2} X^\mu - \delta_{\epsilon_2}\delta_{\epsilon_1} X^\mu$$
$$= a^\alpha \partial_\alpha X^\mu$$

$$a^\alpha = 2i\bar{\epsilon}_1 \rho^\alpha \epsilon_2$$

$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi^\mu = a^\alpha \partial_\alpha \psi^\mu$ (using $\rho^\alpha \partial_\alpha \psi^\mu = 0$ eqs. of motion)

⇒ Commutator of 2 susy trs. gives us translation.

Energy - momentum tensor $T^{\alpha\beta}$

→ Noether current associated with translation invariance.

Supers current J^{α} → 2 comp spinor $\begin{pmatrix} J_+^{\alpha} \\ J_-^{\alpha} \end{pmatrix}$

→ Noether current associated with supersymmetry transformation.

A general procedure for computing Noether current:

Suppose there is a continuous global symmetry and let λ be the infinitesimal parameter of the symmetry trs.

$\Rightarrow \quad \delta_\lambda S = 0.$

Now take λ to depend on space-time coordinates:

$$\lambda \equiv \lambda(\vec{\xi})$$

$$\Rightarrow \quad \delta_\lambda S \neq 0$$
$$\qquad\qquad \parallel$$
$$\int d^2\xi \, (\partial_\alpha \lambda) \, J^\alpha$$

→ general form.

J^α : Noether current

Proof of conservation law of J^α:

$$\partial_\alpha J^\alpha = 0.$$

$$\delta_\lambda S = \int d^2\xi \; \lambda \, \partial_\alpha J^\alpha$$

If the field configuration satisfies eqs. of motion then $\delta S = 0$. to first order for any variation.

$\Rightarrow \quad \delta_\lambda S = 0$. for any $\lambda(\vec{\xi})$

$\Rightarrow \quad \partial_\alpha J^\alpha = 0.$

Use this procedure to determine the Noether currents corresponding to space-time translation symmetry and supersymmetry.

Translation symmetry:

$$\delta X^\mu = a^\alpha(\xi) \, \partial_\alpha X^\mu$$

$$\delta \psi^\mu = a^\alpha(\xi) \, \partial_\alpha \psi^\mu$$

$$\Rightarrow \delta S = -\frac{1}{2\pi\alpha'} \int d^2\xi \left[\eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha (a^\gamma(\vec{\xi}) \partial_\gamma X^\mu) \partial_\beta X^\nu - \frac{i}{2} a^\gamma(\xi) (\partial_\gamma \bar{\psi}^\mu) \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} - \frac{i}{2} \bar{\psi}^\mu \rho^\alpha \partial_\alpha (a^\gamma(\xi) \partial_\gamma \psi^\nu) \eta_{\mu\nu} \right]$$

$$= \int d^2\xi \, \partial_\alpha a^\gamma \, \eta^{\alpha\beta} \, T_{\beta\gamma}$$

$$T_{\alpha\beta} = C^{\phi}\left[\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \frac{i}{4}\bar{\Psi}^\mu \rho_\alpha \partial_\beta \Psi^\nu \eta_{\mu\nu} + \frac{i}{4}\bar{\Psi}^\mu \rho_\beta \partial_\alpha \Psi^\nu \eta_{\mu\nu}\right] - \frac{1}{2}\eta_{\alpha\beta} K$$

K is such that $\eta^{\alpha\beta} T_{\alpha\beta} = 0$.

Similarly we can compute J^α:

$$J^\alpha = \frac{1}{2}\rho^\beta \rho^\alpha \psi^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

We now impose the constraints:

$$T_{\alpha\beta} = 0, \quad J^\alpha = 0.$$

Note: ① If we did not impose the constraint $J^\alpha = 0$ we would ~~not~~ have gotten a compactified bosonic string theory.

ψ^μ's → describe an internal CFT.

$$C_\psi = \frac{D}{2} \quad \text{(can be shown)}.$$

⇒ We would get

$$D + \frac{D}{2} = 26 \quad \to \text{no integer soln. for } D.$$

Even if we had ~~a~~ soln. we would

6 not achieve anything new.

② The condition $J^\alpha = 0$ looks ad hoc.

Proper starting point:

Supersymmetrization of gauge invariant action.

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\gamma_{\alpha\beta}$: 2-dim. metric.

Supersymmetrization ⇒

ψ^μ : Partners of X^μ.

χ^α : Partner of $\gamma_{\alpha\beta}$.

↳ For each α, χ^α is a 2-component spinor.

⇒ $S(X^\mu, \psi^\mu, \gamma_{\alpha\beta}, \chi_\alpha)$

has ① general coordinate inv.

② Weyl.

③ ~~Sup~~ Local supersymmetry

④ Super Weyl

$\Rightarrow$ Fix gauge

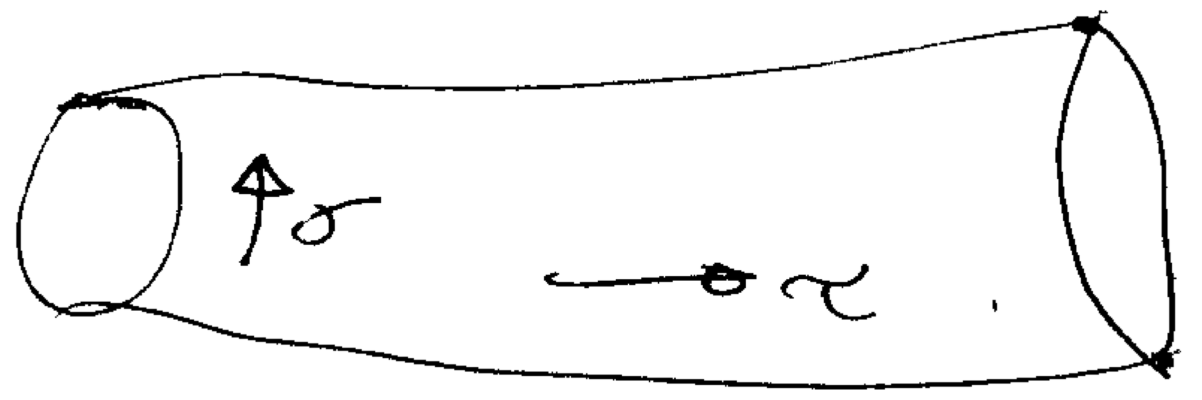
$$\gamma_{\alpha\beta} = \eta_{\alpha\beta}, \quad \chi_\alpha = 0.$$

$\gamma_{\alpha\beta}$ eqn. $\Rightarrow$ $T_{\alpha\beta} = 0$

χ_α eqn. $\Rightarrow$ $J_\alpha = 0$

We shall simply take these constraints as starting points and proceed.

Closed string:



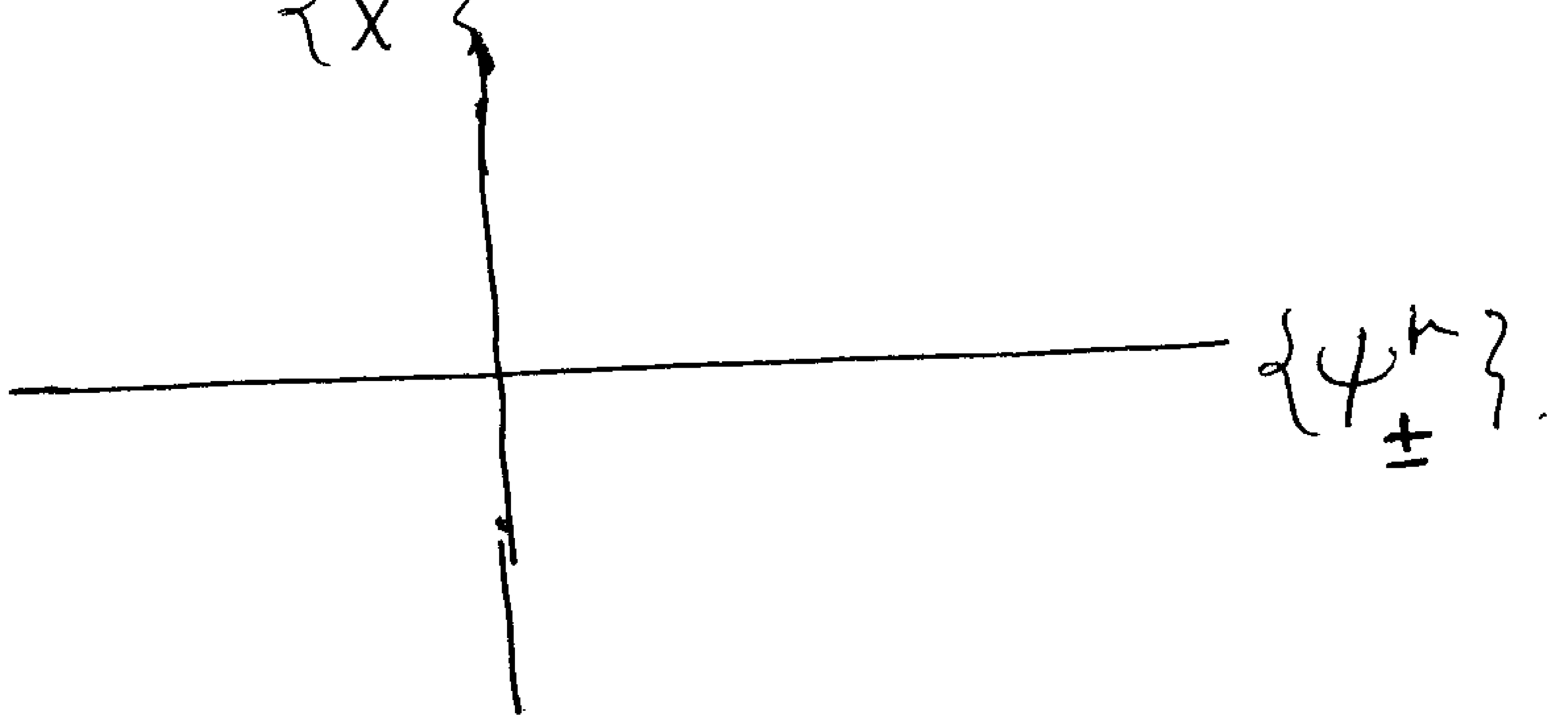
$$0 \le \sigma \le 2\pi$$

$$X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi, \tau)$$

$$\psi^\mu_\pm(\sigma,\tau) = \psi^\mu_\pm(\sigma + 2\pi, \tau)$$

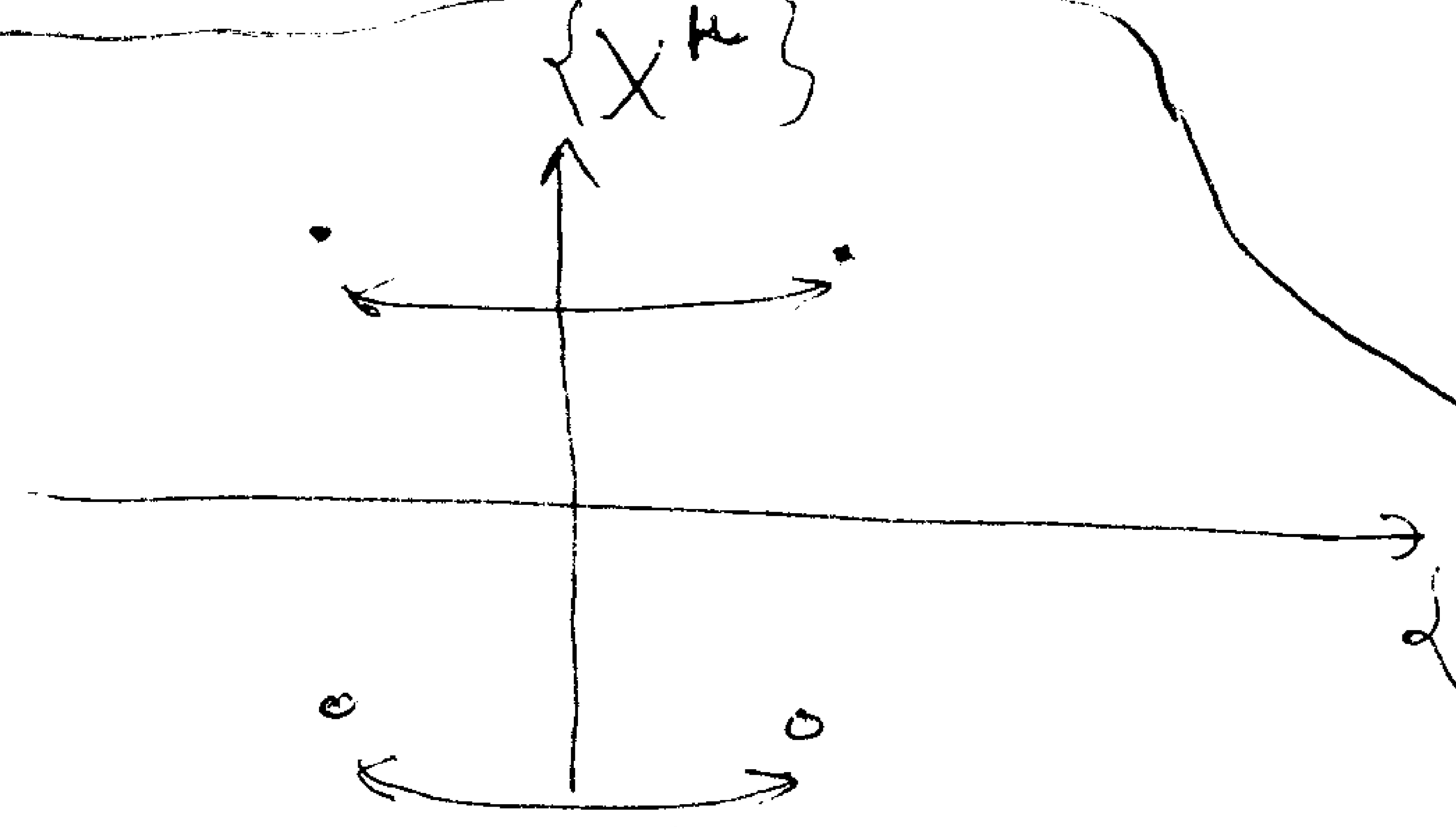
?

Imagine that the string is moving in a space parametrized by X^μ, $\psi^\mu_\pm$.



We shall take the space to be the quotient of the (X, ψ) space by the trs. $\psi_+^\mu \to -\psi_+^\mu$ & $\psi_-^\mu \to -\psi_-^\mu$ $(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Identify the points (X, ψ_+, ψ_-) to $(X, -\psi_+, \psi_-)$, $(X, \psi_+, -\psi_-)$, $(X, -\psi_+, -\psi_-)$



$\Rightarrow$ ① ~~the string~~ A wave fn.

$\Phi(\{X^\mu\}, \{\psi^\mu\})$ must be invariant under $\psi_+^\mu \to -\psi_+^\mu$, $\psi_-^\mu \to -\psi_-^\mu$

$\Phi(\{X^\mu\}, \{\psi_+^\mu\}, \{\psi_-^\mu\}) = \Phi(\{X^\mu\}, \{\pm\psi_+^\mu\}, \{\pm\psi_-^\mu\})$

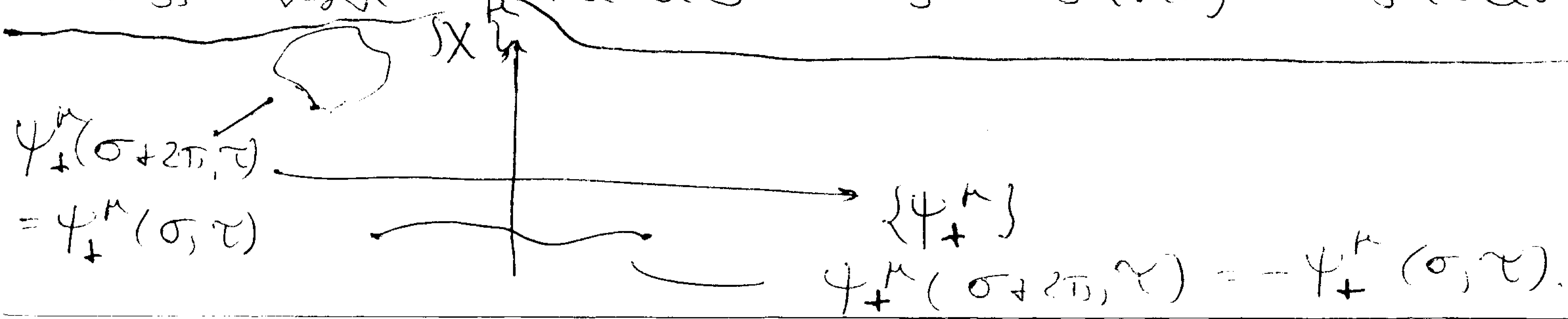
$(-1)^{f_R}$: generator of $\psi_+^\mu \to -\psi_+^\mu$

$(-1)^{f_L}$: generator of $\psi_-^\mu \to -\psi_-^\mu$

$\Rightarrow$ $(-1)^{f_L} |phys\rangle = |phys\rangle$ $\to$ GSO projection

$(-1)^{f_R} |phys\rangle = |phys\rangle$

② We can now consider two/4 different kinds of string states:



9

(144)

Thus we have two sectors for ψ_+^μ

NS-sector: $\psi_+^\mu(\sigma+2\pi,\tau) = -\psi_+^\mu(\sigma,\tau)$

R-sector: $\psi_+^\mu(\sigma+2\pi,\tau) = \psi_+^\mu(\sigma,\tau)$.

Similarly 2 sectors for ψ_-^μ

In each sector we have G.S.O. projection under $(-1)^{F_L}$ and $(-1)^{F_R}$.

The full spectrum is obtained by combining the spectrum from these ~~two~~ four sectors.

Mode expansion:

$X^\mu(\sigma,\tau) \to$ usual.

$\ni \alpha_n^\mu, \tilde{\alpha}_n^\mu, x^\mu, p^\mu$

$[x^\mu, p_\nu] = i\delta^\mu{}_\nu$, $[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$

$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$.

$$S_F = \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\, d\tau\, [\psi_-^\mu(\partial_\tau - \partial_\sigma)\psi_-^\nu + \psi_+^\mu(\partial_\tau + \partial_\sigma)\psi_+^\nu]\,\eta_{\mu\nu}$$

Mode expansion:

$\psi_-^\mu = \sqrt{\alpha'}\sum_r \tilde{b}_r^\mu(\tau)\, e^{-ir\sigma}$

$\psi_+^\mu = \sqrt{\alpha'}\sum_r b_r^\mu(\tau)\, e^{+ir\sigma}$ $\to$ convention.

$r \in \mathbb{Z}$ in the R sector

$\in \mathbb{Z}+\frac{1}{2}$ in the NS sector

$$S_F = \frac{2\pi i}{4\pi} \sum_r \int d\tau \left[b^\mu_{-r} (\partial_\tau + ir) \, b^\nu_{r} + \tilde{b}^\mu_{-r} (\partial_\tau + ir) \, \tilde{b}^\nu_{r} \right] \eta_{\mu\nu}$$

$\Rightarrow$ Momentum conjugate to b^μ_r is

$2 \cdot \frac{i}{2} \, b^\nu_{-r} \eta_{\mu\nu}$ for $r \neq 0$

$2 \, \frac{i}{2} \, b^\nu_0 \eta_{\mu\nu}$ for $r = 0$.

Similarly for $\tilde{b}^\mu_r$

Take $b^\mu_r, \tilde{b}^\mu_r ; r > 0$ as coordinates

$b^\mu_{-r}, \tilde{b}^\mu_{-r} ; r > 0$ as momenta.

$$\left.\begin{aligned} \{b^\mu_r, b^\nu_s\} &= \eta^{\mu\nu} \delta_{r+s,0} \\ \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} &= \eta^{\mu\nu} \delta_{r+s,0} \end{aligned}\right\} r, s \neq 0$$

For $b^\mu_0, \tilde{b}^\mu_0$ we need special treatment.

Momentum conjugate to b^μ_0 is $\frac{i}{2} b^\mu_0$

" " " $\tilde{b}^\mu_0$ is $\frac{i}{2} \tilde{b}^\mu_0$

$\Rightarrow$ Coordinates & Momenta are not independent.

$\Rightarrow$ Constrained system.

) Compute Dirac bracket of b^μ_0, with itself.

) Replace it by $-i \times$ commutator bracket in quantum theory.

Result:

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$$

$$\{\tilde{b}_0^\mu, \tilde{b}_0^\nu\} = \eta^{\mu\nu}$$

Or take:

$$\frac{1}{\sqrt{2}}(\pm b_0^0 + b_0^{D-1}) = (\chi_0^\pm)$$

$$\frac{1}{\sqrt{2}}(b_0^1 \pm i b_0^2) = \chi_1^\pm$$

$$\Rightarrow S_F = i \chi_1^+ \partial_\tau \chi_1^- + \cdots$$

$$\Rightarrow \{\chi_1^+, \chi_1^-\} = 1, \Rightarrow$$

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$$

$\Rightarrow$ General result:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\,\delta_{r+s,0}$$

$$\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\,\delta_{r+s,0}$$

for all r, s

$$\Rightarrow \mathcal{H}_F = \sum_{r>0} r\, \eta_{\mu\nu} (b_{-r}^\mu b_r^\nu + \tilde{b}_{-r}^\mu \tilde{b}_r^\nu) + \text{constant}$$

Construction of the Hilbert space:

NS-NS, NS-R, R-NS, RR sectors.

- NS-NS sector:

Define $|\{k\}\rangle$

$$p_\mu |\{k\}\rangle = k_\mu |\{k\}\rangle$$

$$\alpha_n^\mu |\{k\}\rangle = 0 = \tilde{\alpha}_n^\mu |\{k\}\rangle \quad \text{for } n > 0.$$

$$b_r^\mu |\{k\}\rangle = 0 = \tilde{b}_r^\mu |\{k\}\rangle \quad \text{for } r > 0$$

A generic basis state:

$$\alpha_{-n_1}^{\mu_1} \cdots \alpha_{-n_k}^{\mu_k}\, \tilde{\alpha}_{-l_1}^{\nu_1} \cdots \tilde{\alpha}_{-n_l}^{\nu_l}$$

$$b_{-r_1}^{\tau_1} \cdots b_{-r_m}^{\tau_m}\, \tilde{b}_{-s_1}^{\sigma_1} \cdots \tilde{b}_{-s_n}^{\sigma_n} |\{k\}\rangle$$

GSO projection: $(-1)^{f_L}\, \tilde{b}_{-r}^{\tau} = -\tilde{b}_{-r}^{\tau} (-1)^{f_L}$

$$(-1)^{f_R}\, b_{-s}^{\sigma} = -b_{-s}^{\sigma} (-1)^{f_R}.$$

$(-1)^{f_L}|\{k\}\rangle = -|\{k\}\rangle$, $(-1)^{f_R}|\{k\}\rangle = -|\{k\}\rangle$

$\Rightarrow$ NS sector Fock vacuum is projected out by GSO projection.

$b^{\nu}_{-s}\tilde{b}^{\mu}_{-r}|\{k\}\rangle$ is GSO allowed

Not $b^{\nu}_{-s}|\{k\}\rangle$ or $\tilde{b}^{\mu}_{-r}|\{k\}\rangle$.

(At present $(-1)^{f_L}$, $(-1)^{f_R}$ eigenvalue of $|\{k\}\rangle$ seems ad hoc but it can be derived using BRST formalism).

R sector has some extra complication due to the presence of zero modes.

NS-R sector:

$\tilde{\psi}^{\mu}$: $\frac{1}{2}$ + integer moded

ψ^{μ} : integer moded

$$\{b^{\mu}_0, b^{\nu}_0\} = \eta^{\mu\nu}$$

$\rightarrow$ Clifford algebra.

Quantization $\Rightarrow$ a state in the spinor representation of SO(D-1, 1).

Lorentz group acting on μ.

$\frac{1}{\sqrt{2}}(b_0^1 \pm i\, b_0^2) = \chi_1^\pm$

$\frac{1}{\sqrt{2}}(b_0^3 \pm i\, b_0^4) = \chi_2^\pm$ etc.

$\Rightarrow D/2 \; (5)\; \chi$'s.

$\{\chi_i^-, \chi_j^-\} = \{\chi_i^+, \chi_j^+\} = 0.$

$\{\chi_i^-, \chi_j^+\} = \delta_{ij}$

Define $|0\rangle$ s.t. $\chi_i^- |0\rangle = 0 \quad \forall i.$

Other states:

$\chi_i^+ |0\rangle, \quad \chi_i^+ \chi_j^+ |0\rangle, \; \ldots\ldots$

$\{|0\rangle, \; \chi_i^+ \chi_j^+ |0\rangle, \; \chi_i^+ \chi_j^+ \chi_k^+ \chi_l^+ |0\rangle\}$

$|a\rangle_R \to$ spinor rep s of $SO(D-1,1)$

$\{\chi_i^+ |0\rangle, \; \chi_i^+ \chi_j^+ \chi_k^+ |0\rangle, \; \chi_1^+ \cdots \chi_5^+ |0\rangle\}$

$|\dot{a}\rangle_R \to$ (spinor)$'$ rep. s' of $SO(D-1,1)$

Note:

If s has $(-1)^{f_R} = 1$ then s' has

$(-1)^{f_R} = -1$

& vice versa.

$\Rightarrow$ Two choices.

GSO even NS-R ground state can be in s or s' representation of $SO(D-1,1)$.

Convention: S is the GSO even state.

$|\bar{\alpha}\rangle_R \otimes |\{k\}\rangle \times |a\rangle_R \to$ GSO even Fock ground state.

$|\bar{\dot{\alpha}}\rangle_R \otimes |\{k\}\rangle \times |\dot{a}\rangle_R \to$ GSO odd Fock ground state

A generic basis state:

$$\alpha^{\mu_1}_{-n_1} \cdots \alpha^{\mu_k}_{-n_k} \tilde{\alpha}^{\nu_1}_{-l_1} \cdots \tilde{\alpha}^{\nu_l}_{-n_l}$$

$$b^{\sigma_1}_{-r_1} \cdots b^{\sigma_m}_{-r_m} \tilde{b}^{\tau_1}_{-s_1} \cdots \tilde{b}^{\tau_m}_{-s_m} \quad |\{k\}\rangle \otimes \begin{matrix} |\bar{\alpha}^a\rangle_R \\ |\bar{\alpha}^{\dot{a}}\rangle_R \end{matrix}$$

Pick GSO ~~odd~~ even states only

$(-1)^{f_L}$ even states. $(-1)^{f_R}$ even states.

Similarly in R-NS we have Fock ground state:

$$|\{k\}\rangle \otimes \begin{matrix} |\bar{\alpha}^a\rangle_L \\ |\bar{\alpha}^{\dot{a}}\rangle_L \end{matrix} \times |\{k\}\rangle$$

Once we have chosen $|\bar{\alpha}^a\rangle_R$ to be ~~odd~~ even under $(-1)^{f_R}$, it makes a physical difference whether we chose $|\bar{\alpha}^a\rangle_L$ to be even under $(-1)^{f_L}$ or $|\bar{\alpha}^{\dot{a}}\rangle_L$ to be even under $(-1)^{f_L}$.

IIB : $|\bar{\alpha}^a\rangle_L$ is even under $(-1)^{f_L}$

IIA : $|\bar{\alpha}^{\dot{a}}\rangle_L$ is even under $(-1)^{f_L}$.

R sector:

Fock ground state is

$$|\{k\}\rangle \otimes \left\{ \begin{matrix} |a\rangle_L \\ |\dot{a}\rangle_L \end{matrix} \right\} \otimes \left\{ \begin{matrix} |b\rangle_R \\ |\dot{b}\rangle_R \end{matrix} \right\}.$$

In IIA: $|\{k\}\rangle \otimes |\dot{a}\rangle_L \otimes |b\rangle_R$ is GSO allowed.

In IIB: $|\{k\}\rangle \otimes |\dot{a}\rangle_L \otimes |\dot{b}\rangle_R$ is GSO allowed.

Note: NS-NS, R-R: Bosons NS-R, R-NS: fermions.

Finally, in each sector we need to impose physical state constraint.

$T_{uu} = \sum (L_m + c\,\delta_{m,0}) e^{im\sigma}$

$T_{vv} = \sum (\tilde{L}_m + \tilde{c}\,\delta_{m,0}) e^{-im\sigma}$

$$T_{\alpha\beta} = 0, \quad J^\alpha_\pm = 0$$

$J^\mu_+ = \sum G_n e^{in\sigma}$

$J^\mu_- = \sum \tilde{G}_n e^{-in\sigma}$

$$\langle \text{phys} | (L_n + c\,\delta_{n,0}) | \text{phys}\rangle' = 0$$

$(\tilde{L}_n + \tilde{c}\,\delta_{n,0})$ normal ordering const.

$$\langle \text{phys} | G_n | \text{phys}'\rangle = 0$$

$\tilde{G}_n$

$$L_m = \frac{1}{2} \sum_{n \neq 0, m} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^\mu_m p_\mu + \frac{1}{2}\sum_r \left(r + \frac{1}{2}m\right) b^\mu_{-r} b^\nu_{m+r} \eta_{\mu\nu}$$

$$L_0 = \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{\mu\nu} p_\mu p_\nu + \sum_{r>0} r\, b^\mu_{-r} b^\nu_r \eta_{\mu\nu} + c$$

c normal ordering const.

$\tilde{L}_0, \tilde{L}_m$: $\alpha^\mu_m \to \tilde{\alpha}^\mu_m$, $b^\mu_r \to \tilde{b}^\mu_r$.

$$G_r = \sum_{n\neq 0} \alpha^{\mu}_{-n}\, b^{\nu}_{r+n}\, \eta_{\mu\nu} + \sqrt{\tfrac{\alpha'}{2}}\, p_\mu\, b^{\mu}_{r}$$

$$\tilde{G}_r = \sum_{n\neq 0} \tilde{\alpha}^{\mu}_{-n}\, \tilde{b}^{\nu}_{r+n}\, \eta_{\mu\nu} + \sqrt{\tfrac{\alpha'}{2}}\, p_\mu\, \tilde{b}^{\mu}_{r}$$

Physical states:

$$(L_n + C\delta_{n,0})|phys\rangle = 0 = (\tilde{L}_n + \tilde{C}\delta_{n,0})|phys\rangle \quad \text{for } n \geq 0.$$

$$G_r |phys\rangle = 0 = \tilde{G}_r |phys\rangle \quad \text{for } r \geq 0.$$

Acting on

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k}\, \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell}\, b^{\sigma_1}_{-r_1} \cdots b^{\sigma_m}_{-r_m}$$

$$\tilde{b}^{\tau_1}_{-s_1} \cdots \tilde{b}^{\tau_n}_{-s_n} |\{k\}\rangle \left[\otimes \begin{pmatrix} |a\rangle_L \\ |\dot{a}\rangle_L \end{pmatrix} \otimes \begin{matrix} |b\rangle_R \\ |\dot{b}\rangle_R \end{matrix}\right]$$

L_0 constraint:

$$\frac{\alpha'}{4}\, \eta^{\mu\nu} k_\mu k_\nu + \sum m_i + \sum r_i + C = 0.$$

$\tilde{L}_0$ constraint:

$$\frac{\alpha'}{4}\, \eta^{\mu\nu} k_\mu k_\nu + \sum n_i + \sum s_i + \tilde{C} = 0$$

$$\Rightarrow m^2 = -\eta^{\mu\nu} k_\mu k_\nu$$

$$= \frac{4}{\alpha'}\Big(\underbrace{\sum m_i + \sum r_i}_{N} + C\Big)$$

$$= \frac{4}{\alpha'}\Big(\underbrace{\sum n_i + \sum s_i}_{\tilde{N}} + \tilde{C}\Big)$$

$N, \tilde{N}$: Levels.

Unitarity analysis + Consistency of interacting theory.

$D = 10$

$c = -\frac{1}{2}$ in NS sector
$= 0$ in R sector } Right sector

$\tilde{c} = -\frac{1}{2}$ in NS sector
$= 0$ in R sector } Left sector

Note: $|\{k\}\rangle$ is projected out in the NS-NS sector.

$S_{\mu\nu}\, b^{\mu}_{-1/2}\, \tilde{b}^{\nu}_{-1/2}\, |\{k\}\rangle_{NS\text{-}NS}$ is GSO even

Demanding that there are no -ve norm states gives:

$$c \geq -\tfrac{1}{2}\ , \quad \tilde{c} \geq -\tfrac{1}{2}$$

etc.

Note: lowest $(\text{mass})^2$ state has 0 mass ⇒ no tachyon

Massless states:

Bosons: NS NS & RR.

NSNS

$$m^2 = \frac{4}{\alpha'}\left(\sum m_i + \sum r_i - \frac{1}{2}\right) = 0$$

$$= \frac{4}{\alpha'}\left(\sum n_i + \sum s_i - \frac{1}{2}\right) = 0.$$

$\Rightarrow \sum m_i = 0,\ \sum n_i = 0,\ \sum r_i = \frac{1}{2}$ ~~for one~~ i

$\sum s_i = \frac{1}{2}$ ~~for one i~~.

$\underbrace{\zeta_{\mu\nu}}_{\text{trace less}} b^{\mu}_{-1/2} \tilde{b}^{\nu}_{-1/2} |\{k\}\rangle$

- Sym. part of ζ → graviton

Antisym. part of ζ → Anti-symmetric tensor field.

Trace part of ζ → scalar dilaton.

RR

$\cancel{\zeta_{\alpha\dot\beta}} |\{k\}\rangle$

$$m^2 = \frac{4}{\alpha'} \left(\sum m_i + \sum r_i \right) = 0$$

$$= \frac{4}{\alpha'} \left(\sum n_i + \sum s_i \right) = 0$$

$\Rightarrow \sum m_i = 0, \sum r_i = 0, \sum n_i = 0, \sum s_i = 0$

$\Rightarrow \quad \zeta_{ba} |\{k\}\rangle \otimes |b\rangle_L \otimes |a\rangle_R$ for IIA

$\zeta_{ba} |\{k\}\rangle \otimes |b\rangle_L \otimes |a\rangle_R$ for IIB.

IIA : $s' \otimes s$	$G_r = \sum_{n \neq 0} \alpha^{\mu}_{-n} b^{\nu}_{r+n} \eta_{\mu\nu} + \sqrt{\frac{\alpha'}{2}} p_\mu b^\mu_r$
IIB : $s \otimes s$	$\tilde{G}_r = \sum_{n \neq 0} \tilde{\alpha}^{\mu}_{-n} \tilde{b}^{\nu}_{r+n} \eta_{\mu\nu} + \sqrt{\frac{\alpha'}{2}} p_\mu \tilde{b}^\mu_r$

IIA : vector, rank 3 anti-symmetric tensor.

IIB : scalar, rank 2 anti-symmetric tensor, rank 4 anti-symmetric tensor with self-dual field strength.

~~[illegible]~~

Note: $G_0 = 0$, $\tilde{G}_0 = 0$ constraint.

$\Rightarrow$ Constraint on ξ linear in k^μ.

$\Rightarrow$ Polarization tensor $\xi_{\alpha\bar{\beta}}$ or $\xi_{\bar{\alpha}\bar{\beta}}$ are Fourier transform of field strengths rather than gauge potentials.

IIA: 2 form & 4 form field strength.

IIB: 1 form, 3 form & self-dual 5-form field strength.

Massless fermions

NS-R & R-NS sector:

NS-R

$$\frac{4}{\alpha'}\left(\sum m_i + \sum r_i\right) = 0.$$

$$\frac{4}{\alpha'}\left(\sum n_i + \sum s_i - \frac{1}{2}\right) = 0.$$

$$\Rightarrow \zeta^{a}_{\mu}\, |\{k^\mu\}\rangle \otimes \tilde{b}^{\mu}_{-1/2}\,|0\rangle_L \otimes |a\rangle_R$$

$\rightarrow$ A particle whose polarization tensor has one spinor & one vector index.

$\rightarrow$ Rarita-Schwinger field.

$\Rightarrow$ spin $\frac{3}{2}$ particle in 4-dim.

$\Rightarrow$ Massless spin $\frac{3}{2}$ particle.

$\rightarrow$ gravitino

R-NS

$$\frac{4}{\alpha'}\left(\sum m_i + \sum r_i - \frac{1}{2}\right) = 0$$

$$\frac{4}{\alpha'}\left(\sum n_i + \sum s_i\right) = 0.$$

$$S^{\dot{a}}_{\mu}\,|\{k^\mu\}\rangle \otimes |\dot{a}\rangle_L \otimes b^{\mu}_{-1/2}\,|0\rangle_R \quad \text{for IIA.}$$

$$S^{a}_{\mu}\,|\{k^\mu\}\rangle \otimes |a\rangle_L \otimes b^{\mu}_{-1/2}\,|0\rangle_R \quad \text{for IIB}$$

→ another spin $\frac{3}{2}$ particle.

Note: In IIB the two spin $\frac{3}{2}$ particles have the same chirality

In IIA they have opposite chirality.

One can also show that at each mass level there are equal no. of physical bosonic and fermionic states.

⇒ space-time supersymmetry.

Gravitinoes → supersymmetric partner of gravitons.

2-gravitinoes. ⇒ $\mathcal{N}=2$ supersymmetry.

Massless spectrum → identical to that of type IIA and type IIB supergravity theories respectively.

~~Lightcone gauge~~
~~quantization the string theory~~

Heterotic String Theory

→ Combines superstring theory in the right-moving sector with bosonic string theory in the left moving sector

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Digression: A ~~bosonic~~ string theory in 10 non-compact direction.

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \; \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \quad (\mu, \nu = 0, \ldots 9)$$

$$+ \; S_{CFT}$$

$$\frac{?}{\pi\alpha'} \int d^2\xi \sum_{I=1}^{N} \lambda^I \rho^\alpha \partial_\alpha \lambda^I$$

$$\lambda^I = \begin{pmatrix} \lambda^I_{+} \\ \lambda^I_{-} \end{pmatrix} \to \text{Majorana fermions.}$$

$\Rightarrow$ A conformal field theory

→ Energy momentum tensor → L_n^{CFT}

→ Central Charge from $[L_m^{CFT}, L_n^{CFT}]$

$\Rightarrow$ $C = N/2$.

For a consistent bosonic string theory we need

$$N = 32.$$

What kind of boundary conditions can we put on λ^I?

$\leadsto$ Depends on what kind of identification we make

(Constraints due to requirement of modular invariance)

Example (A) $(\lambda^I_+, \lambda^I_-) \equiv (-\lambda^I_+, \lambda^I_-)$

$$\equiv (\lambda^I_+, -\lambda^I_-) = (-\lambda^I_+, -\lambda^I_-)$$

$\Rightarrow$ (1) Projection: $(-1)^{f^\lambda_L}$ & $(-1)^{f^\lambda_R}$

(2) 4-sectors: $(++), (+-), (-+), (--)$ on $(\lambda^I_+, \lambda^I_-)$

(B) We can divide λ into groups of 16 & call them (χ^M, η^M) $1 \leq M \leq 16$.

Identification: $(\mathbb{Z}_2)^4$.

(1) $\chi^M_+ \to -\chi^M_+$ .. (2) $\chi^M_- \to -\chi^M_-$ .. (3) $\eta^M_+ \to -\eta^M_+$

(4) $\eta^M_- \to -\eta^M_-$

→ ~~Sep~~

① Separate GSO like projection for χ & η

② Allows separately NS & R like b.c. on

χ^M_+, χ^M_-, η^M_+, & η^M_-.

⇒ $2^4 = 16$ sectors.

③ Other choices possible.

Heterotic string theory: ~~...~~

world-sheet Degrees of Freedom

$X^\mu \qquad \mu = 0, \cdots 9$.

$\psi^\mu_{\ominus +} \qquad \mu = 0, \cdots 9 \qquad \psi^M_{\oplus -} = 0$ — left right asymmetry → possible diffeomorphism anomaly.

$\lambda^I_{\oplus -} \qquad I = 1, \cdots 32$

Two possible modular invariant theory:

① $(X^\mu, \psi^\mu_{\ominus +}, \lambda^I_{\oplus -}) = (X^\mu, -\psi^\mu_{\ominus +}, \lambda^I_{\oplus -})$

$= (X^\mu, \psi^\mu_{\ominus +}, -\lambda^I_{\oplus -}) = \cdots$

→ SO(32) heterotic. SO(32) acts on λ^I_+.

② $(X^\mu, \psi^\mu_{\ominus +}, \chi^M_{\oplus -}, \eta^M_{\oplus -}) = (X^\mu, -\psi^\mu_{\ominus +}, \chi^M_{\oplus -}, \eta^M_{\oplus -})$

$= (X^\mu, \psi^\mu_{\ominus +}, -\chi^M_{\oplus -}, \eta^M_{\oplus -}) = (X^\mu, \psi^\mu_{\ominus +}, \chi^M_{\oplus -}, -\eta^M_{\oplus -})$

→ $E_8 \times E_8$ heterotic.

$\supset SO(16) \times SO(16)$ acts on $\chi^M_{\oplus -}$, $\eta^M_{\oplus -}$

Quantization straight forward:

Constraints: $\tilde{L}_m, L_m, G_{m}$, $\cancel{\mathcal{L}}$ J^{α}_{+}

$T_{\alpha\beta}$ Note: No $\tilde{G}_m$.

Supersymmetry to gauge invariant action only on the ~~left~~ right sector.

Examine the spectrum of SO(32) heterotic

Fock vacuum:

$$|\{k^\mu\}\rangle \otimes \left[\begin{pmatrix} |u\rangle \\ |\dot{u}\rangle \end{pmatrix} \otimes \begin{pmatrix} |a\rangle \\ |\dot{a}\rangle \end{pmatrix} \right]$$

spinor rep. of SO(32) in Ramond sector

spinor rep. of SO(9,1) in Ramond sector.

Mass-shell constraint on

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \; \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell} \; b^{\sigma_1}_{-r_1} \cdots b^{\sigma_m}_{-r_m}$$

$$\tilde{d}^{I_1}_{-s_1} \cdots \tilde{d}^{I_n}_{-s_n} \; |\{k^\mu\}\rangle \otimes [\cdots]$$

$$m^2 = \frac{4}{\alpha'} \left(\sum m_i + \sum r_i + c \right)$$

$$= \frac{4}{\alpha'} \left(\sum n_i + \sum s_i + \tilde{c} \right)$$

$c = -\frac{1}{2}$ in NS, 0 in R

$\tilde{c} = -1$ in NS, 1 in R

GSO projection rule on ~~of~~ the right hand sector → usual superstring.

GSO projection rule on the left hand sector (~ oscillators)

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① NS sector Fock vacuum is GSO even.

---

② In R sector, of the two spinor representation, one is GSO even, the other is GSO odd.

Take $|u\rangle$ to be GSO even.

Absence of tachyon:

$$m^2 = \frac{4}{\alpha'}\left(\sum m_i + \sum \lambda_i + c\right).$$

In the R sector of right moving oscillators, $c=0$.

$\Rightarrow m^2 \geq 0$.

NS sector : $c = -\frac{1}{2}$.

$\Rightarrow m^2 \geq -\frac{2}{\alpha'}$. There can be a tachyon.

But the NS sector ground state on the right is odd under $(-1)^{F_R}$. $\Rightarrow$ Projected out $\Rightarrow$ No tachyon.

Massless states:

NS or R sector on the right.

NS sector on the left.

(Since $m^2 \geq \frac{4}{\alpha'} \tilde{c}$ & in the R-sector on the left $\tilde{c} = 1$).

~~$b^\mu_{-1/2} \tilde{\alpha}^\mu_{-1} |\{k^\mu\}\rangle$~~

~~$b^\mu_{-1/2} \psi^I_{-1/2} \psi^J_{-1/2} |\{k^\mu\}\rangle$.~~

(R,NS), (NS-NS) : Bosonic

(NS,R), (R,R) : Fermionic.

Massless states from (NS-NS) & (NS,R).

(NS-NS):

$$\sum m_i + \sum r_i = \frac{1}{2}$$

$$\sum n_i + \sum s_i = 1$$

m_i, n_i integer. $r_i : \frac{1}{2}$ integer.

$b^\mu_{-1/2} \tilde{\alpha}^\nu_{-1} |\{k^\mu\}\rangle \to$ Graviton - dilaton - antisym tensor.

$b^\mu_{-1/2} \psi^I_{-1/2} \psi^J_{-1/2} |\{k^\mu\}\rangle \to SO(32)$ gauge bosons.

~~R~~ (NS R):

$$\sum m_i + \sum r_i = 0.$$

$$\sum n_i + \sum s_i = 1$$

m_i, r_i, n_i : integer, s_i : $\frac{1}{2}$ integer.

$\tilde{\alpha}^{\nu}_{-1} \, |\{k^{\mu}\}\rangle \otimes |a\rangle_R \to$ Gravitino.

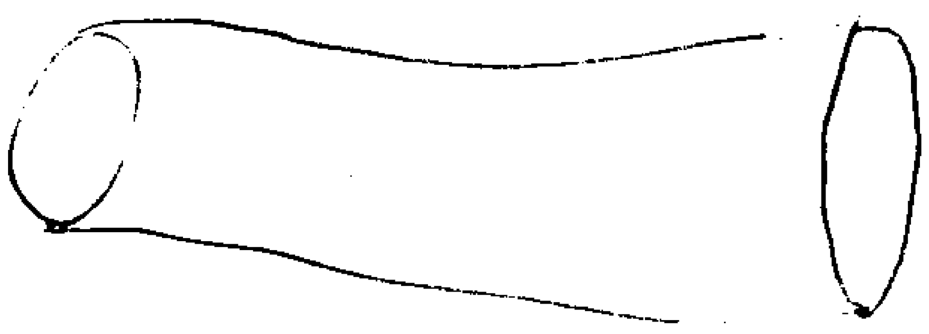
$\psi^{I}_{-1/2} \psi^{J}_{-1/2} \, |\{k^{\mu}\}\rangle \otimes |a\rangle_R \to$ Gaugino.

In this ~~these~~ theory also there is exact ~~dege~~ matching of the degeneracy of bosonic and fermionic states at all mass levels.

$\Rightarrow$ Space-time supersymmetry.

First quantized bosonic string (closed)

Action: $S = -T \int d^2\xi \sqrt{-\det h_{\alpha\beta}}$



$(\xi^0, \xi^1) = (\tau, \sigma)$: World-sheet coordinates

X^μ space-time coordinates.

$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$.

$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1)$ $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots 1)$

$\int d^2\xi \sqrt{\det h_{\alpha\beta}}$ $\to$ area of the world-sheet.

$T = 1/2\pi\alpha'$ $\to$ string tension

Alternative action:

$$\tilde{S} = -\frac{T}{2} \int d^2\xi \sqrt{-\det \gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$\gamma_{\alpha\beta}$: A metric on the world-sheet.

$\to$ New degree of freedom.

$\gamma_{\alpha\beta}$ eqs. of motion:

$$\Rightarrow \cdots \gamma_{\alpha\beta} = \underbrace{F(\vec{\xi})}_{\to \text{ arbitrary}} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$$

~~Substi~~

Substitute in the action $\tilde{S}$

$\Rightarrow$ Old action S.

Classically S & $\tilde{S}$ are equivalent.

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We use S to quantize.

Gauge invariance

① 2 d general coordinate tr.

② Weyl invariance $h_{\alpha\beta} \to \phi(\xi) h_{\alpha\beta}$

$\Rightarrow$ Gauge fix to $h_{\alpha\beta} = \eta_{\alpha\beta}$

~~Be~~ $\to$ $S = -\frac{T}{2} \int d^2\xi \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

$\to$ free scalar field theory

But there are constraints

(Not all the eqs of motion of $\gamma_{\alpha\beta}$ are automatically satisfied due to X^μ eqs of motion)

$$T_{\alpha\beta} \equiv c(\eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu \eta_{\alpha\beta})$$

$$= 0$$

Consequence of residual gauge invariance.

$u = \tau - \sigma$, $v = \tau + \sigma$ $\Rightarrow ds^2 = -d\tau^2 + d\sigma^2 = -du\,dv$

$X'^\mu(u,v) = X^\mu(f(u), g(v))$

arbitrary functions

(A combination of g.c.t. and Weyl trs)

Independent constraints: $T_{uv} = T_{vu} = 0$ identically

$T_{uu} = c\, \eta_{\mu\nu} \partial_u X^\mu \partial_u X^\nu = 0$ | $T_{vv} = c\, \eta_{\mu\nu} \partial_v X^\mu \partial_v X^\nu = 0$

Strategy

① Quantize the theory ignoring the constraints.

② Impose

$$\langle \text{phys} | T_{uu} | \text{phys}' \rangle = 0 = \langle \text{phys} | T_{vv} | \text{phys}' \rangle$$

Step 1

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$$

$$\Rightarrow X^\mu(\tau,\sigma) = \sum_{n\neq 0} \phi^\mu_n(\tau)\, e^{in\sigma} + x^\mu(\tau).$$

a) substitute in the action.

b) Define

$$\Pi_{\nu n} = \frac{\delta \tilde{S}}{\delta(\partial_\tau \phi^\nu_n)} = \frac{1}{\alpha'} \eta_{\mu\nu} \partial_\tau \phi^\mu_{-n}$$

$$p_\nu = \frac{\delta \tilde{S}}{\delta(\partial_\tau x^\nu)} = \frac{1}{\alpha'} \eta_{\mu\nu} \partial_\tau x^\mu$$

conjugate momenta

c) $$\mathcal{H} = \sum_{n\neq 0} \Pi_{\nu n} \partial_\tau \phi^\nu_n + p_\nu \partial_\tau x^\nu - \tilde{S}$$

d) ~~Rea~~ X^μ is real

$$\Rightarrow \Pi^*_{\mu n} = \Pi_{\mu(-n)}, \quad (\phi^\mu_n)^* = \phi^\mu_{-n}$$

$$(x^\mu)^* = x^\mu, \quad (p_\mu)^* = p_\mu$$

Quantization:

$$[\phi^\mu_n, \Pi_{\nu m}] = i\,\delta^\mu_\nu\,\delta_{mn}, \qquad [x^\mu, p_\nu] = i\delta^\mu_\nu$$

$(\phi^\mu_n)^\dagger = \phi^\mu_{(-n)}$ $\Pi^\dagger_{\mu n} = \Pi_{\mu(-n)}$, x^μ, p_ν hermitian

Define:

$$\alpha^\mu_{-n} = \frac{1}{\sqrt{2}}\left(\eta^{\mu\nu}\,\Pi_{\nu n}\sqrt{\alpha'} + \frac{in}{\sqrt{\alpha'}}\,\phi^\mu_{-n}\right)$$

$$(\tilde{\alpha}^\mu_{+n}) = \frac{1}{\sqrt{2}}\left(\eta^{\mu\nu}\,\Pi_{\nu n}\sqrt{\alpha'} - \frac{in}{\sqrt{\alpha'}}\,\phi^\mu_{-n}\right)$$

$$\Rightarrow \quad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}, \qquad (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$$

$$[\alpha^\mu_n, \alpha^\nu_m] = n\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0$$

$$H = \sum_{n>0}\left(\alpha^\mu_{-n}\,\alpha^\nu_n + \tilde{\alpha}^\mu_{-n}\,\tilde{\alpha}^\nu_n\right)\eta_{\mu\nu} + \frac{\alpha'}{2}\,\eta^{\mu\nu}\,p_\mu p_\nu$$

$+ K$

↳ normal ordering constant

Hilbert space

Define $|\{k_\mu\}\rangle$ s.t.

$$p_\mu\,|\{k_\mu\}\rangle = k_\mu\,|\{k_\mu\}\rangle, \qquad \alpha^\mu_m\,|\{k_\mu\}\rangle = 0 = \tilde{\alpha}^\mu_m\,|\{k_\mu\}\rangle$$

General basis of states

$$\alpha^{\mu_1}_{-m_1}\cdots\alpha^{\mu_k}_{-m_k}\,\tilde{\alpha}^{\nu_1}_{-n_1}\cdots\tilde{\alpha}^{\nu_l}_{-n_l}\,|\{k_\mu\}\rangle$$

Constraints

$$T_{uu} = (\text{const.}) \sum L_m e^{im\sigma}$$

$$T_{vv} = (\text{const.}) \sum \tilde{L}_m e^{-im\sigma}$$

$$L_m = \frac{1}{2} \sum_{n \neq 0} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha^\mu_m p_\mu \quad \text{for } m \neq 0$$

$$L_0 = \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} + \frac{\alpha'}{4} \eta^{\mu\nu} p_\mu p_\nu + C$$

C // normal ordering constant $= -1$ (demanding consistency)

Also $D = 26$

$\tilde{L}_m$: $\alpha^\mu_n \to \tilde{\alpha}^\mu_n$.

$\tilde{L}_0$: $\alpha^\mu_n \to \tilde{\alpha}^\mu_n$.

$$L_m^\dagger = L_{-m}, \quad \tilde{L}_m^\dagger = \tilde{L}_{-m}$$

Physical state condition:

$$L_m |\text{phys}\rangle = \tilde{L}_m |\text{phys}\rangle = 0$$

$\Rightarrow$ guarantees $\langle \text{phys} | T_{uu} | \text{phys}' \rangle = 0$

$= \langle \text{phys} | T_{vv} | \text{phys}' \rangle$

Acting on $\alpha^{\mu_1}_{-m_1} \cdot \alpha^{\mu_k}_{-m_k} \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell} |k_\mu\rangle$

$L_0 = 0$ gives:

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'} \left(\Sigma m_i - 1 \right)$$

$\tilde{L}_0 = 0$ gives

$$-\eta^{\mu\nu} k_\mu k_\nu = \frac{4}{\alpha'} \left(\Sigma n_i - 1 \right)$$

$$\begin{array}{c} \parallel\!\mid \\ (\text{Mass})^2 \end{array}$$

$\Rightarrow$ Mass-shell constraint:

$$M^2 = \frac{4}{\alpha'} \left(\Sigma m_i - 1 \right) = \frac{4}{\alpha'} \left(\Sigma n_i - 1 \right)$$

---

① $\Sigma m_i = \Sigma n_i = 0 \rightarrow$ tachyons

---

② $\Sigma m_i = \Sigma n_i = 1 \rightarrow$ graviton, dilaton, anti-symmetric tensor

---

③ $\Sigma m_i = \Sigma n_i > 1 \rightarrow$ Massive states.

We shall make a slight change in notation:

New L_0 = Old $L_0 + 1$ | $L_0 |\{k_\mu = 0\}\rangle = 0$

New $\tilde{L}_0$ = Old $\tilde{L}_0 + 1$ | $\tilde{L}_0 |\{k_\mu = 0\}\rangle = 0$

$\Rightarrow$ ~~the~~ Constraints:

$$(L_m - \delta_{m,0})\,|phys\rangle = 0 = (\tilde{L}_m - \delta_{m,0})\,|phys\rangle$$

for $m \geq 0$

$$\left.\begin{aligned}[L_m, L_n] &= (m-n)\,L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0} \\ [\tilde{L}_m, \tilde{L}_n] &= (m-n)\,\tilde{L}_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}\end{aligned}\right\} c = 26$$

$\to$ 2-dim conformal algebra with central charge $= 26$.

We can construct other string theories by replacing n of the bosonic fields X^μ ($\mu = 26-n, \ldots 25$) by a conformal field theory of central charge n.

$$L_m \equiv L_m^X + L_m^{CFT}, \quad \tilde{L}_m = \tilde{L}_m^X + \tilde{L}_m^{CFT}$$

$\to$ generates ~~the~~ conformal algebras with central charge $= 26$.

Interpretation: $\epsilon_m L_m^{(\tau=0)}$ generates

$$u \to u + \epsilon_m e^{+imu}, \quad v \to v$$

$\tilde{\epsilon}_m \tilde{L}_m^{(\tau=0)}$ generate

$$v \to v + \tilde{\epsilon}_m e^{+imv}, \quad u \to u$$

From now on we shall refer by $L_m, \tilde{L}_m$ by the corresponding operators evaluated at $\tau=0$.

Hamiltonian eq:

$$L_m(\tau) = L_m(0)\, e^{-im\tau}, \quad \tilde{L}_m(\tau) = \tilde{L}_m(0)\, e^{-im\tau}$$

---

if we redefine $L_m(\tau)$, $\tilde{L}_m(\tau)$ by absorbing the factor of $e^{-im\tau}$, then they are time independent and L_m, $\tilde{L}_m$ will refer to these time independent operators.

$$T_{uu} = \sum_m L_m\, e^{im(\tau+\sigma)} = \sum_m L_m\, e^{-imu}$$

$$T_{vv} = \sum_m \tilde{L}_m\, e^{im(\tau-\sigma)} = \sum_m \tilde{L}_m\, e^{-imv}$$

Euclideanize:

$$\tau \to \pm i\tau$$

$$\omega = -(\tau \mp i\sigma) \qquad \bar{\omega} = -(\tau \pm i\sigma)$$

$$u \to -i\omega, \quad v \to -i\bar{\omega}$$

$$T_{\omega\omega} = \sum_m L_m\, e^{-m\omega}, \quad T_{\bar{\omega}\bar{\omega}} = \sum_m \tilde{L}_m\, e^{-m\bar{\omega}}$$

$$\tilde{L}_m \equiv \bar{L}_m$$

$$T_{\omega\bar{\omega}} = T_{\bar{\omega}\omega} = 0$$

$$z = e^{\omega}, \qquad T_{zz} \equiv \left(\frac{d\omega}{dz}\right)^2 T_{\omega\omega} = e^{-2\omega} T_{\omega\omega}$$

$$= \sum_m L_m z^{-m-2}$$

$$T_{\bar z \bar z} = \left(\frac{d\bar\omega}{d\bar z}\right)^2 T_{\bar z \bar z} = \sum_m \bar L_m \bar z^{-m-2}$$

Bosonic string theory

⟷ A 2-dimensional CFT with central charge = 26.

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\,\bar{L}_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}$$

Complex coordinate on the cylinder:

$$\omega = (\tau + i\sigma), \qquad \bar{\omega} = (\tau - i\sigma)$$

* $T_{\omega\omega} = \sum L_m e^{-m\omega}$ $\qquad T_{\bar{\omega}\bar{\omega}} = \sum \bar{L}_m e^{-m\bar{\omega}}$

$$T_{\omega\bar{\omega}} = 0 = T_{\bar{\omega}\omega}$$

τ : Euclidean time.

Physical state :

$$L_n\,|phys\rangle = \bar{L}_n\,|phys\rangle = 0 \quad \text{for } n > 0.$$

$$(L_0 - 1)\,|phys\rangle = 0 = (\bar{L}_0 - 1)\,|phys\rangle.$$

$\{|s_n\rangle\}$: A basis of physical states

* What we want to compute is the S-matrix :

$$S(s_{n_1}, s_{n_2}, \cdots, s_{n_k})$$

Note : We are not distinguishing between incoming and outgoing particles.

* $k_0 > 0$: outgoing

  $k_0 < 0$: incoming.

Suppose $\{\varphi\}$ denotes the collection of all elementary fields in terms of which the CFT is defined.

---

e.g. if we are dealing with the 26 dimensional bosonic string, then $\{\varphi\} = \{X^0, \ldots X^{25}\}$.

Now every state in CFT (physical or not) can be given a wave-function representation.

$\{|\gamma_k\rangle\}$: Complete set of states of CFT

$$|\gamma_k\rangle \leftrightarrow \Psi_{\gamma_k}(\{\varphi(\sigma)\})$$

A loop in φ space.

Configuration space $\leftrightarrow$ Loop space

Note: τ does not enter here.

Analogy with quantum mechanics: A complete set of basis states

$$\leftrightarrow f_n(x)$$

Complete set of functions of the coordinate.

Since $\{|\delta_n\rangle\} \subset \{|\gamma_k\rangle\}$, we have

---

$$|\delta_n\rangle \leftrightarrow \Psi_{\delta_n}(\{\varphi(\sigma)\}).$$

Prescription for N-point S-matrix $S(|s_1\rangle, \cdots |s_N\rangle)$

$$= \int [\mathcal{D}\varphi_1(\sigma)] \cdots [\mathcal{D}\varphi_N(\sigma)] \; \Psi_{s_1}(\{\varphi_1(\sigma)\}) \; \Psi_{s_2}(\{\varphi_2(\sigma)\}) \cdots \Psi_{s_N}(\{\varphi_N(\sigma)\}) \; K(\{\varphi_1(\sigma)\}, \{\varphi_2(\sigma)\}, \cdots \{\varphi_N(\sigma)\})$$

$$K(\{\varphi_1(\sigma)\}, \{\varphi_2(\sigma)\}, \cdots \{\varphi_N(\sigma)\})$$

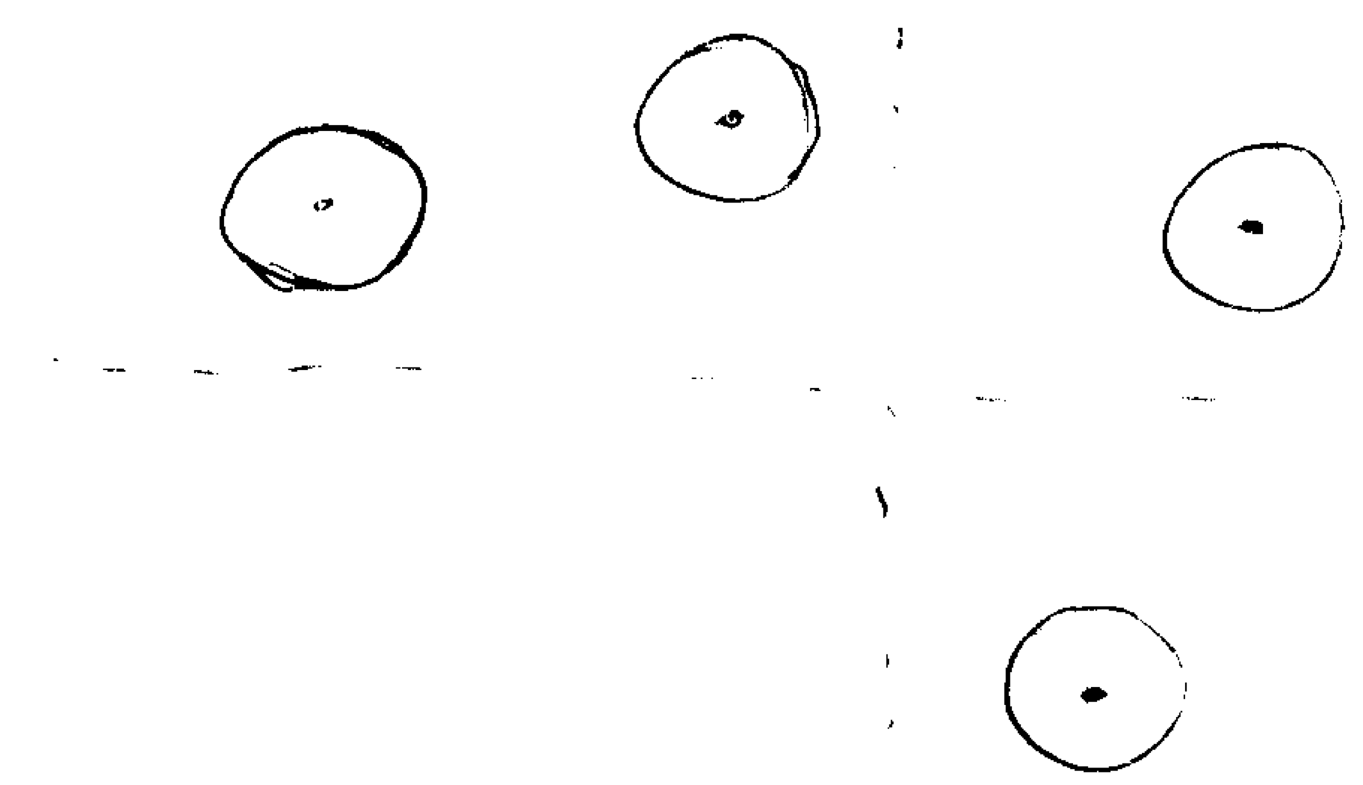
$$= \int \frac{[\mathcal{D}\varphi][\mathcal{D}g]}{V(\text{gauge})} e^{-S(\varphi)} \Bigg|_{\mathbb{C} - \bigcup_{i=1}^{N} D_i}$$

$\varphi = \varphi_i(\sigma)$ on ∂D_i

non-overlapping

D_i : Unit disk

A choice of world-sheet coordinates



(Punctured ~~disk~~ sphere)

Fixed once and for all in a given coordinate system.

(Choice of coordinate system that we use unit disk)

Question: ~~Gauge~~ ~~tot~~ Can we choose a standard form of the metric by gauge fixing?

~~Es~~ We can do it if there are no punctures.

$$ds^2 = |dz|^2$$

z complex coordinate on the plane. | Point at ∞ identified as we want a sphere

Residual gauge invariance

Define an $SL(2, \mathbb{C})$ transformation

$$z = \frac{az' + b}{cz' + d} \qquad a, b, c, d \text{ complex} \quad ad - bc = 1.$$

$1 \leftrightarrow 1$ trs on $\mathbb{C}^2 + \infty$.

$$|dz|^2 = |dz'|^2 \frac{1}{|cz'+d|^4}$$

Weyl trs: $ds^2 \to |cz'+d|^4 \, ds^2$.

$\Rightarrow$ New metric is $|dz'|^2$

Thus $z \to \dfrac{az'+b}{cz'+d}$ together with a Weyl **trs.** is a residual gauge trs. which has not been fixed by the gauge condition $g_{\alpha\beta} = \delta_{\alpha\beta}$.

# of parameters : 3

$(a, b, c, d) \to (\lambda a, \lambda b, \lambda c, \lambda d)$

$\to$ same trs.

Normalize: $ad - bc = 1$

Now return to the case with punctures:

Suppose we have N punctures.

We can always ~~choose~~ use 2-d g.c.t. to have them centered around some predetermined points $z_1, \ldots z_N$.

But after this the allowed group of diffeomorphisms is smaller, since we are only allowed diffeomorphisms which do not move the points $z_1, \ldots z_N$.

$\Rightarrow$ Cannot fix $g_{\alpha\beta} = \delta_{\alpha\beta}$.

$g_{\alpha\beta} \neq \delta_{\alpha\beta} \to$ inconvenient.

Instead we fix $g_{\alpha\beta} = \delta_{\alpha\beta}$ but now integrate over the locations of the disk centers $= z_i$.

(This should really be treated as hidden degrees of freedom of the metric).

Note: We only need to integrate over $(N-3)$ of these coordinates, since we have enough gauge degrees of freedom to fix 3 of the centres and the metric to $\delta_{\alpha\beta}$.

$$\Rightarrow K(\{\varphi_1(\sigma)\}, \cdots \{\varphi_N(\sigma)\})$$

$$= \int \prod_{i=4}^{N} d^2 z_i \; F(z_1, \cdots z_N) \int [\mathcal{D}\varphi] \, e^{-S(\varphi)} \Big|_{\mathbb{C} - \bigcup_{i=1}^{N} D_i}$$

$\varphi = \varphi_i(\sigma)$ on ∂D_i

D_i: Centered around z_i.

Integration measure (Jacobian of trs. from

$$\frac{[\mathcal{D}g]}{\mathrm{Vol}\,[\mathrm{Gauge}]} \to \prod_{i=4}^{N} d^2 z_i \Big)$$

$$F(z_1, \cdots z_N) = |z_1 - z_2|^2 \, |z_2 - z_3|^2 \, |z_1 - z_3|^2$$

Consistency check: The final should be ~~invariant~~ independent of the choice of z_1, z_2, z_3.

R-1

$\{|s_i\rangle\}$: Basis of physical states in CFT

$\Psi_{s_n}(\{\varphi(\sigma)\})$: The functional describing the wave function of the state $|s_n\rangle$

φ : set of basic fields describing the CFT

e.g. $X^0, \cdots X^{25}$ for bosonic string theory in Flat space.

S-matrix:

$$S(|s_1\rangle, \cdots |s_n\rangle)$$

$$= \int [\mathcal{D}\varphi_1(\sigma)] \int [\mathcal{D}\varphi_2(\sigma)] \cdots \int [\mathcal{D}\varphi_n(\sigma)]$$

$$\Psi_{s_1}(\{\varphi_1(\sigma)\}) \cdots \Psi_{s_n}(\{\varphi_n(\sigma)\})$$

$$\underbrace{K(\{\varphi_1(\sigma)\}, \cdots \{\varphi_n(\sigma)\})}$$

$$\|$$

$$\int \frac{[\mathcal{D}\varphi][\mathcal{D}g]}{\text{Vol (gauge)}} \; e^{-S(\varphi, g)} \Big|_{\substack{\mathbb{C} - \bigcup_i D_{z_i} \\ \varphi = \varphi_i(\sigma) \text{ on } \partial D_{z_i}}}$$

Unit disk around z_i

$$\|$$

$$\int d^2 z_4 \cdots d^2 z_N \; |z_1 - z_2|^2 \; |z_2 - z_3|^2 \; |z_1 - z_3|^2$$

$$\int [\mathcal{D}\varphi] \; e^{-S_\varphi} \Big|_{\substack{\mathbb{C} - \bigcup_{i=1}^{N} D_{z_i} \\ \varphi = \varphi_i(\sigma) \text{ on } \partial D_{z_i}}}$$

Next we need to find a convenient description of $\Psi_{\gamma_n}(\{\varphi(\sigma)\})$.

Suppose $\{\mathcal{O}_i(z,\bar{z})\}$ denote a complete set of local operators constructed out of φ and its derivatives at $(z,\bar{z})$.

Define the following ~~path~~ ~~integral~~ object:

$$\Phi_{\mathcal{O}_i}(\{\varphi_0(\sigma)\}) = \int [\mathcal{D}\varphi]\, e^{-S}\, \mathcal{O}_i(0)\Big|$$ Unit disk D around 0, $\varphi = \varphi_0(\sigma)$ on ∂D.

$\Rightarrow$ A functional $\Phi_{\mathcal{O}_i}(\{\varphi_0(\sigma)\})$.

- Interprete this as a wave-function of some state

$|\mathcal{O}_i\rangle$

Fundamental Postulate of CFT: There is $1 \leftrightarrow 1$ correspondence between the set of local operators & set of states.

$|\gamma_k\rangle \leftrightarrow V_{\gamma_k}(z,\bar{z})$ such that

---

$$\Psi_{\gamma_k}(\{\varphi_0(\sigma)\}) = \int \mathcal{D}\varphi\, e^{-S}\, V_{\gamma_k}(0)\Big|$$ Unit disk D around 0 with $\varphi = \varphi_0(\sigma)$ on ∂D.

Special case:

$$\Psi_{s_n}(\{\varphi_0(\sigma)\}) = \int \mathcal{D}\varphi\, e^{-S}\, V_{s_n}(0)\Big|$$ Unit disk D around 0 with $\varphi = \varphi_0(\sigma)$ on ∂D.

Now substitute into the expression for the S-matrix:

$$S(|s_1\rangle, \cdots |s_n\rangle) = \int d^2z_4 \cdots d^2z_N \, |z_1-z_2|^2 |z_2-z_3|^2 |z_1-z_3|^2$$

$$\int [\mathcal{D}\varphi_1(\sigma)]_{\partial D_1} \cdots [\mathcal{D}\varphi_N(\sigma)]_{\partial D_N} \int [\mathcal{D}\varphi]_{D_1} e^{-S(\varphi)|_{D_1}} V_{s_1}(z_1, \bar{z}_1)$$

$$\cdots \int [\mathcal{D}\varphi]_{D_N} e^{-S(\varphi)|_{D_N}} V_{s_N}(z_N, \bar{z}_N)$$

$$\int [\mathcal{D}\varphi]_{\mathbb{C} - \cup_i D_i} e^{-S(\varphi)|_{\mathbb{C} - \cup_i D_i}}$$

Note: $\varphi_i(\sigma)$: Boundary value for φ inside D_i as well as that of φ in $\mathbb{C} - \cup_i D_i$.

Net result: ~~Integ~~ Unrestricted integration over φ over the full complex plane.

$\Rightarrow$

$$S(|s_1\rangle, \cdots |s_n\rangle)$$

$$= \int d^2z_4 \cdots d^2z_N \, |z_1-z_2|^2 |z_2-z_3|^2 |z_1-z_3|^2$$

$$\int [\mathcal{D}\varphi]_{\mathbb{C}} e^{-S(\varphi)|_{\mathbb{C}}} V_{s_1}(z_1, \bar{z}_1) \cdots V_{s_N}(z_N, \bar{z}_N)$$

$$= \int d^2z_4 \cdots d^2z_N \, |(z_1-z_2)(z_2-z_3)(z_1-z_3)|^2$$

$$\underbrace{\langle V_{s_1}(z_1, \bar{z}_1) \cdots V_{s_N}(z_N, \bar{z}_N)\rangle}$$

Correlation function of vertex operators in CFT.

Note. For $N < 2$ the S-matrix element vanishes.

After fixing $z_1, \cdots z_N$, there is a residual gauge invariance.

---

e.g. $N = 2$ $\quad z_1, z_2$ fixed.

$$z \to z_1 + \frac{z - z_1}{z_2 - z_1 + (1-a)(z - z_2)} (z_2 - z_1)$$

a arbitrary complex #.

Integration range of a : $\mathbb{C}$ | Composition law: $a \circ b = ab$

Group invariant integration measure:

$$d^2a / |a|^2$$

$\Rightarrow$ Volume of residual gauge group

$$= \int d^2a / |a|^2 = \infty .$$

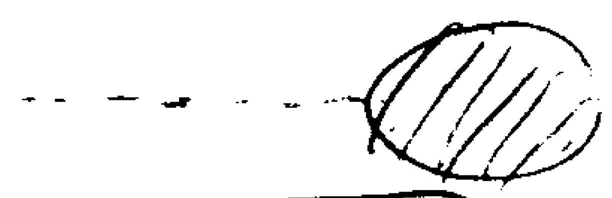
$\Rightarrow$ Amplitude $\to 0$ after being divided by the volume of the gauge group.

Is this expected?

① $N = 0$: no significance.

---

② $N = 1$: One point function vanish.



In the language of second quantized "string field theory" this would mean that we are quantizing the theory around a background which satisfies eqs. of motion.

③ N=2 : Two point function :

Recall the rules for computing S-matrix :

a) Take N-point Green's fn.

b) Multiply it by $(p_i^2 - m_i^2)$ for each external leg

c) Set $p_i^2 = m_i^2$ & ~~take~~

For N=2 :

a) Two pt. Green's fn. $= \frac{1}{p^2 - m^2}$

b) Multiply by $(p^2 - m^2)(p^2 - m^2)$.

$\Rightarrow (p^2 - m^2)$

c) set $p^2 = m^2 \Rightarrow 0$.

Thus on-shell 2-point S-matrix is expected to vanish as is the case here.

How do we find the state $|V\rangle$ corresponding to a given vertex operator V and vice versa?

Consider the correlation fn. of a set of local operators on the plane:

$$\langle \prod_i \mathcal{O}_i(z_i, \bar{z}_i) \rangle \equiv \int \mathcal{D}\varphi \, e^{-S(\varphi)} \prod_i \mathcal{O}_i(z_i, \bar{z}_i)$$

We want to give this an interpretation in terms of ~~~~ v.e.v.

In a normal field theory this would be

$$\langle 0 | T (\prod_i \hat{\mathcal{O}}_i (z_i, \bar{z}_i)) | 0 \rangle$$

Time ordering

Im(z) or Re(z).

In CFT we have other options

$w = \ln$

Defines a cylinder

Take (Im w) = time = $\ln |z|$

Time ordering $\leftrightarrow$ ordering in $|z|$

(radial ordering)

$$\langle \prod_i \mathcal{O}_i(z_i, \bar{z}_i) \rangle = \langle 0 | R(\hat{\mathcal{O}}_i (z_i, \bar{z}_i)) | 0 \rangle$$

operators acting on $\mathcal{H}$

~~Q. How do we find a vertex operator V_s corresponding to a given state $|s\rangle$?~~

~~Answer~~

~~To do this we need to study~~ some general properties of vertex operators:

Define coordinate $z = e^{\omega}$ $\quad \omega = \tau + i\sigma$

$\omega \to -\infty \Rightarrow z \to 0$

$\omega \to \infty \Rightarrow z \to \infty$

$\epsilon_n L_n$ generates: $\omega \to \omega + \epsilon_n e^{n\omega} \Rightarrow \delta z = \epsilon_n z^{n+1}$

$\bar{\epsilon}_n \bar{L}_n$ generates $\bar{\omega} \to \bar{\omega} + \bar{\epsilon}_n e^{n\bar{\omega}} \Rightarrow \delta \bar{z} = \bar{\epsilon}_n \bar{z}^{n+1}$

Consider a vertex operator $V(\omega, \bar{\omega})$

It is a primary vertex operator if under a conformal trs. $\omega \to f(\omega)$, $\bar{\omega} \to \bar{f}(\bar{\omega})$ V transforms to

$$\tilde{V}(\omega, \bar{\omega}) = (f'(\omega))^h (\bar{f}'(\bar{\omega}))^{\bar{h}} \, V(f(\omega), \bar{f}(\bar{\omega}))$$

$$f(\omega) = \omega + \epsilon_n e^{n\omega}$$

$$\tilde{V}(\omega, \bar{\omega}) - V(\omega, \bar{\omega}) = (1 + n\epsilon_n e^{n\omega})^h (1 + \bar{\epsilon}_n e^{n\bar{\omega}})^{\bar{h}} \, V(\omega + \epsilon_n e^{n\omega}, \bar{\omega} + \bar{\epsilon}_n e^{n\bar{\omega}}) - V(\omega, \bar{\omega})$$

$$\underbrace{\qquad}_{\parallel}$$

$$[\epsilon_n L_n + \bar{\epsilon}_n \bar{L}_n, V(\omega, \bar{\omega})]$$

$$\Rightarrow [L_n, V(\omega, \bar{\omega})] = n h e^{n\omega} V(\omega, \bar{\omega}) + e^{n\omega} \partial_\omega V(\omega, \bar{\omega})$$

Define a new coordinate system:

$$z = e^{\omega}$$

~~V_z~~ ~~...~~ $V_z(z,\bar{z}) = \left(\frac{d\omega}{dz}\right)^h \left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{\bar{h}} V(\omega,\bar{\omega})$

$$= e^{-\omega h} e^{-\bar{\omega}\bar{h}} V(\omega,\bar{\omega})$$

$\Rightarrow [L_n, V_z] = e^{-\omega h} e^{-\bar{\omega}\bar{h}} [n h e^{n\omega} e^{+\omega h} e^{\bar{\omega}\bar{h}} V_z$

$+ e^{n\omega} \partial_\omega (e^{\omega h} e^{\bar{\omega}\bar{h}} V_z)]$

$= \{(n+1) h z^n V_{(z,\bar{z})} + z^{n+1} \partial_z V\}$

Similarly,

$[\bar{L}_n, V_z] = [(n+1)\bar{h}\bar{z}^n V(z,\bar{z}) + \bar{z}^{n+1} \partial_{\bar{z}} V]$

Now consider the state $|V\rangle = V_z(0)|0\rangle$

$L_n|V\rangle = [L_n, V(0)]|0\rangle$ for $n \geq 0$

$= 0$ for $n > 0$

$= h\, V(0)|0\rangle = h|V\rangle$ for $n = 0$

$\Rightarrow$ $L_n|V\rangle = 0$ for $n > 0$

$L_0|V\rangle = h|V\rangle$

Similarly: $\bar{L}_n|V\rangle = 0$ for $n > 0$

$\bar{L}_0|V\rangle = \bar{h}|V\rangle$

$\Rightarrow$ Primary states with weight $(h,\bar{h})$

Now suppose we have a correls fn:

$$\int [\mathcal{D}\varphi]\, e^{-S(\varphi)}\, V(0) \prod_i O_i(z_i, \bar z_i) \equiv \langle \prod_i O_i(z_i,\bar z_i)\, V(0) \rangle$$

On the one hand:

$$\langle \prod_i O_i(z_i, \bar z_i)\, V(0) \rangle = \langle 0 | R(\prod_i \hat O_i(z_i, \bar z_i)\, \hat V(0)) | 0 \rangle$$

$$= \langle 0 | R(\prod_i \hat O_i(z_i, \bar z_i))\, \hat V(0) | 0 \rangle$$

On the other hand:

D_0

$$\int [\mathcal{D}\varphi]\, e^{-S[\varphi]} \prod_i O_i(z_i, \bar z_i)\, V(0)$$

$$= \int [\mathcal{D}\varphi_0(\sigma)] \underbrace{\int [\mathcal{D}\varphi]_{D_0}\, e^{-S[\varphi]}\Big|_{\substack{D_0 \\ \varphi = \varphi_0 \text{ on } \partial D_0}}^{V(0)}}_{\Psi_V(\{\varphi_0(\sigma)\})}$$

$$\int [\mathcal{D}\varphi]\, e^{-S[\varphi]} \prod_i O_i(z_i, \bar z_i) \Big|_{\substack{\mathbb{C} - D_0 \\ \varphi = \varphi_0 \text{ on } \partial D_0}}$$

$$= \langle 0 | R(\prod_i \hat O_i(z_i, \bar z_i)) | V \rangle$$

Compare both sides:

$$\Rightarrow \quad | V \rangle = \hat V(0) | 0 \rangle$$

Physical states:

$\rightarrow$ Primary states of weight $(1,1)$

Primary
$\Rightarrow$ Vertex operators of weight $(1,1)$

The S-matrix:

$$S(V_1, \cdots V_n)$$

$$= \int d^2 z_4 \cdots d^2 z_N \, |z_1 - z_2|^2 |z_2 - z_3|^2 \, |z_1 - z_3|^2$$

$$\left\langle \prod_i V_i(z_i, \bar{z}_i) \right\rangle$$

We shall now find the vertex operators for some physical states explicitly in 26-dimensional bosonic string theory in flat space-time.

Mode expansion of X^μ.

$$X^\mu(\sigma,\tau) = \sum_{n\neq 0} \phi^\mu_n(\tau)\, e^{in\sigma} + x^\mu(\tau)$$

---

① Express $\phi^\mu_n(\tau)$ in terms of $\alpha^\mu_n(\tau)$, $\tilde\alpha^\mu_n(\tau)$

---

② Use eqs. of motion to express $\alpha^\mu_n(\tau)$, $\tilde\alpha^\mu_n(\tau)$, $x^\mu(\tau)$ in terms of $\alpha^\mu_n(0) = \alpha^\mu_n$, $\tilde\alpha^\mu_n(0)$, $x^\mu(0)$, $p^\mu(0)$

---

③ Make Euclidean continuation $\tau \to i\tau$

④ $w \equiv \tau - i\sigma$

$\Rightarrow$

$$X^\mu(\sigma,\tau) = x^\mu + \frac{\alpha'}{2} p^\mu (w+\bar w) + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{1}{n}\left(\alpha^\mu_n e^{-nw} + \tilde\alpha^\mu_n e^{-n\bar w}\right)$$

$$= x^\mu + \frac{\alpha'}{2} p^\mu (\ln z + \ln \bar z) + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{1}{n}\left(\alpha^\mu_n z^{-n} + \tilde\alpha^\mu_n \bar z^{-n}\right)$$

Normal ordering x^μ to the left of p_μ since $p_\mu|0\rangle = 0$

$$:e^{ik_\mu \hat X^\mu(z,\bar z)}:$$

$$= e^{ik_\mu \hat X^\mu_<(z,\bar z)}\; e^{ik_\mu \hat X^\mu_0(z,\bar z)}\; e^{ik_\mu \hat X^\mu_>(z,\bar z)}$$

creation operators

$$e^{ik_\mu \hat x^\mu}\, e^{i\frac{\alpha'}{2}\hat p^\mu \ln(z\bar z)}$$

annihilation operators

$$:e^{ik_\mu \hat{X}^\mu(z,\bar{z})}:\Big|_{z=0}|0\rangle$$

$$= e^{ik_\mu \hat{x}^\mu}|0\rangle = |\{k\}\rangle$$

Note: $\hat{X}_>(z,\bar{z})|0\rangle = 0$, $\hat{p}^\mu|0\rangle = 0$

$\hat{X}_<(z,\bar{z}) = 0$ at $z=0$

$\ni$ $e^{ik_\mu \hat{X}^\mu(z,\bar{z})}$ is the vertex operator associated with a tachyon state of momentum k.

$e^{ik_\mu x^\mu}$ is not allowed since it is not local.

$$\partial X^\mu = \frac{\alpha'}{2} p^\mu z^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1}$$

$$\bar{\partial} X^\mu = \frac{\alpha'}{2} p^\mu \bar{z}^{-1} - i\sqrt{\frac{\alpha'}{2}}\sum_n \tilde{\alpha}_n^\mu \bar{z}^{-n-1}$$

Consider

$$S_{\mu\nu} :\partial\hat{X}^\mu \bar{\partial}\hat{X}^\nu e^{ik\cdot\hat{X}}:$$

$$= S_{\mu\nu}\Big(\partial\hat{X}^\mu_< \bar{\partial}\hat{X}^\nu_< :e^{ik\cdot\hat{X}}: + \partial\hat{X}^\mu_< :e^{ik\cdot\hat{X}}: \bar{\partial}\hat{X}^\nu_>$$

$$+ \bar{\partial}\hat{X}^\nu_< :e^{ik\cdot\hat{X}}: \partial\hat{X}^\mu_> + :e^{ik\cdot\hat{X}}: \partial\hat{X}^\mu_> \bar{\partial}\hat{X}^\nu_>$$

$$+ e^{ik\cdot\hat{X}_<} e^{ik\cdot\hat{x}} e^{i\frac{\alpha'}{2}k\cdot\hat{p}}\left(\frac{\alpha'}{2}\hat{p}^\mu z^{-1}\right)\left(\frac{\alpha'}{2}\hat{p}^\nu\bar{z}^{-1}\right) e^{ik\cdot\hat{X}_>}$$

$$+ \partial\hat{X}^\mu_< e^{ik\cdot\hat{X}_<} e^{ik\cdot\hat{x}} e^{i\frac{\alpha'}{2}k\cdot\hat{p}}\left(\frac{\alpha'}{2}\hat{p}^\nu\bar{z}^{-1}\right) e^{ik\cdot\hat{X}_>}$$

$$+ \cdots \Big)$$

$$\zeta_{\mu\nu} :\partial\hat{X}^\mu \bar\partial\hat{X}^\nu e^{ik\cdot\hat{X}}:(z=0,\bar z=0)\,|0\rangle$$

$$= \zeta_{\mu\nu}\, \partial\hat{X}^\mu_<(0)\,\bar\partial\hat{X}^\nu_<(0)\underbrace{:e^{ik\cdot\hat{X}}:(0)\,|0\rangle}$$

$$\zeta_{\mu\nu}\left(-\frac{\alpha'}{2}\right)\alpha^\mu_{-1}\tilde\alpha^\nu_{-1}|k\rangle$$

Thus the vertex operators for the massless states are given by:

$$\zeta_{\mu\nu} :\partial\hat{X}^\mu \bar\partial\hat{X}^\nu : e^{ik\cdot\hat{X}}$$

Physical state condition:

$$\underset{L_1}{k^\mu\zeta_{\mu\nu}} = 0 = \underset{\tilde L_1}{k^\nu\zeta_{\mu\nu}}, \qquad \underset{L_0,\ \tilde L_0}{k^2=0}.$$

We shall now try to compute amplitudes using these vertex operators.

N-tachyon amplitude:

$$A(k_1, \cdots k_N)$$

$$= \int \prod_{i=4}^{N} d^2 z_i\, |z_1-z_2|^2\, |z_2-z_3|^2\, |z_1-z_3|^2$$

$$\left\langle \prod_i :e^{ik_i\cdot X_i(z,\bar z)}: \right\rangle$$

Compute this using path integral.

Strategy :

① Evaluate

$$\langle e^{i \int J_\mu(z,\bar z)\, X^\mu(z,\bar z)\, d^2 z} \rangle$$

$$= \int [\mathcal{D} X^\mu]\, e^{\left[-\frac{1}{4\pi\alpha'} \int d^2 z\, \partial_\alpha X^\mu\, \partial_\alpha X^\nu\, \delta_{\mu\nu} + i \int J_\mu(z,\bar z)\, X^\mu(z,\bar z)\, d^2 z\right]}$$

② Substitute :

$$J_\mu(z,\bar z) = \sum_\ell k_{\ell\mu}\, \underbrace{\delta(z - z_\ell)\, \delta(\bar z - \bar z_\ell)}$$

$\int d^2 z\, \delta(z - z_\ell)\, \delta(\bar z - \bar z_\ell) = 1$

so that $i \int J_\mu(z,\bar z)\, X^\mu(z,\bar z)\, d^2 z$

$$= i \sum_\ell k_\ell \cdot X^\ell(z_\ell, \bar z_\ell)$$

① The evaluation of this integral is standard :

$$\langle e^{i \int J_\mu(z,\bar z)\, X^\mu(z,\bar z)\, d^2 z} \rangle$$

$$= e^{-\frac{1}{2} \int d^2 z\, d^2 z'\, \Delta_{\mu\nu}(z,z')\, J^\mu(z)\, J^\nu(z')}$$

$\Delta_{\mu\nu}(z,z') = -\alpha' \ln|z-z'|\, \delta_{\mu\nu} \rightarrow$ Propagator of 2-dim free scalar fields.

② $$\langle \prod_\ell e^{i k_\ell \cdot X^\ell(z,\bar z)} \rangle = e^{\frac{1}{2}\alpha' \sum_{\ell,\ell'} k_{\ell\mu}\, k_{\ell'\nu}\, \ln|z_\ell - z_{\ell'}|}$$

Normal ordering: Removes the $l = l'$ terms.

$$\langle \prod_l : e^{i k_l \cdot X^l(z,\bar z)} : \rangle$$

$$= e^{\alpha' \sum_{l<l'} k_l \cdot k_{l'} \ln |z_l - z_{l'}|}$$

$$= \prod_{l<l'} |z_l - z_l'|^{\alpha' k_l \cdot k_{l'}}$$

$\Rightarrow$ N tachyon amplitude:

$A(k_1, \ldots k_N)$

$$= \int \prod_{i=4}^{N} d^2 z_i \; |z_1 - z_2|^2 \, |z_2 - z_3|^2 \, |z_1 - z_3|^2$$

$$\prod_{\substack{l \neq l' \\ 1 \leq l \leq N \\ 1 \leq l' \leq N}} |z_l - z_{l'}|^{\frac{1}{2}\alpha' k_l \cdot k_{l'}}$$

Check that the answer is independent of z_1, z_2, z_3

$SL(2,\mathbb{R})$ trs. $y_i \equiv \dfrac{a z_i + b}{c z_i + d}$

$$d^2 y_i = d^2 z_i \frac{1}{|c z_i + d|^4}$$

$$y_i - y_j = (z_i - z_j) \frac{1}{(c z_i + d)(c z_j + d)}$$

$A(k_1, \dots k_N)$

$$= \int \prod_{i=4}^{N} d^2 y_i \prod_{i=4}^{N} |c z_i + d|^4 \; |y_1 - y_2|^2 |y_2 - y_3|^2 |y_1 - y_3|^2$$

$$|c z_1 + d|^4 \; |c z_2 + d|^4 \; |c z_3 + d|^4$$

$$\prod_{l<l'} |y_l - y_{l'}|^{\alpha' k_l \cdot k_{l'}} \; \prod_{l<l'} |(c z_l + d)(c z_{l'} + d)|^{\alpha' k_l \cdot k_{l'}}$$

$$\prod_l |c z_l + d|^{\sum_{l' \neq l} \alpha' k_l \cdot k_l'}$$

$$= \prod_l |c z_l + d|^{\alpha' k_l \cdot (-k_l^*)}$$

$$\prod_l |c z_l + d|^{-4}$$

$\alpha' k_l \cdot k_l = 4$
$\to$ Mass shell constraint.

$\Rightarrow A(k_1, \dots k_N)$

$$= \int \prod_{i=4}^{N} d^2 y_i \; |y_1 - y_2|^2 |y_1 - y_3|^2 |y_1 - y_3|^2$$

$$\prod_{l<l'} |y_l - y_{l'}|^{\alpha' k_l \cdot k_{l'}}$$

Note: y_i's for $i = 4, \dots N$ are dummy variables

$\Rightarrow$ can be changed to z_i's.

But $(y_1, y_2, y_3) \neq (z_1, z_2, z_3)$

Adjusting a, b, c, d we can choose y_1, y_2, y_3 to be 3 arbitrary points.

$\Rightarrow$ The answer is independent of the choice of y_1, y_2, y_3

~~To~~ Permutation symmetry of $1, \ldots N$.

$$v = \frac{py+q}{ry+s}, \qquad ps - qr = 1$$

Make p, q, r, s depend on y_4, y_5, y_6 s.t.

$$v_4 = \frac{pz_4+q}{rz_4+s} = 0, \qquad v_5 = 1, \qquad v_6 = \infty.$$

$$v_1 = \frac{q}{s}, \qquad v_2 = \frac{p+q}{r+s}, \qquad v_3 = \frac{p}{r}$$

now depend on y_4, y_5, y_6

Trade in $d^2y_1 \ldots d^2y_N \to d^2v_1\, d^2v_2\, d^2v_3$
$d^2v_7 \ldots d^2v_N$

Ex. Check that the integrand in terms of v_i have the same for as in terms of y_i except for the replacement

$$|y_1-y_2|^2\,|y_2-y_3|^2\,|y_1-y_3|^2 \to |v_4-v_5|^2\,|v_5-v_6|^2\,|v_4-v_6|^2$$

Convenient choice

$y_1 = 0, \quad y_2 = 1, \quad y_3 = \infty$

Example $\underset{\sim}{N=4}$

$A(k_1, \; k_N)$

$= \int d^2 z_4 \; (0-1)^2 \, |0 - y_3|^2 \, |1 - y_3|^2$

$|0 - z_4|^{\alpha' k_1 \cdot k_4} \; |1 - z_4|^{\alpha' k_2 \cdot k_4} \; |y_3 - z_4|^{\alpha' k_3 \cdot k_4}$

$|0-1|^{\alpha' k_1 \cdot k_2} \; |0 - y_3|^{\alpha' k_1 \cdot k_3} \; |1 - y_3|^{\alpha' k_2 \cdot k_3}$

As $y_3 \to \infty$

$\int d^2 z_4 \; |y_3|^{4 + \alpha' k_3 \cdot (k_1 + k_2 + k_4)} \qquad 4 + \alpha' k_3 \cdot (k_1+k_2+k_4) = 4 - \alpha' k_3^2 = 0$

$|z_4|^{\alpha' k_1 \cdot k_4} \; |1 - z_4|^{\alpha' k_2 \cdot k_4}$

$= \int d^2 z \; |z|^{\alpha' k_1 \cdot k_4} \; |1 - z|^{\alpha' k_2 \cdot k_4}$

$\int d^2 z \; |z|^{2a-2} \, |1-z|^{2b-2}$

$= 2\pi \dfrac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}$

$c = 1 - a - b$

Note : $\alpha' k_1 \cdot k_4 = \frac{\alpha'}{2}\{(k_1 + k_4)^2 - k_1^2 - k_4^2\}$ could be < -2 in some range of momenta.

$\Rightarrow$ The integral diverges from $z \simeq 0$

Similarly there can be divergences from 1 and ∞ for other ranges of momenta

---

We need to define the integral in a range of momenta where it is finite and then do analytic continuation.

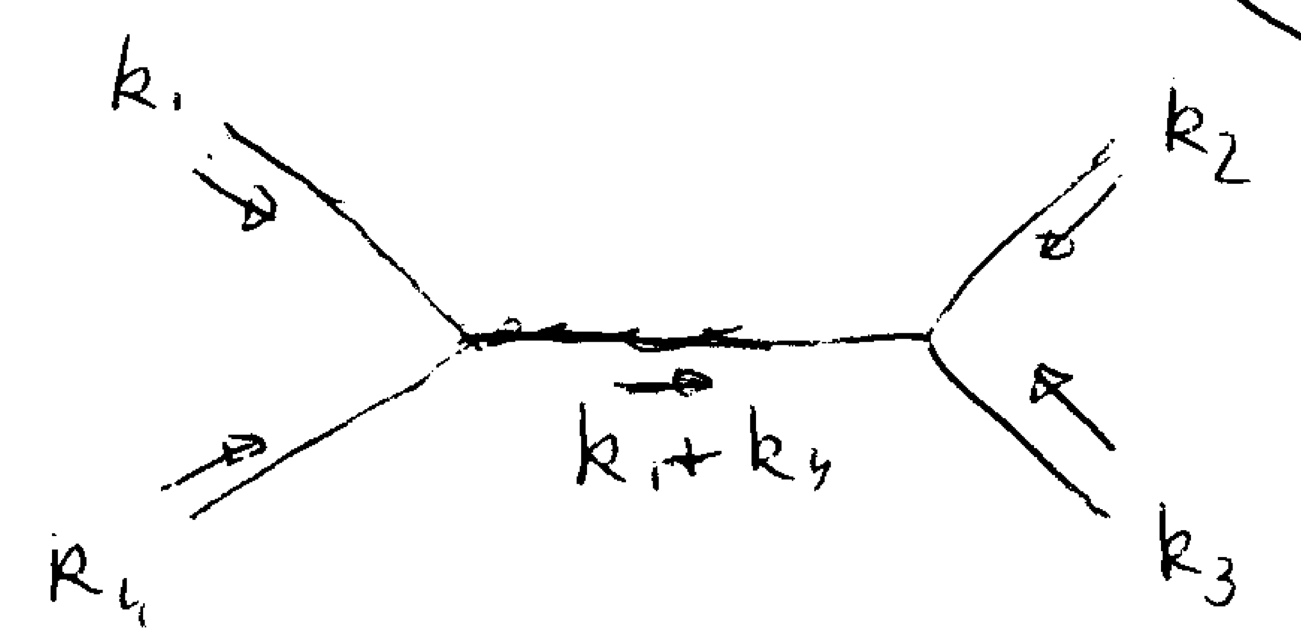
~~For amplitude~~ ~~involving~~ In doing this we shall get poles at

$$\alpha' k_1 \cdot k_4 = -2$$

$$= \frac{\alpha'}{2}\left\{(k_1+k_4)^2 - \underset{4/\alpha'}{k_1^2} - \underset{4/\alpha'}{k_4^2}\right\} = \frac{\alpha'}{2}(k_1+k_4)^2 - 4$$

$$\Rightarrow (k_1+k_4)^2 = \frac{4}{\alpha'}$$

↳ Tachyon pole



$\Rightarrow$ Pole at $(k_1+k_4)^2 = -m_T^2 = \frac{4}{\alpha'}$

The integral also has poles at

$$\alpha' k_1 \cdot k_4 = -4, -6, -8, \ldots$$

$$\Rightarrow (k_1+k_4)^2 = 0, -\frac{4}{\alpha'}, -\frac{8}{\alpha'}, \ldots$$

$$\Rightarrow (m_{\text{intermediate}})^2 = 0, \frac{4}{\alpha'}, \frac{8}{\alpha'}, \ldots$$

massless states — 1st massive states — second massive states etc.

Similarly, divergence from $z \simeq 1$ produces the t channel poles (in $(k_2+k_4)^2 = (k_1+k_3)^2$) and divergence from ∞ produces the u-channel poles (in $(k_1+k_2)^2$)

The same amplitude produces the poles in all three channels

(Unlike in field theory where there are separate Feynman diagrams producing poles in the s, t and u channels)

More generally in an N-point amplitude the singularities as $z_i \to z_j$ produces the poles in $(k_i + k_j)^2$, as $(z_i - z_j)$ has a power of $\alpha' k_i \cdot k_j$

For $1 \le i, j \le 3$, the pole comes from all points other than z_i and z_j coming close together. ~~Close together. going far away~~ → SL(2,C) bringing z_i & z_j together

The origin of these singularities can be traced to the singularities in the operator product expansion of vertex operators in the CFT.

$$V_{r_i}(z, \bar{z})\, V_{r_j}(w, \bar{w}) = \sum_{k} C^{(z, \bar{z}, w, \bar{w})}_{r_i r_j r_k} V_{r_k}(w, \bar{w})$$

sum over complete set of operators

→ A relation which is supposed to hold inside any correlation function

The dependence of C on $z - w$, $\bar{z} - \bar{w}$ can be found from the correlation functions and the ...

Today we shall try to reinterprete these results in terms of singularities in operator product expansion.

But first we need to study carefully the origin of the singularities.

$$\int_0^R d^2z \, |z|^{-n+2\epsilon} = \underset{\text{singular}}{f^{(s)}(n,\epsilon)} + \underset{\text{regular}}{f^{(r)}(R,n)} .$$

$z = \lambda y$

$$\text{l.h.s.} = \lambda^{2-n+2\epsilon} \int_0^{R/\lambda} d^2y \, |y|^{-n+2\epsilon}$$

$$= \lambda^{2-n+2\epsilon} \left(\int_0^R \cdots - \int_{R/\lambda}^R \cdots \right)$$

$$= \lambda^{2-n+2\epsilon} \left(f^{(s)}(n,\epsilon) + f^{(r)}(R,n) \right) + \lambda^{2-n+2\epsilon} \phi^{(r)}(n,R,\lambda)$$

$$\Rightarrow \left(\lambda^{2-n+2\epsilon} - 1\right) f^{(s)}(n,\epsilon) + \left(\lambda^{2-n+2\epsilon} - 1\right) f^{(r)}(R,n)$$
$$+ \lambda^{2-n+2\epsilon} \phi^{(r)}(n,R,\lambda) = 0$$

Singular part must vanish by itself.

$$\Rightarrow \quad f^{(s)}(n,\epsilon) = 0 \quad \text{unless} \quad n = 2 .$$

$$f^{(s)}(2,\epsilon) =_{\text{Sing}} \int_0^R \underset{2\,dx\,dy \,=\, 4\pi r\,dr}{d^2z} \, |z|^{-2+2\epsilon}$$

$$= 4\pi \int_0^R r\,dr \, r^{-2+2\epsilon} = \frac{4\pi}{2\epsilon} r^{-1+2\epsilon}$$

Use this to check the expression:

$$\int d^2 z \, |z|^{2a-2} \, |1-z|^{2b-2} = \frac{2\pi \, \Gamma(a) \, \Gamma(b) \, \Gamma(1-a-b)}{\Gamma(1-a) \, \Gamma(1-b) \, \Gamma(b+a)}$$

Take $a = -n + \epsilon$

$$\text{l.h.s.} = \int d^2 z \, |z|^{-2n+2\epsilon-2} \, |1-z|^{2b-2}$$

$$\Downarrow$$

$$(1-z)^{b-1} \, (1-z^*)^{b-1}$$

$$\Downarrow$$

$$\sum_m \frac{z^m}{m!} \frac{\partial^m}{\partial z^m} (1-z)^{b-1} \Big|_{z=0} \; \sum_p \frac{(z^*)^p}{p!} \frac{\partial^p}{\partial z^{*p}} (1-z^*)^{b-1} \Big|_{z^*=0}$$

Only $m = p = n$ term will contribute to sing.

$$\text{l.h.s.} = \Big(\underbrace{\int d^2 z \, |z|^{2\epsilon-2}}_{(2\pi/\epsilon)} \Big) \sum_n \frac{1}{n!} \{ (b-1)(b-2) \cdots (b-n) \}^2$$

$$\text{r.h.s.} = 2\pi \, \frac{\Gamma(-n+\epsilon) \, \Gamma(b) \, \Gamma(1+n-b-\epsilon)}{\Gamma(1+n-\epsilon) \, \Gamma(1-b) \, \Gamma(-n+\epsilon+b)}$$

$$= \frac{2\pi}{\epsilon} \left\{ \frac{1}{n!} (b-1) \cdots (b-n) \right\}^2 + \text{finite}$$

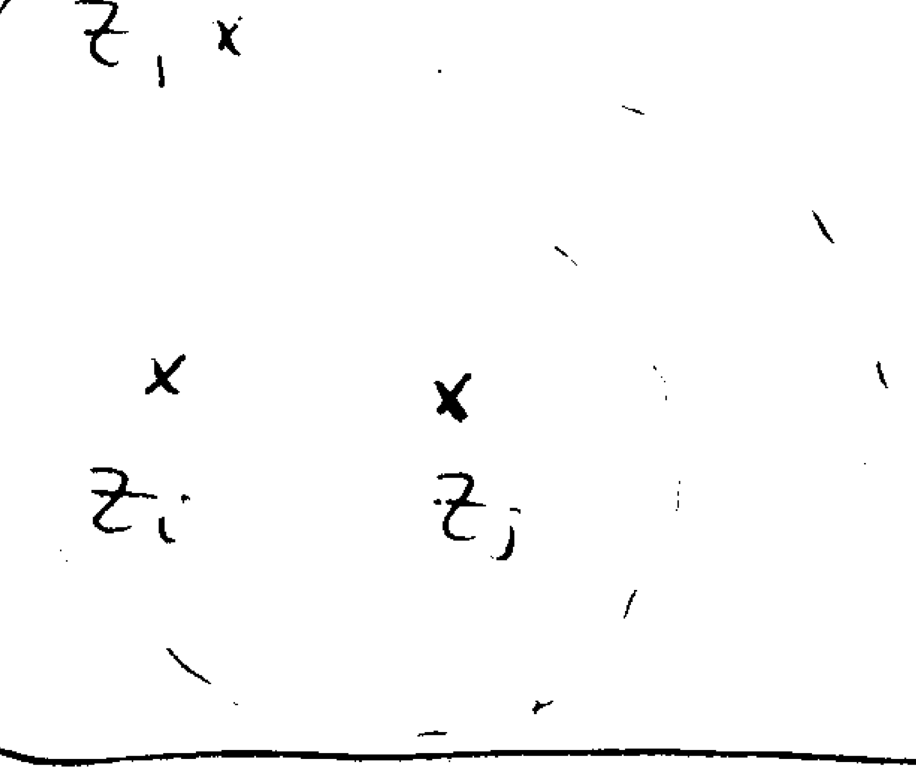
(check)

In general we need to study singularities of

$$\left\langle \prod_i V_{r_i}(z_i, \bar{z}_i) \right\rangle \quad \text{as} \quad z_i \to z_j$$

Take unit disk around z_j

$\Rightarrow D_{z_j}$



$$\left\langle \prod_k V_{r_k}(z_k, \bar{z}_k) \right\rangle$$

$$= \int \mathcal{D}\varphi \; e^{-S(\varphi)} \prod_k V_{r_k}(z_k, \bar{z}_k)$$

$$= \int \mathcal{D}\varphi_0(\sigma) \int \mathcal{D}\varphi \, e^{-S(\varphi)} V_{r_i}(z_i, \bar{z}_i) V_{r_j}(z_j, \bar{z}_j) \Big|_{\substack{D_{z_j} \\ \varphi = \varphi_0 \text{ on } \partial D_{z_j}}}$$

$$\int \mathcal{D}\varphi \; e^{-S(\varphi)} \prod_{k \neq i,j} V_{r_k}(z_k, \bar{z}_k) \Big|_{\substack{\mathbb{C} - D_{z_j} \\ \varphi = \varphi_0 \text{ on } \partial D_{z_j}}}$$

$$\Psi_{r_i, r_j, z_i, \bar{z}_i, z_j, \bar{z}_j}(\varphi_0(\sigma))$$

$$= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j) \, \Psi_{r_k}(\varphi_0(\sigma))$$

$$= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j)$$

Wave fn. associated with state $|r_k\rangle = V_{r_k}(0)|0\rangle$

$$\int \mathcal{D}\varphi \, e^{-S(\varphi)} V_{r_k}(z_j, \bar{z}_j) \Big|_{\substack{D_{z_j} \\ \varphi = \varphi_0 \text{ on } \partial D_{z_j}}}$$

Final result:

$$\frac{\left\langle \prod_k V_{r_k}(z_k, \bar{z}_k) \right\rangle}{= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j)}$$

$$\frac{\int \mathcal{D}\varphi_0(\sigma) \int \mathcal{D}\varphi \, e^{-S(\varphi)} \prod_{k \neq i,j} V_{r_k}(z_k, \bar{z}_k) \Big|_{\substack{\mathbb{C} - D_{z_j} \\ \varphi = \varphi_0(\sigma) \text{ on } \partial D_{z_j}}} \int \mathcal{D}\varphi \, e^{-S(\varphi)} V_{r_k}(z_j, \bar{z}_j) \Big|_{\substack{D_{z_j} \\ \varphi = \varphi_0(\sigma) \text{ on } \partial D_{z_j}}}}{= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j)}$$

$$\frac{\int \mathcal{D}\varphi \, e^{-S(\varphi)} \prod_{l \neq i,j} V_{r_l}(z_l, \bar{z}_l) \, V_{r_k}(z_j, \bar{z}_j)}{= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j)}$$

$$\left\langle \prod_{l \neq i,j} V_{r_l}(z_l, \bar{z}_l) \, V_{r_k}(z_j, \bar{z}_j) \right\rangle$$

Net result:

$$V_{r_i}(z_i, \bar{z}_i) \, V_{r_j}(z_j, \bar{z}_j)$$

$$= \sum_{r_k} C_{r_i r_j r_k}(z_i, \bar{z}_i, z_j, \bar{z}_j) \, V_{r_k}(z_j, \bar{z}_j) \text{ inside } \langle\rangle$$

$$\Downarrow$$

$$z_i - z_j, \; \bar{z}_i - \bar{z}_j$$

New coordinate $z \to f(z)$.

~~In vertex of in the new coordinate~~

~~Chose~~ choose $f(z) = \lambda z$.

$$V_{r_i}(z, \bar{z}) \to \lambda^{h_{r_i}} \bar{\lambda}^{\bar{h}_{r_i}} V_{r_i}(\lambda z, \lambda \bar{z})$$

$$\lambda^{h_{r_i}} \bar{\lambda}^{\bar{h}_{r_i}} V_{r_i}(\lambda z, \lambda \bar{z}) \lambda^{h_{r_j}} \bar{\lambda}^{\bar{h}_{r_j}} V_{r_j}(\lambda w, \lambda \bar{w})$$

$$= \sum_k C_{r_i r_j r_k}(z-w, \bar{z}-\bar{w}) \lambda^{h_{r_k}} \bar{\lambda}^{\bar{h}_{r_k}} V_{r_k}(\lambda w, \lambda \bar{w})$$

$$\Rightarrow V_{r_i}(\lambda z, \lambda \bar{z}) V_{r_j}(\lambda w, \lambda \bar{w})$$

$$= \sum_k C_{r_i r_j r_k}(z-w, \bar{z}-\bar{w}) \lambda^{h_{r_k} - h_{r_i} - h_{r_j}} \bar{\lambda}^{\bar{h}_{r_k} - \bar{h}_{r_i} - \bar{h}_{r_j}} V_{r_k}(\lambda w, \lambda \bar{w}).$$

On the other hand

$$V_{r_i}(\lambda z, \lambda \bar{z}) V_{r_j}(\lambda w, \lambda \bar{w})$$

$$= \sum_k C_{r_i r_j r_k}(\lambda(z-w), \lambda(\bar{z}-\bar{w})) V_{r_k}(\lambda w, \lambda \bar{w})$$

$$\Rightarrow C_{r_i r_j r_k}(\lambda(z-w), \lambda(\bar{z}-\bar{w}))$$

$$= \lambda^{h_{r_k} - h_{r_i} - h_{r_j}} \bar{\lambda}^{\bar{h}_{r_k} - \bar{h}_{r_i} - \bar{h}_{r_j}} C_{r_i r_j r_k}((z-w), \bar{z}-\bar{w})$$

$$\Rightarrow C_{r_i r_j r_k}(z-w, \bar{z}-\bar{w}) = C^{(0)}_{r_i r_j r_k} (z-w)^{h_{r_k} - h_{r_i} - h_{r_j}} (\bar{z}-\bar{w})^{\bar{h}_{r_k} - \bar{h}_{r_i} - \bar{h}_{r_j}}$$

$$\Rightarrow V_{r_i}(z,\bar z)\, V_{r_j}(w,\bar w)$$

$$= \sum_k C^{(0)}_{r_i r_j r_k} (z-w)^{h_{r_k}-h_{r_i}-h_{r_j}} (\bar z-\bar w)^{\bar h_{r_k}-\bar h_{r_i}-\bar h_{r_j}}$$

involve $(\text{momentum})^2$

~~Thus~~ In the N-point amplitude

$$\int \prod_{i=4}^{N} d^2 z_i \; |z_1-z_2|^2 \, |z_2-z_3|^2 \, |z_1-z_3|^2$$

$$\left\langle \prod_{i=1}^{N} V_{s_i}(z_i,\bar z_i) \right\rangle$$

Note: only $h_{r_k} = \bar h_{r_k} = 1$ states contribute to poles in momenta (in critical)
$\Rightarrow$ physical states (string theory)

We can use OPE in any order.

e.g. $V_{s_i} V_{s_j}$ first

$\Rightarrow$ singularities in $(z_i - z_j)$

$\Rightarrow$ Poles in $(k_i + k_j)^2$

Then take the OPE of the result with V_{s_l} $\rightarrow$ singularity in $(z_j - z_l)$

$\Rightarrow$ Poles in $(k_i + k_j + k_l)^2$ etc.

But in the same amplitude, taking the OPE in a different order, we can recover the singularities in ~~other~~ $(z_{i'} - z_{j'})$ for other i', j' and hence poles in $(k_{i'} - k_{j'})^2$

Note: Only $h_{r_k} - h_{s_i} - h_{s_j} = -1$, $\bar{h}_{r_k} - \bar{h}_{s_i} - \bar{h}_{s_j} = -1$ states contribute to the pole.

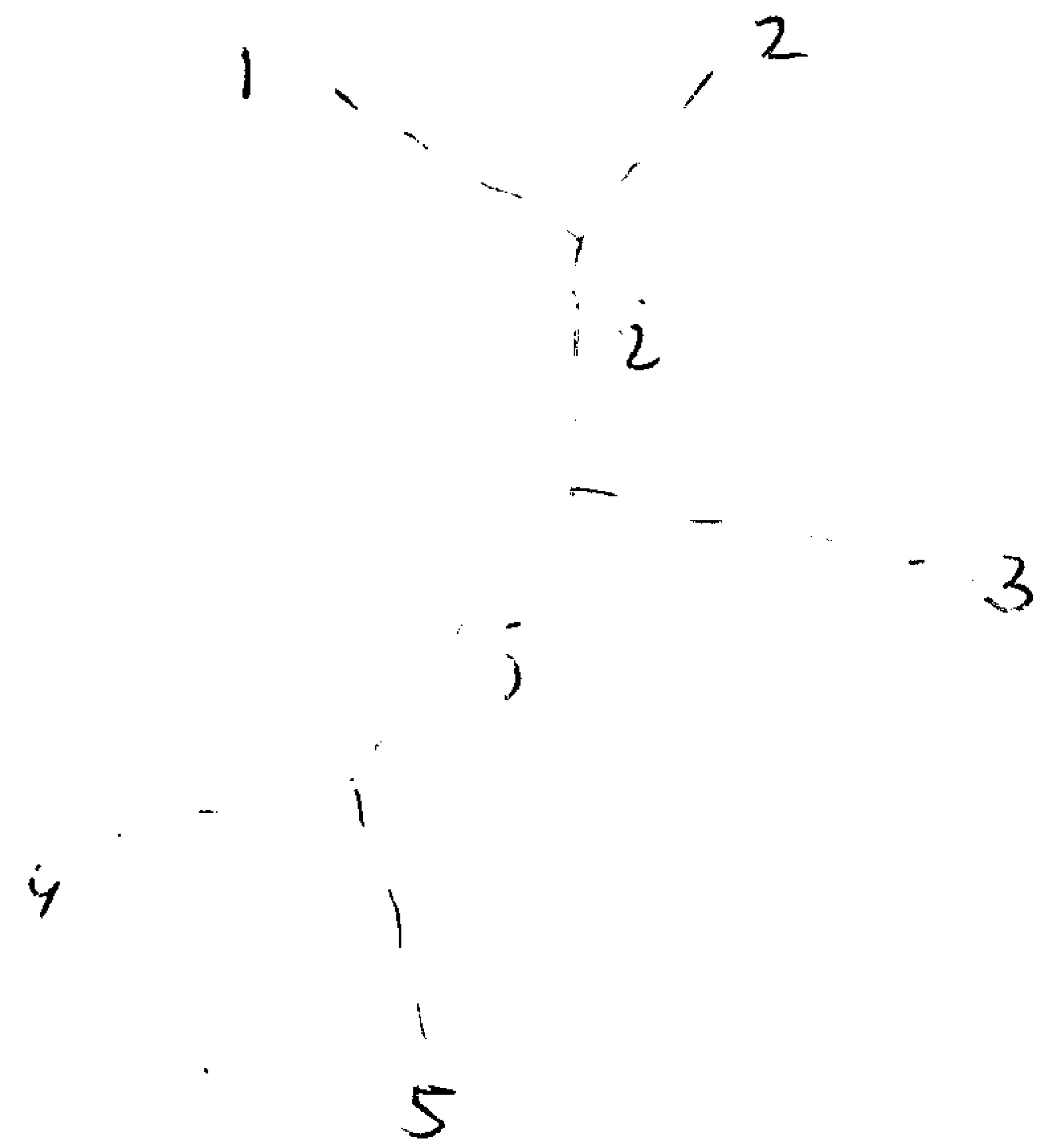
$h_{s_i} = h_{s_j} = 1$, $\bar{h}_{s_i} = \bar{h}_{s_j} = 1$

$\Rightarrow$ $h_{r_k} = 1$, $\bar{h}_{r_k} = 1$.

$\Rightarrow$ $|r_k\rangle$ satisfies physical state condition for L_0

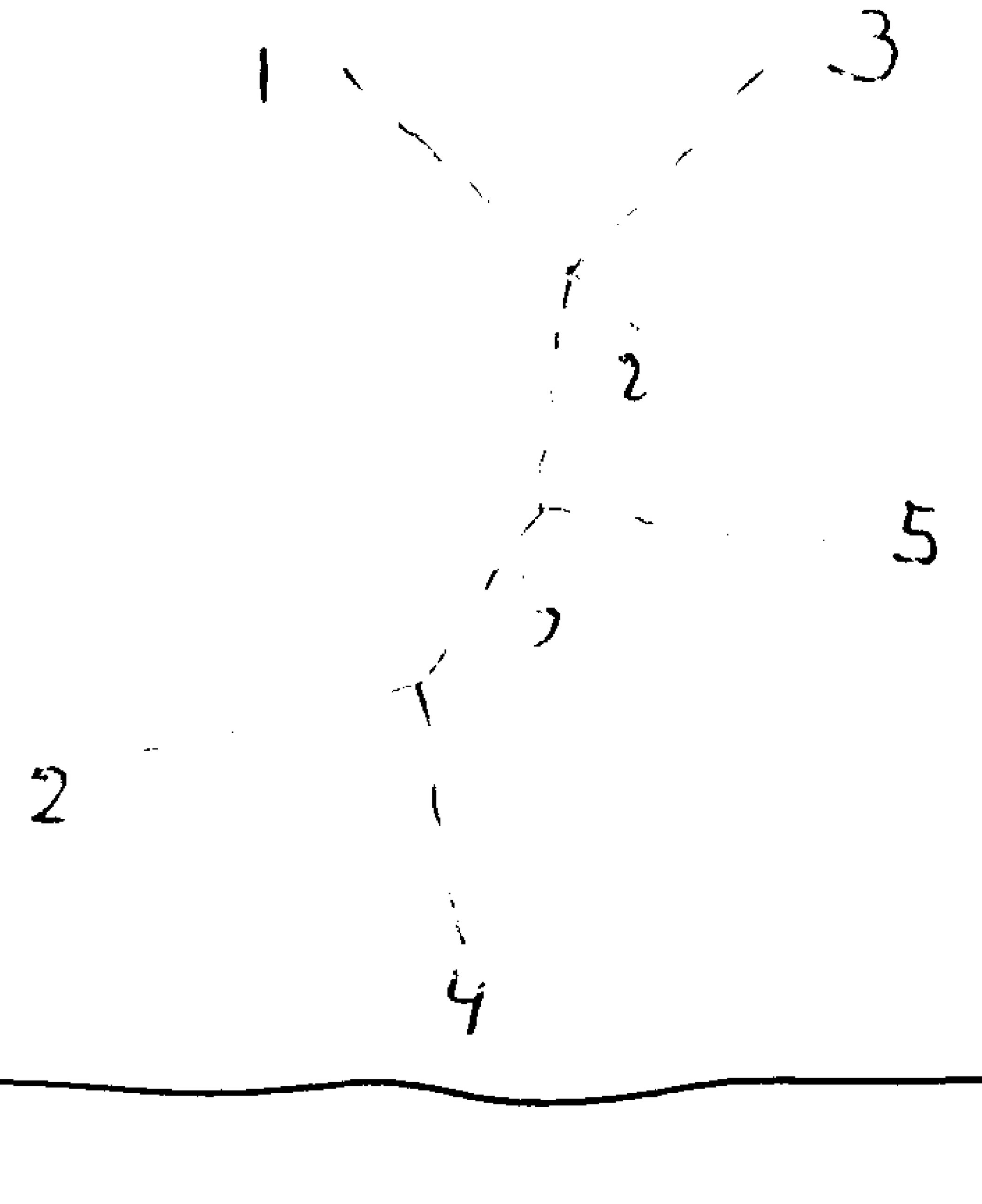
A more careful analysis can be given to show that only physical states contribute to the sum over poles.

The net result: Any tree level N-point amplitude can be expressed as (N-3) sum over intermediate states.



$$\sum_i \sum_j \frac{V_{s_1 s_2 s_i}\, V_{s_i s_j s_3}\, V_{s_j s_4 s_5}}{\{(k_1+k_2)^2 - m_{s_i}^2\}\{(k_1+k_2+k_3)^2 - m_{s_j}^2\}}$$

The same amplitude may be expressed differently:



$$\sum_i \sum_j \frac{V_{s_1 s_3 s_i}\, V_{s_i s_j s_5}\, V_{s_j s_2 s_4}}{\{(k_1+k_3)^2 - m_{s_i}^2\}\{(k_1+k_3+k_5)^2 - m_{s_j}^2\}}$$

→ Known as channel duality.

Coupling constant in string theory.

To N point function we can associate a factor g^{N-2}.

Reflects an ambiguity in defining integral measure over the metric.

$\Rightarrow$ A_N is accompanied by g^{N-2}

3-point fn. g

4-point fn. g^2 etc.

$\to$ consistent with the rules relating ~~different~~ A_N to A_M via sum over intermediate states

g is known as the string coupling constant.

We shall soon see that it can be absorbed into the definition of a field.

S-matrix involving massless fields:

$\zeta_{\mu\nu}\, \alpha^{\mu}_{-1}\, \alpha^{\nu}_{-1}\, |k\rangle$

$\to$ vertex op. $\zeta_{\mu\nu}\, \partial X^{\mu}\, \bar\partial X^{\nu}\, e^{ik\cdot X}$

Can be used to calculate S-matrix.

Need correln. fn. involving

$$\langle \partial X^{\mu_1} \bar\partial X^{\nu_1}(z_1)\, e^{ik_1\cdot X}(z_1)\;\; \partial X^{\mu_2} \bar\partial X^{\nu_2}\, e^{ik_2\cdot X(z_2)}$$

$$\cdots\cdots\rangle$$

$$\partial_{z_1} \partial_{\bar z_1} \partial_{z_2} \partial_{\bar z_1} \frac{\delta}{\delta J^{\mu_1}(z_1 \bar z)} \frac{\delta}{\delta J^{\nu_1}(z_1, \bar z_1)}$$

$$\frac{\delta}{\delta J^{\mu_2}(z_2, \bar z_2)} \frac{\delta}{\delta J^{\nu_2}(z_2, \bar z_2)} \left\langle e^{i \int J_\mu(z,\bar z)\, X^\mu(z,\bar z)\, d^2 z} \right\rangle$$

$$J^\mu = \sum k_i^\mu \, \delta^{(2)}(z - z_i)$$

Known

$$e^{-\frac{1}{2}\int J_\mu(z)\, \Delta^{\mu\nu}_{(z,z')} J_\nu(z')\, d^2 z\, d^2 z'}$$

→ Can be computed

→ S matrix elements involving Massless states

For massless states we can ~~consider~~ take external momenta to be small.

→ expansion in powers of $\alpha' k^2$

→ Notion of low energy effective action.

Find an "effective action" such that the tree level Feynmann rules reproduce the string theory amplitudes.

The low energy effective action will involve expansion in $\alpha' \partial^2$

space-time derivatives (not to be confused with $\partial_z, \partial_{\bar z}$ → world sheet derivatives)

Low energy effective action for bosonic string theory:

$$S = K \int d^{26}x \sqrt{-\det g}\, e^{-2\Phi} \left[R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\, \partial_\mu \Phi\, \partial^\mu \Phi \right]$$

$$K \sim \frac{1}{g^2}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

Recall the procedure for deriving the effective action:

1. Calculate scattering amplitudes
2. Try to write down a field theory action which reproduces the scattering amplitude.

Today we shall discuss another method for computing the S-matrix.

~~Consider for~~ So far we have mostly dealt with propagation of string in flat background.

$$S = -\frac{T}{2}\int d^2\xi\, \partial_\alpha X^\mu \partial_\beta X^\nu \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \eta_{\mu\nu}$$

Now consider its propagation in curved background.

$$S = -\frac{T}{2}\int d^2\xi\, \partial_\alpha X^\mu \partial_\beta X^\nu \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, G_{\mu\nu}(X)$$

$\rightarrow$ an interacting field theory.

Set $T = \frac{1}{2\pi}$ $\alpha' = 1$ $\Rightarrow$ $S = -\frac{1}{4\pi\alpha'}\int$

$G_{\mu\nu}(X) = G_{\mu\nu}(0) + \partial_\rho G_{\mu\nu}(0) X^\rho +$

All operators have dimension 2

$\dim X = 0 \Rightarrow G_{\mu\nu}(X)$ is dimensionless.

$\rightarrow$ The theory is renormalizable.

What other dimension 2 operators can we add to the action without destroying

Result

$$S = -\frac{1}{4\pi}\int d^2\xi \quad \partial_\alpha X^\mu \partial_\beta X^\nu \left[\sqrt{-\gamma}\,\gamma^{\alpha\beta} G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X) \right.$$

$$\left. + \sqrt{-\gamma}\; R^{(2)}_\gamma \; \Phi(X) \right]$$

$\downarrow$

Scalar curvature associated with γ

Note: ① The action is manifestly 2-d general coordinate invariant.

② The first two terms are also Weyl invariant (under $\gamma_{\alpha\beta} \to e^{\phi}\gamma_{\alpha\beta}$) but the last term is not

~~At best only scale invariant~~ $\cancel{\Phi = \text{const}}$

③ When we fix the gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, the last term drops out of the action.

But its effect appears in the definition of the energy momentum tensor $T_{\alpha\beta}$

$T_{\alpha\beta} \propto \dfrac{\delta S}{\delta\gamma^{\alpha\beta}}$ → contains a term

$$\propto \partial_\alpha \partial_\beta \Phi(X)$$

$$= \partial_\mu \partial_\nu \Phi \, \partial_\alpha X^\mu \partial_\beta X^\nu$$

$$+ \partial_\mu \Phi \, \partial_\alpha \partial_\beta X^\mu$$

Contribution from this term to $\eta^{\alpha\beta} T_{\alpha\beta}$ does not vanish

→ reflects lack of Weyl invariance.

... have we included the last term?

The reason is that quantum correction in this ~~theory~~ two dimensional field theory destroys scale and Weyl invariance anyway

→ so we sought as non-vanishing β-functions.

Note: the function $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ & $\Phi(x)$ are the coupling constants in the theory.

e.g. if $G_{\mu\nu}(x) = G_{\mu\nu}(0) + \partial_\rho G_{\mu\nu}(0)\, x^\rho + \ldots$

then the action contains

$$\Big(\underbrace{G_{\mu\nu}(0)}_{\text{coupling constants.}}\, \partial_\alpha X^\mu \partial_\beta X^\nu + \underbrace{\partial_\rho G_{\mu\nu}(0)}\, \partial_\alpha X^\mu \partial_\beta X^\nu X^\rho + \ldots$$

the β-functions, specifying how these coupling constants change with ~~scale~~ the renormalization scale μ, will also be ...tions)

R-G eqn. implicit μ dependence

$$\mu \frac{d\, G_{\mu\nu}(x)}{d\mu} = \beta^G_{\mu\nu}(x)$$

$$\mu \frac{d\, B_{\mu\nu}}{d\mu} = \beta^B_{\mu\nu}(x)$$

$$\mu \frac{d\, \Phi(x)}{d\mu} = \beta^\Phi_{\mu\nu}(x)$$

Put another way,

$$\gamma^{\alpha\beta} T_{\alpha\beta} \propto \Big(\beta^{G}_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \sqrt{-\gamma}\, \gamma^{\alpha\beta}$$

$$+ \beta^{B}_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \epsilon^{\alpha\beta}$$

$$+ \beta^{\Phi} \sqrt{-\gamma}\, R^{(2)}_{\gamma} \Big)$$

β^G, β^B and β^Φ can be computed in perturbation theory.

Now in order to ~~formu~~ formulate a consistent string theory we must demand ~~[illegible]~~ that the two dimensional theory has Weyl invariance at the quantum level

---

(In a gauge fixed version $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ this requires demanding conformal invariance with central charge 26)

$$\Rightarrow \left.\begin{array}{l} \beta^{G}_{\mu\nu} = 0 \\ \beta^{B}_{\mu\nu} = 0 \end{array}\right\} \text{Conformal invariance}$$

$$\beta^{\Phi} = 0 \;\} \text{ central charge } 26$$

In gauge fixed version

β^G, β^B and β^Φ can be computed using perturbation theory.

(Note the perturbation theory is in 2-d coupling constant α'.
Each α' will come with 2 factors of $\partial_\mu \ldots$)

Thus in the $\alpha' = 1$ unit, higher and higher order terms in the perturbation theory will contain more and more derivatives (w.r.t. x^μ).

Result at one loop (for $\alpha' = 1$)

$$\beta^{G}_{\mu\nu} = R^{(26)}_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} = 0$$

$$\beta^{B}_{\mu\nu} = -\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + (\nabla^\rho \Phi) H_{\rho\mu\nu} = 0$$

$$\beta^{\Phi} = -\frac{1}{2}\nabla^2 \Phi + (\nabla_\rho \Phi)(\nabla^\rho \Phi) - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0$$

These equations can be derived from the effective action.

$$S = C\int d^{26}x \left(R^{(26)} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi \right)$$

$\rightarrow$ identical to the effective action described earlier.

Conclusion: eqs of motion of string theory
$\leftrightarrow$ conformal invariance of 2-d field theories.

$\Rightarrow$ solution of the eqs of motion of string theory

$\leftrightarrow$ Conformally invariant 2-d field theories with central charge $= 26$.

This provides a very efficient way of looking for solution of the eqs of motion without even knowing them.

Heterotic string theory:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi \Big[\eta^{\alpha\beta}\,\eta_{\mu\nu}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu$$

$$+ \frac{i}{4\pi\alpha'}\,\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} + \frac{i}{4\pi\alpha'}\,\bar{\lambda}^I\rho^\alpha\partial_\alpha\lambda^I\Big)$$

$\mu, \nu = 0, \cdots 9, \qquad I, J = 1, \cdots 32$

$$\psi^\mu = \begin{pmatrix}\psi_+^\mu \\ 0\end{pmatrix}, \quad \lambda^I = \begin{pmatrix}0 \\ \lambda_-^I\end{pmatrix}, \quad \rho^0 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \quad \rho^1 = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}$$

~~Constraints: $T_{\alpha\beta}$ ..., J_+^μ ...~~

Supersymmetry: $\epsilon = \begin{pmatrix}0 \\ \epsilon_-\end{pmatrix}$ $\quad\epsilon$ real.

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu = \epsilon^+\rho^0\psi^\mu = i\epsilon_-\psi_+^\mu$$

$$\delta_\epsilon\begin{pmatrix}\psi_+^\mu \\ 0\end{pmatrix} = -i\rho^\alpha\partial_\alpha X^\mu\epsilon = \begin{pmatrix}(\partial_\sigma - \partial_\tau)X^\mu\epsilon_- \\ 0\end{pmatrix}$$

$$\Rightarrow \quad \delta_\epsilon\psi_+^\mu = (\partial_\tau - \partial_\sigma)X^\mu\,\epsilon_-$$

Constraints: $\underbrace{T_{\alpha\beta} = 0}_{\text{Energy momentum tensor}}, \quad J_+^\mu = 0$

J_+^μ → Conserved current associated with susy trs.

Energy momentum tensor

$$\Rightarrow \quad (L_n + c\, \delta_{n,0})\,|phys\rangle = 0 \qquad n \geq 0$$

$$(\tilde{L}_n + \tilde{c}\, \delta_{n,0})\,|phys\rangle = 0 \qquad n \geq 0$$

$$G_r\, |phys\rangle = 0 \qquad n \geq 0.$$

We have two sectors NS & R on right

two sectors on left for $SO(32)_H$

four sectors on left for $(E_8 \times E_8)_H$

Mode expansion

$$\psi_+^\mu = \sqrt{\alpha'} \sum_r b_r^\mu(\tau)\, e^{ir\sigma} \qquad r \in \mathbb{Z} \text{ for R}, \ \mathbb{Z}+\tfrac{1}{2} \text{ for NS}$$

For $SO(32)_H$

$$\lambda_-^I = \sqrt{\alpha'} \sum_s \tilde{\lambda}_s^I\, e^{-is\sigma} \qquad s \in \mathbb{Z} \text{ for R}, \ \mathbb{Z}+\tfrac{1}{2} \text{ for NS}$$

$$\{b_r^\mu, b_{r'}^\nu\} = \eta^{\mu\nu}\, \delta_{r+r',0}\,; \qquad \{\tilde{\lambda}_r^I, \tilde{\lambda}_{r'}^J\} = \delta^{IJ}\, \delta_{r+r',0}$$

$$L_n = \frac{1}{2} \sum_{\substack{m \\ n \neq 0}} \alpha_{m-n}^\mu\, \alpha_n^\nu\, \eta_{\mu\nu} + \frac{1}{2}\sqrt{2\alpha'}\, \alpha_m^\mu\, p_\mu$$

$$+ \frac{1}{2} \sum_r \left(r + \frac{1}{2} m\right) b_{-r}^\mu\, b_{m+r}^\nu\, \eta_{\mu\nu}$$

$$L_0 = \sum_{n>0} \alpha_{-n}^\mu\, \alpha_n^\nu\, \eta_{\mu\nu} + \frac{\alpha'}{4}\, \eta^{\mu\nu}\, p_\mu p_\nu$$

$$+ \sum_{r>0} r\, b_{-r}^\mu\, b_r^\nu\, \eta_{\mu\nu}$$

ⓒ

$\tilde{L}_n$, $\tilde{L}_0$: Replace α^μ_n by $\tilde{\alpha}^\mu_n$; b^μ_r by $\tilde{d}^I_r$

$$G_r = \sum_{n\neq 0} \alpha^\mu_{-n} b^\nu_{r+n} \eta_{\mu\nu} + \sqrt{\frac{\alpha'}{2}}\, p_\mu b^\mu_r.$$

No $\tilde{G}_r$.

$c = -\frac{1}{2}$ in NS, 0 in R.

$\tilde{c} = -1$ in NS, 1 in R

GSO projection rules: On the ~~left~~ right it is the same as superstring.

NS vacuum $|\{k\}\rangle$ has $-$ parity ~~projected out~~

R vacuum: $|\{k\}, a\rangle$ has $+$ parity
$|\{k\}, \dot{a}\rangle$ has $-$ parity

GSO projection rule on the left:

NS vacuum: $|\{k\}\rangle$ has $+$ parity.

R vacuum: Two spinors of SO(32):

$|\alpha\rangle$, $|\dot{\alpha}\rangle$

Chiral — $+$ parity; anti-chiral — $-$ parity.

(D)

General State:

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \; \tilde{\alpha}^{\nu_1}_{-n_1} \cdots \tilde{\alpha}^{\nu_\ell}_{-n_\ell} \; b^{\sigma_1}_{-r_1} \cdots b^{\sigma_m}_{-r_m}$$

$$\tilde{d}^{I_1}_{-s_1} \cdots \tilde{d}^{I_n}_{-s_n} \; |\cdots, \{k^\mu\}, \cdots\rangle$$

spinor indices for R sector

L_0 constraint:

$$-\frac{\alpha'}{4}\eta^{\mu\nu}k_\mu k_\nu = \underbrace{\sum m_i + \sum r_i}_{N} + c$$

$\tilde{L}_0$ constraint:

$$-\frac{\alpha'}{4}\eta^{\mu\nu}k_\mu k_\nu = \underbrace{\sum s_i + \sum n_i}_{\tilde{N}} + \tilde{c}$$

$$\Rightarrow \quad m^2 = \frac{4}{\alpha'}(N+c) = \frac{4}{\alpha'}(\tilde{N}+\tilde{c})$$

Is there a tachyon?

Note: $c = -\frac{1}{2}$ in NS, 0 in R

$\tilde{c} = -1$ in NS, 1 in R

$\Rightarrow$ Tachyon may only come from NS-NS.

Need $N - \frac{1}{2} < 0$. Since $N = 0, \frac{1}{2}, \cdots$, we need $N = 0$.

Ⓔ

$\Rightarrow$ Fock vacuum on the right sector.

This is GSO - odd. $\Rightarrow$ No tachyon.

Massless states:

$N + C = 0 = \tilde{N} + \tilde{C}$

Note: If we have R sector on the left, then

$\tilde{C} \to 1 \Rightarrow$ No massless sector.

Choose NS sector on the left.

$\tilde{C} = -1$

$\Rightarrow m^2 = (\tilde{N} - I)\frac{4}{\alpha'}$. Need $\tilde{N} = 1$.

On the right we can have NS or R sector.

NS sector on right: $C = -\frac{1}{2} \Rightarrow$ Need $N = \frac{1}{2}$.

Possible states

$\zeta_{\mu\nu} \; b^{\mu}_{-1/2} \; \tilde{\alpha}^{\nu}_{-1} \; |\{k\}\rangle \to g_{\mu\nu}, B_{\mu\nu}, \Phi$

$\zeta^{IJ}_{\mu} \; b^{\mu}_{-1/2} \; \tilde{\lambda}^{I}_{-1/2} \; \tilde{\lambda}^{J}_{-1/2} \; |\{k\}\rangle \to \frac{32 \times 31}{2}$ combinations (IJ)

$\to SO(32)$ gauge bosons.

(F)

Ramond sector on right: $\tilde{c}=0$. Need $\tilde{N}=0$.

$\tilde{\alpha}^{\nu}_{-1} \, | \{k\}, a \rangle \rightarrow$ gravitino.

$\psi^{I}_{-1/2} \, \psi^{J}_{-1/2} \, |\{k\}, a\rangle \rightarrow$ Fermions in the adjoint of $SO(32)$.

$\rightarrow$ The massless spectrum is that of $N=1$ supergravity in 10-dimensions with $SO(32)$ gauge group.

After imposing physical state constraint, the full spectrum becomes supersymmetric.

$\rightarrow$ Equal number of fermions and bosons at each level.

Note: At the massless level there are no spinors of $SO(32)$ but there are massive states carrying spinor representation, e.g.

$b^{\mu}_{-3/2} \, | \alpha, \{k\} \rangle$	Has $N=\frac{3}{2}$, $\tilde{N}=0$ $c=-\frac{1}{2}$, $\tilde{c}=1$ $\Rightarrow m^2=\frac{4}{\alpha'}(N+c)=\frac{4}{\alpha'}(\tilde{N}+\tilde{c})=\frac{4}{\alpha'}$

$E_8 \times E_8$ heterotic string theory:

On the right there is no change.

On the left we now need to divide the oscillators λ^I into two parts:

$\tilde{d}^M_s$ and $\tilde{f}^M_t$ $M = 1, \cdots 16$

On the left itself we have four sectors

NS - NS : $s, t \in \mathbb{Z} + \frac{1}{2}$
NS - R : $s \in \mathbb{Z} + \frac{1}{2}$, $t \in \mathbb{Z}$
R - NS : $s \in \mathbb{Z}$, $t \in \mathbb{Z} + \frac{1}{2}$
R - R : $s, t \in \mathbb{Z}$

$\tilde{c} = -1$ in NS - NS
$= 0$ in NS - R or R - NS
$= 1$ in R - R

Twisting of each fermion gives a contribution of $\frac{1}{16}$

~~General~~ On the right

$c = -\frac{1}{2}$ in NS - sector

$= 0$ in R - sector

General State

$$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_k}_{-m_k} \; \tilde\alpha^{\nu_1}_{-n_1} \cdots \tilde\alpha^{\nu_k}_{-n_\ell} \; b^{\sigma_1}_{-r_1} \cdots b^{\sigma_m}_{-r_m}$$

$$\tilde d^{M_1}_{-s_1} \cdots \tilde d^{M_n}_{-s_n} \; \tilde f^{N_1}_{-t_1} \cdots \tilde f^{N_p}_{-t_p} \; |\cdots, \{k\}, \cdots\rangle$$

GSO projection on the right: identical

NS: $|\{k\}\rangle$ −

RR: $|\{k\}, a\rangle$ +

$|\{k\}, \dot a\rangle$ −

GSO projection on the left:

	$(-1)^{f_{L_1}}$ parity	$(-1)^{f_{L_2}}$ parity
NSNS $\|\{k\},\ldots\rangle$	+	+
R NS		
$\|\alpha, \{k\}, \ldots\rangle$	+	+
$\|\dot\alpha, \{k\}, \ldots\rangle$	−	+
NS R		
$\|\beta, \{k\}, \ldots\rangle$	+	+
$\|\dot\beta, \{k\}, \ldots\rangle$	−	+
R-R		
$\|\alpha, \beta, \{k\}, \ldots\rangle$	+	+
$\|\alpha, \dot\beta, \{k\}, \ldots\rangle$	+	−
$\|\dot\alpha, \beta, \{k\}, \ldots\rangle$	−	+
$\|\dot\alpha, \dot\beta, \{k\}, \ldots\rangle$	−	−

α : spinor index of $SO(16)_1$

β : spinor index of $SO(16)_2$

(I)

~~Absence of tachyon follows from~~

$$m^2 = \frac{4}{\alpha'}\left(\sum m_i + \sum r_i + c\right) \equiv \frac{4}{\alpha'}(N+c)$$

$$= \frac{4}{\alpha'}\left(\sum n_i + \sum s_i + \sum t_i + \tilde{c}\right) \equiv (\tilde{N}+\tilde{c})\frac{4}{\alpha'}$$

Absence of tachyon : Follows from analysis on the right sector.

$N+c \geq 0$ for GSO even states.

Massless states : Need $N+c = 0 = \tilde{N}+\tilde{c}$

Bosonic states (NS on the right) :

NS - NS - NS : $b^{\mu}_{-1/2}\,\tilde{\alpha}^{\nu}_{-1}\,|\{k\}\rangle$

$c = -\frac{1}{2}$, $\tilde{c} = -1$

$\Rightarrow N = \frac{1}{2}$, $\tilde{N} = 1$ $\rightarrow$ $g_{\mu\nu}$, $B_{\mu\nu}$, Φ

$$b^{\mu}_{-1/2}\,\tilde{d}^{I}_{-1/2}\,\tilde{d}^{J}_{-1/2}\,|\{k\}\rangle$$

$\rightarrow$ $SO(16)_1$ gauge bosons.

$$b^{\mu}_{-1/2}\,\tilde{f}^{I}_{-1/2}\,\tilde{f}^{J}_{-1/2}\,|\{k\}\rangle$$

$\rightarrow$ $SO(16)_2$ gauge bosons.

Ex. What about : $b^{\mu}_{-1/2}\,\tilde{d}^{I}_{-1/2}\,\tilde{f}^{J}_{-1/2}\,|\{k\}\rangle$?

R-NS-NS. $c = -\frac{1}{2}$ $\tilde{c} = 0$

$\rightsquigarrow$ need $N = 1/2$ $\tilde{N} = 0$.

$b^{\mu}_{-1/2} \otimes |\alpha, \{k\}\rangle$

$\rightsquigarrow$ A vector carrying spinor representation of $SO(16)_1$.

How can this be?

The only consistent way we know of coupling a massless vector is using Non-abelian gauge theory.

Answer: The spinor of $SO(16)$ + adjoint of $SO(16)$ $\rightarrow$ adjoint of $E_8 \supset SO(16)$.

The gauge group is actually E_8 and not $SO(16)$

(Analog of self-dual point in circle compactification of bosonic string theory)

NS - R - NS $\quad C = -\frac{1}{2} \quad \tilde{C} = 0.$

→ Need $N = \frac{1}{2}, \quad \tilde{N} = 0.$

$$b^{\mu}_{-1/2} \;| \beta, \{k\}_0 \rangle$$

→ spinor of $SO(16)_2$.

→ Combines with adjoint to give adjoint of $(E_8)_2$.

R - R - NS : $C = -\frac{1}{2} \quad \tilde{C} = 1$ → no massless states.

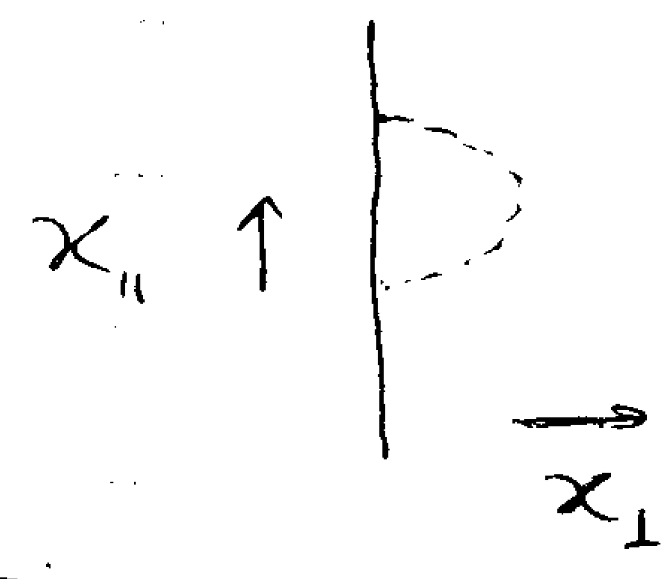
R sector on the right → generates fermionic superpartners of all these bosons.

Result: Again a supersymmetric spectrum

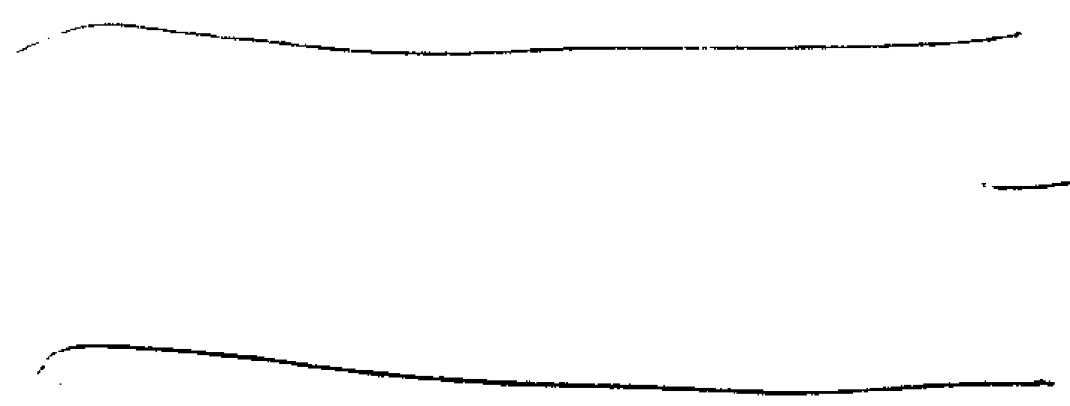
Massless sector: $N=1$ supergravity in $D=10$ with $E_8 \times E_8$ gauge group.

D-branes in superstring theory:

→ Defined in identical manner as in the case of bosonic string theory.



→ Dirichlet b.c. on $X_{\perp}^{i}$
Neumann b.c. on $X^{\mu}_{\parallel}$



→ World-sheet

What about b.c. on ψ^{μ}?

→ should relate left moving ψ^{μ} to right-moving ψ^{μ}

i.e. ψ^{μ}_{+} to ψ^{μ}_{-} at $\sigma = 0, \pi$

choice: $\psi^{\mu}_{+} = \pm \psi^{\mu}_{-}$ at $\sigma = 0, \pi$

~~we can always choose $\psi^{\mu}_{+} = \psi^{\mu}_{-}$ at $\sigma = 0$ by redefining ψ^{μ}~~

World-sheet SUSY relates b.c. on X^{μ} to that on ψ^{μ}.

Consistent choice

$$\vec{\psi}^{\mu}_{\parallel +} = \vec{\psi}^{\mu}_{\parallel -}, \quad \vec{\psi}^{\mu}_{\perp +} = -\vec{\psi}^{\mu}_{\perp -}$$

or $$\vec{\psi}^{\mu}_{\parallel +} = -\vec{\psi}^{\mu}_{\parallel -}, \quad \vec{\psi}^{\mu}_{\perp +} = \vec{\psi}^{\mu}_{\perp -}$$

Note:

$(\partial_\tau + \partial_\sigma)\vec{X}_\parallel = (\partial_\tau - \partial_\sigma)\vec{X}_\parallel$

$(\partial_\tau + \partial_\sigma)\vec{X}_\perp = -(\partial_\tau + \partial_\sigma)\vec{X}_\perp$

At each end $\sigma = 0, \pi$.

Note: We can make a field redefinition

$\psi^{\mu}_{-} \to -\psi^{\mu}_{-}$ to choose at $\sigma = 0$:

$$\vec{\psi}_{\parallel +} = \vec{\psi}_{\parallel -}, \quad \vec{\psi}_{\perp +} = -\vec{\psi}_{\perp -}$$

Then at $\sigma = \pi$ end we still have two choices:

$$\vec{\psi}_{\parallel +} = \vec{\psi}_{\parallel -}, \quad \vec{\psi}_{\perp +} = -\vec{\psi}_{\perp -} \quad \Rightarrow \text{R sector}$$

$$\vec{\psi}_{\parallel +} = -\vec{\psi}_{\parallel -}, \quad \vec{\psi}_{\perp +} = \vec{\psi}_{\perp -} \quad \Rightarrow \text{NS sector.}$$

Quantize using usual procedure.

$\vec{\psi}^{\mu}_{\parallel}$ will have zero modes in R sector

---

$\Rightarrow$ Spinor representation of $SO(p,1)$

$\Rightarrow$ ~~Bos~~ Space-time fermions. ($p+1$-dimensional)

GSO projection: as usual (projects out NS vacuum)

(N)

Note: Construction of D-branes require symmetry between left- and right-moving modes on the world-sheet.

Otherwise we cannot impose boundary condition.

→ No D-branes in heterotic string theory.

Further consistency condition: Collision of open strings may produce closed strings.

→ We need to make sure that all ~~this~~ of those are physical closed string states.

e.g. in type IIA theory, collision of two open strings should ~~only~~ not produce $|a\, b\rangle$ type RR sector state.

In type IIB theory, this process should not produce $|a\, \dot{a}\rangle$ type RR sector state.

→ gives restriction on what kind of D-p-branes we can have.

IIA : Dp-branes with even p.

0, 2, 4, 6, 8.

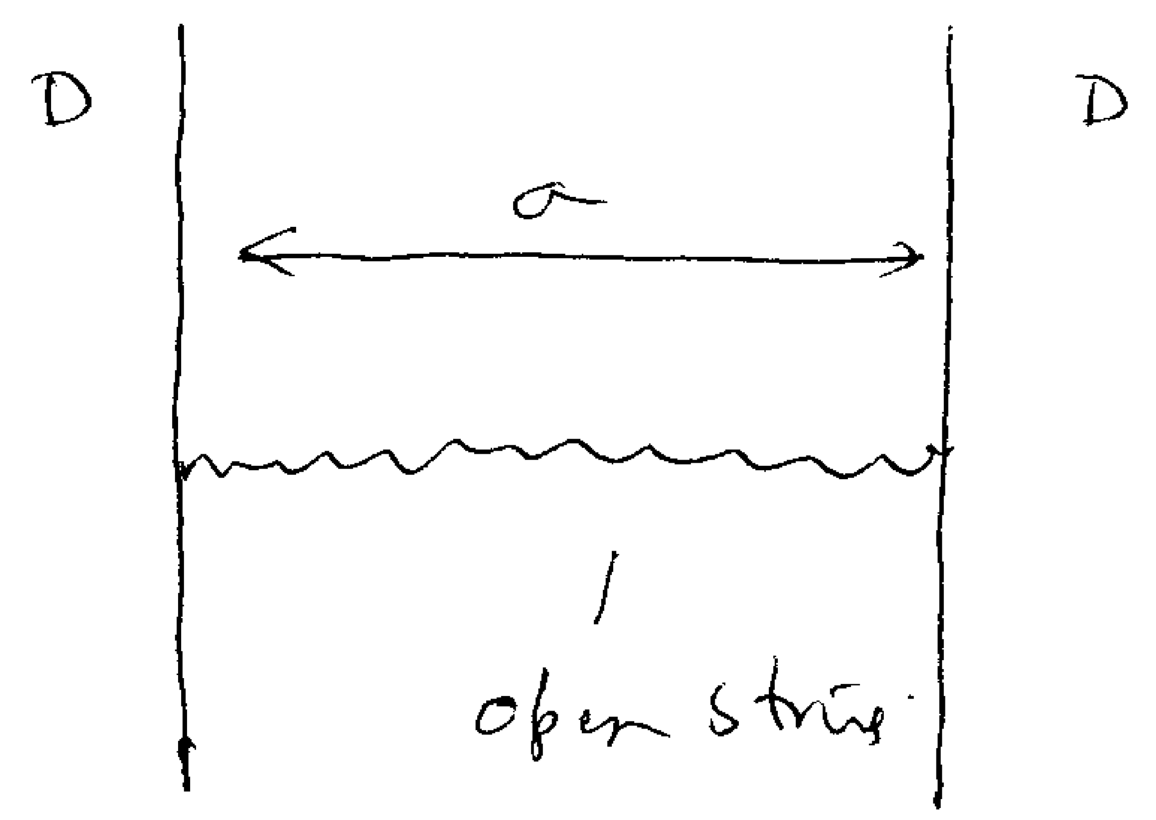
IIB : Dp-branes with odd p

(−1), 1, 3, 5, 7, 9.

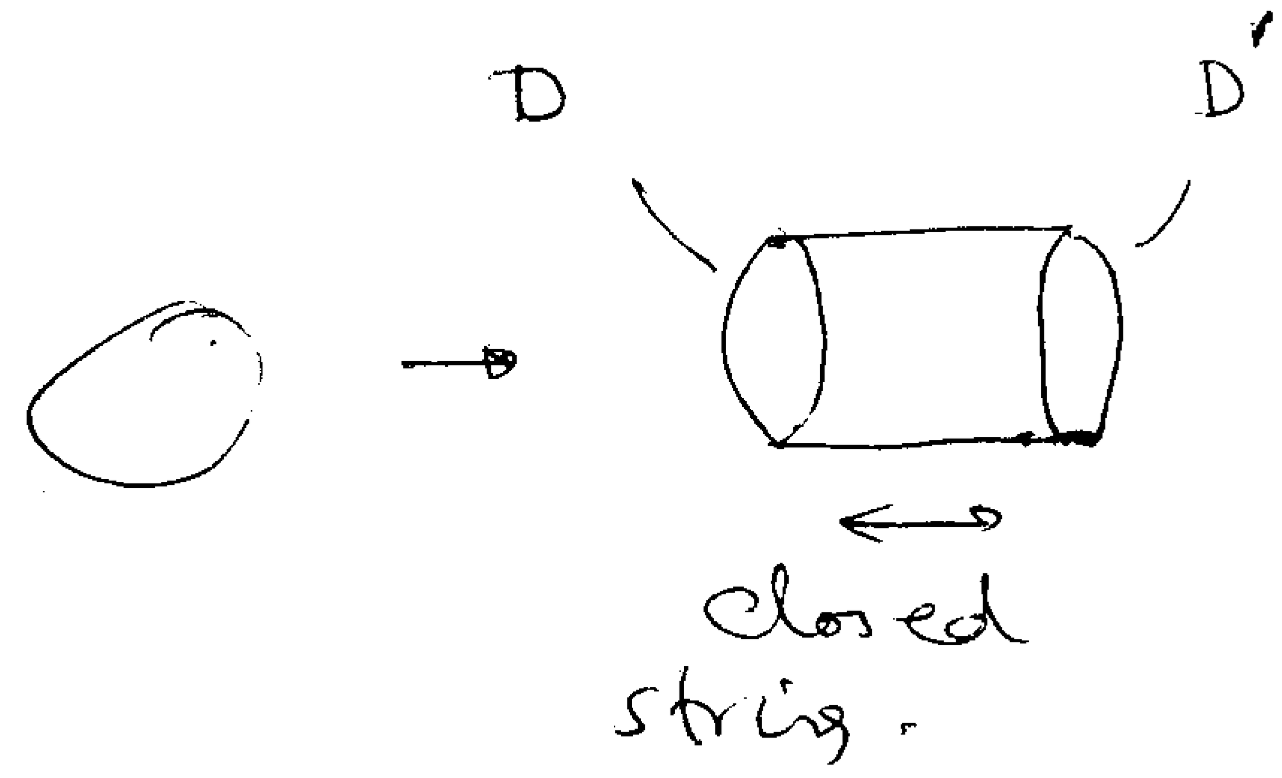
D-instanton

These D-branes carry charge!

Consider a pair of parallel D-branes:



One loop effective action:



Thus if we compute the contribution

$$\sum_n \int d^{p+1}k \, \ln \left(k^2 + m_n^2(a) \right)$$

open string states

(mass)^2 of nth open string state

Can be interpreted as the potential $V(a)$ induced between the two D-branes due to closed string exchange.

Includes graviton exchange.

→ gives tension (energy / area)

$$T_p = \frac{1}{(\alpha')^{\frac{p+1}{2}}} \frac{1}{(2\pi)^p g_c}$$

closed string coupling constant.

Also includes exchange of $(p+1)$ form RR field

$$C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}}$$

In IIA : p = even $\Rightarrow$ $(p+1)$ - odd

→ matches the RR fields.

$C^{(1)}_{\mu} \rightarrow A_\mu$

$C^{(3)}_{\mu\nu\rho} \rightarrow C_{\mu\nu\rho}$

$C^{(5)}_{\mu\nu\rho\sigma\tau} \rightarrow$ Dual + $C^{(3)}_{\mu\nu\rho}$

$dC^{(5)} = * \, dC^{(3)}$

$C^{(7)}_{\mu\nu\rho\sigma\tau} \rightarrow$ Dual to $C^{(1)}_{\mu}$.

$dC^{(7)} = *(dC^{(1)})$

$C^{(9)}_{\mu\nu\rho\sigma\tau}$ = Non-dynamical.

Similarly in IIB:

p = odd, $(p+1)$ = even

→ matches the known RR fields.

- $C^{(0)}_{\mu} \rightarrow a \rightarrow$ RR scalar

$C^{(2)}_{\mu\nu} \rightarrow B'_{\mu\nu}$

$C^{(4)}_{\mu\nu\rho\sigma} \rightarrow D_{\mu\nu\rho\sigma}$

$dC^{(6)} = *(dC^{(2)})$, $dC^{(8)} = *(dC^{(0)})$

$C^{(10)}$ is non-dynamical.

Ⓡ

How does a D-p-brane carry RR (p+1) form charge?

Particle: carries 1 form charge (A_μ).

Coulomb gauge:

$$A_0 \sim \frac{Q}{r^{D-3}} \text{ far away.}$$

D-p-brane along 1, --- p.

$$C^{(p)}_{0\,1,\ldots,p} = \frac{Q}{r^{D-p-3}} \text{ far away from the brane.}$$

How does a D-p-brane feel the effect of p-form field?

Particle: World-line action $\cdots + \int_{\text{world line}} A_\mu \, dx^\mu$

D-p-brane: World volume action:

$$\cdots + \int_{\substack{\text{world}\\\text{volume}}} C^{(p)}_{\mu_1 \cdots \mu_{p+1}} \, dx^{\mu_1} \cdots\cdots dx^{\mu_{p+1}}$$

N- coincident D-branes

→ Supersymmetric U(N) Yang-Mills theory at low energy.

→ generalization of the result from bosonic string theory.

(M)

A D-p brane carries (p+1) form charge

$C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}}$

~~no~~ Act as sources for these fields.

For a charge particle moving along a curve

$X^\mu(s)$

Action $= \cdots + \int A_\mu \frac{dX^\mu}{ds} ds$ + Free particle action + E.m. field action

A D-p brane spans a (p+1) dimensional world-volume labelled by $s_1, \cdots s_{p+1}$: $X^\mu(s_1, \cdots s_{p+1})$

$$\text{Action} = \cdots + \int d^{p+1}s \; C_{\mu_1 \cdots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial s_{\alpha_1}} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial s_{\alpha_{p+1}}} \epsilon^{\alpha_1 \cdots \alpha_{p+1}}$$

We have seen sources up to D7-brane in IIB.

D7-brane carries 8 form ~~charge~~ ~~field~~ charge

Field strength 9 form.

dualize

1 form field strength $\to$ 0 form field

RR scalar a.

(N)

What about D8 & D9-branes in IIA / IIB

9 form $C^{(9)}$ 10-form $C^{(10)}$

If we quantize free field theory of $C^{(9)}$ and $C^{(10)}$ and see what physical states they produce, we find that there are no physical states associated with these fields.

(Just like a gauge field in 1+1 dimension has no physical states)

This is why we have not discovered them so far in analyzing the spectrum of IIA / IIB.

But their existence can be ~~guaranteed~~ seen in scattering amplitudes.

(They appear as internal lines)

Focus on the D9-brane of IIB

(space-filling) $X^{\mu} = s^{\mu}$

Coupling: $N \int C^{(10)}_{\mu_1 \cdots \mu_{10}} \epsilon^{\mu_1 \cdots \mu_{10}} d^{10}x$.

# of D9-branes

→ will act as source of $C^{(10)}$ field.

Unfortunately: this is the only term in the action involving $C^{(10)}$.

No kinetic term.

∵ $dC^{(10)}$ vanishes identically.

Thus $C^{(10)}$ eq. of motion gives

$N = 0$.

Besides, D9-brane also carries energy density.

→ cosmological constant.

Putting D9-branes in flat space-time is inconsistent.

(Note: both involves back reaction of closed strings and is higher order effect in open strings).

(P)

Type I string theory:

~~Recall~~ ~~Copy~~ The type II world-sheet theory has a symmetry under left-right exchange:

$\sigma \to -\sigma \quad \tau \to \tau. \quad \psi_+^\mu \leftrightarrow \psi_-^\mu$

$(\partial_\tau + \partial_\sigma) \leftrightarrow (\partial_\tau - \partial_\sigma)$.

→ World-sheet parity Ω.

Are ~~the~~ GSO projection rules invariant under such a symmetry?

NS-NS ↔ NS-NS; NS-R ↔ R-NS, RR ↔ RR

↓

No problem.

$(-1)^{f_L} \leftrightarrow (-1)^{f_R} \qquad |\{k\}\rangle \to |\{k\}\rangle$

(NS-R) & (R-NS): $(-1)^{f_L} \leftrightarrow (-1)^{f_R}$

$\tilde{b}^\mu_{-1/2} |\{k\}, a\rangle \to b^\mu_{-1/2} |a, \{k\}\rangle$

↳ exists only in IIB and not in IIA.

In IIA we have $b^\mu_{-1/2} |\dot{a}, \{k\}\rangle$

Thus the Ω - symmetry is broken by the GSO projection rules of IIA but preserved by the GSO projection rules of IIB.

We can now construct a new theory: (Unoriented string theory)

Mod out IIB by Ω.

---

i.e. keep only those states of IIB which are invariant under Ω.

Massless Bosons:

NS-NS sector: $G_{\mu\nu}$, $\cancel{B_{\mu\nu}}$, Φ

R-R sector: $\cancel{\chi}$, $B'_{\mu\nu}$, $\cancel{D_{\mu\nu\rho\sigma}}$, $C^{(10)}_{\mu_1 \cdots \mu_{10}}$

Massless fermions: Come from linear combination of states in (NS-R) and (R-NS) sectors.

(R)

Is that ~~att~~ a consistent theory?

~~It~~ ~~turns~~ ~~out~~ No.

Calculate scattering amplitudes and ~~one~~

~~keep~~ effective action

$\Rightarrow$ (1) A term $\propto \int C^{(10)}_{\mu_1 \cdots \mu_{10}} \epsilon^{\mu_1 \cdots \mu_{10}} d^{10}x$

(2) A negative cosmological constant

$-\Lambda \int \sqrt{-\det g}$

$\Rightarrow$

(a) $C^{(10)}$ eq. of motion in not satisfied.

(b) Flat background is not a consistent solution.

Possible remedy: Can we cancel these terms in the effective action by putting in some D9-branes?

Ans. Yes, with 32 D9-branes.

$= 2^{10/2}$

(5)

In (9+1) dimension D9-branes are always coincident.

⇒ U(32) gauge group + Massless Fermions in adjoint

But we now need to determine which of these gauge bosons survive Ω projection

⇒ SO(32) gauge group with fermions in the adjoint representation of the gauge group.

Type I string theory.

⇒ 5 consistent string theories in (9+1) dim.

IIA, IIB, $(E_8 \times E_8)_H$, $(SO(32))_H$, ~~Type~~ I

Self-Duality

$g_{IIB} \to g_{IIB}^{-1}$

Fundamental string ⟷ D-string

Non-perturbatively equivalent.

$g_{SO(32)} = g_I^{-1}$