String compactification:

String theory is a theory of quantum gravity + other forces & particles.

~ defined in (9+1) dimensional space-time.

Goal: Look for (9+1) dimensional space-times of the form:

$$\mathbb{R}^{3,1} \times M_6 \longrightarrow \text{a 6-dim. compact manifold of small size}$$

(3+1) dim. Minkowski space
(nearly Minkowski)

① M_6 has to be chosen so that it is a valid soln. to the eqns. coming from string theory.

(Generalization of Einstein's eq).

② Ultimate goal: Choose M_6 so that the resulting theory on $\mathbb{R}^{3,1}$ describe what we see in nature.

- Standard model based on $SU(3) \times SU(2) \times U(1)$

gauge theory + gravity at low energy

Thus the subject of string compactification
→ ~~most~~ direct connection to
the physical universe.
At the same time it is ~~one~~ the part ~~of~~ of string theory that is ~~the~~ most mathematically complex.
Why?
~~First of all~~ Classifying possible choices of M_6 satisfying all consistency conditions ~~involve considerable~~ and deriving (3+1) dimensional physics from it involve considerable mathematical complexity.
→ requires us to go beyond classical ~~a~~ geometry and explore stringy geometry.

String theory in (9+1) dim.

→ some massless states + ∞ # of massive ~~states~~ states

Massive states have mass $\sim 10^{19}$ GeV.

$\gg$ ~~o~~ observable mass today

($\sim 10^{3}$ GeV)

Naive guess: We can keep only the massless states in (9+1) dim & study their dynamics on $\mathbb{R}^{3,1} \times M_6$

→ involves harmonic analysis on M_6

→ complicated but classical mathematics.

However it turns out that this naive approach some time fails.

Some apparently massive states in 9+1 dimension can become massless in $\mathbb{R}^{3,1} \times M_6$.

$\rightsquigarrow$ stringy magic.

We cannot always first restrict to massless states in (9+1) dim. and proceed.

Another way of saying this is that in string theory there are no point-like objects to probe classical geometry.

All objects have finite size

$\rightsquigarrow$ ordinary geometry will have to be replaced by stringy geometry.

$\rightsquigarrow$ Unusual things can happen there.

Conventions: (differential forms

$\Leftrightarrow$ anti-symmetric tensor).

$F^{(p)}_{\mu_1 \cdots \mu_p} \to$ rank p anti-symmetric tensor.

$$F^{(p)} = \frac{1}{p!} F_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p}.$$

$$dF = \frac{1}{\cancel{(p+1)}!} \partial_{\mu_0} F_{\mu_1 \cdots \mu_p} dx^{\mu_0} \wedge \cdots dx^{\mu_p}.$$

$$= \frac{1}{(p+1)!} \left\{ \partial_{\mu_0} F_{\mu_1 \cdots \mu_p} + (-1)^p \times \text{cyclic perm} \right\} dx^{\mu_0} \wedge \cdots dx^{\mu_p}.$$

$$\varepsilon^{\mu_1 \cdots \mu_d} = (\sqrt{-\det g})^{-1} \in^{\mu_1 \cdots \mu_d}$$

$$\Downarrow$$

$$\in^{0 \cdots (d-1)} = 1$$

+ totally anti-symmetric.

$\varepsilon^{\mu_1 \cdots \mu_p}$ transforms as a tensor.

$$\int F^{(d)} = \int d^d x \sqrt{-\det g} \in^{\mu_1 \cdots \mu_d} F^{(d)}_{\mu_1 \cdots \mu_d}.$$

$$= \int d^d x \, F_{01 \cdots (d-1)}$$

Given $F^{(p)}$, $(*F^{(p)})_{\mu_1 \cdots \mu_{d-p}} = \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_{d-p}}{}^{\nu_1 \cdots \nu_p} F_{\nu_1 \cdots \nu_p}$

II A supergravity

$\hbar = c = 1$

Bosonic fields:

$$\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{l_s^8} = \frac{1}{(2\pi)^7(\alpha')^4}$$

$$l_s = 2\pi\sqrt{\alpha'}$$

String tension $\frac{1}{2\pi\alpha'}$

$$g_s = e^{\langle\Phi\rangle}$$

NS-NS : $G_{\mu\nu}$, $B^{(2)}_{\mu\nu}$, Φ

R-R : $C^{(1)}_{\mu}$, $C^{(3)}_{\mu\nu\rho}$

$$F^{(p)} \equiv dC^{(p-1)} \qquad \mathcal{H}^{(3)} = dB^{(2)}$$

$$\tilde{F}^{(2)} = F^{(2)}, \quad \tilde{F}^{(4)} = F^{(4)} - C^{(1)} \wedge \mathcal{H}^{(3)}$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,(-\det G)^{1/2} e^{-2\Phi}\Big(R_G + 4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}\mathcal{H}^{(3)}_{\mu\nu\rho}\mathcal{H}^{(3)}_{\mu'\nu'\rho'}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}\Big)$$

$$S_R = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\,(-\det G)^{1/2}\Big(\frac{1}{4}F^{(2)}_{\mu\nu}F^{(2)}_{\mu'\nu'}G^{\mu\mu'}G^{\nu\nu'} + \frac{1}{2\times 4!}\tilde{F}^{(4)}_{\mu\nu\rho\sigma}\tilde{F}^{(4)}_{\mu'\nu'\rho'\sigma'}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}G^{\sigma\sigma'}\Big)$$

Note: κ_{10} can be scaled by $\Phi \to \Phi + \ln\lambda$, $C^{(k)}_\mu \to \frac{1}{\lambda}C^{(k)}_\mu$

~~Standard~~ Canonical metric: $g_{\mu\nu}$

$$G_{\mu\nu} = e^{\Phi/2} g_{\mu\nu}$$

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,(-\det g)^{1/2}\Big(R_g - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}e^{-\Phi}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}\mathcal{H}_{\mu\nu\rho}\mathcal{H}_{\mu'\nu'\rho'} - \frac{1}{4}e^{-\frac{\Phi}{2}}g^{\mu\mu'}g^{\nu\nu'}F^{(2)}_{\mu\nu}F^{(2)}_{\mu'\nu'} - \frac{1}{2\times 4!}e^{-\frac{3\Phi}{2}}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}g^{\sigma\sigma'}\tilde{F}^{(4)}_{\mu\nu\rho\sigma}\tilde{F}^{(4)}_{\mu'\nu'\rho'\sigma'}\Big)$$

Note: p-form field strength action

$$-\frac{1}{2K_{10}^2}\frac{1}{p!}\int d^{10}x\sqrt{-\det g}\;\tilde{F}^{(p)}_{\mu_1\cdots\mu_p}\tilde{F}^{(p)}_{\nu_1\cdots\nu_p}\,e^{\alpha\Phi}$$

$$\underset{\text{Ex.}}{=}-\frac{1}{2K_{10}^2}\int e^{\alpha\Phi}\,F^{(p)}\wedge *_g F^{(p)}$$

Gauge transformation laws:

g.c.t usual.

$$\delta B = d\underbrace{\bar{\Lambda}^{(1)}}_{1\text{ form}}$$

$$\delta C^{(1)} = d\underbrace{\Lambda^{(0)}}_{0\text{ form}}$$

$$\delta C^{(3)} = d\Lambda^{(2)} + \Lambda^{(0)}\wedge H$$

$$\delta\tilde{F}^{(4)} = \delta(dC^{(3)} - C^{(1)}\wedge H)$$

$$= d\Lambda^{(0)}\wedge H - d\Lambda^{(0)}\wedge H = 0$$

$\Rightarrow$ The action is gauge invariant.

Fermions:

$\psi_\mu^{(i)}$ $i=1,2$ $\to$

$\to$ For each μ, i, $\psi_\mu^{(i)}$ is a Majorana-Weyl spinor $\to$ 16 component real

$\psi_\mu^{(1)}, \psi_\mu^{(2)}$: opposite chirality

$\chi^{(1)}, \chi^{(2)}$: opposite chirality $\to$ dilatino

IIA: Could define:

$$C_3' = C_3 + C^{(1)} \wedge B$$

$$\delta C_3' = \delta C_3 + d\Lambda^{(0)} \wedge B + C^{(1)} \wedge d\tilde{\Lambda}^{(1)}$$

$$= d\Lambda^{(2)} + \Lambda^{(0)} \wedge H + d\Lambda^{(0)} \wedge B$$

$$+ C^{(1)} \wedge d\Lambda^{(p)}$$

$$= d\Lambda^{(2)} + d(\Lambda^{(0)} \wedge B) - d(C^{(1)} \wedge \tilde{\Lambda}^{(1)})$$

$$+ dC^{(1)} \wedge \tilde{\Lambda}^{(1)}$$

$$dC_3' = dC_3 + F^{(2)} \wedge B - C^{(1)} \wedge H$$

$$\tilde{F}^{(4)} = dC_3' - F^{(2)} \wedge B + C^{(1)} \wedge H - C^{(1)} \wedge H$$

$$= dC_3' - F^{(2)} \wedge B$$

$$\delta C_3' = d\Lambda^{(2)} + \tilde{\Lambda}^{(1)} \wedge F^{(2)}$$

Note: Φ has no potential.

$g_{\mu\nu} = \eta_{\mu\nu}$, $\Phi = c \rightarrow$ sol^{n} of arbitrary c.

$\Downarrow$

moduli field.

String theory also gives ∞ set of higher derivative corrections which we ignore for now.

__IIB__ NSNS- sector identical.

RR sector: $C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\nu\rho\sigma}$

$$F^{(1)} = dC^{(0)}, \quad F^{(3)} = dC^{(2)}$$

$$F^{(5)} = dC^{(4)}$$

$$\tilde{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)}$$

$$\tilde{F}^{(5)} = F^{(5)} - \frac{1}{2} C^{(2)} \wedge H^{(3)} + \frac{1}{2} B^{(2)} \wedge F^{(3)}$$

$$S = \underbrace{S_{NSNS}}_{\text{identical}} + S_{RR}$$

$$S_{RR} = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\det G}$$

$$\Big[\frac{1}{2} F^{(1)}_{\mu} F^{(1)}_{\nu} G^{\mu\nu} + \frac{1}{2\times 3!} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho} \tilde{F}^{(3)}_{\mu'\nu'\rho'} + \frac{1}{2\times 5!} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} G^{\sigma\sigma'} G^{\tau\tau'} \tilde{F}^{(5)}_{\mu\nu\rho\sigma\tau} \tilde{F}^{5}_{\mu'\nu'\rho'\sigma'\tau'} \Big] - \frac{1}{4\kappa_{10}^2} \int C^{(4)} \wedge H^{(3)} \wedge F^{(3)}.$$

Constraints: $\tilde{F}^{(5)} = * \tilde{F}^{(5)}$.

Gauge Transformation:

$$\delta C^{(2)} = d\Lambda^{(1)}$$

$$\delta C^{(4)} = d\Lambda^{(3)} - \frac{1}{2} \Lambda^{(1)} \wedge H^{(3)} + \frac{1}{2} \tilde{\Lambda}^{(1)} \wedge F^{(3)}$$

Canonical metric: $G_{\mu\nu} = e^{\frac{\Phi}{2}} g_{\mu\nu}$

$$S_{RR} = -\frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det g}$$

$$\Big[\frac{1}{2} e^{2\phi} g^{\mu\nu} \partial_\mu C^{(0)} \partial_\nu C^{(0)}$$

$$+ \frac{1}{2\times 3!} e^{\phi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho} \tilde{F}^{(3)}_{\mu'\nu'\rho'}$$

$$+ \frac{1}{2\times 5!} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} g^{\tau\tau'} \tilde{F}^{(5)}_{\mu\nu\rho\sigma\tau} \tilde{F}^{(5)}_{\mu'\nu'\rho'\sigma'\tau'} \Big]$$

$$\tau = C^{(0)} + i e^{-\Phi} \qquad G^{(3)} = F^{(3)} - \tau H^{(3)}$$

$$S_{IIB} = \frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det g}$$

$$\Big[R_g - \frac{1}{2(\mathrm{Im}\,\tau)^2} g^{\mu\nu} \partial_\mu \tau \partial_\nu \bar{\tau}$$

$$- \frac{1}{2(\mathrm{Im}\,\tau)} \frac{1}{3!} G^{(3)}_{\mu\nu\rho} \bar{G}^{(3)}_{\mu'\nu'\rho'} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'}$$

$$- \frac{1}{2} \frac{1}{5!} \tilde{F}^{(5)}_{\mu\nu\rho\sigma\tau} \tilde{F}^{(5)}_{\mu'\nu'\rho'\sigma'\tau'} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} g^{\tau\tau'} \Big]$$

$$- \frac{1}{4K_{10}^2} \int C^{(4)} \wedge H^{(3)} \wedge F^{(3)}$$

$$\frac{1}{2K_{10}^2} = \frac{2\pi}{l_s^8}, \quad l_s = 2\pi\sqrt{\alpha'}$$

$$\frac{1}{2K_{10}^2} = \frac{1}{(2\pi)^7 (\alpha')^4}$$

Duality symmetry:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

$$\begin{pmatrix} \mathcal{H}^{(3)} \\ F^{(3)} \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} \mathcal{H}^{(3)} \\ F^{(3)} \end{pmatrix}$$

$$\tilde{F}^{(5)} \to \tilde{F}^{(5)} \qquad g_{\mu\nu} \to g_{\mu\nu}$$

Fermions: $\psi_{\mu}^{(i)}$, $i = 1, 2$

→ Two Majorana-Weyl gravitinos of same chirality.

$\chi^{(i)}$ $i = 1, 2$ → Two Majorana Weyl dilatinos of same chirality but opposite to that of the gravitino.

$SL(2, \mathbb{Z})$ duality is conjectured to be an exact symmetry. → strong-weak coupling

$\mathcal{H}^{(3)}$ and $F^{(3)}$ are both quantized
↓ F-string charge ↘ D-string charge.

$SL(2, \mathbb{R}) \to SL(2, \mathbb{Z})$.

SO(32) and $E_8 \times E_8$ Heterotic string theory:

NSNS - sector : identical to IIA & IIB

No RR sector.

Instead we have gauge fields A_μ^a

$a = 1, \dots, 496$

\# of generators of $E_8 \times E_8$ and $SO(32)$

$248 + 248$ $\qquad$ $\frac{32 \times 31}{2}$

$$S = \frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det G}\, e^{-2\Phi}$$

$$\Big[R_G + 4 G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

$$- \frac{1}{2 \times 3!} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \tilde{\mathcal{H}}^{(3)}_{\mu\nu\rho} \tilde{\mathcal{H}}^{(3)}_{\mu'\nu'\rho'}$$

$$- \frac{C_H}{2!} G^{\mu\mu'} G^{\nu\nu'} \mathrm{Tr}_V (F_{\mu\nu} F_{\mu'\nu'}) \Big] \; ; \; \cancel{\mathrm{Tr}(T^a T^b) = \delta_{ab}}$$

$$\tilde{\mathcal{H}}^{(3)} = \underbrace{dB^{(2)}}_{\mathcal{H}^{(3)}} - C_H \, \omega^{(3)}(A)$$

$$\omega^{(3)}(A) = \mathrm{Tr}_V \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$A_\mu = A_\mu^a T^a \, , \quad F_{\mu\nu} = F_{\mu\nu}^a T^a$$

$Tr_v \to$ Trace in the vector representation of $SO(32)$

$= \frac{1}{30} Tr_A$ └ Trace in the adjoint representation of $SO(32)$

For $E_8 \times E_8$ there is no vector representation

$\Rightarrow$ Use adjoint representation

$$Tr_v = \frac{1}{30} Tr_A \quad \text{for } E_8 \times E_8.$$

$$C_{\mathcal{H}} = \frac{\alpha'}{4}$$

(Follows from string theory / supersymmetry)

Gauge trs:

$$\delta A = d\Lambda_{\mathcal{H}} + [A, \Lambda_{\mathcal{H}}]$$

$\Lambda_{\mathcal{H}} \to \Lambda_{\mathcal{H}}^a T^a$

$$\delta \omega^{(3)}(A) = d\, Tr_v (\Lambda_{\mathcal{H}} F)$$

$$\delta B^{(2)} = d\tilde{\Lambda}^{(1)} + C_{\mathcal{H}}\, Tr_v (\Lambda_{\mathcal{H}} F)$$

$$\delta \mathcal{H}^{(3)} = C_{\mathcal{H}}\, d\, Tr_v (\Lambda_{\mathcal{H}} F)$$

$$\delta \tilde{\mathcal{H}}^{(3)} = \delta \mathcal{H}^{(3)} - C_{\mathcal{H}}\, \delta \omega^{(3)}(A) = 0$$

Canonical metric $g_{\mu\nu}$

$$G_{\mu\nu} = e^{\frac{\Phi}{2}} g_{\mu\nu}$$

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\det g} \Big[R_g - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

$$- \frac{1}{2 \times 3!} e^{-\Phi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \tilde{H}^{(3)}_{\mu\nu\rho} \tilde{H}^{(3)}_{\mu'\nu'\rho'}$$

$$- \frac{c_H}{2!} e^{-\Phi/2} g^{\mu\mu'} g^{\nu\nu'} \mathrm{Tr}_V (F_{\mu\nu} F_{\mu'\nu'}) \Big]$$

Type I superstring theory:

$IIB/\Omega \to$ world-sheet parity transformation.

NS-sector: $G_{\mu\nu}$ & Φ survive.

R-sector: $C^{(2)}_{\mu\nu}$ survives. $F^{(3)} = dC^{(2)}$

We need to add 32 D-9 branes for tadpole cancellation.

$\Rightarrow$ SO(32) gauge field.

$$S_{NS-NS} = \frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det G}\, e^{-2\Phi}$$

$$[R_G + 4 G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi]$$

$$S_{RR} = -\frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det G}\, \frac{1}{2\times 3!}$$

$$G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho} \tilde{F}^{(3)}_{\mu'\nu'\rho'}$$

$$S_{Gauge} = -\frac{1}{2K_{10}^2} \int d^{10}x \sqrt{-\det G}\, e^{-\Phi}$$

$$\frac{c_I}{2!} G^{\mu\mu'} G^{\nu\nu'} \mathrm{Tr}_V (F_{\mu\nu} F_{\mu'\nu'})$$

$$\tilde{F}^{(3)} = dC^{(2)} - c_I \omega_3(A)$$

$$\Downarrow$$

$$\mathrm{Tr}_V (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

Canonical metric: $g_{\mu\nu}$

$$G_{\mu\nu} = e^{\frac{\Phi}{2}} g_{\mu\nu}.$$

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\det g} \, [R_g - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

$$- \frac{1}{2 \times 3!} e^{-\Phi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho} \tilde{F}^{(3)}_{\mu'\nu'\rho'}$$

$$- \frac{1}{2\kappa_{10}^2} \frac{c_I}{2!} \int d^{10}x \, e^{-\frac{\Phi}{2}} g^{\mu\mu'} g^{\nu\nu'} \mathrm{Tr}_v (F_{\mu\nu} F_{\mu'\nu'})]$$

$$c_I = \frac{\alpha'}{4}$$

D-string tension: $\frac{1}{2\pi\alpha' g_I}$ in string scale

$g_I = e^{\langle \Phi_I \rangle}$.

Gauge transformation:

$$\delta A = d\Lambda_I + [A, \Lambda_I]$$

$$\Lambda_I^a T^a$$

$$\delta C^{(2)} = d\Lambda^{(1)} + c_I \mathrm{Tr}(\Lambda_I F)$$

$$\delta \tilde{F}^{(3)} = \delta (dC^{(2)} - c_I \omega_3(A)) = 0$$

since $\delta \omega_3(A) = d \mathrm{Tr}_v (\Lambda_I F)$

Compare the canonical actions of heterotic & type I for same K_{10}:

sets overall scale.

Same if we choose

$$\Phi_{H} = -\Phi_{I}$$

$$B^{(2)}_{H} = C^{(2)}_{I}$$

$$A_{H} = A_{I}$$

Heterotic Type I

Conjecture: These two string theories are dual to each other.

Heterotic coupling $e^{\langle\Phi_H\rangle}$ strong - weak

Type I coupling $e^{\langle\Phi_I\rangle}$

Note: We have shifted Φ to match overall normalization.

In explicit calculation we have to take into account extra factors if this shift has not been done.

Another important classical field theory:

11-dimensional supergravity.

Fields: $\underbrace{g_{\mu\nu}, C^{(3)}_{\mu\nu\rho}}_{\text{Bosonic}}, \psi_\mu \to$ Gravitino.

32 real components.

(Majorana).

Bosonic action:

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x \sqrt{-\det g}\,\Big(R_g - \frac{1}{2}\times\frac{1}{4!} \times g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} F^{(4)}_{\mu\nu\rho\sigma} F^{(4)}_{\mu'\nu'\rho'\sigma'}\Big)$$

$$-\frac{1}{6}\int C^{(3)}\wedge F^{(4)}\wedge F^{(4)}$$

$$F^{(4)} = dC^{(3)}.$$

This does not come from the low energy limit of any string theory but is nevertheless going to be important.

Supersymmetry:

δ(boson) = fermionic combination of fields

δ(fermion) = bosonic combination of fields.

In particular: in heterotic, type I

$\delta \psi_\mu = D_\mu \epsilon^{(x)} + \underbrace{---}_{\text{terms involving other bosonic background fields.}}$

$\epsilon(x)$: Arbitrary x dependent Majorana-Weyl spinor.

Type II: $\epsilon(x) \to \epsilon^{(i)}(x)$: $i = 1, 2$.

Global supersymmetry $\to$ refers to a background.

If δ(field) = 0 for some $\epsilon(x)$ in a background then that $\epsilon(x)$ describes a global SUSY.

<u>Heterotic / type I:</u>

$g_{\mu\nu} = \eta_{\mu\nu}$, All other fields = 0.

$\delta \psi_\mu = \partial_\mu \epsilon = 0$ if $\epsilon(x)$ = constant.

$\Rightarrow$ 16 independent ϵ: $N=1$ SUSY.

Type IIA / IIB:

$\delta\psi_\mu^{(\alpha)} = D_\mu \epsilon^{(\alpha)} + \cdots \quad \alpha = 1, 2.$

$\epsilon^{(1)}, \epsilon^{(2)}$ have opposite chirality in IIA

same " in IIB

$g_{\mu\nu} = \eta_{\mu\nu}$, all other fields $= 0$.

$\delta\psi_\mu^{(\alpha)} = \partial_\mu \epsilon^{(\alpha)} = 0$ for constant $\epsilon^{(\alpha)}$.

$\epsilon^{(1)} =$ const. $\quad \epsilon^{(2)} = 0$

$\epsilon^{(2)} =$ const. $\quad \epsilon^{(1)} = 0$.

$\Rightarrow$ 32 independent components.

$N = 2$ SUSY.

(Note: SUSY is different in IIA & IIB]

IIA: opposite chirality $\epsilon^{(\alpha)}$.

IIB: same chirality $\epsilon^{(\alpha)}$.

11-d Sugra:

32 independent components $\Rightarrow N = 1$ in 11-D.

Compactification

Take the 9+1 dimensional space to be

$\mathbb{R}^{d,1} \times M_{9-d}$ → Some $(9-d)$ dimensional compact space.

⇓

$(d+1)$ dimensional Minkowski space

⇓

Could also be dS or AdS.

Physically relevant situation:

$d=3$.

We need dS but its curvature is so low that to first approximation we can look for Minkowski.

Instead to attempting to go directly at $d=3$ we shall first study compactification in a more general context.

Simplest compactification

$M_{9-d} = S^1 \quad (d=8).$

circle.

$0 \leq y < 2\pi R$

Convention: $0 \leq y < 2\pi \quad (y \equiv y + 2\pi)$

The physical radius is parametrized by G_{yy} (or g_{yy}).

Simple example: Scalar field ϕ.

~~$\partial_\mu \partial^\mu \phi = 0$~~

Convention: $\{\hat{x}^{\hat{\mu}}\}$: Full set of coordinates

$\hat{\mu} = 0, \ldots 9.$

$\{x^\mu\}$: Coordinates of $R^{d,1}$.

$\mu = 0, \ldots d$

y^m: coordinate of M_{9-d} — compact space.

~~Conv~~ In 10-d only massless states are relevant.

Consider a massless scalar ϕ:

$$\hat{\Box}\,\phi = 0$$

$\Downarrow$ in flat background $(\mathbb{R}^{8,1} \times S^1)$

$$\left(-\partial_0^2 + \vec{\nabla}^2 + g^{yy}\partial_y^2\right)\phi = 0 \qquad g_{yy} = R^2,\; g^{yy} = \frac{1}{R^2},$$

$$\phi = \sum \phi_n\, e^{iny}$$

$$\left(-\partial_0^2 + \vec{\nabla}^2 - \frac{n^2}{R^2}\right)\phi_n = 0.$$

ϕ_n : a ~~mass~~ scalar field of

$$\text{mass}^2 = \frac{n^2}{R^2}$$

We assume that R is small

$\sim l_s$

$\Rightarrow$ Only $n=0$ modes describe massless fields in 9-dim.

— Keep only this mode.

Conclusion: Massless scalar in 10-d

$\Rightarrow$ single massless scalar in 9-d.

Massless vector $\hat{A}_{\hat{\mu}}$

$$\hat{A}_y = \sum_n \hat{A}_{yn} \, e^{iny}$$

$$\hat{A}_\mu = \sum_n A_{\mu n} \, e^{iny}$$

Keep only $n=0$ mode:

$A_{y0} \rightarrow$ a massless scalar.

$A_{\mu 0} \rightarrow$ a massless vector.

Similarly from the metric:

$g_{yy} \rightarrow$ massless scalar

$g_{y\mu} \rightarrow$ massless vector

$g_{\mu\nu} \rightarrow$ metric

} in 9-dim

The action involving these 9 dimensional fields can be evaluated by taking the original action and evaluating it for y-independent fields.

This procedure is good for a quick counting of the fields we have.

e.g. 11-d sugra on $S^1 \times \mathbb{R}^{9,1}$

Original fields $g_{\hat{\mu}\hat{\nu}}$, $C^{(3)}_{\hat{\mu}\hat{\nu}\hat{\rho}}$

$$\Big\downarrow \text{ on } S^1$$

$g_{\mu\nu}$, $g_{y\mu}$, g_{yy}, $C_{\mu\nu\rho}$, $C_{y\nu\rho}$

$\downarrow$ Metric, $\downarrow$ Vector, $\downarrow$ Scalar, $\downarrow$ 3-form, $\downarrow$ 2-form

→ Same as the spectrum of IIA supergravity.

But one can actually demonstrate that the actions are equal.

$\downarrow$

For this we have to cleverly identify 11-d fields ~~&~~ to fields of IIA.

Strategy: Choose ~~...~~ ~~...~~ appropriate combination of fields whose gauge trs. laws are simple.

$$ds^2 = \tilde{g}_{\mu\nu}\, dx^\mu dx^\nu + \tilde{g}_{mn}(dy^m + A^m_\mu dx^\mu)$$
$$(dy^n + A^n_\nu dx^\nu)$$

$\Rightarrow$ The ~~[illegible]~~ ordinal metric is:

$$\underbrace{\left(\tilde{g}_{\mu\nu} + A^m_\mu A^n_\mu \tilde{g}_{mn}\right)}_{g_{\mu\nu}} dx^\mu dx^\nu$$
$$+ \tilde{g}_{mn} A^m_\mu dy^n dx^\mu + \tilde{g}_{mn} A^n_\nu dy^m dx^\nu$$
$$+ \tilde{g}_{mn} dy^m dy^n$$

$$g_{n\mu} = \tilde{g}_{mn} A^m_\mu$$

$$g_{mn} = \tilde{g}_{mn}$$

Why this parametrization?

Consider y independent g.c.t.

$$y^m \to y^m + f^m(x)$$
$$x^\mu \to \xi^\mu(x)$$

$$ds^2 = \tilde{g}_{\mu\nu}\, \partial_\rho \xi^\mu \partial_\sigma \xi^\nu dx^\rho dx^\sigma$$
$$+ \tilde{g}_{mn}\,(dy^m + \partial_\mu f^m dx^\mu$$
$$+ A^m_\mu \partial_\rho \xi^\mu dx^\rho)(dy^n + \partial_\nu f^n dx^\nu$$
$$+ A^n_\nu \partial_\sigma \xi^\nu dx^\sigma)$$

$\cancel{\tilde{g}'_{\mu\nu}\, dx^\mu\, dx^\nu}$

$\tilde{g}_{\rho\sigma} \to \tilde{g}_{\mu\nu}\, \partial_\rho \xi^\mu\, \partial_\sigma \xi^\nu$

$A_\rho \to \partial_\rho \xi^\mu A^m_\mu + \partial_\rho f^m$

$\tilde{g}_{mn} \to \tilde{g}_{mn}.$

With this parametrization,

$\left.\begin{array}{ll} \tilde{g}_{\rho\sigma} & : \text{tensor of rank 2} \\ A_\rho & : \text{vector} \end{array}\right\}$ under g.c.t.

$\tilde{g}_{mn}$: scalar

$A_\rho \to A_\rho + \partial_\rho f^n$: gauge trs. law of A_ρ.

→ standard transformations.

→ Makes it easy to compare.

11-d sugra on $(S^1)^{9-d}$:

$$C^{(3)} = \frac{1}{6}\tilde{C}^{(3)}_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$+ \frac{1}{6}\times 3\ \tilde{C}^{(3)}_{m\nu\rho}\left(dy^m + A^m_\mu dx^\mu\right)\wedge dx^\nu \wedge dx^\rho$$

$$+ \frac{1}{6}\times 3\ \tilde{C}^{(3)}_{mn\rho}\left(dy^m + A^m_\mu dx^\mu\right)$$

$$\wedge\left(dy^n + A^n_\nu dx^\nu\right)\wedge dx^\rho$$

$$+ \frac{1}{6}\ \tilde{C}^{(3)}_{mnp}\left(dy^m + A^m_\mu dx^\mu\right)\wedge\left(dy^n + A^n_\nu dx^\nu\right)$$

$$\wedge\left(dy^p + A^p_\rho dx^\rho\right).$$

Under $y^m \to y^m + f^m(x)$

$dy^m + A^m_\mu dx^\mu$ is invariant.

$\Rightarrow$ With this defn.

$\tilde{C}^{(3)}_{\mu\nu\rho}$, $\tilde{C}^{(3)}_{m\nu\rho}$, $\tilde{C}^{(3)}_{mn\rho}$ & $\tilde{C}^{(3)}_{mnp}$ are gauge neutral.

3-form, 2-form, 1-form, scalar.

Useful for making field identification between different formulations.

• Gauge trs. parameter $\Lambda^{(2)}$.

$$\Lambda^{(2)} = \frac{1}{2}\tilde{\Lambda}^{(2)}_{\mu\nu}\, dx^\mu \wedge dx^\nu + \tilde{\Lambda}^{(2)}_{m\mu}\,(dy^m + A_\nu dx^\nu) \wedge dx^\mu + \tilde{\Lambda}^{(2)}_{mn}(dy^m + A^m_\mu dx^\mu) \wedge (dy^n + A^n_\nu dx^\nu)$$

$$\delta C^{(3)} = d\Lambda^{(2)}$$

$$= \frac{1}{6}(d\tilde{\Lambda}^{(2)})_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$+ \partial_\rho \tilde{\Lambda}^{(2)}_{m\mu}\, dx^\rho \wedge (dy^m + A_\nu dx^\nu) \wedge dx^\mu$$

$$+ \tilde{\Lambda}^{(2)}_{m\mu}\, \partial_\rho A_\nu\, dx^\rho \wedge dx^\nu \wedge dx^\mu$$

$$+ \partial_\rho \tilde{\Lambda}^{(2)}_{mn}\, dx^\rho \wedge (dy^m + A^m_\mu dx^\mu) \wedge (dy^n + A^n_\nu dx^\nu)$$

$$+ \tilde{\Lambda}^{(1)}_{mn} \wedge \partial_\rho A^m_\mu\, dx^\rho \wedge dx^\mu \wedge (dy^n + A^n_\nu dx^\nu)$$

$$+ \tilde{\Lambda}^{(2)}_{mn}(dy^m + A^m_\mu dx^\mu) \wedge \partial_\sigma A^n_\nu\, dx^\sigma \wedge dx^\nu$$

$$\Rightarrow \delta \tilde{C}^{(3)} = d\tilde{\Lambda}^{(2)} + \tilde{\Lambda}^{(1)}_m \wedge F^{(2)m}$$

$$\delta \tilde{C}^{(2)}_n = -d\tilde{\Lambda}^{(1)}_m + \tilde{\Lambda}^{(0)}_{mn} \cdot F^{(2)m}$$

$$\delta \tilde{C}^{(1)}_{mn} = d\tilde{\Lambda}^{(2)}_{mn}$$

on S^1: $\delta\tilde{C}^{(3)} = d\tilde{\Lambda}^{(2)} + \tilde{\Lambda}^{(1)} \wedge F^{(2)}$

$\delta\tilde{C}^{(2)} = -d\tilde{\Lambda}^{(1)}$

$\tilde{C}^{(1)}$ does not exist.

Recall in IIA string theory:

$\delta B = d\tilde{\Lambda}^{(1)}$ → agrees with $\tilde{C}^{(2)}$

$\delta C^{(1)} = d\Lambda^{(0)}$ → agrees with A_μ

$\delta C^{(3)} = d\Lambda^{(2)} + \Lambda^{(0)} H.$

→ does not agree with $\tilde{C}^{(3)}$ from sugra.

$C'^{(3)} = C^{(3)} + C^{(1)} \wedge B$

$\delta C'^{(3)} = d\Lambda^{(2)} + \bar{\Lambda}^{(1)} \wedge F^{(2)}$ ~

$C'^{(3)}$ of IIA $= \tilde{C}^{(3)}$ of 11-d sugra.

Ex. Check that the two actions are equal if we choose:

$$ds^2_{11} = e^{-\frac{2\Phi}{3}} \underbrace{G_{\mu\nu}}_{\text{IIA string metric}} dx^\mu dx^\nu$$

$$+ e^{\frac{4\Phi}{3}} (dy + A^{(1)}_\mu dx^\mu)^2$$

$$\sqrt{\det g}\, g^{\mu\nu} = \sqrt{e^{-10\times\frac{2\Phi}{3} + \frac{4\Phi}{3}}}\; e^{+\frac{2\Phi}{3}} = e^{-\frac{8\Phi}{3} + \frac{2\Phi}{3}} = e^{-2\Phi}$$

Note: 11-d limit ⇒ G_{yy} large

⇒ Φ large ⇒ IIA strongly coupled.

Conjecture: strongly coupled IIA ⇒ 11-d sugra.

Other duality relations:

$\mathrm{II}A$ on S^1 = $\mathrm{II}B$ on S^1.

$$G^{\mathrm{II}A}_{yy} = 1/G^{\mathrm{II}B}_{yy} \qquad \alpha' = 1$$

$$e^{-2\Phi_{\mathrm{II}A}} \sqrt{G^{\mathrm{II}A}_{yy}} = e^{-2\Phi_{\mathrm{II}B}} \sqrt{G^{\mathrm{II}B}_{yy}}$$

Note: Small $\Phi_{\mathrm{II}A} \Rightarrow$ Small $\Phi_{\mathrm{II}B}$

→ Weak coupling duality:

→ testable in string perturbation theory.

Small $G^{\mathrm{II}A}_{yy}$ ⇒ small radius.

⇓

Large $G^{\mathrm{II}B}_{yy}$ ⇒ Large radius.

Surprising in the full theory!

Expectation: Small radius ⇒ effectively (8+1)-dim theory.

Large radius ⇒ effectively (9+1) dim theory

How can the two be equivalent?

A more precise formulation of the puzzle:

Recall that the n-th Fourier mode of ~~the~~ a ~~massless~~ massless particle in (9+1) d.

$\Rightarrow$ a particle of mass $\frac{n}{R}$ is the (8+1) d theory

∞ as $R \to 0$.

$\Rightarrow$ Only a finite # of light modes remain.

(One from each field).

In $R \to \infty$ limit: they become massless.

$\Rightarrow \infty$ # of light modes in (8+1) d.

How can the two theories be the same?

Stringy magic (T-duality).

String wound along circle n times has mass $\propto nR$

As $R \to \infty$ they become light.

Fourier modes of the dual theory.

Other duality relations in (8+1) dim.

$E_8 \times E_8$ heterotic on S^1 of radius R

$= E_8 \times E_8$ " " " " " α'/R.

Same for $SO(32)$ heterotic.

Both the theories on S^1 have additional moduli fields.

$A^a_y \neq 0$ ~~for an Cartan algebra~~

We can choose A^a_y s.t.

$E_8 \times E_8 \to SO(16) \times SO(16)$

$SO(32) \to SO(16) \times SO(16)$

In each case choose $\langle A^a_y \rangle$ such that it is half-way towards the full period.

$$\underbrace{e^{i\int A^a_y T^a dy}}_{\text{a group element which squares to identity.}} \cancel{= \text{identity} \Rightarrow \text{no effect}}.$$

In this case: $E_8 \times E_8$ on circle of radius R = $SO(32)$ heterotic on circle of radius α'/R.

Another stringy surprise: Consider either $SO(32)$ or $E_8 \times E_8$ heterotic on S^1 with $A^a_y = 0$.

At $R = \sqrt{\alpha'}$ the gauge group

$\underset{\text{from } G_{y\mu}}{E_8 \times E_8 \times U(1)}$ is enhanced to $E_8 \times E_8 \times SU(2)$

Same for $SO(32)$ heterotic.

Some stringy winding modes become massless to provide $W^\pm$ bosons of $SU(2)$.

Combine the duality trs:

M theory on $S^1 \times S^1$ = IIA on S^1

= IIB on S^1.

$SL(2,\mathbb{Z})$ of IIB $\equiv$ geometric trs.

$$\begin{pmatrix} y^8 \\ y^9 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y^8 \\ y^9 \end{pmatrix} \qquad \begin{matrix} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{matrix}$$

Compactification on T^6:

Type IIB/IIA: 32 supercharges

= $\mathcal{N}=8$ SUSY.

Heterotic; Type I: 16 supercharges

= $\mathcal{N}=4$ SUSY

Action: Can be worked out by dimensional reduction.

— Also fixed by SUSY.

Good 4-d theories but not phenomenologically interesting.

→ No chiral fermions.

A 10-d spinor (16-component Majorana-Weyl)

→ 4 4-d Majorana spinors.

→ left-right symmetric interaction with gauge fields.

(Form real representation).