Clarification on Axiom-detector eq:

\[ S \propto \int d\tau \sqrt{e^{\tau} + \frac{e^{-\tau}}{c-2}} \]

Vary \( \tau \):

\[ S^2 = \int \sqrt{e^{\tau} \left[ \frac{\partial}{\partial \tau} \left( \frac{e^{\tau}}{c-2} \right) + \frac{\partial}{\partial \tau} \left( \frac{e^{-\tau}}{c-2} \right) \right]} \]

\[ = \int \sqrt{e^{\tau} \left[ \frac{1}{(c-2)^2} \left( e^{2\tau} + 2e^{\tau} \omega \nu + e^{-\tau} \omega \nu \right) \right]} \frac{e^{\tau}}{c-2} \]

\[ = \int d\tau \left[ -\frac{e^{\tau}}{c-2} \left( \frac{e^{\tau}}{c-2} \right) + \frac{2e^{\tau} \omega \nu}{c-2} \right] \frac{e^{\tau}}{c-2} \]

\[ \Rightarrow \frac{\omega^2}{c^2} - \frac{2}{c^2} \omega \nu = 0 \]

Flat metric, holomorphic \( \tau \):

\[ \frac{\partial^2}{\partial \tau^2} = 0 \quad \frac{\partial^2}{\partial \tau^2} \omega + 2 \frac{\partial}{\partial \tau} \omega = 0 \]

\[ \Rightarrow \text{holomorphic } \tau \text{ satisfies eq. of motion}. \]
Some clarification about the coordinate system:

**Metric:** Flat in torus coordinates $\omega$.

- Identified under $\omega \to -\omega$.

Recall $ds^2 = |d\omega|^2$.

Near $\omega = 0$, single valued coordinate is $z^2 = \omega^2$, $\omega = z^{1/2}$

$z^2 \to 0 \Rightarrow z \to 0 \Rightarrow \omega \to 0$ identified.

$ds^2 = |d\omega|^2 = \frac{1}{4} \frac{dz}{\sqrt{z^{1/2} - \omega^{1/2}}} \frac{d\bar{z}}{\sqrt{z^{1/2} - \omega^{1/2}}}

Not relevant.

On tetrahedron:

$ds^2 = \frac{1}{4} \sum_{x=1}^{4} \frac{dz_x d\bar{z}_x}{(z_x - \omega_x)^{1/2}(\bar{z}_x - \bar{\omega}_x)^{1/2}}$

$\omega_1 \to \omega_4$: Locations of 07-planes.

Note: as $z \to \infty$, $ds^2 = \frac{dz d\bar{z}}{z^{1/2} \bar{z}^{1/2}} = dy d\bar{y}$, $y = \frac{1}{2}$.

- Metric is regular at $\infty$.

In $z$-coordinate system:

$z = \frac{1}{2^{1/5}} \left( \frac{16}{\sqrt{2} \sqrt{2}} \left( \frac{z}{z_0} + i \frac{\omega}{\sqrt{2} \omega_0} \right) \right)$
F-theory:

Consider some Calabi-Yau manifold $M$ of $2n$ real dimensions which is "elliptically fibered".

→ A fiber bundle with base a complex manifold $B$ and the fiber a torus $T^2$ with modular parameter $\tau$.
→ $\tau$ varies holomorphically with the base coordinate $z$.

→ $T^2$ can degenerate at codimension 2 subspace when $\tau \to \infty$ or one of its $SL(2,\mathbb{Z})$ images.

(Total space can still be non-singular)

→ Consider M-theory on $M$.

Locally M-theory on $T^2 \cong TB$ in $S^4$:

$m^2$ site $\to 0$ $\Rightarrow$ $S^1$ site $\to \infty$. 

Thus M-theory on $M$

$\approx$ IIB on a manifold $N$ obtained by fiber

$S^1$ over $B$.

- axion-dilaton modulus of IIB
- varies as we move on the base.

Now consider $M$ and adjust its Kahler modulus such that $H^2$ site $\to 0$ at every point in $B$.

$\Rightarrow$ in IIB, $S^1$ site $\to \infty$ at every point in $B$.

$\Rightarrow$ effectively IIB on $B$ with $\tau$ varying over $B$.

$\approx$ effectively $(2n-2)$ dimensional.

$\Rightarrow$ F-theory compactification.

$\Rightarrow$ A mechanism by which we use $2n$-dim. elliptically fibered (y to smooth a compactification of IIB on a $(2n-2)$-dim

manifold.
Example: $C_{Y_1} = T^2 \Rightarrow$ Base is a point.

$C_{Y_2} = k^3$

Not all $k^3$ are elliptically fibered but a subset is.

Base: $S^2 = \mathbb{CP}^1$

Coordinate $z$.

$z$ varies over $T$

$z(t)$ is not arbitrary but chosen so that the total space is $k^3$.

How to do this?

Weierstrass form of $T^2$:

A torus can be described by the eq:

$$y^2 = x^3 + f(x) + g$$

Constants

$x, y$: Complex variables $\in \mathbb{C}$

1 eq. in 2 variables $\Rightarrow 1$ complex dim.
\[ y^2 = (x-x_1)(x-x_2)(x-x_3) \]

Away from \( x_1, x_2, x_3 \), \( x \) is a local good coordinate system.

Near \( x = x_i \), we have two points for each \( x \). (\( x, y \) & \( x, -y \))

\[ \square \text{ A sphere with 4 square root branch points at } x_1, x_2, x_3, x. \]

\[ \mu \quad \mu \quad \mu \quad \mu \]

\[ \square \text{ Riemann surface with two branches. } \Rightarrow \text{ a torus.} \]

8. How is \( \eta \) related to \( f \) and \( g \)?

\[ j(\tau) = \frac{4 \cdot (24f)^3}{27g^2 + 4f^3} \]

\[ \frac{\Theta_2(\tau)^8 + \Theta_3(\tau)^8 + \Theta_4(\tau)^8}{\eta(\tau)^{24}} \rightarrow q^{-1} \text{ as } q = e^{2\pi i \tau} \rightarrow 0 \]

\[ j(i) = (24)^3. \quad j(e^{i\pi/3}) = 0 \]
Now consider an elliptically fibred K3.

t(t) varies as a fn of t

⇒ f(t), g(t) vary as fn of t.

\[ y^2 = x^3 + f(t)x + g(t) \]

One eq in 3 variables  ⇒ 2 complex dim manifold.

Under what condition does this describe a K3 (CICY)?

Ans: f(t) in a polynomial of degree 8 and g(t) is a polynomial of degree 12.

Proof: Use homogeneous coordinates on CP^1

\[ z = \frac{z_1}{z_2} \]

\[ z_1^{12} \cdot y^2 = x^3 \cdot z_2^{12} + z_2^{12} \cdot x \cdot f(z_1, z_2) + g(z_1, z_2) \]

\[ y = z_2^6 \cdot y, \quad x = z_2^4 \cdot y \]

\[ y^2 \cdot \tilde{z}^3 + \tilde{z} \cdot f(z_1, z_2) + g(z_1, z_2) \]

Scaling \((z_1, z_2, \tilde{z}, y) \Rightarrow (\lambda z_1, \lambda z_2, \lambda^{12} \tilde{z}, \lambda^6 y)\)
A degree $d$ eq. in weighted projective space of weight $(\omega_1, \ldots, \omega_n)$:

$$C_i^0 = \text{Coefficient of } x^i \text{ in }$$

$$\sum \frac{1}{(1 + \omega_i)} \left( \frac{1}{1 + dJ} \right)^{\omega_i - d}$$

Here $\sum \omega_i = 1 + 1 + 4 + 6 = 12$

$d = 12$

$\Rightarrow C_1 = 0$.

$$j(x^2) = \frac{\Delta^3}{\det^3 + 27 \Delta}$$

$$\Delta(x) = 9 f(x)^3 + 27 g(x)^2$$

Zeros of $\Delta$ are poles of $j$ as fiber degenerates.

How many such zeroes?

$\Delta : \deg 24 \Rightarrow 24$ zeroes.

We shall see that degenerate fiber $\Rightarrow 7$ branch locations on $B$ (not necessarily D7).
Now consider a special case:

\[ f(t) = a \prod_{j=1}^4 \left( t - \omega_j \right)^2 \quad \Delta = (4a^3 + 27b^2) \]

\[ g(t) = b \prod_{j=1}^3 \left( t - \omega_j \right)^3 \quad \Delta = (2 - \omega_j)^6 \]

\[ j(\tau) = \frac{4 \cdot (2\pi a)^3}{4a^3 + 27b^2} \Rightarrow \tau \text{ independent.} \]

\[ \tau(t) \] is constant on \( S^2 \).

Monodromy: What happens if \( t \) goes around a \( \omega_j \)?

\[ y^2 = x^3 + 4f(t)x + g(t) \]

\( t \) \textbf{saturates eq.}

\[ z = z_0 + \epsilon \in \mathbb{C} \]

\[ f(t) = \epsilon^2, \quad z = z_0 + \epsilon \quad g(t) = z_0 + \epsilon \]

To satisfy the eqn.

\[ x = z_0 + K_3 \epsilon \in \mathbb{C} \quad y = K_4 \epsilon \in \mathbb{C} \]

As \( \phi \to \phi + 2\pi \), \( z \to z \), \( x \to x \), \( y \to -y \)
What is the $y = -y$ on $\mathbb{T}^2$?

Reversing the sign of both coordinates:

$\mathbb{SL}(2, \mathbb{Z}) \times \mathbb{R} \setminus \{0\} \cong \mathbb{C} \setminus \{0\}$

Metric on the base:

$$ds^2 \propto \frac{dz \, d\bar{z}}{(z - \frac{1}{z})^{1/2} (z - \frac{1}{w})^{1/2} (\frac{w}{z})^{1/2}}$$

- In general,

- In this special case $\Delta$, the tensor of $\Delta$ are grouped in 6 at the place.

$$ds^2 \propto \frac{dz \, d\bar{z}}{(z - \frac{1}{z})^{1/2} (z - \frac{1}{w})^{1/2} (\frac{w}{z})^{1/2} (\frac{w}{z})^{1/2}}$$

$\Rightarrow$ Tetrahedron.

Thus at this point in the moduli space $F$-theory on elliptically fibered $K3 \equiv \Omega$ II B on $\mathbb{T}^2 / \{-1\}^2 \mathbb{Z}_2$

What happens when we deform it?